# Quasi-Universality of Reeb Graph Distances

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## Abstract

We establish bi-Lipschitz bounds certifying quasi-universality (universality up to a constant factor) for various distances between Reeb graphs: the interleaving distance, the functional distortion distance, and the functional contortion distance. The definition of the latter distance is a novel contribution, and for the special case of contour trees we also prove strict universality of this distance. Furthermore, we prove that for the special case of merge trees the functional contortion distance coincides with the interleaving distance, yielding universality of all four distances in this case.

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## 1 Introduction

The *Reeb graph* is a topological signature of real-valued functions, first considered in the context of Morse theory [11] and subsequently applied to the analysis of geometric shapes [9, 13]. It describes the connected components of level sets of a function, and for Morse functions on compact manifolds or PL functions on compact polyhedra it turns out to be a finite topological graph with a function that is monotonic on the edges. If the domain of the function is simply-connected, then the Reeb graph is contractible, hence a tree, and is therefore often called a *contour tree*. In topological data analysis, Reeb graphs are used for surveying functions, and also in a discretized form termed *Mapper* [14] for the analysis of point cloud data, typically high-dimensional or given as an abstract finite metric space.

An important requirement for topological signatures is the ability to quantify their similarity, which is typically achieved by means of an extended pseudometric on the set of isomorphism classes of signatures under consideration, referred to as a distance. In order for such a distance to be practical, it should be resilient to noise and perturbations of the input data, which is formalized by the property of *stability*: small perturbations of the data lead to small perturbations of the signature. Mathematically speaking, the signature is a Lipschitz-continuous map between metric spaces, and often the Lipschitz constant is assumed to be 1, meaning that the map is non-expansive. Previous examples of distances between Reeb graphs satisfying stability include the *functional distortion distance* [2], the *interleaving distance* [8], and the *Reeb graph edit distance* [4]. While stability guarantees that similarity


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of data sets is preserved, it does not provide any guarantees regarding the *discriminativity* of the distance on the signature. Indeed, a certain loss of information is inherent and even desired for most signatures; in fact, a key strength of topological signatures is their invariance to reparametrizations or isometries of the input data, independent of the metric used to distinguish non-isomorphic signatures. Thus, given any signature map defined on some metric space of possible data, such as the space of real-valued functions on a fixed domain with the uniform metric, one stable distance is considered more discriminative than another if it assigns larger or equal distances to all possible pairs of signatures. For example, the functional distortion distance is an upper bound for the interleaving distance and thus more discriminative in that sense; in fact, the two distances are bi-Lipschitz equivalent [5]. One may now ask if a given distance is *universal*, meaning that it is both stable and an upper bound for all stable distances, and thus the most discriminative among all stable distances. This can be expressed by the universal property of a quotient metric [7, 12], giving rise to the name “universal”. Since there is only one such distance, we refer to it as *the* universal distance. Perhaps surprisingly, neither the interleaving distance nor the functional distortion distance are universal, while the *Reeb graph edit distance* [4] turns out to be universal.

These results raise the question of whether the mentioned distances are *quasi-universal*, i.e., bi-Lipschitz equivalent to the universal distance. We address this question by proving lower and upper Lipschitz bounds relating all three mentioned distances, together with the novel *functional contortion distance*, a slight variation of the functional distortion distance. It has a simple definition, is more discriminative than the functional distortion and interleaving distances while still being stable, and in fact coincides with the universal distance when restricted to contour trees, as we also show in this paper. Furthermore, we show that the interleaving distance of merge trees, considered as a special case of Reeb graphs, coincides with the functional contortion distance, establishing the universality of all four distances in this particular setting.

Previous results relating the distances considered in this paper were obtained in [2, 3, 5, 8]. We discuss these results in detail in Section 2. In [4], an edit distance is introduced and shown to be universal, and an example is given showing that the functional distortion distance is not universal. Furthermore, in [6], an $\ell^p$-generalization of the interleaving distance is introduced for unbounded merge trees with finitely many nodes, for all $p \in [1, \infty]$. It satisfies a universal property analogous to the one considered. The case $p = \infty$ yields a variant of our universality result for the interleaving distance between unbounded merge trees with finitely many nodes.

## 2 Preliminaries

▶ **Definition 1** (Reeb graph). *A Reeb graph is a pair $(F, f)$ where $F$ is a non-empty connected topological space and $f\colon F \to \mathbb{R}$ a continuous function, such that $F$ admits the structure of a 1-dimensional CW complex for which*

- *$f$ restricts to an embedding on each 1-cell, and*
- *for every bounded interval $I \subset \mathbb{R}$, the preimage $f^{-1}(I)$ intersects a finite number of cells.*

We often refer to $F$ as a Reeb graph without referring explicitly to the function $f$. A morphism (isomorphism) of Reeb graphs is a value-preserving continuous map (homeomorphism).

▶ Remark 2. Suppose $(F, f)$ is a Reeb graph, let $I \subseteq \mathbb{R}$ be a closed interval, and fix the structure of a CW complex on $F$ as in Definition 1. As we may subdivide any 1-cell whose interior intersects $f^{-1}(\partial I)$, the preimage $f^{-1}(I)$ also admits the structure of a CW complex. Thus, the preimage $f^{-1}(I)$ is locally path-connected and therefore the connected components and the path-components of $f^{-1}(I)$ coincide.

▶ **Definition 3** (Contour tree). *A contour tree is a contractible Reeb graph.*

Contour trees further specialize to *merge trees*, which can be thought of as upside down trees in the sense that its root is at the top and its branches grow from top to bottom.

▶ **Definition 4** (Merge tree). *A* merge tree *is a Reeb graph $(F, f)$, such that $F$ admits the structure of a 1-dimensional CW complex as in Definition 1 with the additional property that each 0-cell is the lower boundary point of at most a single 1-cell.*

We note that this definition allows for both unbounded merge trees, which is an implied necessity of the definition in [10], and bounded merge trees.

▶ **Definition 5** (Induced Reeb graph). *Let $X$ be a topological space, $f\colon X \to \mathbb{R}$ a continuous function. Let $\mathcal{R}X$ be the quotient space $X/{\sim_f}$, with $x \sim_f y$ iff $x$ and $y$ belong to the same connected component of some level set of $f\colon X \to \mathbb{R}$, and let $q_{\mathcal{R}X}\colon X \to \mathcal{R}X$ be the natural quotient map, and let $\mathcal{R}f\colon \mathcal{R}X \to \mathbb{R}$ be the unique continuous map such that the diagram*

$$\begin{tikzcd} X \arrow[rr, bend left, "q_{\mathcal{R}X}"] \arrow[dr, "f"'] & & \mathcal{R}X \arrow[dl, "\mathcal{R}f"] \\ & \mathbb{R} & \end{tikzcd}$$

*commutes. If $(\mathcal{R}X, \mathcal{R}f)$ is a Reeb graph, we say that it is the Reeb graph* induced by $(X, f)$.

We say that two or more points are *connected in* some space if there is a connected component of that space containing all those points. For $t \in \mathbb{R}$ and $\delta \geq 0$, let

$$I_\delta(t) := [t - \delta, t + \delta] \subset \mathbb{R}$$

denote the closed interval of radius $\delta$ centered at $t$.

Let $f\colon X \to \mathbb{R}$ be a continuous function and let $\delta \geq 0$. We define the $\delta$-*thickening* of $X$ as

$$\mathcal{T}_\delta X := X \times [-\delta, \delta], \qquad \mathcal{T}_\delta f\colon \mathcal{T}_\delta X \to \mathbb{R},\ (p, t) \mapsto f(p) + t.$$

Moreover, let

$$\tau_X^\delta\colon X \to \mathcal{T}_\delta X,\ p \mapsto (p, 0)$$

be the natural embedding of $X$ into its $\delta$-thickening. Now let $g\colon G \to \mathbb{R}$ be a Reeb graph and consider its $\delta$-thickening $\mathcal{T}_\delta g\colon \mathcal{T}_\delta G \to \mathbb{R}$. We define the $\delta$-*smoothing* of $G$ as

$$\mathcal{U}_\delta G := \mathcal{R}\mathcal{T}_\delta G, \qquad \mathcal{U}_\delta g := \mathcal{R}\mathcal{T}_\delta g\colon \mathcal{R}\mathcal{T}_\delta G \to \mathbb{R}.$$

Moreover, let

$$q_{\mathcal{U}_\delta G}\colon \mathcal{T}_\delta G \to \mathcal{U}_\delta G = \mathcal{R}\mathcal{T}_\delta G$$

be the natural quotient map as in Definition 5, and let

$$\zeta_G^\delta\colon G \xrightarrow{\tau_G^\delta} \mathcal{T}_\delta G \xrightarrow{q_{\mathcal{U}_\delta G}} \mathcal{U}_\delta G$$

be the natural map of $G$ into its $\delta$-smoothing, which is the composition of $\tau_G^\delta$ and $q_{\mathcal{U}_\delta G}$.

Now suppose $f\colon F \to \mathbb{R}$ is another Reeb graph and that $\phi\colon F \to \mathcal{U}_\delta G$ is a continuous map. Identifying points in $\mathcal{U}_\delta G = \mathcal{R}\mathcal{T}_\delta G$ with subsets of $\mathcal{T}_\delta G$ via the quotient map $q_{\mathcal{R}G}\colon \mathcal{T}_\delta G \to \mathcal{U}_\delta G$, the map $\phi$ induces a set-valued map

$$\Phi := \mathrm{pr}_G \circ \phi\colon F \to \mathcal{P}(G)$$

from $F$ to the power set of $G$, where $\mathrm{pr}_G\colon \mathcal{T}_\delta G = G \times [-\delta, \delta] \to G$, $(q,t) \mapsto q$ is the projection onto $G$. Moreover, suppose $\psi\colon G \to \mathcal{U}_\delta F$ is another continuous map, and define the set-valued map analogously as

$$\Psi := \mathrm{pr}_F \circ \psi\colon G \to \mathcal{P}(F).$$

▶ **Definition 6** (Interleaving distance $d_I$ [8]). *We say that the pair of maps $\phi\colon F \to \mathcal{U}_\delta G$ and $\psi\colon G \to \mathcal{U}_\delta F$ is a* $\delta$-interleaving *of $(F, f)$ and $(G, g)$ if the triangles*

$$\begin{tikzcd} F \arrow[rr, bend left, "\phi"] \arrow[dr, "f"'] & & \mathcal{U}_\delta G \arrow[dl, "\mathcal{U}_\delta g"] \\ & \mathbb{R} & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[rr, bend left, "\psi"] \arrow[dr, "g"'] & & \mathcal{U}_\delta F \arrow[dl, "\mathcal{U}_\delta f"] \\ & \mathbb{R} & \end{tikzcd}$$

*commute and the following two conditions are satisfied:*

- *For any $x \in F$, $x$ and $\Psi(\Phi(x))$ are connected in $f^{-1}(I_{2\delta}(f(x)))$.*
- *For any $y \in G$, $y$ and $\Phi(\Psi(y))$ are connected in $g^{-1}(I_{2\delta}(g(y)))$.*

*The* interleaving distance*, denoted $d_I(F, G)$, is defined as the infimum of the set of $\delta$ admitting a $\delta$-interleaving between $(F, f)$ and $(G, g)$.*

Note that for any $t \in \mathbb{R}$ the map $f^{-1}(I_\delta(t)) \to (\mathcal{T}_\delta f)^{-1}(t)$ given by $x \mapsto (x, t - f(x))$ is a homeomorphism, with the inverse given by the restriction of $\mathrm{pr}_F\colon \mathcal{T}_\delta F \to F$. In particular, the points of $\mathcal{U}_\delta F$, which are the connected components of level sets of $\mathcal{T}_\delta f$, are in bijection with connected components of interlevel sets of $f$. Hence, the connectedness condition for an interleaving is equivalent to requiring that $\psi_\delta \circ \phi = \tau_F^{2\delta}$ and $\phi_\delta \circ \psi = \tau_G^{2\delta}$, where $\phi_\delta$ is the induced map $\mathcal{U}_\delta F \to \mathcal{U}_{2\delta} G$, $[x, s] \mapsto [\phi(x), s]$ and similarly for $\psi_\delta$.

▶ **Definition 7** (Functional distortion distance $d_{FD}$ [2]). *Let $(F, f)$ and $(G, g)$ be two Reeb graphs. Given a pair $(\phi, \psi)$ of maps $\phi\colon F \to G$ and $\psi\colon G \to F$, consider the correspondence*

$$C(\phi, \psi) = \{(x, y) \in F \times G \mid \phi(x) = y \text{ or } x = \psi(y)\}.$$

*The pair $(\phi, \psi)$ is a* $\delta$-distortion pair *if $\|f - g \circ \phi\|_\infty \le \delta$, $\|f \circ \psi - g\|_\infty \le \delta$, and for all $(x, y), (x', y') \in C(\phi, \psi)$ we have*

$$|d_f(x, x') - d_g(y, y')| \le 2\delta,$$

*where $d_f(x, x')$ denotes the infimum length of any interval $I$ such that $x$ and $x'$ are connected in $f^{-1}(I)$, and similarly for $d_g$. The* functional distortion distance*, denoted $d_{FD}(F, G)$, is defined as the infimum of all $\delta$ admitting a $\delta$-distortion pair between $(F, f)$ and $(G, g)$.*

▶ **Definition 8** (Functional contortion distance $d_{FC}$). *Let $(F, f)$ and $(G, g)$ be two Reeb graphs. A pair $(\phi, \psi)$ of maps $\phi\colon F \to G$ and $\psi\colon G \to F$ is a* $\delta$-contortion pair *between $(F, f)$ and $(G, g)$ if the following symmetric conditions are satisfied.*

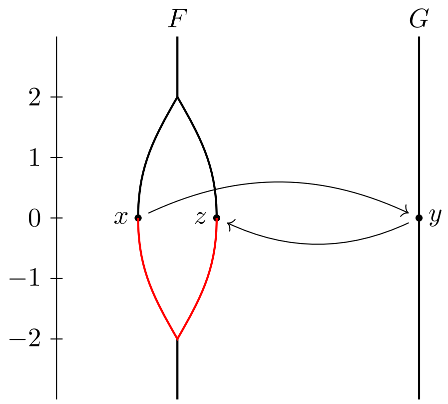



**Figure 1** Reeb graphs $(F, f)$ and $(G, g)$. If $\phi\colon F \to G$ and $\psi\colon G \to F$ are a 1-distortion pair, we allow $\phi(x) = y$ and $\psi(y) = z$ because there is a path from $x$ to $z$ in $f^{-1}[-2, 0]$. However, in this case $(\phi, \psi)$ is not a 1-contortion pair, because $x$ and $z$ are not connected in $f^{-1}(I_1(g(y))) = f^{-1}[-1, 1]$.

- *For any $x \in F$ and $y \in \psi^{-1}(x)$, the points $\phi(x)$ and $y$ are connected in $g^{-1}(I_\delta(f(x)))$.*
- *For any $y \in G$ and $x \in \phi^{-1}(y)$, the points $\psi(y)$ and $x$ are connected in $f^{-1}(I_\delta(g(y)))$.*

*The* functional contortion distance*, denoted $d_{FC}(F, G)$, is defined as the infimum of the set of $\delta$ admitting a $\delta$-contortion pair between $(F, f)$ and $(G, g)$.*

The definition of $d_{FC}$ is arguably easier to work with than that of $d_{FD}$, since to verify that $(\phi, \psi)$ is a $\delta$-contortion pair, one only has to check one condition for each element of $C(\phi, \psi)$, while to verify that $(\phi, \psi)$ is a $\delta$-distortion pair, one needs to check a condition for each pair of elements of $C(\phi, \psi)$. We prove that $d_{FC}$ satisfies the triangle inequality in [1, Appendix A]. In [1, Appendix B], we give a simple example showing that $d_{FC}$ and $d_{FD}$ are not the same.

▶ Remark 9. Let $(\phi, \psi)$ be a $\delta$-contortion pair between $(F, f)$ and $(G, g)$. For any $x \in F$ we have $\phi(x) \in g^{-1}(I_\delta(f(x)))$, which implies $\|f(x) - g \circ \phi(x)\| \le \delta$. It follows that $\|f - g \circ \phi\|_\infty \le \delta$, and by a symmetric argument we also get $\|g - f \circ \psi\|_\infty \le \delta$.

▶ **Definition 10** (Universal distance $d_U$ [7, 4]). *Let $(F, f)$ and $(G, g)$ be two Reeb graphs. The* universal distance*, denoted $d_U(F, G)$, is defined as the infimum of $\|\tilde{f} - \tilde{g}\|_\infty$ taken over all spaces $Z$ and functions $\tilde{f}, \tilde{g}\colon Z \to \mathbb{R}$ such that $(\mathcal{R}Z, \mathcal{R}\tilde{f}) \cong (F, f)$ and $(\mathcal{R}Z, \mathcal{R}\tilde{g}) \cong (F, g)$.*

The distance $d_U$ is readily seen to be universal. Recall that the *Reeb graph edit distance* [4] is also universal, providing an alternative explicit construction for the universal distance.

If $d$ and $d'$ are distances on Reeb graphs and $c \in [0, \infty)$, we use the notation $d \le cd'$ to express that for all Reeb graphs $(F, f)$ and $(G, g)$, the inequality $d(F, G) \le cd'(F, G)$ holds.

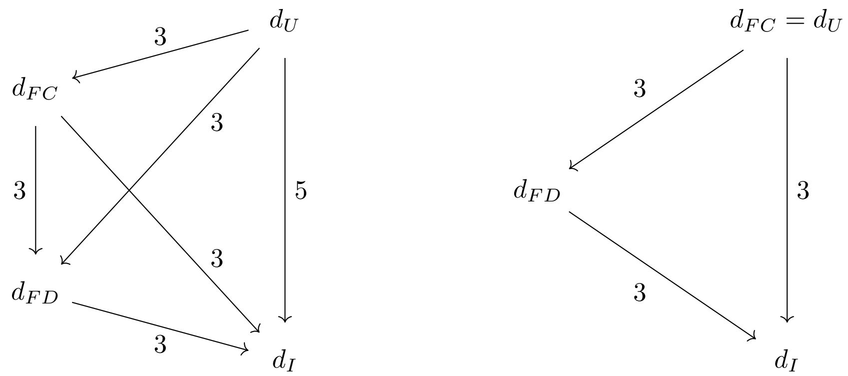
▶ **Theorem 11** (Quasi-universality of Reeb graph distances). *The functional contortion distance $d_{FC}$, the functional distortion distance $d_{FD}$, and the interleaving distance $d_I$ on Reeb graphs are quasi-universal (bi-Lipschitz equivalent to the universal distance). Specifically, we have*

$$d_{FC} \le d_U \le 3d_{FC} \qquad d_{FD} \le d_U \le 3d_{FD} \qquad d_I \le d_U \le 5d_I$$
$$d_{FD} \le d_{FC} \le 3d_{FD} \qquad d_I \le d_{FC} \le 3d_I \qquad d_I \le d_{FD} \le 3d_I.$$

Of these, only $d_I \le d_{FD} \le 3d_I$ [5], $d_I \le d_U$, and $d_{FD} \le d_U$ were known, the latter two being equivalent to stability of $d_I$ [8] and of $d_{FD}$ [2]. See Figure 2a for a visualization of the inequalities in Theorem 11. In fact, one can easily check that all the inequalities in the theorem follow from only six of them, namely

$$d_I \le d_{FD} \le d_{FC} \le d_U \qquad d_U \le 3d_{FD} \qquad d_U \le 5d_I \qquad d_{FC} \le 3d_I.$$

In future work, we plan to show that all these bounds are indeed tight.




**(a)** Inequalities for Reeb graphs. **(b)** Inequalities for contour trees.

**Figure 2** An arrow from $d$ to $d'$ labeled $c$ means that the inequalities $d'(F, G) \leq d(F, G) \leq cd'(F, G)$ hold for all Reeb graphs (in (a)) or contour trees (in (b)).

▶ **Theorem 12** (Universality of the functional contortion distance for contour trees). *The functional contortion distance is universal for contour trees: given two contour trees $(F, f)$ and $(G, g)$, we have*

$$d_{FC}(F, G) = d_U(F, G).$$

This theorem gives us a simpler set of inequalities for contour trees than what we have for general Reeb graphs; see Figure 2b. Finally, for merge trees, the hierarchy collapses:

▶ **Theorem 13** (Universality of the interleaving distance for merge trees). *The interleaving distance is universal for merge trees: given two merge trees $(F, f)$ and $(G, g)$, we have*

$$d_I(F, G) = d_{FD}(F, G) = d_{FC}(F, G) = d_U(F, G).$$

Previously, only the equality $d_I(F, G) = d_{FD}(F, G)$ was known [3]. We prove Theorem 11 in Section 3, Theorem 12 in Section 4, and Theorem 13 in Section 5.

## 3 Bi-Lipschitz bounds for Reeb graph distances

This section is devoted to proving Theorem 11. In Section 3.1 we prove $d_I \leq d_{FD} \leq d_{FC} \leq d_U$, in Section 3.2 we prove $d_U \leq 3d_{FD}$, in Section 3.3 we prove $d_U \leq 5d_I$, and in Section 3.4 we prove $d_{FC} \leq 3d_I$. As mentioned before, these inequalities together imply the theorem.

### 3.1 The inequalities $d_I \leq d_{FD} \leq d_{FC} \leq d_U$

The following lemma and its proof are similar to [4, Proposition 3], but without the assumption that $X$ is compact.

▶ **Lemma 14.** *Let $(F, f)$ be a Reeb graph induced by a map $\hat{f}\colon X \to \mathbb{R}$, let $q\colon X \to F$ be the associated quotient map, and suppose $K \subseteq F$ is connected. Then $q^{-1}(K)$ is connected.*

**Proof.** Suppose the lemma is false. Then $q^{-1}(K) = X_1 \sqcup X_2$, where $X_1$ and $X_2$ are nonempty and contained in disjoint open subsets of $X$. Since $F$ is equipped with the quotient topology of $q$, $q(X_1)$ and $q(X_2)$ are both open subsets of $q(X_1) \cup q(X_2) = K$. Because $K$ is connected, we must have $q(X_1) \cap q(X_2) \neq \emptyset$.

Let $x \in q(X_1) \cap q(X_2)$, so $V_1 := q^{-1}(x) \cap X_1$ and $V_2 := q^{-1}(x) \cap X_2$ are both nonempty. Since $X_1$ and $X_2$ are open and disjoint subsets of $X_1 \cup X_2$, $V_1$ and $V_2$ are open and disjoint subsets of $V_1 \cup V_2 = q^{-1}(x)$. But by definition of induced Reeb graph, $q^{-1}(x)$ is connected, so we have a contradiction. ◀

▶ **Theorem 15.** *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_{FC}(F, G) \leq d_U(F, G).$$

**Proof.** Let $X$ be a topological space with functions $\hat{f}, \hat{g} \colon X \to \mathbb{R}$ that induce Reeb graphs $(F, f)$ and $(G, g)$, respectively, and let $q_F \colon X \to F$, $q_G \colon X \to G$ denote the corresponding Reeb quotient maps. Suppose $\|\hat{f} - \hat{g}\|_\infty = \delta \geq 0$. For any $\epsilon > 0$, we will construct functions $\phi \colon F \to G$ and $\psi \colon G \to F$ that form a $(\delta + 2\epsilon)$-contortion pair; the theorem follows.

Fix $\epsilon > 0$. Pick a discrete subset $S \subseteq F$ containing all the 0-cells of $F$ such that for each 1-cell $C$ and each connected component $K$ of $C \setminus S$, the image $f(K)$ is contained in some interval $[a, b]$ of length $b - a = \epsilon$. Pick a subset $T \subseteq G$ analogously. Define a map $\phi \colon S \to G$ by picking an element $\phi(s) \in q_G(q_F^{-1}(s))$ for each $s \in S$.

Let $L$ be the closure of a connected component of $F \setminus S$. Observe that $L$ is contained in a single 1-cell $C$ and is homeomorphic to a closed interval, with endpoints $z, z' \in S$. By our assumptions on $S$, $L$ is contained in a connected component of $f^{-1}[a, b]$ for some $a < b$ with $b - a = \epsilon$. By Lemma 14, $q_F^{-1}(L)$ is connected, and by continuity of $q_G$, $J := q_G(q_F^{-1}(L))$ is connected, too. Since $J$ is connected, we can extend $\phi$ continuously to $L$ by choosing a path from $\phi(z)$ to $\phi(z')$ in $J$. Moreover, because $\|\hat{f} - \hat{g}\|_\infty = \delta$, we have $J \subseteq g^{-1}[a - \delta, b + \delta] \subseteq g^{-1}(I_{\delta+\epsilon}(f(x)))$. It follows that for every $x \in L$, $\phi(x)$ and $q_G(q_F^{-1}(x))$ are connected in $g^{-1}(I_{\delta+\epsilon}(f(x)))$. We do this for every $L$ as described and get a continuous map $\phi \colon F \to G$. Analogously, we get a continuous map $\psi \colon G \to F$ such that for any $y \in G$, $\psi(y)$ and $q_F(q_G^{-1}(y))$ are connected in $f^{-1}(I_{\delta+\epsilon}(g(y)))$.

Pick an $x \in L$, where $L$ is as in the previous paragraph, and let $y = \phi(x)$. By construction, $y \in q_G(q_F^{-1}(x'))$ for some $x' \in L$. Thus, $x' \in q_F(q_G^{-1}(y))$, which, as noted, is in the same connected component of $f^{-1}(I_{\delta+\epsilon}(g(y)))$ as $\psi(y)$. But $x$ and $x'$ are connected in $f^{-1}[a, b]$ for some $a < b$ with $b - a = \epsilon$, so it follows that $x$ and $\psi(y)$ are connected in $f^{-1}(I_{\delta+2\epsilon}(g(y)))$. Along with the symmetric statement that follows by a similar argument, this is exactly what is needed for $(\phi, \psi)$ to be a $(\delta + 2\epsilon)$-contortion pair. ◀

▶ **Theorem 16.** *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_{FD}(F, G) \leq d_{FC}(F, G).$$

**Proof.** Suppose $\phi \colon F \to G$ and $\psi \colon G \to F$ form a $\delta$-contortion pair for some $\delta \geq 0$. Then

$$||f - g \circ \phi||_\infty, ||g - f \circ \psi||_\infty \leq \delta \tag{1}$$

by Remark 9. Let $(x, y), (x', y') \in C(\phi, \psi)$, where $C(\phi, \psi)$ is as in Definition 7. We claim that if $x$ and $x'$ are connected in $f^{-1}[a, b]$ for some $a \leq b$, then $y$ and $y'$ are connected in $g^{-1}[a - \delta, b + \delta]$. Together with the symmetric statement and Equation (1), this is enough to show that $(\phi, \psi)$ is a $\delta$-distortion pair, from which the lemma follows.

Assume that $x$ and $x'$ lie in the same connected component $K$ of $f^{-1}[a, b]$. We have that $\phi(K)$ is connected, and it follows from Equation (1) that $\phi(K) \subseteq g^{-1}[a - \delta, b + \delta]$. Since $\phi(x), \phi(x') \in \phi(K)$, $\phi(x)$ and $\phi(x')$ are connected in $g^{-1}[a - \delta, b + \delta]$. By definition of $C(\phi, \psi)$, either $y$ is equal to $\phi(x)$, or $y \in \psi^{-1}(x)$. In the latter case, $y$ and $\phi(x)$ are connected in

$g^{-1}(I_\delta(f(x))) \subseteq g^{-1}[a-\delta, b+\delta]$ by definition of $\delta$-contortion. Similarly, $y'$ and $\phi(x')$ are also connected in $g^{-1}[a-\delta, b+\delta]$. Putting everything together, $y$ and $y'$ are connected in $g^{-1}[a-\delta, b+\delta]$, which completes the proof. ◀

▶ **Theorem 17** ([5, Lemma 8]). *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_I(F, G) \leq d_{FD}(F, G).$$

The setting of [5] is slightly different than ours, but the proof of the result carries over.

## 3.2 Relating universal and functional distortion distance

We denote the connected component of a point $p$ in a space $X$ by $K_p(X)$.

▶ **Theorem 18.** *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_U(F, G) \leq 3d_{FD}(F, G).$$

**Proof.** Assume that $\phi : F \to G$ and $\psi : G \to F$ form a $\delta$-distortion pair. We construct a subspace $Z \subseteq F \times G$ such that the canonical projections $\mathrm{pr}_F : F \times G \to F$, $\mathrm{pr}_G : F \times G \to G$ restrict to Reeb quotient maps $q_F : Z \to F$, $q_G : Z \to G$ of $f \circ q_F$ and $g \circ q_G$, and $\|f \circ q_F - g \circ q_G\|_\infty \leq 3\delta$, proving that $d_U \leq 3d_{FD}$.

For $x \in F$, let

$$C(x) = K_x(f^{-1}[a, a+2\delta]),$$

where $a$ is chosen such that $C(x)$ contains $\psi \circ \phi(x)$. By definition of $\delta$-distortion, such an $a$ always exists, though it does not have to be unique. We define $C(y)$ analogously for $y \in G$:

$$C(y) = K_y(g^{-1}[a', a'+2\delta])$$

for some $a'$, and $C(y)$ contains $\phi \circ \psi(y)$. Now define

$$Z = \bigcup_{x \in F} C(x) \times \phi(C(x)) \cup \bigcup_{y \in G} \psi(C(y)) \times C(y) \subseteq F \times G$$

and the functions $\hat{f} = f \circ \mathrm{pr}_F$, $\hat{g} = g \circ \mathrm{pr}_G : Z \to \mathbb{R}$.

To show that $\|\hat{f} - \hat{g}\|_\infty \leq 3\delta$, by symmetry it suffices to show that for every $x \in F$ and every $(z, y) \in C(x) \times \phi(C(x))$ we have $|f(z) - g(y)| \leq 3\delta$. Pick $w \in C(x)$ such that $\phi(w) = y$. We have $|f(z) - f(w)| \leq 2\delta$ by construction of $C(x)$, and $|f(w) - g(y)| \leq \delta$ by definition of $\delta$-distortion. Together, we have $|f(z) - g(y)| \leq 3\delta$ as claimed.

To show that $q_F : Z \to F$ is surjective, simply observe that for any $x \in F$,

$$(x, \phi(x)) \in C(x) \times \phi(C(x)) \subseteq Z.$$

A similar argument shows that also $q_G : Z \to G$ is surjective.

It remains to show that the fibers of $q_F$ are connected; by symmetry, the same is then true for $q_G$ as well. The fiber of $z \in F$ is of the form $q_F^{-1}(z) = \{z\} \times G_z \subseteq Z$, where $G_z = q_G(q_F^{-1}(z)) \subseteq G$ is a subspace, homeomorphic to the fiber. Note that $G_z$ has the explicit description

$$G_z = \bigcup_{\substack{x \in F \\ z \in C(x)}} \phi(C(x)) \cup \bigcup_{\substack{y \in G \\ z \in \psi(C(y))}} C(y).$$

Now $\phi(z)$ is contained in any $\phi(C(x))$ with $x \in F$ and $z \in C(x)$, and in any $C(y)$ with $y \in \psi^{-1}(z)$, and each of these subspaces is connected. Thus,

$$G'_z = \bigcup_{\substack{x\in F\\ z\in C(x)}} \phi(C(x)) \cup \bigcup_{\substack{y\in\psi^{-1}(z)\\ z\in\psi(C(y))}} C(y)$$

is connected and contains $\psi^{-1}(z)$ as a subset. Clearly, if $z \in \psi(C(y))$, then $C(y)$ contains an element of $\psi^{-1}(z)$, so $C(y)$ intersects $G'_z$. As $C(y)$ is connected, it follows that

$$G'_z \cup \bigcup_{\substack{y\in G\\ z\in\psi(C(y))}} C(y) = G_z$$

is connected. ◀

## 3.3 Relating universal and interleaving distance

▶ **Lemma 19.** *Let $(\phi, \psi)$ be a $\delta$-interleaving of $(F, f)$ and $(G, g)$ for some $\delta \geq 0$. If $K \subseteq F$ ($K' \subseteq G$) is connected, then $\Phi(K)$ ($\Psi(K')$) is connected.*

**Proof.** By continuity of $\phi$, $\phi(K) \subseteq \mathcal{U}_\delta G$ is connected. Thus, by Lemma 14,

$$C := q^{-1}_{\mathcal{U}_\delta G}(\phi(K)) \subseteq \mathcal{T}_\delta G$$

is connected. We have that $\Phi(K)$ is exactly the image of $C$ under the projection $\mathrm{pr}_G \colon \mathcal{T}_\delta G \to G$. Since this projection is continuous and $C$ is connected, $\Phi(K)$ is connected.

The statement for $K'$ and $\Psi$ follows by symmetry. ◀

▶ **Theorem 20.** *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_U(F, G) \leq 5 d_I(F, G).$$

**Proof.** Let $(\phi, \psi)$ be a $\delta$-interleaving of $(F, f)$ and $(G, g)$, so to any $x \in F$, there is associated a subset $\Phi(x) \subseteq G$ that is a connected component of $g^{-1}(I_\delta(f(x)))$. Similarly, for any $y \in G$, $\Psi(y)$ is a connected component of $f^{-1}(I_\delta(g(y)))$. We construct a subspace $Z \subseteq F \times G$ and two functions $\hat{f}, \hat{g} \colon Z \to \mathbb{R}$ with $\|\hat{f} - \hat{g}\|_\infty \leq 5\delta$ such that the canonical projections $\mathrm{pr}_F \colon F \times G \to F$, $\mathrm{pr}_G \colon F \times G \to G$ restrict to Reeb quotient maps $q_F \colon Z \to F$ of $\hat{f}$ and $q_G \colon Z \to G$ of $\hat{g}$, proving that $d_U \leq 5 d_I$.

For $x \in F$ and $y \in G$, let

$$C(x) = K_x(f^{-1}(I_{2\delta}(f(x)))), \qquad C(y) = K_y(g^{-1}(I_{2\delta}(g(y)))),$$

and let

$$Z = \bigcup_{x\in F} C(x) \times \Phi(C(x)) \cup \bigcup_{y\in G} \Psi(C(y)) \times C(y) \subseteq F \times G.$$

To show that $\|\hat{f} - \hat{g}\|_\infty \leq 5\delta$, by symmetry it suffices to show that for every $x \in F$ and every $(z, y) \in C(x) \times \Phi(C(x))$ we have $|f(z) - g(y)| \leq 5\delta$. Pick $w \in C(x)$ such that $y \in \Phi(w)$. We have $|f(z) - f(w)| \leq 4\delta$ by construction of $C(x)$, and $|f(w) - g(y)| \leq \delta$ by definition of $\delta$-interleaving. Together, we have $|f(z) - g(y)| \leq 5\delta$ as claimed.

To show that $q_F \colon Z \to F$ is surjective, simply observe that for any $x \in F$ and $y \in \Phi(x)$,

$$(x, y) \in C(x) \times \Phi(C(x)) \subseteq Z.$$

A similar argument shows that also $q_G \colon Z \to G$ is surjective.

It remains to show that the fibers of $q_F$ are connected; by symmetry, the same is then true for $q_G$ as well. The fiber of $z \in F$ is of the form $q_F^{-1}(z) = \{z\} \times G_z \subseteq Z$, where $G_z = q_G(q_F^{-1}(z)) \subseteq G$ is a subspace, homeomorphic to the fiber. Note that $G_z$ has the explicit description

$$G_z = \bigcup_{\substack{x \in F \\ z \in C(x)}} \Phi(C(x)) \cup \bigcup_{\substack{y \in G \\ z \in \Psi(C(y))}} C(y).$$

Clearly, $\Phi(z)$ is contained in any $\Phi(C(x))$ with $x \in F$ and $z \in C(x)$. In addition, $\Phi(z) \subseteq C(y)$ for any $y \in G$ such that $z \in \Psi(y)$, as $\Phi(\Psi(y)) \subseteq C(y)$ by definition of interleaving. By Lemma 19, $\Phi(C(x))$ is connected. Thus,

$$G'_z = \bigcup_{\substack{x \in F \\ z \in C(x)}} \Phi(C(x)) \cup \bigcup_{\substack{y \in G \\ z \in \Psi(y)}} C(y) \subseteq G_z$$

is connected, since it is a union of connected sets that all contain $\Psi(z)$. To complete the proof that $G_z$ is connected, it suffices to show that $C(y)$ intersects $G'_z$ for all $y \in G$ such that $z \in \Psi(C(y))$. To see this, observe that there is a $w \in C(y)$ such that $z \in \Psi(w)$, and $w \in C(w) \subseteq G'_z$. ◀

## 3.4 Relating functional contortion and interleaving distance

▶ **Theorem 21.** *Given any two Reeb graphs $(F, f)$ and $(G, g)$, we have*

$$d_{FC}(F, G) \leq 3d_I(F, G).$$

**Proof.** Let $(\phi, \psi)$ be a $\delta$-interleaving of $(F, f)$ and $(G, g)$ for some $\delta \geq 0$. We will show that for an arbitrary $\epsilon > 0$, there are $\mu \colon F \to G$ and $\nu \colon G \to F$ with functional contortion $3\delta + 3\epsilon$.

Pick a discrete subset $S \subset F$ containing all the 0-cells of $F$ such that for each 1-cell $C$ and connected component $I$ of $C \setminus S$, the interval $f(I)$ has length less than $\epsilon$. Pick a discrete subset $T \subset G$ with the same properties.

For each $x \in S$, pick an arbitrary $y \in \Phi(x)$ and define $\mu(x) = y$. Similarly, for each $y \in T$, let $\nu(y) \in \Psi(y)$. Let $I$ be a connected component of $F \setminus S$. Observe that $I$ is contained in a single 1-cell $C$, and that its closure $\overline{I}$ contains two points $z, z' \in S$. By our assumptions on $I$, $\overline{I}$ is contained in a connected component of $f^{-1}[a, b]$ for some $a < b$ with $b - a = \epsilon$. By Lemma 19, it follows that $\Phi(\overline{I})$ is contained in a connected component $K$ of $g^{-1}[a - \delta, b + \delta]$. We can therefore use a a path from $\mu(z)$ to $\mu(z')$ in $K$ to extend $\mu$ continuously to $\overline{I}$; see Figure 3. This implies that for all $x \in \overline{I}$, $\mu(x)$ and $\Phi(x)$ are both contained in $K$ and thus also in the same component of $g^{-1}(I_{\delta+\epsilon}(f(x))$, as

$$g^{-1}[a - \delta, b + \delta] \subseteq g^{-1}(I_{\delta+\epsilon}(f(x)).$$

We do the same for all connected components of $F \setminus S$ and define $\nu$ similarly on $G \setminus T$.

Let $\delta' = \delta + \epsilon$. We now prove that $(\mu, \nu)$ is a $3\delta'$-contortion pair. By symmetry, it is enough to show that for any $x \in F$, $x$ and $\nu(\mu(x))$ are connected in $f^{-1}(I_{3\delta'}(g(\mu(x))))$. The $\delta$-interleaving $(\Phi, \Psi)$ induces a $\delta'$-interleaving $(\Phi', \Psi')$ canonically: For $x \in F$, $\Phi'(x)$ is the connected component of $g^{-1}(I_{\delta'}(f(x))$ containing $\Phi(x)$ as a subset, and $\Psi'(y)$ is defined similarly for $y \in G$. We observed that $\mu(x)$ and $\Phi(x)$ are connected in $g^{-1}(I_{\delta'}(f(x))$, so

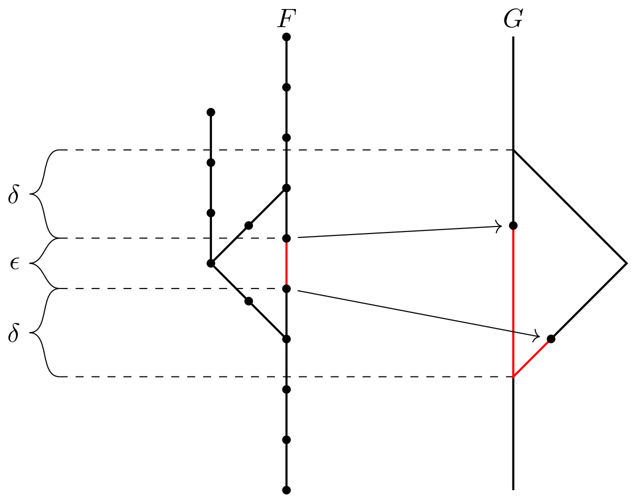



**Figure 3** Construction of a functional contortion pair for Theorem 21. The points of $S$ are shown as black dots. The arrows and the red segments in $F$ and $G$ show $\mu$ applied to two points in $S$ and how we can extend $\mu$ to the segment between the points.

$\mu(x) \in \Phi'(x)$. Similarly, $\nu(\mu(x)) \in \Psi'(\mu(x))$. Putting the two together, we get $\nu(\mu(x)) \in \Psi'(\Phi'(x))$. By definition of interleaving, we have $\Psi'(\Phi'(x)) \subseteq K_x(f^{-1}(I_{2\delta'}(f(x))))$. Since $|f(x) - g(\mu(x))| \leq \delta'$, we have

$$f^{-1}(I_{2\delta'}(f(x))) \subseteq f^{-1}(I_{3\delta'}(g(\mu(x)))),$$

so

$$\nu(\mu(x)) \in \Psi'(\Phi'(x)) \subseteq K_x(f^{-1}(I_{3\delta'}(g(\mu(x))))),$$

which is what we wanted to prove. ◀

## 4 Contour trees

**Proof of Theorem 12.** We know $d_{FC}(F,G) \leq d_U(F,G)$ by Theorem 15, so it remains to prove $d_{FC}(F,G) \geq d_U(F,G)$.

Assume that there is a $\delta$-contortion $(\phi,\psi)$ between $F$ and $G$. We construct a subspace $Z \subseteq F \times G$ and two functions $\hat{f}, \hat{g} \colon Z \to \mathbb{R}$ with $\|\hat{f} - \hat{g}\|_\infty \leq \delta$ such that the canonical projections $\mathrm{pr}_F \colon F \times G \to F$, $\mathrm{pr}_G \colon F \times G \to G$ restrict to Reeb quotient maps $q_F \colon Z \to F$ of $\hat{f}$ and $q_G \colon Z \to G$ of $\hat{g}$, proving that $d_{FC}(F,G) \geq d_U(F,G)$.

Let $x \neq x' \in F$, and let $\rho \colon [0,1] \to F$ be an injective path from $x$ to $x'$. Since $F$ is a contour tree and therefore a contractible 1-dimensional CW complex, the image of this path is uniquely determined, so we can define $B(x,x') = \mathrm{im}(\rho)$. Observe that $z \in B(x,x') \setminus \{x,x'\}$ if and only if $x$ and $x'$ are in different connected components of $F \setminus \{z\}$. $B(y,y')$ for $y,y' \in G$ is defined similarly.

Let $Z \subseteq F \times G$ be given by

$$Z = \left[ \bigcup_{x \in F, y \in \psi^{-1}(x)} \{x\} \times B(\phi(x), y) \right] \cup \left[ \bigcup_{y \in G, x \in \phi^{-1}(y)} B(\psi(y), x) \times \{y\} \right].$$

To show that $\|\hat{f} - \hat{g}\|_\infty \leq \delta$, by symmetry it suffices to show that for every $x \in F$, $y \in \psi^{-1}(x)$ and $y' \in B(\phi(x), y)$, $|f(x) - g(y')| \leq \delta$. But by definition of $\delta$-contortion, $\phi(x)$ and $y$ are connected in $g^{-1}[f(x) - \delta, f(x) + \delta]$, so $B(\phi(x), y) \subseteq g^{-1}[f(x) - \delta, f(x) + \delta]$, and the statement follows.

For any $x \in F$, we have

$$(x, \phi(x)) \in B(\psi \circ \phi(x), x) \times \{\phi(x)\} \subseteq Z,$$

and it follows immediately that $q_F \colon Z \to F$ is surjective, and similarly for $q_G \colon Z \to G$.

It remains to show that the fibers of $q_F$ and $q_G$ are connected. By symmetry, we only need to prove this for $q_F$. The fiber of $x \in F$ is of the form $q_F^{-1}(x) = \{x\} \times G_x \subseteq Z$, where $G_x = q_G(q_F^{-1}(x)) \subseteq G$ is a subspace, homeomorphic to the fiber. To follow the arguments that follow, we suggest keeping an eye on Figure 4. Let

$$U_x = \{\phi(x)\} \cup \bigcup_{y \in \psi^{-1}(x)} B(\phi(x), y) \subseteq G.$$

(Including $\{\phi(x)\}$ only makes a difference if $\phi^{-1}(x)$ is empty.) Since $U_x$ is a union of connected sets intersecting in the point $\phi(x)$, $U_x$ is connected. Moreover, $U_x \subseteq G_x$ by construction, as we have already observed that $(x, \phi(x)) \in q_F^{-1}(x)$.

Let $D_x = \{\phi(x)\} \cup \psi^{-1}(x)$. Note that $D_x \subseteq U_x$. Let $y \in G_x \setminus U_x$. Then there is an $x' \in \phi^{-1}(y)$ such that $x \in B(\psi(y), x')$. Equivalently, $x'$ and $\psi(y)$ are in different connected components $F_1$ and $F_2$, respectively, of $F \setminus \{x\}$. (Since $y \notin D_x$, neither $x'$ nor $\psi(y)$ is equal to $x$.) Since $D_x$ is closed, so is $\phi^{-1}(D_x)$. This means that we can pick an $x'' \in \phi^{-1}(D_x)$ such that

$$B \coloneqq B(x', x'') \setminus \{x''\}$$

does not intersect $\phi^{-1}(D_x)$. It follows that $B \subseteq F_1$, since $x \notin B$. It also follows that $\psi \circ \phi(B) \subseteq F_2$, since $x \notin \psi \circ \phi(B)$ and $\psi(y) \in \psi \circ \phi(B)$. Thus, for all $z \in B$, we have $x \in B(\psi \circ \phi(z), z)$; i.e.,

$$(x, \phi(z)) \in B(\psi \circ \phi(z), z) \times \{\phi(z)\} \subseteq Z,$$

so $\phi(B) \subseteq G_x$. This means that there is a path in $G_x$ from $y = \phi(x')$ to $\phi(x'') \in U_x$. Since $y$ was an arbitrary point in $G_x$ and $U_x$ is connected, it follows that $G_x$ is connected. ◀

## 5 Merge trees

In this section, we focus on merge trees, which are a special case of contour trees that also arise from the connected components of the sublevel set filtration of a function.

The merge trees obtained this way carry a function that is unbounded above, and they are characterized by the property that the canonical map from the merge tree to the Reeb graph of its epigraph is an isomorphism [10]. Our definition is more general and also admits bounded functions, and in Section 5.1 we develop an analogous characterization for these general merge trees via the property that said canonical map is an embedding. Relating this property to our definition is not straightforward, and we defer the proofs to [1, Appendix C].

Our goal in Section 5.2 is to prove that the interleaving distance for merge trees is universal. By Theorem 12, it suffices to construct a $\delta$-contortion pair from a $\delta$-interleaving of merge trees. Summarizing the idea for the simpler special case of a merge tree $G$ unbounded above, the key insight behind the proof is that the $\delta$-smoothing of $G$ is isomorphic to an upward $\delta$-shift of $G$. Composing the interleaving morphisms with the isomorphisms obtained this way yields the desired $\delta$-contortion pair in the unbounded case.

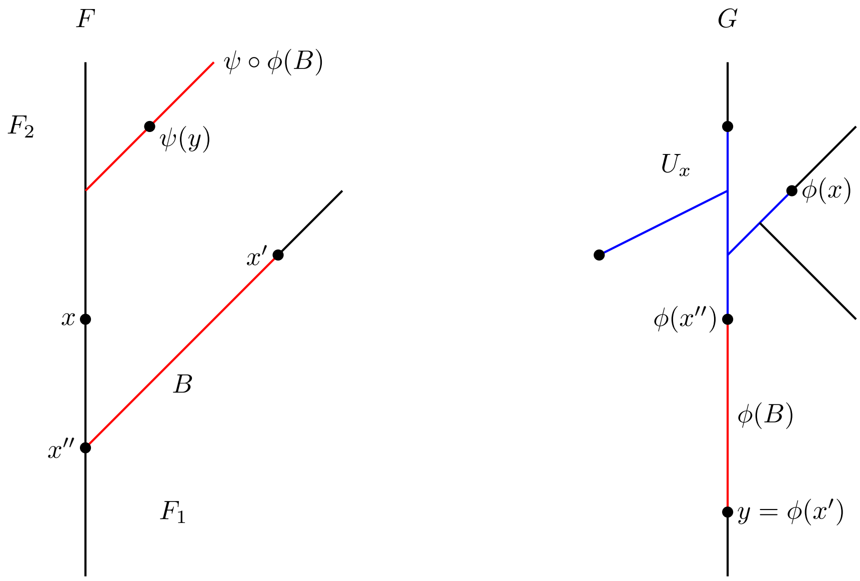



**Figure 4** Illustration of constructions used to prove connectedness of fibers of $q_F$.

## 5.1 The epigraph and merge trees

We formally characterize merge trees using a construction based on *epigraphs*, as previously suggested by Morozov et al. [10].

▶ **Definition 22** (Epigraph). *Let $f\colon X \to \mathbb{R}$ be a continuous function. We define the* epigraph *of $f$ as the space $\mathcal{E}X := X \times [0,\infty)$ equipped with the function $\mathcal{E}f\colon \mathcal{E}X \to \mathbb{R}$, $(p,t) \mapsto f(p) + t$.*

While this is not the usual definition of the epigraph $\{(p,y) \in X \times \mathbb{R} \mid f(p) \leq y\}$, we note that the map $\mathcal{E}f\colon \mathcal{E}X \to \mathbb{R}$ and the projection of the ordinary epigraph to the second component are isomorphic as $\mathbb{R}$-spaces. Our definition has the benefit that we have the strict equality $\delta + \mathcal{E}f = \mathcal{E}(\delta + f)\colon \mathcal{E}X \to \mathbb{R}$ for any $\delta \in \mathbb{R}$.

Now suppose that $(G,g)$ is a Reeb graph, and let $m := \sup_{p\in G} g(p) \in (-\infty,\infty]$. We now define the map $\tilde{\rho}_G\colon \mathcal{E}G \to \mathcal{E}G$, $(p,t) \mapsto (p, \min\{t, m - g(p)\})$, which makes the diagram

$$\begin{tikzcd}
(\mathcal{E}g)^{-1}(-\infty,m] \arrow[r, equal] \arrow[d, hook] & (\mathcal{E}g)^{-1}(-\infty,m] \arrow[d, hook] \\
\mathcal{E}G \arrow[r, dashed, "\tilde{\rho}_G"] \arrow[d, "\mathcal{E}g"'] & \mathcal{E}G \arrow[d, "\mathcal{E}g"] \\
\mathbb{R} \arrow[r, "\min\{-,m\}"'] & \mathbb{R}
\end{tikzcd}$$

commute. We state the following immediate consequence of this definition.

▶ **Lemma 23.** *For each $t \in \mathbb{R}$ the map $\tilde{\rho}_G\colon \mathcal{E}G \to \mathcal{E}G$ restricts to a homeomorphism between the fibers $(\mathcal{E}g)^{-1}(t)$ and $(\mathcal{E}g)^{-1}(\min\{t,m\})$.*

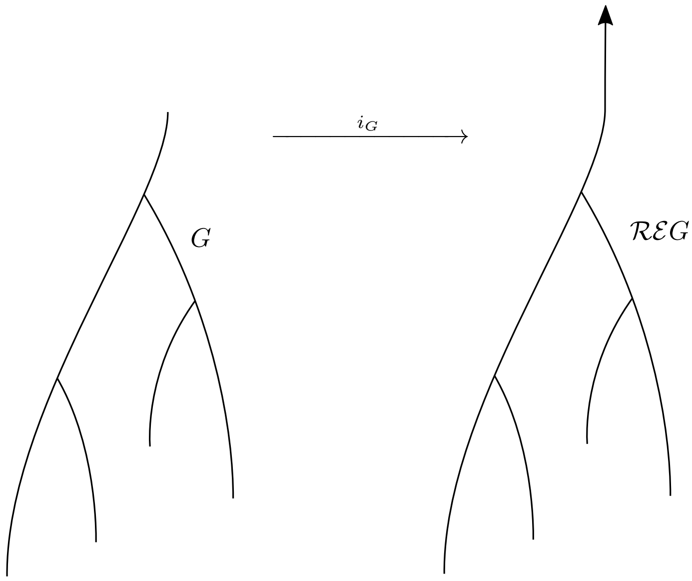



**Figure 5** The embedding $i_G\colon G \to \mathcal{R}\mathcal{E}G$ of a merge tree $G$ into its unbounded variant $\mathcal{R}\mathcal{E}G$.

By the universal property of the quotient topology, there is a unique continuous map $\mathcal{R}\tilde{\rho}_G\colon \mathcal{R}\mathcal{E}G \to \mathcal{R}\mathcal{E}G$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{E}G \arrow[rr, "\tilde{\rho}_G"] \arrow[dr, "q_{\mathcal{R}\mathcal{E}G}"] \arrow[dd, "\mathcal{E}g"'] & & \mathcal{E}G \arrow[dr, "q_{\mathcal{R}\mathcal{E}G}"] \arrow[dd, "\mathcal{E}g"' near start] & \\
& \mathcal{R}\mathcal{E}G \arrow[rr, "\mathcal{R}\tilde{\rho}_G" near start] \arrow[dl, "\mathcal{R}\mathcal{E}g"] & & \mathcal{R}\mathcal{E}G \arrow[dl, "\mathcal{R}\mathcal{E}g"] \\
\mathbb{R} \arrow[rr, "\min\{-,m\}"'] & & \mathbb{R} &
\end{tikzcd} \tag{2}$$

▶ **Corollary 24.** *For each $t \in \mathbb{R}$ the map $\mathcal{R}\tilde{\rho}_G\colon \mathcal{R}\mathcal{E}G \to \mathcal{R}\mathcal{E}G$ restricts to a bijection between the fibers $(\mathcal{R}\mathcal{E}g)^{-1}(t)$ and $(\mathcal{R}\mathcal{E}g)^{-1}(\min\{t, m\})$.*

Let $\kappa_X\colon X \to X \times [0,\infty) = \mathcal{E}X$, $p \mapsto (p, 0)$ denote the natural embedding of $X$ into the epigraph of $f$. We state several properties (see [1, Appendix C] for the proofs) of the map

$$i_G\colon G \xrightarrow{\kappa_G} \mathcal{E}G \xrightarrow{q_{\mathcal{R}\mathcal{E}G}} \mathcal{R}\mathcal{E}G.$$

▶ **Lemma 25.** *The images of the maps $i_G\colon G \to \mathcal{R}\mathcal{E}G$ and $\mathcal{R}\tilde{\rho}_G\colon \mathcal{R}\mathcal{E}G \to \mathcal{R}\mathcal{E}G$ are identical.*

▶ **Lemma 26.** *A Reeb graph $(G, g)$ is a merge tree iff the map $i_G\colon G \to \mathcal{R}\mathcal{E}G$ is an embedding.*

Now suppose that $(G, g)$ is a merge tree. The composite map

$$i_G\colon G \xrightarrow{\kappa_G} \mathcal{E}G \xrightarrow{q_{\mathcal{R}\mathcal{E}G}} \mathcal{R}\mathcal{E}G$$

is non-surjective iff $g\colon G \to \mathbb{R}$ is bounded above. We define $\rho_G\colon \mathcal{R}\mathcal{E}G \to G$ to be the unique continuous map – which exists by Lemmas 25 and 26 – making the diagram

$$\begin{tikzcd} G \arrow[d, "\kappa_G"'] & \mathcal{R}\mathcal{E}G \arrow[l, dashed, "\rho_G"'] \arrow[d, "\mathcal{R}\tilde{\rho}_G"] \\ \mathcal{E}G \arrow[r, "q_{\mathcal{R}\mathcal{E}G}"'] & \mathcal{R}\mathcal{E}G \end{tikzcd} \tag{3}$$

commute. As an immediate corollary of Corollary 24, we obtain the following observation.

▶ **Corollary 27.** *For each $t \in \mathbb{R}$, the map $\rho_G\colon \mathcal{R}\mathcal{E}G \to G$ restricts to a bijection between the fibers $(\mathcal{R}\mathcal{E}g)^{-1}(t)$ and $g^{-1}(\min\{t, m\})$.*

## 5.2 Interleavings, contortions, and merge trees

Let $f\colon X \to \mathbb{R}$ be an arbitrary continuous function and let $\delta \geq 0$. We define the map

$$\kappa_X^\delta\colon \mathcal{T}_\delta X \to \mathcal{E}X,\ (p, t) \mapsto (p, t + \delta),$$

which makes the diagram

$$\begin{tikzcd} \mathcal{T}_\delta X \arrow[r, "\kappa_X^\delta"] \arrow[d, "\mathcal{T}_\delta f"'] & \mathcal{E}X \arrow[d, "\mathcal{E}f"] \\ \mathbb{R} \arrow[r, "(-)+\delta"'] & \mathbb{R} \end{tikzcd}$$

commute. Now let $(G, g)$ be a merge tree. By the universal property of the quotient topology, there is a unique continuous map $\mathcal{R}\kappa_G^\delta\colon \mathcal{U}_\delta G \to \mathcal{R}\mathcal{E}G$ making the diagram

$$\begin{tikzcd} \mathcal{T}_\delta G \arrow[rrr, "\kappa_G^\delta"] \arrow[dr, "q_{\mathcal{U}_\delta G}"] \arrow[dd, "\mathcal{T}_\delta g"'] & & & \mathcal{E}G \arrow[dr, "q_{\mathcal{R}\mathcal{E}G}"] \arrow[dd, "\mathcal{E}g"'] & \\ & \mathcal{U}_\delta G \arrow[rrr, "\mathcal{R}\kappa_G^\delta"] \arrow[dl, "\mathcal{U}_\delta g"] & & & \mathcal{R}\mathcal{E}G \arrow[dl, "\mathcal{R}\mathcal{E}g"] \\ \mathbb{R} \arrow[rrr, "(-)+\delta"'] & & & \mathbb{R} & \end{tikzcd} \tag{4}$$

commute.

▶ **Lemma 28.** *The map $\mathcal{R}\kappa_G^\delta\colon \mathcal{U}_\delta G \to \mathcal{R}\mathcal{E}G$ is injective.*

As in the previous subsection, let $m := \sup_{p\in G} g(p) \in (-\infty, \infty]$, let $p \in G$, and let $\delta' = \delta$ or $\delta' \in [-\delta, \delta]$ if $g(p) = m$.

▶ **Lemma 29.** *The composite map*

$$G \times [-\delta, \delta] = \mathcal{T}_\delta G \xrightarrow{q_{\mathcal{U}_\delta G}} \mathcal{U}_\delta G \xrightarrow{\mathcal{R}\kappa_G^\delta} \mathcal{R}\mathcal{E}G \xrightarrow{\rho_G} G$$

*maps $(p, -\delta')$ to $p$.*

**Proof of Theorem 13.** Suppose $(F, f)$ and $(G, g)$ are merge trees and that

$$\phi\colon F \to \mathcal{U}_\delta G \quad \text{and} \quad \psi\colon G \to \mathcal{U}_\delta F$$

form a $\delta$-interleaving (of Reeb graphs). We show that the composite maps

$$\tilde{\phi}\colon F \xrightarrow{\phi} \mathcal{U}_\delta G \xrightarrow{\mathcal{R}\kappa_G^\delta} \mathcal{R}\mathcal{E}G \xrightarrow{\rho_G} G, \qquad \tilde{\psi}\colon G \xrightarrow{\psi} \mathcal{U}_\delta F \xrightarrow{\mathcal{R}\kappa_F^\delta} \mathcal{R}\mathcal{E}F \xrightarrow{\rho_F} F$$

form a $\delta$-contortion pair. Together with Theorem 12, this proves the claim.

Let $x \in F$ and let $y \in \tilde{\psi}^{-1}(x)$. We have to show that $y$ and $\tilde{\phi}(x)$ are connected in $g^{-1}(I_\delta(f(x)))$. By the symmetry of Definition 8, this is also sufficient. By the commutativity of the lower parallelogram in (4) the value of $\mathcal{R}\kappa_F^\delta(\psi(y))$ under $\mathcal{R}\mathcal{E}f$ is

$$(\mathcal{U}_\delta f)(\psi(y)) + \delta = g(y) + \delta.$$

In conjunction with the commutativity of (3) and the lower parallelogram in (2) we obtain

$$f(x) = \big(f \circ \tilde{\psi}\big)(y) = \min\{g(y) + \delta, m'\},$$

where $m' := \sup_{p \in F} f(p)$, and hence

$$f(x) - g(y) = \min\{g(y) + \delta, m'\} - g(y) = \min\{\delta, m' - g(y)\}.$$

Moreover, $g(y) = (\mathcal{U}_\delta f)(\psi(y)) \le m' + \delta$, so in conjunction with Lemma 29 we obtain that

$$G \times [-\delta, \delta] = \mathcal{T}_\delta G \xrightarrow{q_{\mathcal{U}_\delta G}} \mathcal{U}_\delta G \xrightarrow{\mathcal{R}\kappa_G^\delta} \mathcal{R}\mathcal{E}G \xrightarrow{\rho_G} G$$

maps $(x, g(y) - f(x))$ to $x$. Thus, the composite map

$$\mathcal{U}_\delta G \xrightarrow{\mathcal{R}\kappa_G^\delta} \mathcal{R}\mathcal{E}G \xrightarrow{\rho_G} G$$

maps both $q_{\mathcal{U}_\delta F}(x, g(y) - f(x))$ and $\psi(y)$ to $x$. By Lemma 28 and Corollary 27 this implies

$$q_{\mathcal{U}_\delta F}(x, g(y) - f(x)) = \psi(y).$$

Completely analogously we obtain that $q_{\mathcal{U}_\delta G}\big(\tilde{\phi}(x), f(x) - \big(g \circ \tilde{\phi}\big)(x)\big) = \phi(x)$. Thus, $y$ and $\tilde{\phi}(x)$ are connected in $g^{-1}(I_{2\delta}(g(y)))$ by Definition 6. It remains to show that $y$ and $\tilde{\phi}(x)$ are connected in $g^{-1}(I_\delta(f(x)))$. To this end, let $t := \min\{g(y) + 2\delta, m\}$, where $m := \sup_{p \in G} g(p)$.

▷ Claim 30. We have $\big(g \circ \tilde{\phi}\big)(x) = t$.

Proof. We first consider the case $\big(f \circ \tilde{\psi}\big)(y) = f(x) < m'$. In this case, we have $f(x) = \big(f \circ \tilde{\psi}\big)(y) = g(y) + \delta$ and thus $\big(g \circ \tilde{\phi}\big)(x) = t$. Now suppose $f(x) = m'$. Since $\tilde{\phi}(x) \in g^{-1}(I_{2\delta}(g(y)))$, we must have $\big(g \circ \tilde{\phi}\big)(x) \le t$. Now suppose $\big(g \circ \tilde{\phi}\big)(x) < t \le m$. Then $\big(g \circ \tilde{\phi}\big)(x) = f(x) + \delta = m' + \delta$. In particular, we have $m' + \delta < t \le m$. Now let $s \in (m' + \delta, m)$. Then we have $(\mathcal{U}_\delta f)^{-1}(s) = \emptyset$ while $g^{-1}(s) \neq \emptyset$, a contradiction to the existence of the map $\psi|_{g^{-1}(s)}\colon g^{-1}(s) \to (\mathcal{U}_\delta f)^{-1}(s)$. ◁

We obtain the connectivity of $y$ and $\tilde{\phi}(x)$ in $g^{-1}(I_\delta(f(x)))$ from their connectivity in $g^{-1}(I_{2\delta}(g(y)))$ by define a retraction $\sigma\colon g^{-1}(I_{2\delta}(g(y))) \to g^{-1}(t)$ as a composition of maps

$$\begin{tikzcd}
g^{-1}(I_{2\delta}(g(y))) \arrow[d, hook] \\
g^{-1}(-\infty, g(y)+2\delta] \arrow[d, "\kappa_G"] \\
(\mathcal{E}g)^{-1}(-\infty, g(y)+2\delta] \arrow[d, "\tilde{\sigma}"] \\
(\mathcal{E}g)^{-1}[t, g(y)+2\delta] \arrow[d, "q_{\mathcal{R}\mathcal{E}G}"] \\
(\mathcal{R}\mathcal{E}g)^{-1}[t, g(y)+2\delta] \arrow[d, "\rho_G"] \\
g^{-1}(t),
\end{tikzcd}$$

where

$$\tilde{\sigma}\colon (\mathcal{E}g)^{-1}(-\infty, g(y)+2\delta] \to (\mathcal{E}g)^{-1}[t, g(y)+2\delta],\ (p,s) \mapsto (p, \max\{s, t-g(p)\}).$$

By the defintion of $\tilde{\sigma}$ the map $\sigma\colon g^{-1}(I_{2\delta}(g(y))) \to g^{-1}(t)$ is indeed a retraction. As $\tilde{\phi}(x) \in g^{-1}(t)$ by Claim 30 the points $\sigma(y)$ and $\tilde{\phi}(x)$ are connected in the fiber $g^{-1}(t)$. Since the fibers of $g$ are discrete, this implies that $\sigma(y) = \tilde{\phi}(x)$. Defining the path

$$\gamma\colon [0,1] \to \mathcal{E}G,\ s \mapsto (y, s(t-g(y)))$$

the composition

$$[0,1] \xrightarrow{\gamma} \mathcal{E}G \xrightarrow{q_{\mathcal{R}\mathcal{E}G}} \mathcal{R}\mathcal{E}G \xrightarrow{\rho_G} G$$

yields a path from $y$ to $\sigma(y) = \tilde{\phi}(x)$ in $g^{-1}(I_\delta(f(x)))$. ◀

## References


1 Ulrich Bauer, Håvard Bakke Bjerkevik, and Benedikt Fluhr. Quasi-universality of Reeb graph distances. Preprint, 2021. arXiv:2112.00720.

2 Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. Measuring distance between Reeb graphs. In *Computational geometry (SoCG'14)*, pages 464–473. ACM, New York, 2014.

3 Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. Measuring distance between Reeb graphs. Extended version of conference paper, 2016. arXiv:1307.2839v2.

4 Ulrich Bauer, Claudia Landi, and Facundo Mémoli. The Reeb graph edit distance is universal. *Found. Comput. Math.*, 21(5):1441–1464, 2021. doi:10.1007/s10208-020-09488-3.

5 Ulrich Bauer, Elizabeth Munch, and Yusu Wang. Strong equivalence of the interleaving and functional distortion metrics for Reeb graphs. In *31st International Symposium on Computational Geometry*, volume 34 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 461–475. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015.

6 Robert Cardona, Justin Curry, Tung Lam, and Michael Lesnick. The universal $\ell^p$-metric on merge trees. Preprint, 2021. arXiv:2112.12165.

7 Michele d'Amico, Patrizio Frosini, and Claudia Landi. Natural pseudo-distance and optimal matching between reduced size functions. *Acta Appl. Math.*, 109(2):527–554, 2010. doi:10.1007/s10440-008-9332-1.

8 Vin de Silva, Elizabeth Munch, and Amit Patel. Categorified Reeb graphs. *Discrete Comput. Geom.*, 55(4):854–906, 2016. doi:10.1007/s00454-016-9763-9.

9 Masaki Hilaga, Yoshihisa Shinagawa, Taku Komura, and Tosiyasu L. Kunii. Topology matching for fully automatic similarity estimation of 3d shapes. In Lynn Pocock, editor, *Proceedings of the 28th Annual Conference on Computer Graphics and Interactive Techniques, SIGGRAPH 2001, Los Angeles, California, USA, August 12-17, 2001*, pages 203–212. ACM, 2001. doi:10.1145/383259.383282.

10 Dmitriy Morozov, Kenes Beketayev, and Gunther Weber. Interleaving distance between merge trees. Presented at TopoInVis'13. Manuscript, 2013. URL: https://www.mrzv.org/publications/interleaving-distance-merge-trees/.

11 Georges Reeb. Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. *C. R. Acad. Sci. Paris*, 222:847–849, 1946.

12 Luis N. Scoccola. *Locally persistent categories and metric properties of interleaving distances.* PhD thesis, The University of Western Ontario, 2020. URL: https://ir.lib.uwo.ca/etd/7119.

13 Yoshihisa Shinagawa and Tosiyasu L. Kunii. Constructing a Reeb graph automatically from cross sections. *IEEE Computer Graphics and Applications*, 11(6):44–51, 1991. doi:10.1109/38.103393.

14 Gurjeet Singh, Facundo Mémoli, and Gunnar E. Carlsson. Topological methods for the analysis of high dimensional data sets and 3d object recognition. In Mario Botsch, Renato Pajarola, Baoquan Chen, and Matthias Zwicker, editors, *4th Symposium on Point Based Graphics, PBG@Eurographics 2007, Prague, Czech Republic, September 2-3, 2007*, pages 91–100. Eurographics Association, 2007. doi:10.2312/SPBG/SPBG07/091-100.