A Solution to Ringel's Circle Problem

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Abstract

We construct families of circles in the plane such that their tangency graphs have arbitrarily large girth and chromatic number. This provides a strong negative answer to Ringel's circle problem (1959). The proof relies on a (multidimensional) version of Gallai's theorem with polynomial constraints, which we derive from the Hales-Jewett theorem and which may be of independent interest.

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1 Introduction

A *constellation* (see [10]) is a finite collection of circles in the plane in which no three circles are tangent at the same point. The *tangency graph* $G(\mathcal{C})$ of a constellation $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and edges comprising of the pairs of tangent circles in $\mathcal{C}$. In this paper, graph-theoretic terms such as chromatic number or girth (i.e., the minimum length of a cycle) applied to a constellation $\mathcal{C}$ refer to the tangency graph $G(\mathcal{C})$.

Jackson and Ringel [10] discussed four problems regarding the chromatic number of constellations. The problems are illustrated in Figure 1.

(a) *The penny problem.* What is the maximum chromatic number of a constellation of non-overlapping unit circles?

(b) *The coin problem.* What is the maximum chromatic number of a constellation of non-overlapping circles (of arbitrary radii)?

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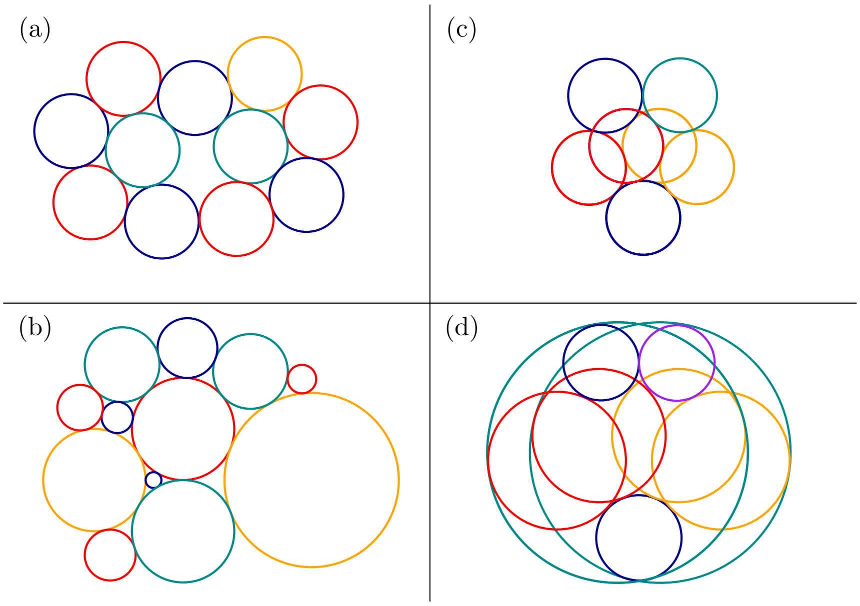

Figure 1 An illustration of the four coloring problems of tangency graphs of constellations: (a) a penny graph, (b) a coin graph, (c) an overlapping penny graph, and (d) a general constellation as in the circle problem.

(c) *The overlapping penny problem.* What is the maximum chromatic number of a (possibly overlapping) constellation of unit circles?

(d) *The circle problem.* What is the maximum chromatic number of a general constellation of circles?

Jackson and Ringel provided a simple proof that the answer to the *penny problem* is 4. The claim that the answer for the *coin problem* is also 4 is equivalent to the *four color theorem* [1, 2]. Indeed, on the one hand, if the circles are non-overlapping, then $G(\mathcal{C})$ is planar and thus 4-colorable by the four-color theorem. On the other hand, by the Koebe-Andreev-Thurston *circle packing theorem* [13], every planar graph can be realized as $G(\mathcal{C})$ for some constellation $\mathcal{C}$ of non-overlapping circles, and hence, the assertion that every such constellation $\mathcal{C}$ is 4-colorable implies the four color theorem.

The *overlapping penny problem* is equivalent to the celebrated Hadwiger-Nelson problem, which asks what is the minimum number of colors needed for a coloring of the plane such that no two points at distance 1 get the same color. Indeed, if all circles in $\mathcal{C}$ have a radius of $1/2$, then two circles are tangent if and only if the distance between their centers is 1. For this setting, Isbell [20] observed about 60 years ago that 7 colors suffice, and only recently de Grey [5] showed that 4 colors are not sufficient, and hence, the chromatic number of the plane lies between 5 and 7.

Unlike for the first three problems, in which a finite upper bound was known already when they were stated, for the *circle problem* no finite upper bound was known. This open problem was introduced for the first time by Ringel [19] in 1959 and appeared in several places as either a question (e.g., [10, 11, 15]) or a conjecture that there is a finite upper bound (e.g., [12]). For lower bounds, Jackson and Ringel [10] presented an example that requires 5 colors, see Figure 1(d). Another such example follows from de Grey's 5-chromatic unit distance graph. No construction requiring more than 5 colors has been known so far.

In this paper, we solve Ringel's circle problem in a strong sense by showing that the chromatic number is unbounded, even if we require high girth.

▶ **Theorem 1.** *There exist constellations of circles in the plane with arbitrarily large girth and chromatic number.*

The constellation condition (that no three circles are tangent at a point) is crucial for Ringel's circle problem to be interesting – otherwise one could drive the chromatic number arbitrarily high by taking a set of circles all tangent at one point. In Theorem 1, however, the condition is redundant because it follows from the stronger condition that the girth of the tangency graph is larger than 3. Actually, we prove an even stronger statement (Theorem 9) in which we additionally forbid pairs of internally tangent circles.

The first author [4] recently proved that there are intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ with arbitrarily large girth and chromatic number. The main tool for this result is a "sparse" version of Gallai's theorem due to Prömel and Voigt [17] (see Theorem 3), whose applications include a modification of Tutte's construction of triangle-free graphs with large chromatic number [6, 7].

To prove Theorem 1, we also use a "sparse" version of Gallai's theorem. However, it is crucial in our context to guarantee that there are no "unwanted" tangencies in the resulting collection of circles. To this end, we develop a refined "sparse" version of Gallai's theorem with additional (polynomial) constraints (Theorem 4). We believe that this version may be applicable to obtaining lower bound constructions for other geometric coloring problems, in which some specific form of algebraic independence is requested.

Tangent circles can be thought of as circles intersecting at zero angle. We extend Theorem 1 to graphs defined by pairs of circles intersecting at an arbitrary fixed angle. Specifically, we say that two intersecting circles C_1 and C_2 intersect at angle θ if at any intersection point of C_1 and C_2, the angle between the tangent line to C_1 and the tangent line to C_2 equals θ. For any $\theta \in [0, \pi/2]$, the *θ-graph* $G_\theta(\mathcal{C})$ of a collection of circles $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and edges comprising the pairs of circles in $\mathcal{C}$ that intersect at angle θ. In particular, the 0-graph is the tangency graph. We extend Theorem 1 as follows.

▶ **Theorem 2.** *For every $\theta \in [0, \pi/2]$, there exist θ-graphs of circles in the plane with arbitrarily large girth and chromatic number.*

The proof of Theorem 2 for $\theta > 0$ is significantly simpler than the proof for $\theta = 0$ corresponding to Theorem 1. Therefore, the remainder of the paper is organized as follows. In Section 2, we introduce Gallai's theorem and prove a version of it with additional constraints as needed for the proof of Theorem 1. In Section 3, we prove Theorem 2 for $\theta > 0$. As the underlying ideas and tools are similar but simpler, this can be considered as a warm-up for the proof of the more involved case $\theta = 0$, which follows in Section 4.

2 Gallai's theorem with constraints

We start by introducing results from Ramsey theory – Gallai's theorem and its versions that we need for the proofs of Theorems 1 and 2.

A *homothetic map* in $\mathbb{R}^d$ is a map $h\colon \mathbb{R}^d \to \mathbb{R}^d$ of the form $h(p) = p^* + \lambda p$ for some $p^* \in \mathbb{R}^d$ and $\lambda > 0$. In other words, a homothetic map is a composition of (positive) uniform scaling and translation. A set $T' \subseteq \mathbb{R}^d$ is a *homothetic copy* of a set $T \subseteq \mathbb{R}^d$ if there is a homothetic map h in $\mathbb{R}^d$ such that $T' = h(T)$.

The following beautiful theorem, which is a generalization of the well-known van der Waerden's theorem on arithmetic progressions [21], was first discovered by Gallai in the 1930s, as reported by Rado [18].

▶ **Gallai's Theorem.** *For every finite set $T \subset \mathbb{R}^d$, there exists a finite set $X \subset \mathbb{R}^d$ such that every k-coloring of X contains a monochromatic homothetic copy of T.*

A *cycle* of length $\ell \geqslant 2$ on a set X is a tuple $(T_1, \ldots, T_\ell)$ of distinct subsets of X such that there exist distinct elements $x_1, \ldots, x_\ell \in X$ with $x_i \in T_i \cap T_{i+1}$ for $i \in [\ell - 1]$ and $x_\ell \in T_\ell \cap T_1$.

In order to guarantee high girth in the proofs of Theorems 1 and 2, we need an appropriate "sparse" version of Gallai's theorem, which excludes short cycles among all homothetic copies of T in X (one of which is guaranteed to be monochromatic). In particular, the following strengthening of Gallai's theorem suffices for the purpose of proving Theorem 2 for $\theta > 0$.

▶ **Theorem 3** (Prömel, Voigt [17]). *For every finite set $T \subset \mathbb{R}^d$ of size at least 3 and for any integers $g \geqslant 3$ and $k \geqslant 1$, there exists a finite set $X \subset \mathbb{R}^d$ such that every k-coloring of X contains a monochromatic homothetic copy of T and no tuple of fewer than g homothetic copies of T in X forms a cycle on X.*

Because Theorem 3 only guarantees the existence of a set X, it is not specific enough to prove Theorem 1. Roughly speaking, in our proof of Theorem 1, we apply a (refined version of) Gallai's theorem to a family of circles in the plane (with $d = 3$, the third coordinate representing the radius) such that the resulting family of circles satisfies a number of additional conditions, e.g., it does not contain two internally tangent circles. To guarantee the additional properties, we develop a refined "sparse" version of Gallai's theorem, which imposes polynomial constraints on the resulting set.

We say that a family $\mathcal{F}$ of $2d$-variate real polynomials *respects* a set $X \subset \mathbb{R}^d$ if $f(p, q) \neq 0$ for all $f \in \mathcal{F}$ and all pairs of distinct points $p, q \in X$.

▶ **Theorem 4.** *Let T be a finite subset of $\mathbb{R}^d$ of size at least 3, let $\mathcal{F}$ be a countable family of $2d$-variate real polynomials that respects T, and let g and k be positive integers. Then there exist a finite set $X \subset \mathbb{R}^d$ and a collection $\mathcal{T}$ of homothetic copies of T in X satisfying the following conditions:*

1. *$\mathcal{F}$ respects X,*
2. *no tuple of fewer than g homothetic copies of T in $\mathcal{T}$ form a cycle,*
3. *every k-coloring of X contains a monochromatic homothetic copy of T in $\mathcal{T}$.*

One of the standard ways of proving Gallai's theorem is to derive it from the Hales-Jewett theorem [9]. Our proof of Theorem 4 goes along the same line.

For $m, n \in \mathbb{N}$, a subset L of the n-dimensional m-cube $[m]^n$ is called a *combinatorial line* if there exist a non-empty set of indices $I = \{i_1, \ldots, i_k\} \subseteq [n]$ and a choice of $x_i^* \in [m]$ for every $i \in [n] \setminus I$ such that

$$L = \big\{(x_1, \ldots, x_n) \in [m]^n : x_{i_1} = \cdots = x_{i_k} \text{ and } x_i = x_i^* \text{ for } i \notin I\big\}.$$

The indices in I are called the *active coordinates* of L.

▶ **Hales-Jewett Theorem.** *For any $m, k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that every k-coloring of $[m]^n$ contains a monochromatic combinatorial line.*

We need the following "sparse" version of the Hales-Jewett theorem.

▶ **Theorem 5** (Prömel, Voigt [16]). *For any $m, g, k \in \mathbb{N}$ with $m \geqslant 3$, there exist $n \in \mathbb{N}$ and a set $H \subseteq [m]^n$ such that every k-coloring of H contains a monochromatic combinatorial line of $[m]^n$ and no tuple of fewer than g combinatorial lines of $[m]^n$ contained in H forms a cycle.*

We also need the following simple algebraic fact.

▶ **Lemma 6.** *For every countable family $\mathcal{F}$ of n-variate real polynomials that are not identically zero, the union of their zero sets $\bigcup_{f\in\mathcal{F}} Z(f)$, where $Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$, has empty interior.*

Proof. Fix $f \in \mathcal{F}$. Clearly, $Z(f)$ is a closed set in $\mathbb{R}^n$. Suppose for the sake of contradiction that there is a point x in the interior of $Z(f)$. Let $y \in \mathbb{R}^n$ be such that $f(y) \neq 0$. The univariate polynomial $f_{x,y}$ given by $f_{x,y}(t) = f(x + t(y - x))$ is not identically zero, because $f_{x,y}(1) = f(y) \neq 0$, so it has finitely many roots. However, $f_{x,y}(t) = 0$ whenever $|t|$ is sufficiently small for the point $x + t(y - x)$ to fall into an open neighborhood of x contained in $Z(f)$. There are infinitely many such values t, which is a contradiction. Hence, $Z(f)$ has empty interior. The lemma now follows by the Baire category theorem – a standard tool from topology, which asserts that a countable union of closed sets with empty interior in a complete metric space (such as $\mathbb{R}^n$ with the Euclidean metric) has empty interior. ◀

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. We can assume without loss of generality that $\mathcal{F}$ contains the polynomial δ defined by

$$\delta(p_1, \ldots, p_d, q_1, \ldots, q_d) = (p_1 - q_1)^2 + \cdots + (p_d - q_d)^2,$$

because distinct points $p, q \in \mathbb{R}^d$ satisfy $\delta(p, q) \neq 0$. Put $T = \{t_1, \ldots, t_m\}$, where $m = |T|$. Let n and $H \subseteq [m]^n$ be as claimed in Theorem 5 applied to $[m]$, g, and k. For a given vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, define a map $\zeta_\gamma \colon H \to \mathbb{R}^d$ by $\zeta_\gamma(x) = \sum_{i=1}^n \gamma_i t_{x_i}$, and put $X_\gamma = \zeta_\gamma(H) = \{\zeta_\gamma(x) : x \in H\} \subset \mathbb{R}^d$.

We aim to find a vector $\gamma \in \mathbb{R}^n$ with positive coordinates such that $\mathcal{F}$ respects the set X_γ. For any $f \in \mathcal{F}$ and any distinct $x, y \in H$, let $F_{f,x,y}$ be the n-variate polynomial defined by

$$F_{f,x,y}(\gamma_1, \ldots, \gamma_n) = f\bigl(\textstyle\sum_{i=1}^n \gamma_i t_{x_i},\ \sum_{i=1}^n \gamma_i t_{y_i}\bigr).$$

Given $f \in \mathcal{F}$ and distinct points $x, y \in H$, let $i \in [n]$ be an index such that $x_i \neq y_i$. Setting $\gamma_i = 1$ and $\gamma_j = 0$ for $j \neq i$, we obtain $F_{f,x,y}(\gamma) = f(t_{x_i}, t_{y_i}) \neq 0$, which shows that $F_{f,x,y}$ is not identically zero. Apply Lemma 6 to the family $\{F_{f,x,y} : f \in \mathcal{F},\ x, y \in H,\ x \neq y\}$ to conclude that the union $\bigcup_{f\in\mathcal{F},x,y\in H,x\neq y} Z(F_{f,x,y})$ of the zero sets of the polynomials $F_{f,x,y}$ has empty interior. In particular, there exists a vector $\gamma \in \mathbb{R}^n$ with positive coordinates such that $F_{f,x,y}(\gamma) \neq 0$ for all $f \in \mathcal{F}$ and all distinct $x, y \in H$, so that $\mathcal{F}$ respects the set X_γ.

Fix such a vector γ, and let $\zeta = \zeta_\gamma$ and $X = X_\gamma$. Condition 1 thus follows. Furthermore, by our assumption that $\delta \in \mathcal{F}$, we have $\delta(\zeta(x), \zeta(y)) = F_{\delta,x,y}(\gamma) \neq 0$ for any distinct $x, y \in H$, which shows that ζ is injective.

Let $\mathcal{L}$ be the set of combinatorial lines that are contained in H. Every combinatorial line $L \in \mathcal{L}$ gives rise to a homothetic copy of T in X as follows: if I is the set of active coordinates of L and the coordinates $i \notin I$ are fixed to x_i in L, then the set

$$\zeta(L) = \{\zeta(x) : x \in L\} = \bigl\{\textstyle\sum_{i\notin I} \gamma_i t_{x_i} + \bigl(\sum_{i\in I} \gamma_i\bigr) t_j : j \in [m]\bigr\}$$

is a homothetic copy of T. Specifically, we have $\zeta(L) = h(T)$ for the homothetic map h given by $h(p) = p^* + \lambda p$ with $p^* = \sum_{i\notin I} \gamma_i t_{x_i}$ and $\lambda = \sum_{i\in I} \gamma_i > 0$. Let $\mathcal{T} = \{\zeta(L) : L \in \mathcal{L}\}$.

We show that conditions 2 and 3 hold for X and $\mathcal{T}$. Since no tuple of fewer than g combinatorial lines in $\mathcal{L}$ form a cycle and ζ is injective, no tuple of fewer than g members of $\mathcal{T}$ form a cycle, which is condition 2. For the proof of condition 3, consider a k-coloring ϕ of X. It induces a k-coloring $x \mapsto \phi(\zeta(x))$ of H, in which, by Theorem 5, there is a monochromatic combinatorial line $L \in \mathcal{L}$. We conclude that the homothetic copy $\zeta(L)$ of T in $\mathcal{T}$ is monochromatic in ϕ. ◀

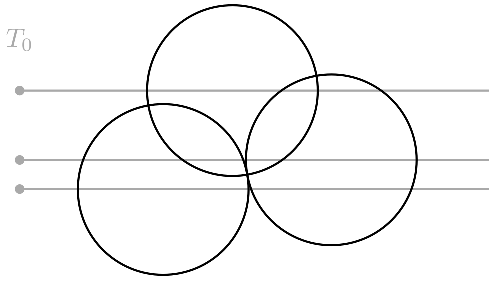

Figure 2 Construction of the set T_0 for $\theta = \pi/2$.

The above proof method can also be used to prove other versions of Gallai's theorem with constraints. On the one hand, we can describe constraints using functions other than polynomials if they satisfy a suitable analogue of Lemma 6, e.g., real analytic functions. On the other hand, we can use other versions of the Hales-Jewett theorem, e.g., the density Hales-Jewett theorem due to Furstenberg and Katznelson [8], which asserts that for any $m \in \mathbb{N}$ and $\alpha > 0$, there is $n \in \mathbb{N}$ such that every subset of $[m]^n$ of size at least αm^n contains a combinatorial line. Then, the same proof leads to the following result.

▶ **Theorem 7.** *Let T be a finite subset of $\mathbb{R}^d$, let $\mathcal{F}$ be a countable family of $2d$-variate real polynomials that respects T, and let $\alpha > 0$. Then there exists a finite set $X \subset \mathbb{R}^d$ such that $\mathcal{F}$ respects X and every subset of X of size at least $\alpha|X|$ contains a homothetic copy of T.*

3 Proof of Theorem 2 for $\theta > 0$

In this section, we prove Theorem 2 for all $\theta \in (0, \pi/2]$. Here is the precise statement.

▶ **Theorem 8.** *For every $\theta \in (0, \pi/2]$ and any integers $g \geqslant 3$ and $k \geqslant 1$, there is a collection of circles $\mathcal{C}$, no two concentric, such that the θ-graph $G_\theta(\mathcal{C})$ has girth at least g and chromatic number at least k.*

Proof. We fix $\theta \in (0, \pi/2)$ and $g \geqslant 3$, and construct the families of circles by induction on k. The base case $k \leqslant 3$ is easy, because all odd cycles can be represented as θ-graphs of circles satisfying the conditions of the theorem.

For the induction step, assume we have already constructed a family of circles $\mathcal{C}_k$, no two concentric, such that the θ-graph $G_\theta(\mathcal{C})$ has girth at least g and chromatic number at least k, where $k \geqslant 3$. Let $\mathfrak{L}(y)$ denote the horizontal line at coordinate $y \in \mathbb{R}$, that is, $\mathfrak{L}(y) = \mathbb{R} \times \{y\}$. To construct a family $\mathcal{C}_{k+1}$, we perform the following process.

1. *Constructing a "template" set from the family $\mathcal{C}_k$.* For each circle $C \in \mathcal{C}_k$, we pick all horizontal lines that intersect C at angle θ (meaning that the angle between the horizontal line and the tangent line to C at either of the intersection points is θ), see Figure 2. There are two such lines when $\theta \in (0, \pi/2)$ and only one (through the center of the circle) when $\theta = \pi/2$. By slightly rotating the family $\mathcal{C}$ if needed, we can guarantee that these lines are all distinct. When $\theta = \pi/2$, this requires the additional assumption that no two circles in $\mathcal{C}$ are concentric (which is otherwise superfluous). Let $T_0 \subset \mathbb{R}$ be the set of y-coordinates of the lines.
2. *Applying Gallai's theorem.* Theorem 3 in $\mathbb{R}$ applied to the set T_0 yields a finite set $Y \subset \mathbb{R}$ such that every k-coloring of Y contains a monochromatic homothetic copy of T_0 and no tuple of fewer than $\lceil g/2 \rceil$ homothetic copies of T_0 in Y form a cycle.

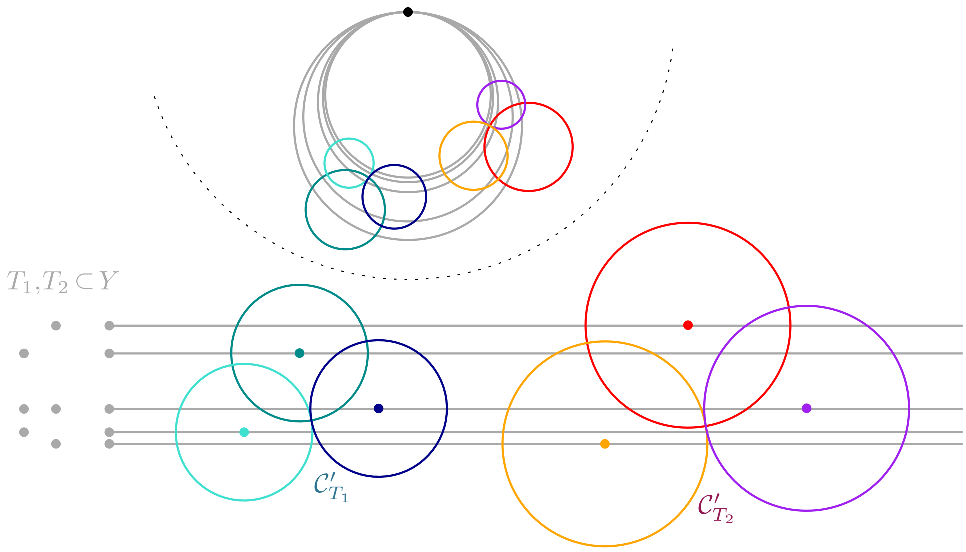

Figure 3 Illustration for steps 3–5 in the proof of Theorem 8 for $\theta = \pi/2$: Construction of a preliminary family from Y and the result after inversion with respect to the dotted circle. Note that the set Y consists of two homothetic copies of the set T_0 from Fig. 2 that have one element in common.

3. *Geometric interpretation of the resulting set.* The set Y gives rise to the family of horizontal lines $\mathcal{L}' = \{\mathfrak{L}(y) : y \in Y\}$. Let $\mathcal{T}$ be the family of homothetic copies of T_0 in Y.
4. *Attaching a copy of $\mathcal{C}_k$ for each homothetic copy of the "template".* For each $T \in \mathcal{T}$, we consider the set of horizontal lines $\mathcal{L}'_T = \{\mathfrak{L}(y) : y \in T\}$ and construct a homothetic copy $\mathcal{C}'_T$ of $\mathcal{C}_k$ such that each line in $\mathcal{L}'_T$ intersects a single circle $\mathcal{C}'_T$ at angle θ; see Figure 3. (Each circle in $\mathcal{C}'_T$ intersects two lines in $\mathcal{L}'_T$ at angle θ when $\theta \in (0, \pi/2)$ and only one when $\theta = \pi/2$.) This is possible because the set of lines $\{\mathfrak{L}(y) : y \in T_0\}$ has this property with respect to $\mathcal{C}_k$, and T is a homothetic copy of T_0. We spread the copies $\mathcal{C}'_T$ horizontally so that a vertical line separates $\mathcal{C}'_{T_1}$ from $\mathcal{C}'_{T_2}$ for any distinct $T_1, T_2 \in \mathcal{T}$. Let $\mathcal{C}' = \bigcup_{T \in \mathcal{T}} \mathcal{C}'_T$.
5. *Constructing the final family $\mathcal{C}_{k+1}$ via inversion.* Finally, we construct the family $\mathcal{C}_{k+1}$ by applying a geometric inversion to the lines and circles in $\mathcal{L}' \cup \mathcal{C}'$, where the center of inversion is chosen not to lie on any of these lines or circles. See Figure 3 for an illustration. By basic properties of inversion, the resulting family consists only of circles; in particular, the lines in $\mathcal{L}'$ turn into a bunch of circles tangent to the horizontal line at the center of inversion [3, chapter 6]. To ensure that the inversion does not create concentric circles, we choose the center of inversion not to lie on any line passing through the centers of two circles in $\mathcal{C}'$ or any vertical line passing through the center of a circle in $\mathcal{C}'$.

We claim that the θ-graph $G_\theta(\mathcal{C}_{k+1})$ has girth at least g and chromatic number at least $k+1$. Since inversion preserves angles, this graph is isomorphic to the θ-graph of $\mathcal{L}' \cup \mathcal{C}'$ (which is defined analogously to the θ-graph for a collection of circles). Let G denote the latter θ-graph. It is thus sufficient to prove that G has girth at least g and chromatic number at least $k+1$.

To this end, we observe that by the construction, G has the following structure: for every $T \in \mathcal{T}$, the subgraph of G induced on the vertices in $\mathcal{C}'_T$ is isomorphic to $G_\theta(\mathcal{C}_k)$, and the remaining edges form a collection of bipartite subgraphs between the vertices in $\mathcal{C}'_T$ and the vertices in $\mathcal{L}'_T$, where each vertex in $\mathcal{C}'_T$ is adjacent to two corresponding vertices in $\mathcal{L}'_T$ if $\theta \in (0, \pi/2)$ and only one if $\theta = \pi/2$.

We exploit the structure above in the proofs of the final two claims. They are standard when applying generalizations of Tutte's construction; see, e.g., [4, 14].

▷ Claim 8.1. The graph G has girth at least g.

Proof. For every $T \in \mathcal{T}$, every cycle in G that lies entirely within $\mathcal{C}'_T$ has length at least g because the subgraph of G induced on the vertices in $\mathcal{C}'_T$ is isomorphic to $G_\theta(\mathcal{C}_k)$, the girth of which is at least g by the induction hypothesis. It thus remains to consider a cycle in G of length $\ell \geqslant 3$ that does not lie entirely within $\mathcal{C}'_T$ for any $T \in \mathcal{T}$. It must contain vertices from $\mathcal{L}'$, say, $L_1, \ldots, L_m$ in this order along the cycle. For each $i \in [m]$, since L_i has no edges to the rest of $\mathcal{L}'$ and at most one edge to $\mathcal{C}'_T$ for each $T \in \mathcal{T}$, the neighbors of L_i on the cycle lie in two different sets of the form $\mathcal{C}'_T$. For each $i \in [m]$, let $T_i \in \mathcal{T}$ be such that the part of the cycle between L_i and L_{i+1} (or L_1 when $i = m$) lies within $\mathcal{C}'_{T_i}$.

It follows that $(T_1, \ldots, T_m)$ is a cycle in $\mathcal{T}$ of length m or contains such a cycle if some members of $\mathcal{T}$ repeat among $T_1, \ldots, T_m$. Hence, Theorem 3 yields $m \geqslant \lceil g/2 \rceil$. Since the cycle contains at least one vertex from $\mathcal{C}'$ between L_i and L_{i+1} (or L_1 when $i = m$) for any $i \in [m]$, we conclude that $\ell \geqslant 2m \geqslant g$. ◁

▷ Claim 8.2. The graph G has chromatic number at least $k + 1$.

Proof. Suppose for the sake of contradiction that the graph G is k-colorable. Pick a proper k-coloring of G, and consider its restriction to the vertices in $\mathcal{L}'$. It induces a k-coloring of Y via the correspondence $Y \ni y \leftrightarrow \mathfrak{L}(y) \in \mathcal{L}'$. It follows from the application of Theorem 3 that there is a monochromatic homothetic copy T of T_0 in $\mathcal{T}$, which means that the set of lines $\mathcal{L}'_T$ is monochromatic. Since the edges of G that connect these lines with $\mathcal{C}'_T$ match all of $\mathcal{C}'_T$, their common color does not occur on the circles in $\mathcal{C}'_T$. Therefore, the given k-coloring of G induces a proper $(k-1)$-coloring of the subgraph of G induced on the vertices in $\mathcal{C}'_T$, which is isomorphic to $G_\theta(\mathcal{C}_k)$. This contradicts the assumption that the graph $G_\theta(\mathcal{C}_k)$ has chromatic number at least k. ◁

This completes the proof of Theorem 8 by induction. ◀

Observe that the proof above cannot be used for $\theta = 0$ to prove Theorem 1, because the inversion at step 5 turns all lines in $\mathcal{L}'$ into circles tangent at one point, so the resulting collection of circles is not a constellation (and the resulting tangency graph has girth 3).

4 Proof of Theorem 1

In this section, we prove Theorem 1. For the purpose of induction, we prove the following stronger statement, which directly implies Theorem 1.

▶ **Theorem 9.** *For any integers $g \geqslant 3$ and $k \geqslant 1$, there exists a collection of circles $\mathcal{C}$, no two concentric and no two internally tangent, such that the tangency graph $G(\mathcal{C})$ has girth at least g and chromatic number at least k.*

For the sake of clarity, we first present the construction of the families $\mathcal{C}$ and then we prove that the construction satisfies the requirements of the theorem.

4.1 High-level description of the construction

We fix $g \geqslant 3$ and prove the theorem by induction on k. The base case $k \leqslant 3$ is easy, because all odd cycles can be represented as tangency graphs of circles satisfying the conditions of the theorem.

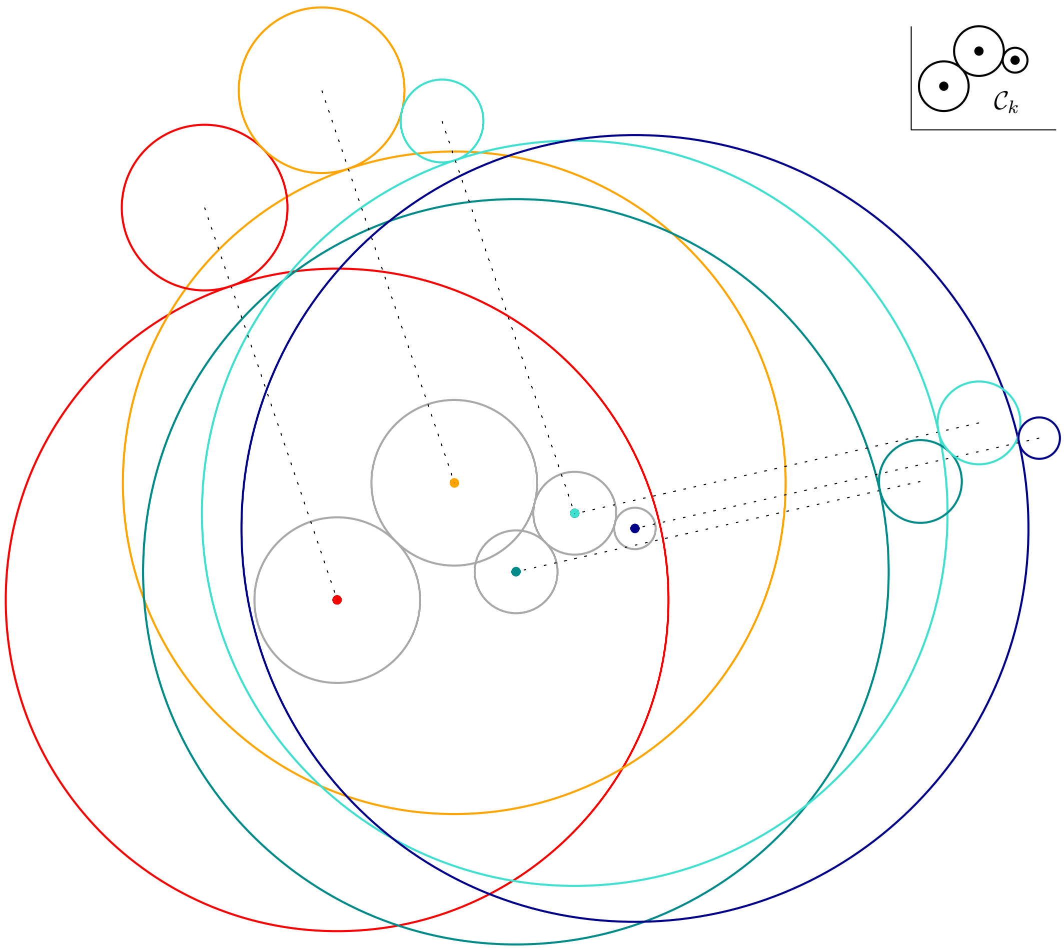

Figure 4 Illustration for the construction of the family $\mathcal{C}_{k+1}$ from $\mathcal{C}_k$. Gray circles represent a part of the set $X \subset \mathbb{R}^3$ containing two homothetic copies of T_0 in $\mathcal{T}$. The family $\mathcal{C}_{k+1}$ contains a (large) circle for each point in X and a homothetic copy of $\mathcal{C}_k$ for each homothetic copy of T_0 in $\mathcal{T}$.

For the induction step, assume we have already constructed a family of circles $\mathcal{C}_k$, no two concentric and no two internally tangent, such that the tangency graph $G(\mathcal{C}_k)$ has girth at least g and chromatic number at least k. Let $\mathfrak{C}(x, y, r)$ denote the circle with center $(x, y) \in \mathbb{R}^2$ and radius $r > 0$. To construct a family $\mathcal{C}_{k+1}$ with girth g and chromatic number $k + 1$, we perform the following process. See Figure 4 for an illustration.

1. *Constructing a "template" set from the family $\mathcal{C}_k$.* We represent each circle $\mathfrak{C}(x, y, r) \in \mathcal{C}_k$ by the point $(x, y, r) \in \mathbb{R}^3$, to obtain the set

$$T_0 = \big\{(x, y, r) \in \mathbb{R}^3 : \mathfrak{C}(x, y, r) \in \mathcal{C}_k\big\} \subset \mathbb{R}^3.$$

2. *Applying Gallai's theorem with constraints.* Theorem 4 in $\mathbb{R}^3$ applied to the set T_0 with appropriate constraints to be detailed below yields a finite set $X \subset \mathbb{R}^3$ and a collection $\mathcal{T}$ of homothetic copies of T_0 in X such that every k-coloring of X contains a monochromatic homothetic copy of T_0 in $\mathcal{T}$ and no tuple of fewer than $\lceil g/3 \rceil$ homothetic copies of T_0 in $\mathcal{T}$ form a cycle. For each $T \in \mathcal{T}$, let h_T be the homothetic map from T_0 to T in $\mathbb{R}^3$. It has the following form for some $x^*_T, y^*_T, r^*_T \in \mathbb{R}$ and $\lambda_T > 0$:

$$h_T : \mathbb{R}^3 \ni (x, y, r) \mapsto \big(x^*_T + \lambda_T x,\ y^*_T + \lambda_T y,\ r^*_T + \lambda_T r\big) \in \mathbb{R}^3.$$

3. *Geometric interpretation of the resulting set.* Let $R_0 = \max\{r' : (x', y', r') \in X\}$, and let $R \in \mathbb{R}$ satisfy $R > R_0$. (In the sequel, R is going to be "large".) The set X gives rise to a family of "large" circles $\mathcal{C}'_R$, parameterized by R, defined as follows:

$$\mathcal{C}'_R = \{\mathfrak{C}(x', y', R - r') : (x', y', r') \in X\}.$$

The use of the stronger Theorem 4 with appropriate constraints, rather than Theorem 3, allows us to infer that no two circles in $\mathcal{C}'_R$ are concentric or internally tangent.

4. *Attaching a copy of $\mathcal{C}_k$ for each homothetic copy of the "template".* We pick a set $\{\phi_T\}_{T\in\mathcal{T}}$ of distinct angles in $[0, \pi)$ that satisfy a certain condition to be detailed below. For every $T \in \mathcal{T}$ and every circle $C = \mathfrak{C}(x, y, r) \in \mathcal{C}_k$, we define the following two circles, where $(x', y', r') = h_T(x, y, r) \in T$:

$$\mu_{R,T}(C) = \mathfrak{C}(x', y', R - r'), \quad \text{which is a circle in } \mathcal{C}'_R,$$
$$\nu_{R,T}(C) = \mathfrak{C}\big(x' + (R - r^*_T)\cos(\phi_T),\ y' + (R - r^*_T)\sin(\phi_T),\ \lambda_T r\big).$$

In words, $\mu_{R,T}(C)$ is a "large" circle with center (x', y'), and $\nu_{R,T}(C)$ is a "small" circle with center translated from (x', y') in direction ϕ_T, externally tangent to $\mu_{R,T}(C)$. For every $T \in \mathcal{T}$, we set

$$\mathcal{C}'_{R,T} = \{\nu_{R,T}(C) : C \in \mathcal{C}_k\}.$$

Since the angles ϕ_T are distinct and the radii of the circles $\nu_{R,T}(C)$ do not depend on R, when R is sufficiently large, the circles $\mathcal{C}'_{R,T_1}$ are disjoint from those in $\mathcal{C}'_{R,T_2}$ for any distinct $T_1, T_2 \in \mathcal{T}$.

5. *Constructing the final family $\mathcal{C}_{k+1}$.* Finally, we define

$$\mathcal{C}''_R = \mathcal{C}'_R \cup \bigcup_{T\in\mathcal{T}} \mathcal{C}'_{R,T}.$$

We will show that $\mathcal{C}_{k+1} := \mathcal{C}''_R$ satisfies all claimed properties if R is sufficiently large.

Comparison with the construction in Section 3

This construction follows the general strategy of the construction for the θ-graph presented in Section 3. Notable differences result from the need to avoid multiple circles mutually tangent at one point, which arise in the last step of that construction when applying inversion.

1. While in Section 3, we have $T_0 \subset \mathbb{R}$, here we have to resort to the more complex choice of $T_0 \subset \mathbb{R}^3$.
2. While in Section 3, the standard "sparse" version of Gallai's theorem is sufficient, here we need the stronger version with constraints, to be able to infer that the resulting set of circles avoids concentricities and internal tangencies.
3. The construction of "large" circles here is explicit and uses a parameter R that must be chosen appropriately, while in Section 3, horizontal lines are used instead.
4. The construction of "small" circles here is more involved and uses a set of parameters $\{\phi_T\}_{T\in\mathcal{T}}$ that must be chosen appropriately.
5. In Section 3, a final application of inversion is required to transform the horizontal lines into circles, which is no longer needed in the construction here.

The proof of validity of the construction is somewhat more complex, accordingly.

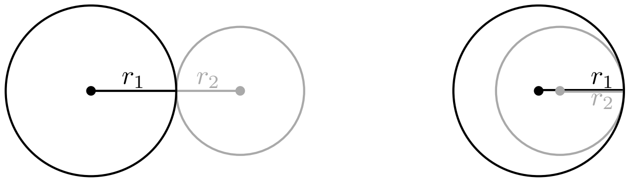

Figure 5 External and internal tangency of circles.

4.2 Proof of Theorem 9

In the proof, we use the following two simple lemmas in addition to Theorem 4.

▶ **Lemma 10.** *Circles $\mathfrak{C}(x_1, y_1, r_1)$ and $\mathfrak{C}(x_2, y_2, r_2)$ are*
- *externally tangent if and only if $(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 + r_2)^2$,*
- *internally tangent if and only if $(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 - r_2)^2$.*

Proof. Circles $\mathfrak{C}(x_1, y_1, r_1)$ and $\mathfrak{C}(x_2, y_2, r_2)$ are externally tangent if and only if the segment connecting their centers (x_1, x_2) and (y_1, y_2) has length $r_1 + r_2$, and they are internally tangent if and only if it has length $|r_1 - r_2|$. See Figure 5 for an illustration. ◀

▶ **Lemma 11.** *If $a, b, c, \varphi \in \mathbb{R}$, $(a, b) \neq (0, 0)$, and the vectors (a, b) and $(\cos\varphi, \sin\varphi)$ are not parallel in $\mathbb{R}^2$, then the equality $(a + R\cos\varphi)^2 + (b + R\sin\varphi)^2 = (c + R)^2$ holds for at most one value $R \in \mathbb{R}$.*

Proof. Consider the univariate polynomial f defined by

$$\begin{aligned} f(R) &= (a + R\cos\varphi)^2 + (b + R\sin\varphi)^2 - (c + R)^2 \\ &= (a^2 + b^2 - c^2) + 2R(a\cos\varphi + b\sin\varphi - c). \end{aligned}$$

It is identically zero only if $a^2 + b^2 = c^2$ and $a\cos\varphi + b\sin\varphi = c$. However, if $a^2 + b^2 = c^2$ and the vectors (a, b) and $(\cos\varphi, \sin\varphi)$ are not parallel, then the Cauchy-Schwarz inequality yields $|a\cos\varphi + b\sin\varphi| < \sqrt{a^2 + b^2} \cdot \sqrt{\cos^2\varphi + \sin^2\varphi} = |c|$, so $a\cos\varphi + b\sin\varphi \neq c$. Since f is not identically zero and has degree at most 1, it has at most one root in $\mathbb{R}$. ◀

We are now ready to present the details of the proof of Theorem 9.

Proof of Theorem 9. We proceed by induction on k and, for the induction step, construct the family of circles $\mathcal{C}_{k+1}$ from a family of circles $\mathcal{C}_k$ as was described above. The following claim describes the property of the set X constructed by applying our enhanced version of Gallai's theorem, namely Theorem 4, with appropriate polynomial constraints to T_0.

▷ Claim 9.1. There exists a finite set $X \subset \mathbb{R}^3$ and a collection $\mathcal{T}$ of homothetic copies of T_0 in X with the following properties:

1. for any two distinct points $(x_1, y_1, r_1), (x_2, y_2, r_2) \in X$, we have
   a. $(x_1, y_1) \neq (x_2, y_2)$,
   b. $(x_1 - x_2)^2 + (y_1 - y_2)^2 \neq (r_1 - r_2)^2$,
2. no tuple of fewer than $\lceil g/3 \rceil$ homothetic copies of T_0 in $\mathcal{T}$ form a cycle,
3. every k-coloring of X contains a monochromatic homothetic copy of T_0 in $\mathcal{T}$.

Proof. Consider the following two 6-variate polynomials:

$$\begin{aligned} f_{\mathsf{a}}(x_1, y_1, r_1, x_2, y_2, r_2) &= (x_1 - x_2)^2 + (y_1 - y_2)^2, \\ f_{\mathsf{b}}(x_1, y_1, r_1, x_2, y_2, r_2) &= (x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2. \end{aligned}$$

We have $f_{\mathsf{a}}(x_1, y_1, r_1, x_2, y_2, r_2) \neq 0$ if and only if $(x_1, y_1) \neq (x_2, y_2)$, which holds in particular for distinct points $(x_1, y_1, r_1), (x_2, y_2, r_2) \in T_0$, by the assumption that no two circles in $\mathcal{C}_k$ are concentric. Lemma 10 and the assumption that no two circles in $\mathcal{C}$ are internally tangent imply $f_{\mathsf{b}}(x_1, y_1, r_1, x_2, y_2, r_2) \neq 0$ for any distinct points $(x_1, y_1, r_1), (x_2, y_2, r_2) \in T_0$. Theorem 4 applied to T_0, $\mathcal{F} = \{f_{\mathsf{a}}, f_{\mathsf{b}}\}$, $\lceil g/3 \rceil$, and k directly yields the requested set X and collection $\mathcal{T}$. ◁

For the construction of the families of circles $\{\nu_{R,T}\}_{T\in\mathcal{T}}$, we let $\{\phi_T\}_{T\in\mathcal{T}}$ be a set of distinct angles in $[0, \pi)$ such that for every $T \in \mathcal{T}$, the unit vector $(\cos\phi_T, \sin\phi_T) \in \mathbb{R}^2$ is not parallel to the vector $(x_1 - x_2, y_1 - y_2)$ for any distinct points $(x_1, y_1, r_1), (x_2, y_2, r_2) \in X$, where the latter vector is non-zero, by condition 1a of Claim 9.1.

We now claim that for a sufficiently large R, the family $\mathcal{C}''_R$ defined above satisfies the following conditions on concentricity and tangency.

▷ Claim 9.2. The following holds when R is sufficiently large:

1. no two circles in $\mathcal{C}''_R$ are concentric,
2. no two circles in $\mathcal{C}''_R$ are internally tangent,
3. a pair of circles in $\mathcal{C}''_R$ is externally tangent if and only if it belongs to one of the two following types:
   a. $\mu_{R,T}(C)$ and $\nu_{R,T}(C)$ for any $T \in \mathcal{T}$ and any $C \in \mathcal{C}_k$,
   b. $\nu_{R,T}(C_1)$ and $\nu_{R,T}(C_2)$ for any $T \in \mathcal{T}$ and any $C_1, C_2 \in \mathcal{C}_k$ that are externally tangent.

Proof. First, consider two distinct circles $C' = \mathfrak{C}(x', y', R - r')$ and $C'' = \mathfrak{C}(x'', y'', R - r'')$ in $\mathcal{C}'_R$, where $(x', y', r'), (x'', y'', r'') \in X$. By condition 1a of Claim 9.1, the circles C' and C'' are not concentric. By Lemma 10, the circles C' and C'' are internally tangent if and only if $(x' - x'')^2 + (y' - y'')^2 = (r' - r'')^2$, which does not hold due to condition 1b of Claim 9.1. Also by Lemma 10, the circles C' and C'' are externally tangent if and only if $(x' - x'')^2 + (y' - y'')^2 = (2R - r' - r'')^2$, which does not hold when R is sufficiently large.

Next, let $T \in \mathcal{T}$, and consider two distinct circles C'_1 and C'_2 such that for $i \in [2]$, we have $C'_i = \mathfrak{C}\big(x'_i + (R - r^*_T)\cos\phi_T,\ y'_i + (R - r^*_T)\sin\phi_T,\ \lambda_T r_i\big) = \nu_{R,T}(C_i)$, where $C_i = \mathfrak{C}(x_i, y_i, r_i) \in \mathcal{C}_k$ and $h_T(x_i, y_i, r_i) = (x'_i, y'_i, r'_i) \in T$. The assumption that the circles C_1 and C_2 are not concentric, that is, $(x_1, y_1) \neq (x_2, y_2)$, yields $(x'_1, y'_1) \neq (x'_2, y'_2)$, which implies that the circles C'_1 and C'_2 are not concentric. By Lemma 10, the circles C'_1 and C'_2 are internally tangent if and only if $(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = \lambda_T^2(r_1 - r_2)^2$, which is equivalent to $(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 - r_2)^2$, which does not hold due to the assumption that C_1 and C_2 are not internally tangent. Also by Lemma 10, the circles C'_1 and C'_2 are externally tangent if and only if $(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = \lambda_T^2(r_1 + r_2)^2$, which is equivalent to $(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 + r_2)^2$, which means that C_1 and C_2 are externally tangent.

Next, for two distinct $T_1, T_2 \in \mathcal{T}$, consider circles of the form $C'_1 = \nu_{R,T_1}(C_1)$ and $C'_2 = \nu_{R,T_2}(C_2)$, where $C_1, C_2 \in \mathcal{C}_k$. Since $\phi_{T_1} \neq \phi_{T_2}$ and the radii of C'_1 and C'_2 are independent of R, the circles C'_1 and C'_2 are arbitrarily far apart as R grows. In particular, they are disjoint and not nested when R is sufficiently large.

Finally, consider a circle $C' = \mathfrak{C}(x', y', R - r') \in \mathcal{C}'_R$, where $(x', y', r') \in X$, and a circle $C'' = \mathfrak{C}\big(x'' + (R - r^*_T)\cos\phi_T,\ y'' + (R - r^*_T)\sin\phi_T,\ \lambda_T r\big) = \nu_{R,T}(C)$, where $C = \mathfrak{C}(x, y, r) \in \mathcal{C}_k$ and $h_T(x, y, r) = (x'', y'', r'') \in T$. When $R > r^*_T$, since the vectors $(x'' - x', y'' - y')$ and $(\cos\phi_T, \sin\phi_T)$ are not parallel unless $(x', y') = (x'', y'')$, the circles C' and C'' are not concentric. By Lemma 10, the circles C' and C'' are externally tangent if and only if

$$\big(x'' - x' + (R - r^*_T)\cos\phi_T\big)^2 + \big(y'' - y' + (R - r^*_T)\sin\phi_T\big)^2 = (R - r' + \lambda_T r)^2,$$

which is equivalent to

$$\big((x'' - x') + R'\cos\phi_T\big)^2 + \big((y'' - y') + R'\sin\phi_T\big)^2 = \big((r'' - r') + R'\big)^2,$$

where $R' = R - r_T^*$. By Lemma 11, the equality above holds for at most one value of R' (so at most one value of R) unless $(x', y', r') = (x'', y'', r'')$, in which case it clearly holds for every value of R' (so every value of R). The latter case means that $C'' = \mu_{R,T}(C)$, as requested in case 3a. Also by Lemma 10, the circles C' and C'' are internally tangent if and only if

$$\big(x'' - x' + (R - r_T^*)\cos\phi_T\big)^2 + \big(y'' - y' + (R - r_T^*)\sin\phi_T\big)^2 = (R - r' - \lambda_T r)^2,$$

which is equivalent to

$$\big((x'' - x') + R'\cos\phi_T\big)^2 + \big((y'' - y') + R'\sin\phi_T\big)^2 = \big((2r_T^* - r' - r'') + R'\big)^2,$$

where $R' = R - r_T^*$. By Lemma 11, the equality above holds for at most one value of R' (so at most one value of R) unless $(x', y', r') = (x'', y'', r'')$. In the latter case, C' and C'' are externally tangent (as we have shown previously), so they cannot be internally tangent. ◁

Let $R > R_0$ be sufficiently large for the conclusions of Claim 9.2 to hold. Conditions 2 and 3 of Claim 9.2 imply the following structure of the tangency graph $G(\mathcal{C}''_R)$: for every $T \in \mathcal{T}$, the subgraph induced on the vertices in $\mathcal{C}'_{R,T}$ is isomorphic to $G(\mathcal{C}_k)$ and the remaining edges form a collection of matchings between the vertices in $\mathcal{C}'_{R,T}$ (which are of the form $\nu_{R,T}(C)$ for $C \in \mathcal{C}_k$) and the vertices in $\mathcal{C}'_R$ of the form $\mu_{R,T}(C)$ for $C \in \mathcal{C}_k$. We exploit this structure in the proofs of the final two claims, which are analogous to Claims 8.1 and 8.2.

▷ Claim 9.3. The tangency graph $G(\mathcal{C}''_R)$ has girth at least g.

Proof. Let $G = G(\mathcal{C}''_R)$. For every $T \in \mathcal{T}$, since the subgraph of G induced on the vertices in $\mathcal{C}'_{R,T}$ is isomorphic to $G(\mathcal{C})$, the girth of which is at least g by the induction hypothesis, every cycle in G that lies entirely within $\mathcal{C}'_{R,T}$ has length at least g. Consider now a cycle in G of length $\ell \geqslant 3$ that does not lie entirely within $\mathcal{C}'_{R,T}$ for any $T \in \mathcal{T}$. It must contain vertices from $\mathcal{C}'_R$, say, $C_1, \ldots, C_m$ in this order along the cycle. For each $i \in [m]$, since C_i has no edges to the rest of $\mathcal{C}'_R$ and at most one edge to $\mathcal{C}'_{R,T}$ for each $T \in \mathcal{T}$, the neighbors of C_i on the cycle lie in two different sets of the form $\mathcal{C}'_{R,T}$. For each $i \in [m]$, let $T_i \in \mathcal{T}$ be such that the part of the cycle between C_i and C_{i+1} (or C_1 if $i = m$) lies within $\mathcal{C}'_{R,T_i}$. It follows that $(T_1, \ldots, T_m)$ is a cycle in $\mathcal{T}$ of length m or contains such a cycle if some members of $\mathcal{T}$ repeat among $T_1, \ldots, T_m$. Condition 2 of Claim 9.1 yields $m \geqslant \lceil g/3 \rceil$. Since there are at least two vertices from $\mathcal{C}'_{R,T_i}$ between C_i and C_{i+1} (or C_1 when $i = m$) for any $i \in [m]$, we conclude that $\ell \geqslant 3m \geqslant g$. ◁

▷ Claim 9.4. The tangency graph $G(\mathcal{C}''_R)$ has chromatic number at least $k+1$.

Proof. Suppose for the sake of contradiction that the graph $G = G(\mathcal{C}''_R)$ is k-colorable. Pick a proper k-coloring of G, and consider its restriction to the vertices in $\mathcal{C}'_R$. It induces a k-coloring of X via the correspondence $X \ni (x', y', r') \leftrightarrow \mathfrak{C}(x', y', R - r') \in \mathcal{C}'_R$. By condition 3 of Claim 9.1, there is a monochromatic homothetic copy T of T_0 in $\mathcal{T}$, which means that the set of circles $\{\mu_{R,T}(C) : C \in \mathcal{C}_k\}$ is monochromatic. Since these circles are connected to $\mathcal{C}'_{R,T}$ by a perfect matching in G, their common color does not occur on the circles in $\mathcal{C}'_{R,T}$. Therefore, the given k-coloring of G induces a proper $(k-1)$-coloring of the graph $G(\mathcal{C}'_{R,T})$, which is isomorphic to $G(\mathcal{C}_k)$. This contradicts the assumption that the graph $G(\mathcal{C}_k)$ has chromatic number at least k. ◁

We complete the proof of the induction step by setting $\mathcal{C}_{k+1} = \mathcal{C}''_R$ and observing that the induction statement follows from Claims 9.2 (conditions 1 and 2), 9.3, and 9.4. ◀

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