Universal Properties of Catalytic Variable Equations

Michael Drmota
TU Wien, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

Eva-Maria Hainzl
TU Wien, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

Abstract

Catalytic equations appear in several combinatorial applications, most notably in the enumeration of lattice paths and in the enumeration of planar maps. The main purpose of this paper is to show that under certain positivity assumptions the dominant singularity of the solution function has a universal behavior. We have to distinguish between linear catalytic equations, where a dominating square-root singularity appears, and non-linear catalytic equations, where we – usually – have a singularity of type 3/2.

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1 Introduction

Catalytic equations have their origin mostly in map enumeration [12] and in lattice path enumeration [3]. Such equations were first solved with the help of the *kernel method* [3, 11, 1] (in the linear case) and with the help of the *quadratic method* [12, 5] (in the quadratic case). Both approaches were unified and extended by Bousquet-Mélou and Jehanne [4]. They considered *general catalytic equations* of the form

$$P(z, u, M(z, u), M_1(z), \ldots, M_k(z)) = 0, \tag{1}$$

where $P(z, u, x_0, x_1, \ldots x_k)$ is a polynomial and all power series $M(z, u), M_1(z), \ldots, M_k(z)$ are characterized by this equation.

The variable "u" is called *catalytic* since it is usually an auxiliary variable that counts an additional (usually combinatorial) parameter which simplifies the recursive decomposition of the structure of interest. In general, one is just interested in the function $M(z, 0)$, $M(z, 1)$ or in $M_1(z)$.

One of the most prominent examples is the counting problem of rooted planar maps that goes back to Tutte [12]. Let $M_k(z)$ denote the generating function of those maps, where the root face has valency $k \ge 0$. Then we have $M_0(z) = 1$ and

$$M_k(z) = z \sum_{j=0}^{k-2} M_j(z) M_{k-j-2}(z) + z \sum_{j=k-1}^{\infty} M_j(z) \qquad (k \ge 1) \tag{2}$$

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where the right sum arises if the root edge is not a bridge and the left sum if the deletion of the root edge results in decomposing the map into two components. One is interested in the generating function $M(z) = \sum_{k\ge 0} M_k(z)$ of all planar maps. By introducing the variable u and setting $M(z,u) = \sum_{k\ge 0} M_k(z)u^k$, the infinite system (2) rewrites to the *catalytic equation*

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}. \tag{3}$$

By using the above mentioned quadratic method the equation can be explicitly solved:

$$M(z) = M(z,1) = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2}.$$

This leads to an explicit formula $M_n = [z^n]\, M(z,1) = \frac{2(2n)!}{(n+2)!n!}3^n$ and to an asymptotic one: $M_n \sim (2/\sqrt{\pi})12^n n^{-5/2}$. Note that the asymptotic behavior is reflected by the dominant singular behavior of $M(z,1)$ at $z_0 = 1/12$. The type of the singularity is $3/2$ which translates to the critical exponent $-5/2 = -1 - 3/2$ by the well known Transfer Lemma [10].

In [4] several applications mostly from map enumeration (different classes of planar maps, constellations, hard particles in planar maps etc.) are given. Bousquet-Mélou and Jehanne [4] considered in particular equations of the form

$$M(z,u) = F_0(z,u) + zQ\left(z, u, M(z,u), \Delta^{(1)}(M(z,u)), \ldots, \Delta^{(k)}(M(z,u))\right), \tag{4}$$

where $F_0(z,u)$ and $Q(z,u,\alpha_0,\alpha_1,\ldots,\alpha_k)$ are polynomials and where we have used the abbreviations

$$\Delta^{(j)}(M(z,u)) = \frac{M(z,u) - M(z,0) - u\, M_u(z,0) - \cdots u^{j-1} M_{u^{j-1}}(z,0)}{u^j} \qquad (j \ge 1).$$

It is convenient to consider just the catalytic variable u at 0. In the above case of planar maps we substitute u by $u+1$ to reduce it to this case.

One main result of [4] it that equations of the form (1) can be solved with the help of proper systems of polynomial equations. Hence, the solutions are always algebraic functions and consequently for every singularity we have a Puiseux expansion. However, this approach does not specify the kind of the Puiseux expansion. There is in principle no restriction on the rational exponents that might occur.

However, if we consider the special case $k = 1$ (in (4))

$$M(z,u) = F_0(z,u) + zQ\left(z, u, M(z,u), \frac{M(z,u) - M(z,0)}{u}\right), \tag{5}$$

where $F_0(z,u)$ and $Q(z,u,\alpha_0,\alpha_1)$ are polynomials with *non-negative coefficients*, then Drmota, Noy, and Yu [7] showed that there is a dichotomy (under natural conditions on Q). If Q is linear in α_0 and α_1 then the dominant singularity is of type $\frac{1}{2}$, that is, a square-root singularity which leads to an asymptotic behavior for the coefficients of the form $\sim c\rho^n n^{-3/2}$. However, in the non-linear case the dominant singularity is of type $\frac{3}{2}$ (as in the above mentioned example of planar maps) which means that the coefficients are asymptotically of the form $\sim c\rho^n n^{-5/2}$.

In what follows we will focus on the case $k = 2$, where F_0 and Q are polynomials with non-negative coefficients and we will show that the results for the case $k = 1$ can be extended. However, there are several (major) differences. Whereas in the case $k = 1$ the catalytic equation can be solved with the help of a so-called positive system of polynomial equations (see [2, 7]) which determines directly a dominant square-root singularity for the involved solution function this property is widely lost for the cases $k \geq 2$. Thus, it is necessary to develop new methods and concepts in order to deduce the universal singular behavior. Clearly we expect similar properties for all equations of the form (4) as well as for systems of positive catalytic equations but the cases $k > 2$ are even more involved.

2 Main results

The solution method by Bousquet-Mélou and Jehanne [4] for an equation of the form (1) works as follows. One considers the algebraic system of $3k$ equations

$$\begin{aligned} P(z,u_i(z),f_i(z),M_1(z),\ldots,M_k(z)) &= 0, \quad 1 \leq i \leq k, \\ P_{x_0}(z,u_i(z),f_i(z),M_1(z),\ldots,M_k(z)) &= 0, \quad 1 \leq i \leq k, \\ P_u(z,u_i(z),f_i(z),M_1(z),\ldots,M_k(z)) &= 0, \quad 1 \leq i \leq k, \end{aligned} \tag{6}$$

for the $3k$ unknown functions $u_1(z),\ldots,u_k(z)$, $f_1(z),\ldots,f_k(z)$, $M_1(z),\ldots,M_k(z)$. In general it is not clear that such a system is solvable. However, if the catalytic equation is of the form (4) then this is granted and leads to the unknown functions $M_1(z),\ldots,M_k(z)$ (see [4]).

In our context we reformulate the catalytic equation slightly to

$$u^2\Delta(z,u) + uM_1(z) + M_0(z) \tag{7}$$

$$= zQ\left(z,u,u^2\Delta(z,u)+uM_1(z)+M_0(z),u\Delta(z,u)+M_1(z),\Delta(z,u)\right)$$

$$=: R(z,u,\Delta(z,u),M_1(z),M_0(z)) \tag{8}$$

where $\Delta(z,u) = \Delta^{(2)}M(z,u)$, $M_1(z) = M_u(z,0)$, $M_0(z) = M(z,0)$ and $Q(z,u,\alpha_0,\alpha_1,\alpha_2)$ and consequently $R(z,u,y_0,y_1,y_2)$ are polynomials with non-negative coefficients. W.l.o.g. the polynomial part $F_0(z,u)$ can be omitted by substituting $M(z,u) = \widetilde{M}(z,u) + F(0,u)$. In particular we have

$$\begin{aligned} P(z,u,x_0,x_1,x_2) &= zQ\left(z,u,u^2x_0+ux_1+x_2,ux_0+x_1,x_0\right) - u^2x_0 - ux_1 - x_2 \\ &= R(z,u,x_0,x_1,x_2) - u^2x_0 - ux_1 - x_2. \end{aligned}$$

The system (6) now rewrites to

$$\begin{aligned} u_i(z)^2\Delta_i(z) + u_i(z)M_1(z) + M_0(z) &= R\left(z,u_i(z),\Delta_i(z),M_1(z),M_0(z)\right), \quad i=1,2, \\ u_i(z)^2 &= R_{y_0}\left(z,u_i(z),\Delta_i(z),M_1(z),M_0(z)\right), \quad i=1,2, \\ 2u_i(z)\Delta_i(z) + M_1(z) &= R_u\left(z,u_i(z),\Delta_i(z),M_1(z),M_0(z)\right), \quad i=1,2, \end{aligned} \tag{9}$$

for the six indeterminate functions $M_1(z), M_0(z)$, $u_{1,2}(z)$ and $\Delta_{1,2}(z)$ (which correspond to the functions $\Delta_{1,2}(z) = \Delta(z,u_{1,2}(z))$). In order to distinguish between $i=1$ and $i=2$ we assume that $u_1(z) > 0$ and $u_2(z) < 0$ for $z > 0$ that are sufficiently small.

We now state our main results that generalize the results of [7] to the case $k=2$. We say that an algebraic function has a square-root singularity at z_0 if the dominating term in the Puiseux expansion at z_0 is of the form $(z-z_0)^{1/2}$. Similiarly we say that a singulariy at z_0 has type $3/2$ if the dominating term is of the form $(z-z_0)^{3/2}$. Recall that all solutions of (7) are algebraic.

▶ **Theorem 1.** *Suppose that the polynomial Q in the catalytic equation (7) is linear in $(\alpha_0,\alpha_1,\alpha_2)$ and has non-negative coefficients. Suppose further that Q_{α_0} is not a polynomial in u^2 and that u does not divide Q_{α_1}. Then the functions $M(z,0)$ and $M_u(z,0)$ have a common radius of convergence z_0 and a square-root singularity at z_0.*

For the second theorem we will need an extra condition of the term

$$T := R_{uuu} + (3R_{uuy_0} - 6)\frac{2u - R_{uy_0}}{R_{y_0y_0}} + 3R_{uy_0y_0}\left(\frac{2u - R_{uy_0}}{R_{y_0y_0}}\right)^2 + R_{y_0y_0y_0}\left(\frac{2u - R_{uy_0}}{R_{y_0y_0}}\right)^3. \quad (10)$$

▶ **Theorem 2.** *Suppose that the polynomial Q in the catalytic equation (7) is non-linear in $(\alpha_0, \alpha_1, \alpha_2)$ and has non-negative coefficients. Suppose further that $Q_{\alpha_0 u} \neq 0$ and that u does not divide Q_{α_1}. Then the functions $M(z, 0)$ and $M_u(z, 0)$ have a common radius of convergence z_0, and if $T \neq 0$ at $(z, u) = (z_0, u_1(z_0))$ then both $M(z, 0)$ and $M_u(z, 0)$ have a singularity of type $3/2$.*

We first comment on the conditions on the polynomial Q. They are just put to simplify the presentation. They exclude degenerate cases that reduce to finite systems or systems or to cases where at least one solution to the curve equation is constant 0 (which can further be reduced to cases with universal laws).

Secondly the condition $T \neq 0$ at $(z, u) = (z_0, u_1(z_0))$ in Theorem 2 seems to be artificial. Actually one always has $T \geq 0$ and we have not found an example yet, where $T = 0$, but it is unclear how the zero case could be excluded. A corresponding condition for the case $k = 1$ always holds, since in this case all (corresponding) summands are positive. Nevertheless, the case $T = 0$ can be also discussed and we would get a dominating singularity of the form $(z - z_0)^{4/3}$.

Finally, as mentioned above for the planar map counting problem, the type of the dominating singularity is reflected in the asymptotic behavior of the coefficients. In order to keep the presentation simple we do not go into these details. We just mention that in the linear case the square-root singularity corresponds to asymptotics of the form $c\, z_0^{-n} n^{-3/2}$ whereas in the non-linear case the singularity of type $3/2$ corresponds to asymptotics of the form $c\, z_0^{-n} n^{-5/2}$. However, in general these kinds of asymptotics hold only in residue classes (compare with the results of [7]).

▶ **Example 3.** Let us consider one-dimensional non-negative lattice paths, where we allow steps of the form ± 1 and ± 2. The generating functions $E_k(z)$ of walks that start at 0 and end at level k satisfy the system of equations

$$\begin{aligned}
E_0(z) &= 1 + z(E_1(z) + E_2(z)),\\
E_1(z) &= z(E_0(z) + E_1(z) + E_2(z)),\\
E_k(z) &= z(E_{k-2}(z) + E_{k-1}(z) + E_{k+1}(z) + E_{k+2}(z)) \qquad (k \geq 2).
\end{aligned}$$

Hence, the generating function $E(z, u) = \sum_{k \geq 0} E_k(z) u^k$ satisfies

$$E(z, u) = 1 + z(u + u^2)E(z, u) + z\frac{E(z, u) - E(z, 0)}{u} + z\frac{E(z, u) - E(z, 0) - uE_v(u, 0)}{u^2}. \quad (11)$$

This is precisely a linear equation of the form (4) with $k = 2$. Theorem 1 applies directly and implies that the generating function $E_0(z) = E(z, 0)$ of excursions has a square-root singularity, compare also with [3] or with the discussion in Section 6.

▶ **Example 4.** 3-Constellations are Eulerian maps, where the faces are bi-colored, black faces have valency 3 whereas white faces have a valency that is a multiple of 3 (more generally one considers m-constellations, see [4]). The corresponding (catalytic) equation for 3-constellations is given by

$$\begin{aligned}
C(z, u) = 1 &+ zuC(z, u)^3 + zu(2C(z, u) + C(z, 1))\frac{C(z, u) - C(z, 1)}{u - 1}\\
&+ zu\frac{C(z, u) - C(z, 1) - (u - 1)C_u(z, 1)}{(u - 1)^2}
\end{aligned}$$

This catalytic equation is almost of the form, where we can apply Theorem 2 due to the additional appearance of $C(z,1)$. However, the polynomial P in (8) has still non-negative coefficients. Thus a slight extension of Theorem 2 applies, where we require the determinant of A in the calculations of Section 5 to be positive at z_0. Furthermore, (10) is satisfied. Consequently, the function $C(z,1)$ has a dominant singularity of type $3/2$, see also the discussion in Section 6

Further examples can be found in [4, 7]. It should be also mentioned that Theorems 1 and 2 can be extended to prove central limit theorems for several parameters that are encoded by an additional variable (see [8, 9, 7]).

3 The Curve Equation

Our first observation is that $M(z,u)$ which is defined by equation (7) is analytic by considering the equation as a fixed point problem in the sequence space of the coefficients. The factor z on the right hand side can be chosen small enough such that the map is a contraction and yields a unique solution that is analytic by uniform convergence. Furthermore, by rewriting (7) into an infinite system (by considering the expansion with respect to u and by iterating this system) it follows that the solution function has non-negative coefficients.

Given that we know there is a unique solution $M(z,u)$ fulfilling the equation, we may regard equation (7) (or (8)) as an equation in z and u and differentiate the equation with respect to u and group the terms with a factor $\partial_u \Delta(z,u)$ into the equation

$$u^2 = R_{y_0}\left(z, u, \Delta(z,u), M_1(z), M_0(z)\right) =: C(z,u)$$

which was proven to have two unique solutions $u_1(z)$ and $u_2(z)$, with $u_1(0) = u_2(0) = 0$. We will refer to $u^2 = C(z,u)$ as the *curve equation*. Note that $C(z,u)$ has non-negative coefficients.

Next we consider $u_1(z)$ and $u_2(z)$. In general, the two series only have a Puiseux expansion at 0 but applying the Weierstrass preparation theorem to equation (7), we can see that locally both $u_i(z)$ are zeros of

$$u_i^2 - C(z,u_i) = K(z,u_i)\left(u_i^2 + a_1(z)u_i + a_2(z)\right) = 0$$

where $K(z,u)$, $a_1(z), a_2(z)$ are analytic functions at 0 with $a_1(0) = a_2(0) = 0$ and $K(0,0) \neq 0$. Note that all these functions are uniquely given. Hence, we can express

$$u_{1,2}(z) = -\frac{a_1(z)}{2} \pm \sqrt{\frac{a_1(z)^2}{4} - a_2(z)} =: g(z) \pm \sqrt{h(z)}. \tag{12}$$

Now the idea is to split u and all power series in u into two parts: one with a factor $\sqrt{h}$ and the other without. That is, for

$$u_{1,2}^2 = (g \pm \sqrt{h})^2 = g^2 + h \pm \sqrt{h}\, 2g.$$

we define $(u^2)^+ = g^2 + h$ and $(u^2)^- = 2g$ and further we split

$$\Delta(z, u_{1,2}) = \Delta^+(z,g,h) \pm \sqrt{h}\,\Delta^-(z,g,h).$$

By doing the same with $R(z, u_{1,2}, \Delta_{1,2}, M_1, M_0)$ and the curve equation

$$g^2 + h \pm \sqrt{h} 2g = C^+(z,g,h) \pm \sqrt{h}\, C^-(z,g,h)$$

and considering the unique solutions to the system

$$h = C^{+}(z,g,h) - g^2, \quad g = \frac{1}{2} C^{-}(z,g,h) \tag{13}$$

that consequently have to be exactly g and h as defined in (12) we may derive the following result.

▶ **Lemma 5.** *Suppose that $C(z,u)$ is a power series with non-negative coefficients such that z divides $C(z,u)$. Furthermore let $u_{1,2}(z) = g(z) \pm \sqrt{h(z)}$ be the two solutions with $u(0) = 0$ of the equation $u^2 = C(z,u)$. Then $g(z)$ and $h(z)$ are power series with $g(0) = h(0) = 0$ and non-negative coefficients.*

The non-negativity of the coefficients does not follow immediately. In fact, we have to verify that $h'(0), g'(0) > 0$ and subsequently that all higher derivatives $h^{(n)}(0), g^{(n)}(0) > 0$ individually. We will leave out the detailed proof, since it is long and technical and most of the rest of our results do not rely on this fact. The important part is that $u_1(z)$ is positive and monotone increasing for $z > 0$ and that $|u_2(z)| < u_1(z)$ if the curve equation is not a power series in u^2.

4 Proof of Theorem 1 (The Linear Case)

If Q is linear in α_0, α_1 and α_2, we can rewrite (7) to

$$M(z,u) = R_0(z,u) + zR_1(z,u)M(z,u) + zR_2(z,u)\Delta M(z,u) + zR_3(z,u)\Delta^{(2)} M(z,u)$$

where $R_0(z,u), R_1(z,u), R_2(z,u), R_3(z,u)$ are polynomials with non-negative coefficients. Equivalently we have

$$\begin{aligned} M(z,u)\left(1 - \left(R_1(z,u) + \frac{1}{u}R_2(z,u) + \frac{1}{u^2}R_3(z,u)\right)\right) \\ = R_0(z,u) - zR_2(z,u)\frac{M(z,0)}{u} - zR_3(z,u)\left(\frac{M_u(z,0)}{u} + \frac{M(z,0)}{u^2}\right). \end{aligned} \tag{14}$$

In this case, the curve equation is a polynomial equation in u and z

$$u^2 - C(z,u) = u^2 - \left(u^2 R_1(z,u) + uR_2(z,u) + R_3(z,u)\right)$$

and can be independently solved (and is actually the basic equation of the original kernel method). Subsequently, by using (14) and the two solutions $u_{1,2}(z)$ of the curve equation we get the following linear system of equations

$$M(z,0) + \left(u_1(z) - z\frac{R_2(z,u_1(z))}{1 - zR_1(z,u_1(z))}\right) M_u(z,0) = \frac{R_0(z,u_1(z))}{1 - zR_1(z,u_1(z))}, \tag{15}$$

$$M(z,0) + \left(u_2(z) - z\frac{R_2(z,u_2(z))}{1 - zR_1(z,u_2(z))}\right) M_u(z,0) = \frac{R_0(z,u_2(z))}{1 - zR_1(z,u_2(z))}, \tag{16}$$

to calculate $M(z,0)$ and $M_u(z,0)$. (Of course if these functions are given we can use them to obtain the full solution function $M(z,u)$.)

We start by determining the singular expansions of $u_1(z)$ and $u_2(z)$, where we will use the following lemma to show that $g(z), h(z)$ have a common square root singularity at their radius of convergence z_0 and $u_2(z)$ is regular at z_0.

▶ **Lemma 6.** *Let $C(z,u) = \sum_{k,j\ge 0} c_{k,j} z^k u^j$ be an analytic function with non-negative coefficients and k_1, k_2 and $j_1 < j_2$ such that $c_{k_1,j_1}, c_{k_2,j_2} \neq 0$. Then, for $z, u > 0$ inside the region of convergence, it holds that*

$$C(z,u)C_{uu}(z,u) - C_u(z,u)^2 + \frac{C_u(z,u)C(z,u)}{u} > 0$$

Proof. By assumption we clearly have $|C(z,ue^{i\theta})| \le C(z,u)e^{-c\theta^2}$ for $z, u > 0$, θ sufficiently close to 0 and some constant $c = c(z,u) > 0$ if C is a power series with positive coefficients. Further, by using an exp-log scheme and the Taylor expansion of the logarithm at $\theta = 0$, it holds that

$$C(z,ue^{i\theta}) = \exp\left(\log(C(z,u)) + iu\frac{C_u(z,u)}{C(z,u)}\theta + \frac{u^2}{2\,C(z,u)^2} b(z,u)\theta^2 + o(\theta^2)\right)$$

where $b(z,u) = C_u(z,u)^2 - \frac{C_u(z,u)C(z,u)}{u} - C_{uu}(z,u)C(z,u)$. Thus,

$$|C(z,ue^{i\theta})| = C(z,u)e^{\frac{u^2}{2\,C(z,u)^2}b(z,u)\theta^2 + o(\theta^2)} \le C(z,u)e^{-c\theta^2}$$

and consequently, that factor $b(z,u)$ has to be negative. ◀

▶ **Lemma 7.** *Suppose that $C(z,u)$ is not a power series in u^2, but a power series with non-negative coefficients with degree ≥ 3 in u, and where z divides $C(z,u)$. Further denote by $u_{1,2}(z) = g(z) \pm \sqrt{h(z)}$ the two solutions with $u(0) = 0$ of the equation $u^2 = C(z,u)$. If $(z_0, u_1(z_0))$ is inside the region of convergence of $C(z,u)$, where z_0 is the smallest $z > 0$ such that $2u_1(z_0) = C_u(z_0, u_1(z_0))$, then the critical exponent of $u_1(z)$ at z_0 is $1/2$, while $u_2(z)$ is regular at z_0. That is, $g(z), h(z)$ have a common square-root singularity at their radius of convergence z_0, and their square-root singularities cancel in the representation $u_2(z) = g(z) - \sqrt{h(z)}$.*

Proof. We certainly have

$$u_1(z_0)^2 = C(z_0, u_1(z_0)), \quad 2u_1(z_0) = C_u(z, u_1(z_0)), \quad C_z(z, u_1(z_0)) > 0.$$

By Lemma 6 it also follows that $2 < C_{uu}(z, u_1(z_0))$. Then by standard arguments using the Weierstrass preparation theorem (compare with [6, Remark 2.20]), we can derive that $u_1(z)$ is locally equal to $u_1(z) = g_1(z) + h_1(z)\sqrt{z - z_0}$, where $g_1(z), h_1(z)$ are analytic functions around z_0 and $h_1(z_0) \neq 0$. Now let us assume that $2u_2(z_0) = C_u(z_0, u_2(z_0))$ as well. This would imply that

$$2(g_0 + \sqrt{h_0}) = C_u(z_0, g_0 + \sqrt{h_0}) \quad \text{and} \quad 2(g_0 - \sqrt{h_0}) = C_u(z_0, g_0 - \sqrt{h_0}),$$

where $g_0 = g(z_0)$ and $h_0 = h(z_0)$. Since

$$\frac{C_u(z, g + \sqrt{h}) + C_u(z, g - \sqrt{h})}{2} = C_g^+(z, g, h)$$

it would follow that $2g_0 = C_g^+(z_0, g_0, h_0)$ and therefore $C_g^+(z_0, g_0, h_0) - C^-(z_0, g_0, h_0) = 0$. At this point we mention that

$$\binom{k}{2\ell}(k - 2\ell) - \binom{k}{2\ell+1} = \binom{k}{2\ell+1}2\ell \ge 0$$

which ensures that

$$
\begin{aligned}
&C_g^+(z,g,h) - C^-(z,g,h) = \\
&\qquad = \sum_{k,\ell} C_k(z)\binom{k}{2\ell}(k-2\ell)g^{k-2\ell-1}h^\ell - \sum_{k,\ell} C_k(z)\binom{k}{2\ell+1}g^{k-2\ell-1}h^\ell \\
&\qquad = \sum_{k,\ell} C_k(z)\binom{k}{2\ell+1}2\ell g^{k-2\ell-1}h^\ell
\end{aligned} \tag{17}
$$

Since C has degree ≥ 3 in u and $g \neq 0$ if $C(z,u)$ is not a power series in u^2 this is a contradiction to being 0. Hence, $2u_2(z_0) \neq C_u(z_0, u_2(z_0))$ and consequently $u_2(z)$ is regular at z_0. This implies further that $g(z) = (u_1(z)+u_2(z))/2$ and $h(z) = (u_1(z)-u_2(z))^2/4$ share a square root singularity at z_0. ◀

In a final step we can also detect the singular behavior of $M(z,0)$ and $M_u(z,0)$.

▶ **Lemma 8.** *Suppose that the assumptions of Theorem 1 are satisfied and let $M(z,0)$ and $M_u(z,0)$ be the solutions of the linear system (15)–(16). Then $M(z,0)$ and $M_u(z,0)$ have square-root singularities at z_0.*

Proof. We recall that $M(z,0)$ and $M_u(z,0)$ are given by (15)–(16) and that $u_1(z)$ and $u_2(z)$ are the solutions to the curve equation, where $u_1(z)$ has a square-root singularity at z_0, whereas $u_2(z)$ is regular at z_0. We recall that (15)–(16) can be rewritten as

$$
M(z,0) + \left(u_{1,2}(z) - \frac{zR_2(z,u_{1,2}(z))}{1 - zR_1(z,u_{1,2}(z))}\right) M_u(z,0) = \frac{R_0(z,u_{1,2}(z))}{1 - zR_1(z,u_{1,2}(z))}. \tag{18}
$$

At this point we rewrite $u_{1,2}(z)$ as $u_{1,2}(z) = g(z) \pm \sqrt{h(z)}$ and split up between the $+$-part and the $-$-part. In particular we have

$$
(1)^- = 0 \quad \text{and} \quad \left(u - \frac{zR_2(z,u)}{1-zR_1(z,u)}\right)^- = 1 - \left(\frac{zR_2(z,u)}{1-zR_1(z,u)}\right)^-,
$$

which leads to

$$
\left(1 - \left(\frac{zR_2(z,u)}{1-zR_1(z,u)}\right)^-\right) M_u(z,0) = \left(\frac{R_0(z,u)}{1-zR_1(z,u)}\right)^-
$$

Now notice that by our conditions that Q_{α_0} is not a polynomial in u^2 the negative part on the right hand side is non-zero. Finally, we obtain

$$
M_u(z,0) = \frac{\left(\frac{R_0(z,u)}{1-zR_1(z,u)}\right)^- (z,g(z),h(z))}{1 - \left(\frac{zR_2(z,u)}{1-zR_1(z,u)}\right)^- (z,g(z),h(z))},
$$

where the right hand side depends on z, g and h and has non-negative coefficients. Therefore, it immediately follows that $M_u(z,0)$ has a square-root singularity at z_0.

Now we can use the equation (18) to deduce that $M(z,0)$ has *at most* a square-root-singularity at z_0 – it might be that the singularity cancels. However, by considering the original catalytic equation (7) for constant $u=0$ we have

$$
M(z,0) = R_0(z,0) + zR_1(z,0)M(z,0) + zR_2(z,0)M_u(z,0) + zR_3(z,0)M_{uu}(z,0)
$$

Thus, it follows that $M(z,0)$ has *at least* a square-root-singularity at z_0, as u is not a factor of $R_2(z,0)$. Consequently, $M(z,0)$ has square-root-singularity at z_0. This completes the proof of the lemma. ◀

5 Proof of Theorem 2 (The Non-Linear Case)

In this section we use the following notation. If an expression like $R(z, u, \Delta(z, u), M_1(z), M_0(z))$ is evaluated along $u_1(z)$ (and $\Delta_1(z) = \Delta(z, u_1(z))$), we just write R. If the expression is evaluated along $u_2(z)$, we will write $\overline{R}$. We also assume that (10) holds and that $C(z, u)$ is not a power series in u^2.

The proof of Theorem 2 itself will, again, mostly be concerned about the singularity of $u_1(z)$. We will first show this at z_0, where the determinant of the Jacobian of the system (9) equals 0. By considering just 5 equations we can compute functions $\Delta_1(z, u_1), M_1(z, u_1), M_0(z, u_1), u_2(z, u_1), \Delta_2(z, u_1)$ that are analytic at $z_0, u_1(z_0)$. By substituting these functions into the 6th equation (the curve equation) we finally get a single equation for the unknown function $u_1 = u_1(z)$. Next we prove that $u_1(z)$ has a square root singularity at z_0 (provided that (10) holds) and that the functions $M_1(z), M_0(z), \Delta_2(z), u_2(z)$ have at most a 3/2 singularity. Finally we will confirm the 3/2 singularity analogously to the linear case.

At several points, the derivative of equation (7) with respect to z plays a crucial role. Along $u_{1,2}(z)$ the terms with factor $\partial_z\Delta(z, u)$ cancel again, and we are left with the system

$$\begin{pmatrix} 1 - R_{y_2} & u - R_{y_1} \\ \overline{1 - R_{y_2}} & \overline{u - R_{y_1}} \end{pmatrix} \begin{pmatrix} M_0'(z) \\ M_1'(z) \end{pmatrix} = \begin{pmatrix} R_z \\ \overline{R_z} \end{pmatrix} \tag{19}$$

We will denote the matrix on the left hand side by A and use it in particular to prove that the critical exponent of $M_1(z)$ and $M_0(z)$ is 3/2.

The matrix A appears also right in the first step of the proof. We consider the Jacobian matrix of system (9)

$$\begin{pmatrix} A & 0 & 0 \\ C_1 & B_1 & 0 \\ C_2 & 0 & B_2 \end{pmatrix}$$

where A, C_1, C_2, B_1, B_2 are 2×2 matrices such that its determinant decomposes into three factors. These are the determinants of the submatrices

$$A, \quad B_1 = \begin{pmatrix} 2u - R_{uy_0} & -R_{y_0y_0} \\ 2\Delta - R_{uu} & 2u - R_{uy_0} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \overline{2u - R_{uy_0}} & \overline{-R_{y_0y_0}} \\ \overline{2\Delta - R_{uu}} & \overline{2u - R_{uy_0}} \end{pmatrix}$$

and we denote them by

$$D_0 = \det A, \quad D_1 = \det B_1, \quad D_2 = \det B_2.$$

In a first step, we show that D_0 is never 0 and that the submatrix B_2 which corresponds to the equations for $u_2(z)$ and $\Delta_2(z)$ is invertible, if $D_1 = 0$. Further note that it is fairly obvious from the curve equation that the smallest positive z_0 where $u_1(z)$ is singular is bounded by the convergence radius of $M_0(z)$ and $M_1(z)$ and that $u_2(z)$ will be regular for all $0 < z < z_0$.

▶ **Lemma 9.** *Let z_0 be the smallest positive z, where $u_1(z)$ is singular. Then, the determinants D_0, D_1, D_2 evaluated at z_0 satisfy $D_0 \neq 0$, $D_1 = 0$, and $D_2 \neq 0$.*

Proof. First, we consider equation (19) and note that $R_{y_2} = Q_{\alpha_0}$ and $R_{y_1} = uQ_{\alpha_0} + Q_{\alpha_1}$. Since u_1, u_2 fulfill the curve equation, $Q_{u\alpha_0} \neq 0$ and $u_1 > u_2$, we know that

$$0 < 1 - R_{y_2} < 1 - \overline{R_{y_2}}.$$

Assuming that $D_0 = 0$, it would also have to hold that $0 < u - R_{y_1} < \overline{u - R_{y_1}}$. But $M_1(z)$ and $M_2(z)$ have non-negative coefficients. Hence, (19) would imply that $R_z < \overline{R_z}$ which is certainly wrong. Next in order to prove that D_1 and D_2 cannot be both 0 we consider the curve equation first. Its partial derivative with respect to u equals

$$2u = R_{uy_0} + R_{y_0y_0} \cdot \partial_u \Delta$$

If we differentiate equation (7) twice with respect to u, we can see that the terms with factor $\partial_u^2 \Delta(z,u)$ add up again to the curve equation.

$$\left(u^2 - R_{y_0}\right) \partial_u^2 \Delta + 2\left(2u - R_{uy_0}\right) \partial_u \Delta + 2\Delta = R_{uu} + R_{y_0y_0} \left(\partial_u \Delta\right)^2 \tag{20}$$

Hence, we may compute $\partial_u \Delta$ along $u_1(z)$ as

$$\partial_u \Delta = \frac{2u - R_{uy_0} \pm \sqrt{D_1}}{R_{y_0y_0}} \tag{21}$$

and along $u_2(z)$ analogously. That is, along $u_1(z)$ and $u_2(z)$ it holds that

$$C_u(z, u_i(z)) = 2u_i(z) \pm \sqrt{D_i}.$$

But if $D_1 = D_2 = 0$, then we could repeat the calculations at the end of the proof of Lemma 7 and show that $C_u^+ - C^- = 0$ which is impossible. ◀

Equation (21) concerning the partial derivative of $\Delta(z,u)$ further tells us that $2u > R_{uy_0}$ since $\Delta(z,u)$ has non-negative coefficients and $u_1(z) > 0$ for $z > 0$. This means that the following submatrix of the Jacobian of system (9) is invertible and its inverse equals

$$\begin{pmatrix} A & 0 & 0 \\ C_1 & 2u - R_{uy_0} & 0 \\ C_2 & 0 & B_2 \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 & 0 \\ D_1 & (2u - R_{uy_0})^{-1} & 0 \\ -B_2^{-1} C_2 A^{-1} & 0 & B_2^{-1} \end{pmatrix}$$

where C_1, C_2 are generally non-zero matrices that contain the partial derivatives with respect to M_0 and M_1 of the third, fifth and sixth equation of system (9) respectively and D_1 can be computed appropriately. The implicit function theorem yields analytic functions $M_0(z, u_1), M_1(z, u_1), u_2(z, u_1), \Delta_1(z, u_1)$, and $\Delta_2(z, u_2)$ which have partial derivatives with respect to u that are equal to

$$\partial_{u_1} \begin{pmatrix} M_0(z,u_1) \\ M_1(z,u_1) \\ \Delta_1(z,u_1) \\ u_2(z,u_1) \\ \Delta_2(z,u_1) \end{pmatrix} = -\begin{pmatrix} A^{-1} & 0 & 0 \\ D_1 & (2u - R_{uy_0})^{-1} & 0 \\ -B_2^{-1} C_2 A^{-1} & 0 & B_2^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2\Delta - R_{uu} \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

Hence, depending on the critical exponent of $u_1(z)$ in its singular expansion at z_0, we can already state that $M_0(z), M_1(z)$ have at most 2α-singularities. The following lemma proves $\alpha = 1/2$ under the vital condition (10).

▶ **Lemma 10.** *Let $u_{1,2}(z) = g(z) \pm \sqrt{h}$ be the two solutions to the curve equation, z_0 be a the smallest positive z where $u_1(z)$ is singular. If (10) holds at $(z, u) = (z_0, u_1(z_0))$ then $u_1(z)$ has a square root singularity at z_0.*

Proof. First we prove the case where a square root singularity appears. By the computations above we have analytic functions $M_0(z,u_1)$, $M_1(z,u_1)$, and $\Delta_1(z,u_1)$ that we can plug into our equation $u^2 = R_{y_0}(z,u,\Delta,M_1,M_0)$. Now we want to show that for $u_0 = \lim_{z\nearrow z_0} u(z)$, it holds that

$$\begin{aligned}
&0 = u_0^2 - R_{y_0}(z_0,u_0,\Delta_1(z_0,u_0),M_1(z_0,u_0),M_0(z_0,u_0))\big|_{z=z_0,u_1=u_0},\\
&0 = 2u_0 - \partial_{u_1} R_{y_0}(z,u_1,\Delta_1(z,u_1),M_1(z,u_1),M_0(z,u_1))\big|_{z=z_0,u_1=u_0},\\
&0 \neq 2 - \partial_{u_1}^2 R_{y_0}(z,u_1,\Delta_1(z,u_1),M_1(z,u_1),M_0(z,u_1))\big|_{z=z_0,u_1=u_0},\\
&0 \neq -\partial_z R_{y_0}(z,u_1,\Delta_1(z,u_1),M_1(z,u_1),M_0(z,u_1))\big|_{z=z_0,u_1=u_0}.
\end{aligned}$$

The square root singularity of $u_1(z)$ then follows by standard arguments. So, any of the following computations are evaluated at (z_0,u_0).

We already computed that the first partial derivative with respect to u of all plugged in functions is 0 except for $\partial_{u_1}\Delta_1(z,u_1)$. Hence, if we consider the second equation which we want to prove and use the fact that $D_1 = 0$ then

$$\partial_{u_1}\Delta_1 = -\frac{2\Delta_1 - R_{uu}}{2u_1 - R_{uy_0}} = \frac{2u_1 - R_{uy_0}}{R_{y_0y_0}} = \partial_u \Delta$$

and further,

$$\begin{aligned}
&2u_1 - R_{uy_0} - R_{y_0y_0}\partial_{u_1}\Delta_1(z,u_1) - R_{y_0y_2}\partial_{u_1}M_0(z,u_1) - R_{y_0y_1}\partial_{u_1}M_1(z,u_1)\\
&\quad = 2u_1 - R_{uy_0} + R_{y_0y_0}\frac{2\Delta_1 - R_{uu}}{2u_1 - R_{uy_0}} = 0.
\end{aligned}$$

For the third equation, we compute the second partial derivatives of $M_1(z,u_1), M_0(z,u_1)$ and $\Delta_1(z,u_1)$ analogously to our computations above. If we only consider the submatrix for the relevant derivatives, we obtain that

$$\begin{aligned}
\partial_{u_1}^2 \begin{pmatrix} M_0\\ M_1\\ \Delta_1\end{pmatrix} &= \begin{pmatrix} A^{-1} & 0\\ D_1 & (2u_1 - R_{uy_0})^{-1}\end{pmatrix}\\
&\quad\cdot \begin{pmatrix} R_{uu} + (2R_{uy_0} - 4u_1)(\partial_{u_1}\Delta_1) + R_{y_0y_0}(\partial_{u_1}\Delta_1)^2 - 2\Delta_1\\ 0\\ R_{uuu} + 2R_{uuy_0}\partial_{u_1}\Delta_1 + R_{uy_0y_0}(\partial_{u_1}\Delta_1)^2 - 4\partial_{u_1}\Delta_1\end{pmatrix}
\end{aligned}$$

Now by dividing the first entry in the right vector above by $R_{y_0y_0}$, it is easy to see that at (z_0,u_0) it is equal to 0. That is,

$$\frac{R_{uu} - 2\Delta_1}{2u - R_{uy_0}}\frac{2u - R_{uy_0}}{R_{y_0y_0}} - 2\frac{2u - R_{uy_0}}{R_{y_0y_0}}(\partial_{u_1}\Delta_1) + (\partial_{u_1}\Delta_1)^2 = 0$$

Consequently $(\partial_{u_1}^2 M_0)(z_0,u_0) = (\partial_{u_1}^2 M_1)(z_0,u_0) = 0$ and

$$(\partial_{u_1}^2\Delta_1)(z_0,u_0) = \left(\frac{R_{uuu} + 2R_{uuy_0}\partial_{u_1}\Delta_1 + R_{uy_0y_0}(\partial_{u_1}\Delta_1)^2 - 4\partial_{u_1}\Delta_1}{2u_1 - R_{uy_0}}\right)$$

The expression from the third equation is thus equal to

$$2 - R_{uuy_0} - 2R_{uy_0y_0}(\partial_{u_1}\Delta_1) - R_{y_0y_0y_0}(\partial_{u_1}\Delta_1)^2 - R_{y_0y_0}(\partial_{u_1}^2\Delta_1).$$

If we plug in the expression for $\partial^2_{u_1}\Delta_1$ and multiply the equation by $\partial_{u_1}\Delta_1$, we obtain

$$6(\partial_{u_1}\Delta_1) - R_{uuu} - 3R_{uuy_0}\partial_{u_1}\Delta_1 - 3R_{uy_0y_0}(\partial_{u_1}\Delta_1)^2 - R_{y_0y_0y_0}(\partial_{u_1}\Delta_1)^3$$

which is non-zero by assumption.

What is left to show is that the derivative with respect to z is non-zero. In a first step we compute

$$\partial_z \begin{pmatrix} M_0(z,u_1) \\ M_1(z,u_1) \\ \Delta_1(z,u_1) \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ D_1 & (2u - R_{uy_0})^{-1} \end{pmatrix} \begin{pmatrix} R_z \\ \overline{R_z} \\ R_{zy_0} \end{pmatrix} \tag{23}$$

Note that $A^{-1}(R_z, \overline{R_z})^T = (M_0'(z), M_1'(z))^T > 0$ by (19) and the fact that $M_0(z), M_1(z)$ have non-negative coefficients. Further we have to compute $D_1 = (d_{11}, d_{12})$ which equal

$$\begin{aligned} d_{11} &= \frac{1}{(2u - R_{uy_0})\det A}\Big(R_{uy_2}\overline{(u - R_{y_1})} + (1 - R_{uy_1})\overline{(1 - R_{y_2})}\Big) \\ d_{12} &= \frac{-1}{(2u - R_{uy_0})\det A}\Big(R_{uy_2}(u - R_{y_1}) + (1 - R_{uy_1})(1 - R_{y_2})\Big). \end{aligned}$$

So the right hand side of the fourth inequality that we want to prove is

$$\begin{aligned} &R_{zy_0} + R_{y_0y_0}\partial_z\Delta_1(z,u_1) + R_{y_0y_2}\partial_z M_0(z,u_1) + R_{y_0y_1}\partial_z M_1(z,u_1) \\ &\quad = R_{zy_0} + R_{y_0y_2}M_0'(z) + R_{y_0y_1}M_1'(z) \\ &\qquad\quad + \frac{R_{y_0y_0}}{(2u - R_{uy_0})\det A}\Big(R_{uy_2}\overline{(u - R_{y_1})} + (1 - R_{uy_1})\overline{(1 - R_{y_2})}\Big) R_z \\ &\qquad\quad - \frac{R_{y_0y_0}}{(2u - R_{uy_0})\det A}\Big(R_{uy_2}(u - R_{y_1}) + (1 - R_{uy_1})(1 - R_{y_2})\Big)\overline{R_z} \\ &\quad = R_{zy_0} + R_{y_0y_2}M_0'(z) + R_{y_0y_1}M_1'(z) \\ &\qquad\quad + \frac{R_{y_0y_0}}{(2u - R_{uy_0})}\left(R_{uy_2}M_0'(z) - (1 - R_{uy_1})M_1'(z)\right). \end{aligned} \tag{24}$$

Now let us do a similar trick as in the computation of $\partial_u\Delta(z,u)$. We consider

$$\begin{aligned} &\partial_z\partial_u\left(u^2\Delta(z,u) + uM_1(z) + M_0(z) - R(z,u,\Delta(z,u),M_1(z),M_0(z))\right) \\ &\quad = (2u - R_{uy_0} - R_{y_0y_0}\partial_u\Delta(z,u))\,\partial_z\Delta(z,u) + \left(u^2 - R_{y_0}\right)\partial_u\partial_z\Delta(z,u) \\ &\qquad + M_1'(z) - R_{zu} - R_{zy_0}\partial_u\Delta(z,u) - R_{uy_1}M_1'(z) - R_{uy_2}M_0'(z) \\ &\qquad - R_{y_0y_1}\partial_u\Delta(z,u)M_1'(z) - R_{y_0y_2}\partial_u\Delta(z,u)M_0'(z) \\ &\quad = (1 - R_{uy_1} - R_{y_0y_1}\partial_u\Delta(z,u))M_1'(z) - (R_{uy_2} + R_{y_0y_2}\partial_u\Delta(z,u))M_0'(z) \\ &\qquad - R_{zu} - R_{zy_0}\partial_u\Delta(z,u) = 0 \end{aligned} \tag{25}$$

The terms with factor $\partial_u\partial_z\Delta(z,u)$ add up to the curve equation and cancel, the ones with factor $\partial_z\Delta(z,u)$ add up to 0 since by (20)

$$\left(2u - R_{uy_0} - \frac{1}{2}R_{y_0y_0}\partial_u\Delta\right) = \frac{R_{uu} - 2\Delta}{2\,\partial_u\Delta} = \frac{1}{2}R_{y_0y_0}\partial_u\Delta$$

Now if we multiply (24) with $\partial_u\Delta(z,u) = \partial_{u_1}\Delta_1(z,u_1)$ we can see by our computation of (25) that it is equal to $R_{zu} > 0$ which was left to show. ◀

We can determine the singularities of $M_0(z)$ and $M_1(z)$ analogously to the linear case. Obviously if $u_1(z)$ has a square root singularity both of them can have at most 3/2-singularities by our computation of $\partial_{u_1} M_0(z, u_1)$ and $\partial_{u_1} M_1(z, u_1)$.
The negative part of the equations of (19) gives a positive equation for $M_1'(z)$

$$M_1'(z) = \partial_z \left(R(z, u, \Delta(z, u), M_1(z), M_0)\right)^-$$

where the right hand side does indeed confirm this 3/2 singularity. Similarly by the positive equation $M_0(z) = zQ(z, 0, M_0(z), M_1(z), M_2(z))$ the 3/2 singularity of $M_0(z)$ is proved as well.

Finally, we comment on the case where the condition (10) is not satisfied. In this case, the third equation that we stated in the beginning is satisfied and one may compute analogously to above that $\partial^3_{u_1} R_{y_0}(z, u_1, \Delta_1(z, u_1), M_1(z, u_1), M_0(z, u_1)) \neq 0$. By the Weierstrass preparation theorem $u_1(z)$ therefore satisfies a cubic equation

$$(u_1(z) - u_0)^3 + a_2(z)(u_1(z) - u_0)^2 + a_1(z)(u_1(z) - u_0) + a_0(z) = 0$$

where $a_i(z), i = 0, 1, 2$ are analytic functions at z_0 with $a_i(z_0) = 0$ and, since the fourth equation that we stated in the beginning is satisfied, $a_0(z) = (z - z_0)b(z)$ with $b(z_0) \neq 0$. By considering the critical exponents of each of the summands it follows that $u_1(z)$ has a 1/3-singularity, which implies then that the critical exponent of $M_0(z)$ and $M_1(z)$ is 4/3.

6 Examples

In this section, we will illustrate our generic computations in the proof of Theorems 1 and 2 on the examples given in Section 2.

▶ **Example 11** (Example 3 continued). For one-dimensional non-negative lattice paths where we allow steps of the form ± 1 and ± 2 we obtained the functional equation

$$E(z,u) = 1 + z(u+u^2)E(z,u) + z\frac{E(z,u) - E(z,0)}{u} + z\frac{E(z,u) - E(z,0) - uE_v(u,0)}{u^2}. \quad (26)$$

We know that the curve equation

$$u^2 = z(1+u)u^3 + zu + z.$$

has two solutions $u_1(z)$, $u_2(z)$ with $u_1(0) = u_2(0) = 0$ and $u_1(z)$ is singular at $z_0 > 0$. The common zeros (z_0, u_0) of this equation and its partial derivative with respect to u are

$$\left\{(0,0), \left(\frac{1}{4}, 1\right), \left(-\frac{4}{9}, -\frac{1-\sqrt{15}i}{4}\right), \left(-\frac{4}{9}, -\frac{1+\sqrt{15}i}{4}\right)\right\}$$

Hence, it follows that $z_0 = \frac{1}{4}$ and $u_1(z_0) = 1$. Furthermore, the local expansion of $u_1(z)$ at $z = z_0$ is given by

$$u_1(z) = 1 - \sqrt{8}\sqrt{1-4z} + \cdots.$$

Next we consider the system of equations for $g(z)$ and $h(z)$:

$$\begin{aligned} g^2 + h &= z(g^4 + h^2 + g^3 + 3(2g^2 + g)h + g + 1), \\ 2g &= z((4g+1)h + (4g^3 + 3g^2 + 1)). \end{aligned}$$

At $z_0 = \frac{1}{4}$ there are only finitely many solutions, however, the only ones with $g_0 + \sqrt{h_0} = 1$ are $g_0 = (\sqrt{5}-1)/4$ and $h_0 = (15 - 5\sqrt{5})/8$. Consequently we have

$$u_2(z_0) = g_0 - \sqrt{h_0} = \frac{\sqrt{5}-3}{2}.$$

Finally we use the linear system (15)–(16) to obtain the local expansion for $M(z,0)$ and $M_u(z,0)$:

$$\begin{aligned} M(z,0) &= (6-2\sqrt{5}) - \frac{101\sqrt{2}-45\sqrt{10}}{19}\sqrt{1-4z} + \cdots, \\ M_u(z,0) &= (4\sqrt{5}-8) - \frac{28\sqrt{10}-62\sqrt{2}}{19}\sqrt{1-4z} + \cdots. \end{aligned}$$

Since there are no periodicities this implies that

$$[z^n]E(z,0) \sim \frac{101\sqrt{2}-45\sqrt{10}}{38\sqrt{\pi}}\, n^{-3/2}\, 4^n$$

▶ **Example 12** (Example 4 continued)**.** The functional equation for 3-Constellations can be transformed to the equation

$$\begin{aligned} C(z,v) = 1 &+ z(v+1)C(z,v)^3 + z(v+1)(2C(z,v)+C(z,0))\frac{C(z,v)-C(z,v)}{v} \\ &+ z(v+1)\frac{C(z,v)-C(z,0)-vC_v(z,0)}{v^2} \end{aligned}$$

by substituting $v = u-1$. The equations for the unknowns $u_i(z), d_i = \Delta^{(2)}C(z,u_i(z)), i = 1,2$ and $m_1(z) = C_v(z,u_i(z)), m_0 = C(z,u_i(z))$ are given by

$$\begin{aligned} u_i^2 d_i + u_i m_1 + m_0 =& z(u_i+1)((u_i^2 d_i + u_i m_1 + m_0)^3) \\ &+ z(u_i+1)((2u_i^2 d_i + 2u_i m_1 + 3m_0 + 3)(u_i d_i + m_1) + d_i) \\ u_i^2 =& z(u_i+1)(3(u_i^2 d_i + u_i m_1 + m_0 + 1)^2 u_i^2 + 2(u_i d_i + m_1)u_i^2) \\ &+ z(u_i+1)((2u_i^2 d_i + 2u_i m_1 + 3m_0 + 3)u_i + 1), \\ 2u_i d_i + m_1 =& z(u_i+1)(3(u_i^2 d_i + u_i m_1 + m_0 + 1)^2(2u_i d_i + m_1) + 2(2u_i d_i + m_1)(u_i d_i + m_1) \\ &+ z(u_i+1)(2u_i^2 d_i + 2u_i m_1 + 3m_0 + 3)d_i) + ((u_i^2 d_i + u_i m_1 + m_0 + 1)^3) \\ &+ z(u_i+1)((2u_i^2 d_i + 2u_i m_1 + 3m_0 + 3)(d_i u_i + m_1) + d_i). \end{aligned}$$

Numerical computations show, that the smallest positive z_0 where the Jacobian of this system is invertible equals $z_0 \approx 0.0494$. Indeed, the exact value for the singularity is $4/81 = 0,04938....$ The other variables take the approximate values

$$u_1 \approx 0.6867, \quad u_2 \approx -0.1562, \quad d_1 \approx 0.1070, \quad d_2 \approx 0.0433, \quad m_1 \approx 0.1134, \quad m_0 \approx 0.0833.$$

Note that all computations can be worked out although the scheme of Theorem 2 is not strictly satisfied. We need to check the values of the determinants which equal

$$\det A \approx -0.2588, \quad \det B_1 \approx 0, \quad \det B_2 \approx 0.1828.$$

The neccessary condition of Theorem 2 is also satisfied, since the value of the expression equals $T \approx 2.7209$ (after cancellation of a positive factor to simplify computations). Finally we get the asymptotics

$$[z_n]C(z,0) \sim c\, n^{-5/2}\left(\frac{81}{4}\right)^n$$

for $c \approx 0.0731$.

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