A Perfect Sampler for Hypergraph Independent Sets

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Abstract
The problem of uniformly sampling hypergraph independent sets is revisited. We design an efficient perfect sampler for the problem under a similar condition of the asymmetric Lovász local lemma.

When specialized to $d$-regular $k$-uniform hypergraphs on $n$ vertices, our sampler terminates in expected $O(n \log n)$ time provided $d \leq c \cdot 2^{k^2}$ where $c > 0$ is a constant, matching the rapid mixing condition for Glauber dynamics in Hermon, Sly and Zhang [10]. The analysis of our algorithm is simple and clean.

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1 Introduction

The problem of uniformly sampling hypergraph independent sets, or equivalently the solutions of monotone CNF formulas, has been well-studied in recent years. Consider a hypergraph $\Phi = (V, C)$ on $|V| = n$ vertices. A set $S \subseteq V$ is an independent set if $C \cap S \neq C$ for all $C \in C$.

Assuming the hypergraph is $d$-regular and $k$-uniform, [10] showed that the natural Glauber dynamics mixes in $O(n \log n)$ time when $d \leq c \cdot 2^{k^2}$ for some constant $c > 0$. The sampler implies a fully polynomial-time randomized approximation scheme (FPRAS) for counting hypergraph independent sets [13]. On the other hand, it was shown in [1] that there is no FPRAS for the problem when $d \geq 5 \cdot 2^{k^2}$, unless $\mathsf{NP} = \mathsf{RP}$. Therefore, the result of [10] is tight up to a multiplicative constant.

The proof in [10] analyzes the continuous-time Glauber dynamics under the framework of information percolation developed in [14] for studying the cutoff phenomenon of the Ising model. In this framework, one can view the coupling history of a Markov chain as time-space slabs, and the failure of the coupling at time $t$ as a discrepancy path percolating from $t$ back to the beginning. The analysis of this structure can result in (almost) optimal bounds in many interesting settings.

The percolation analysis in [10] is technically complicated due to the continuous nature of the chain which leads to involved dependencies in both time and space. Recently, discrete analogs of the time-space slabs have been introduced in sampling solutions of general CNF

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1 Part of the work was done while the author was an undergraduate student at Shanghai Jiao Tong University.
formulas in the Lovász local lemma regime [11, 8]. Notably in [8], an elegant discrete
time-space structure, tailored to *systematic scan*, was introduced to support the analysis of
coupling from the past (CFTP) paradigm [17]. In contrast to simulating Glauber dynamics,
the method of CFTP allows to produce *perfect samples* (i.e. without approximation error)
from stationary distribution.

In this article, we refine the discrete time-space structure in [8], which we call the *witness graph*, and apply it to the problem of sampling hypergraph independent sets. This leads to an
efficient CFTP sampler that (1) outputs perfect samples; (2) matches the bound in [10] and
operates also in the asymmetric case under Lovász local lemma-like condition; and (3) has
significantly simpler analysis. More specifically, we study a natural grand coupling of the
systematic scan for sampling hypergraph independent sets. To apply the method of CFTP,
one needs to detect the coalescence of the grand coupling efficiently at each stage of the
algorithm. The monotone property of hypergraph independent sets allows us to reduce the
detection to arguing about percolation on the witness graph. This observation provides
sufficient flexibility for us to carefully bound related probabilities. We first show that, under
a condition similar to the *asymmetric* Lovász local lemma, a perfect sampler exists.

\begin{theorem}
Let $\mathcal{G}$ collects all hypergraphs $\Phi = (V, \mathcal{C})$ such that
\[ \forall C \in \mathcal{C} : 2|C| \cdot 2^{-|C|} \leq (1 - \varepsilon) \cdot x(C) \cdot \prod_{C' \in \Gamma_{2}(C)} (1 - x(C')) , \]
for some constant $\varepsilon \in (0, 1)$ and function $x : \mathcal{C} \to (0, 1)$. Here $\Gamma_{2}(C)$ denotes the set of
hyperedges within distance 2 to $C$ in $\Phi$ (excluding $C$ itself).

There exists an algorithm that inputs a hypergraph $\Phi \in \mathcal{G}$ and outputs an independent set of $\Phi$ uniformly at random, in expected time
\[ O \left( \frac{1}{\log(1 - \varepsilon)} \cdot \log \left( \sum_{C \in \mathcal{C}} \frac{x(C)}{1 - x(C)} \right) \cdot \sum_{C \in \mathcal{C}} \sum_{v \in C} d_v |C| \right) , \]
where $d_v$ is number of hyperedges in $\Phi$ that contains $v$.
\end{theorem}

In the proof of Theorem 1, we view a discrepancy path of the coupling percolating from
time $t$ back to the beginning as an object similar to the *witness tree* in [16] for certifying the
non-termination of a randomized algorithm. This is an interesting analogy between an
algorithm that samples the solutions of CNF formulas and an algorithm that finds them. We
map the discrepancy paths in the witness graph to connected trees generated by *multi-type
Galton-Watson process* in the hypergraph. As a result, a certain spatial mixing property
implies the rapid mixing of the chain.

For $k$-uniform $d$-regular hypergraphs, the result in Theorem 1 translates to the condition
$d \leq \frac{c \cdot 2^{k/2}}{k^{1.5}}$ for some constant $c > 0$ by choosing $x(C) = \frac{1}{4 \pi k^2}$. We give a refined analysis for
this symmetric case to remove the denominator:

\begin{theorem}
Let $\mathcal{G}$ collects all $k$-uniform $d$-regular hypergraphs $\Phi = (V, \mathcal{C})$ such that
\[ d \leq \left( \frac{1}{4} \cdot \frac{9 - \varepsilon}{2} - \frac{1}{2} \right) \cdot 2^{k/2} . \]
for some constant $\varepsilon \in (0, 1)$. There exists an algorithm that inputs a hypergraph $\Phi \in \mathcal{G}$
and outputs a hypergraph independent set of $\Phi$ uniformly at random, in expected time
\[ O \left( \frac{1}{\log(1 - \varepsilon)} \cdot k^2 d^2 n \cdot (\log n + \log d) \right) \text{ where } n := |V| . \]
\end{theorem}
Compared to the result in [10], our sampler has the advantage of being perfect. Our analysis inductively enumerates discrepancy paths, taking into account the structure of the overlapping between hyperedges. Thanks to the clean structure of the witness graph, our proof is much simpler than the one in [10].

Relation to sampling solutions of general CNF formulas

The problem of sampling hypergraph independent sets is a special case of sampling the solutions of general CNF formulas or CSP instances which draw a lot of recent attention [15, 3, 11, 8, 12, 7, 6, 5, 4, 2, 9]. For general $k$-CNF formulas where each variable is of degree $d$ (corresponding to the $k$-uniform $d$-regular hypergraphs here), the best condition needed for an efficient sampler is $d < \frac{1}{\text{poly}(k)} \cdot 2^{\frac{d}{40}}$ [11, 8] which is much worse than the condition $d < c \cdot 2^k$ for hypergraph independent sets. A main reason is that for general CNF formulas, the state space of the local Markov chain is no longer connected, and therefore one needs to project the chain onto a sub-instance induced by some special set of variables. The loss of the projection step, however, is not well-understood. On the other hand, this is not an issue for hypergraph independent sets and therefore (almost) optimal bounds can be obtained.

We remark that the technique developed here for non-uniform graphs can also be applied to general CNF formulas to analyze the projection chain.

2 Preliminaries

2.1 Hypergraph independent sets

Recall in the introduction we mentioned that a set $S \subseteq V$ is an independent set of a hypergraph $\Phi = (V, C)$ if $S \cap C \neq C$ for every $C \in C$. Sometimes we represent it as a (binary) coloring $\sigma \in \{0, 1\}^V$, which is the indicator for $S$. That is, we say $\sigma$ is an hypergraph independent set if $\forall C \in C, \exists v \in C: \sigma(v) = 0$. Denoting by $\Omega_{\Phi}$ the collection of all independent sets of $\Phi$, our goal is to efficiently produce a sample from the uniform measure $\mu$ on $\Omega_{\Phi}$.

Let us fix some notations used throughout our discussion:

- We assume $V := [n]$ and $m := |C|$. We denote the degree of a vertex $v \in V$ as $d_v := |\{C \in C : v \in C\}|$ and the maximum degree as $d := \max_{v \in V} d_v$. A hypergraph is $d$-regular if every vertex is of degree $d$ and is $k$-uniform if every hyperedge is of size $k$.

- For any $C \in C$, we use $\Gamma_{\Phi}(C)$ to denote the set of hyperedges $C'$ such that either $C \cap C' \neq \emptyset$ or there exists $C^*$ such that $C \cap C^* \neq \emptyset$ and $C^* \cap C' \neq \emptyset$. Furthermore, let $\Gamma_{\Phi}(C) := \Gamma_{\Phi}(C) \setminus \{C\}$.

- For any coloring $\sigma \in \{0, 1\}^V$, we use $\sigma^{v \leftarrow r}$ to denote the coloring obtained after recoloring $v \in V$ by $r \in \{0, 1\}$ in $\sigma$.

2.2 Systematic scan and coupling from the past

Fix a hypergraph $\Phi = (V, C)$ and the uniform distribution $\mu$ over $\Omega_{\Phi}$. We use the so-called systematic scan Markov chain to sample independent sets from $\mu$.

Let us define a transition map $f : \Omega_{\Phi} \times V \times \{0, 1\} \rightarrow \Omega_{\Phi}$ by

$$f(\sigma; v, r) := \begin{cases} 
\sigma^{v \leftarrow r} & \text{if } \sigma^{v \leftarrow r} \in \Omega_{\Phi}; \\
\sigma & \text{otherwise.}
\end{cases}$$
It takes the given coloring $\sigma \in \Omega_\Phi$ and tries to recolor vertex $v$ with the proposed $r$. Next, we fix a deterministic scan sequence $v_1, v_2, \ldots$ where $v_i := (i \mod n) + 1$ and write

$$F(\sigma; r_1, \ldots, r_t) := f(f(\cdots f(\sigma; v_1, r_1)\cdots); v_t, r_t).$$

Basically, it begins with the given coloring $\sigma \in \Omega_\Phi$ and runs $f$ for $t$ steps to update vertex colors. The vertices are updated periodically as specified by the scan sequence; the proposed colors for every steps are provided by the arguments $r_1, \ldots, r_t$. For a collection of initial states $S \subseteq \Omega_\Phi$, we use $F(S; r_1, \ldots, r_t)$ to denote the set $\{F(\sigma; r_1, \ldots, r_t) : \sigma \in S\}$.}

**Definition 3.** The systematic scan is a Markov chain $(X_t)$ defined by

$$X_t := F(\sigma; R_1, \ldots, R_t) \quad \forall t$$

for some $\sigma \in \Omega_\Phi$ and some independent Bernoulli$(1/2)$ variables $R_1, R_2, \ldots$.

The systematic scan is not a time-homogeneous Markov chain. However, we can (and will) bundle every $n$ steps into an atomic round so that the bundled version — denoting its transition matrix $P_\Phi$ — is homogeneous. It is easy to check that $P_\Phi$ is irreducible and aperiodic with stationary distribution $\mu$.

Given a Markov chain with stationary $\mu$, in the usual Markov chain Monte Carlo method, one obtains a sampler for $\mu$ as follows: Starting from some initial $X_0$, simulate the chain for $t$ steps with sufficiently large $t$ and output $X_t$. The fundamental theorem of Markov chains says that if the chain is finite, irreducible and aperiodic, then the distribution of $X_t$ converges to $\mu$ when $t$ approaches infinity. However, since we always terminate the simulation after some fixed finite steps, the sampler obtained is always an “approximate” sampler instead of a “perfect” one.

The work of [17] proposed an ingenious method called coupling from the past (CFTP) to simulate the given chain in a reverse way with a random stopping time. A perfect sampler for $\mu$ can be obtained in this way. Roughly speaking, it essentially simulates an infinite long chain using only finitely many steps. To achieve this, it relies upon a routine to detect whether the (finite) simulation coalesces with the virtual infinite chain. Once the coalescence happens, one can output the result of the simulation – which is also the result of the infinite chain and thus follows the stationary distribution perfectly.

We use CFTP to simulate our systematic scan and obtain a desired perfect sampler for hypergraph independent sets. The key ingredient to apply CFTP is how to detect the coalescence. We describe and analyze our algorithm in Section 3.

### 3 Perfect Sampler via Information Percolation

In this section, we describe our perfect sampler for hypergraph independent sets. With the help of a data structure introduced in [8], we apply the argument of information percolation to analyze the algorithm. We utilize the monotonicity of the hypergraph independent sets and establish a sufficient condition for the algorithm to terminate.

#### 3.1 The witness graph

In this section, we introduce the notion of witness graph $H_T = (V_T, E_T)$ for the systematic scan up to some time $T \in \mathbb{N}$. A similar structure was used in [8] for sampling general CNF formulas.
Given a vertex \( v \in V \), we denote its last update time up to moment \( t \) as

\[
\text{UpdTime}(v, t) = \max \{ t' \leq t : v_{t'} = v \}.
\]

Clearly \( \text{UpdTime}(v, t) \in (t - n, t) \) is a deterministic number.

For \( C \in \mathcal{C} \) and \( t \in \mathbb{T} \), let \( e_{C,t} := \{ \text{UpdTime}(v, t) : v \in C \} \) be timestamps when the elements in \( C \) got their latest updates up to time \( t \). Conversely, we say \( e_{C,t} \) has label \( C \) and denote it by \( C(e_{C,t}) := C \).

The vertex set of witness graph is given by

\[
V_T := \{ e_{C,t} : t \in [T], v_t \in C \in \mathcal{C} \}.
\]

We put a directed edge \( e_{C,t} \rightarrow e_{C',t'} \) into \( E_T \) if and only if \( t' \in (e_{C,t} \cap e_{C',t'}) \setminus \{ t \} \). Note that \( H_T \) is acyclic, since \( \max e > \max e' \) for any directed edge \( e \rightarrow e' \). We remark that the witness graph is a deterministic object which by itself does not incorporate any randomness.

The following lemma measures the number of vertices labelled by a certain \( C \in \mathcal{C} \) that are 2-distances from a given vertex in the witness graph. It is useful throughout the enumeration in later discussions.

\[\textbf{Lemma 4.} \text{ For any } e_{C,t} \in V_T \text{ and } C' \in \mathcal{C}, \text{ we have } |e_{C',t'} \in V_T : \text{dist}(e_{C,t}, e_{C',t'}) = 2| \leq 2|C'|.\]

\[\textbf{Proof.} \text{ If } C' \notin \Phi_2(C) \text{ then } |e_{C',t'} \in V_T : \text{dist}(e_{C,t}, e_{C',t'}) = 2| = 0. \text{ Otherwise, } t' \in [t - 2n + 2, t), \text{ and there are at most } 2|C'| \text{ timestamps } e_{C',t'} \text{ satisfying } t' \in [t - 2n + 2, t). \]

### 3.2 Coalescence and percolation

For any \( L \in \mathbb{N} \) we fix \( T = T(L) := n(L + 1) \). On top of the witness graph \( H_T \) we define a probability space as follows. We tie an independent \textbf{Bernoulli}(1/2) variable \( B_t \) to each time point \( t \in \mathbb{T} \). We say a vertex \( v \in V_T \) is open if \( B_t = 1 \) for all \( t \in e \), and call a set of vertices \( P \subseteq V \) open if all \( e \in P \) are open. The event \( \mathcal{B}_L \) is defined as

“\( H_T \) contains an induced open path \( P = (e_1, \ldots, e_L) \subseteq V_T \) of length \( L \) where \( e_1 \cap (T - n, T] \neq \emptyset \).”

For the coming few pages, the notation \( \mathcal{B}_L(R_1, \ldots, R_T) \) indicates that we are using (external) random variables \( R_1, \ldots, R_T \) as concret \( e_{C,t} = \{1\} \quad e_{C,2} = \{1, 2\} \quad e_{D,2} = \{2\} \quad e_{D,3} = \{2, 3\} \quad e_{C,4} = \{1, 2, 4\} \quad e_{D,4} = \{2, 3, 4\} \quad e_{C,5} = \{5, 2, 4\} \quad e_{D,5} = \{6, 3, 4\} \quad e_{C,6} = \{5, 6, 4\} \quad e_{D,6} = \{6, 3, 4\} \quad e_{C,8} = \{5, 6, 8\} \quad e_{D,8} = \{6, 7, 8\} \quad e_D,7 = \{6, 7, 4\} \quad e_{C,9} = \{5, 6, 8\} \quad e_{D,9} = \{6, 7, 8\} \quad e_{C,10} = \{5, 6, 8\} \quad e_{D,10} = \{6, 7, 8\} \quad e_{C,11} = \{5, 6, 8\} \quad e_{D,11} = \{6, 7, 8\} \quad \ldots \quad \ldots \]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{witness_graph.pdf}
\caption{An illustration of the witness graph. In this example, the hypergraph \( \Phi \) has four vertices and two hyperedges. We list the vertices of the witness graph in the table on the right. The lower left picture visualizes three vertices in \( H_T \); all of them have label \( C \).}
\end{figure}
Starting from Section 4, however, we will switch back to the abstract setting and sweep the concrete realization under the rug.

Recall $F$ is the transition map for our systematic scan.

**Lemma 5.** Assume $L \in \mathbb{N}$, $T := n(L+1)$ and $R_1, \ldots, R_T$ are independent Bernoulli($1/2$) variables. If $|F(\Omega_F; R_1, \ldots, R_T)| > 1$ then $\mathcal{B}_L = \mathcal{B}_L(R_1, \ldots, R_T)$ happens.

To prove Lemma 5 we specify a grand coupling, namely a family of Markov chains that share the same random sequence $R_1, \ldots, R_T$ provided by the lemma.

For every $\sigma \in \Omega_F$, define a copy of systematic scan $(X_{\sigma,t})_{0 \leq t \leq T}$ by

$$X_{\sigma,t} := F(\sigma, R_1, \ldots, R_t) \quad \forall 0 \leq t \leq T.$$  

In addition, we define an auxiliary Markov chain $(Y_t)_{0 \leq t \leq T}$ by

$$Y_0 := \{1\}^V, \quad Y_t := Y_{t-1}^{v_{i} \leftarrow R_t} \quad \forall 1 \leq t \leq T.$$  

The chain $(Y_t)_{0 \leq t \leq T}$ dominates the execution of $(X_{\sigma,t})_{0 \leq t \leq T}$ by monotonicity:

**Proposition 6.** For all $\sigma \in \Omega_F$ and $0 \leq t \leq T$, we have $X_{\sigma,t} \leq Y_t$.

**Proof.** Initially, $X_{\sigma,0} = \sigma \leq \{1\}^V = Y_0$ for all $\sigma \in \Omega_F$. At any time $t \geq 1$, all the chains update the same vertex $v_t$ and (i) if $R_t = 0$ then $X_{\sigma,t}(v_t) = 0$; (ii) if $R_t = 1$ then $Y_t(v_t) = 1$. So the ordering $X_{\sigma,t} \leq Y_t$ is preserved throughout.

**Proposition 7.** Let $0 \leq t \leq T$ be a time point. If there exist $\sigma, \tau : X_{\sigma,t}(v_t) \neq X_{\tau,t}(v_t)$, then there is a hyperedge $C \in \mathcal{C}$ containing $v_t$ such that $C \setminus \{v_t\}$ was fully colored by “1” in exactly one of $X_{\sigma,t}$ and $X_{\tau,t}$. Furthermore, $Y_t(C) = \{1\}^{C,t}$.  

**Proof.** Since $X_{\sigma,t}(v_t) \neq X_{\tau,t}(v_t)$, we know that $R_t = 1$ and thus $Y_t(v_t) = 1$. According to the definition of transition map $f$, one of the two chains failed to recolor $v_t$ by “1” because there exists a hyperedge $C$ such that $C \setminus \{v_t\}$ was fully colored “1” in that chain just before time $t$. Hence $Y_t(C \setminus \{v_t\}) = \{1\}^{C,t \setminus \{v_t\}}$ by Proposition 6. Putting together we have $Y_t(C) = \{1\}^{C,t}$.

We are now ready to prove Lemma 5.

**Proof of Lemma 5.** Assume $|F(\Omega_F; R_1, \ldots, R_T)| > 1$; that is, there exist $\sigma, \tau \in \Omega_F$ and $v \in V$ such that $X_{\sigma,T}(v) \neq X_{\tau,T}(v)$. We inductively construct a path in $H_T$ as follows:

1. Set $i \leftarrow 1$. Let $t_i := \text{UpdTime}(v,T) \in (T-n, T]$.

2. While $t_i \geq n$ do the following. Regarding Proposition 7 and the fact $X_{\sigma,t_i}(v_{t_i}) \neq X_{\tau,t_i}(v_{t_i})$, there is a hyperedge $C_t \in \mathcal{C}$ containing $v_{t_i}$ such that $C_t \setminus \{v_{t_i}\}$ was fully colored by “1” in exactly one of $X_{\sigma,t_i}$ and $X_{\tau,t_i}$. So we may find an earliest time $t_{i+1} \in e_i := C_{t_i}$ such that $X_{\sigma,t_{i+1}}(v_{t_{i+1}}) \neq X_{\tau,t_{i+1}}(v_{t_{i+1}})$ and $t_{i+1} < t_i$. Moreover, $Y_{t_i}(C_t) = \{1\}^{C_t}$ implies that $e_i$ is open. Let $i \leftarrow i + 1$ and repeat.

Note that $t_{i+1} = e_i$ by definition. On the other hand $t_{i+1} \in e_{i+1} = C_{t_{i+1}}$ since $v_{t_{i+1}} \in C_{t_{i+1}}$. Combining with the condition $t_{i+1} < t_i$, we see $t_{i+1} \in (e_i \cap e_{i+1}) \setminus \{t_i\}$ and consequently $e_i \rightarrow e_{i+1}$ is indeed an edge in $H_T$. Therefore the above procedure returns an open (not necessarily induced) path $P^\circ = (e_1, \ldots, e_r) \subseteq V_T$ where $r$ is the number of rounds. It starts at vertex $e_1$ that intersects $(T-n, T]$, and ends at vertex $e_r$ that intersects $[1, n]$.

Let $P \subseteq P^\circ$ be a shortest path from $e_1$ to $e_r$ in $H_T[P^\circ]$. It must be an induced path in $H_T$ since $H_T$ is acyclic. It is open since $P \subseteq P^\circ$. Finally, every vertex $e \in P$ “spans” a time interval of at most $n$, so the length of $P$ is at least $T/n - 1 = L$.  

\[\blacksquare\]
3.3 The perfect sampler

We are ready to introduce our perfect sampler for hypergraph independent sets. Let \( L \in \mathbb{N} \) be a parameter to be fixed later (roughly in the order of \( O(\log m) \)) and \( T = T(L) := n(L + 1) \) as usual. The algorithm follows the standard framework of CFTP.

**Algorithm 1** The CFTP sampler of hypergraph independent sets.

**Input:** A hypergraph \( \Phi = (V, C) \).

**Output:** An independent set of \( \Phi \) sampled uniformly from \( \mu \).

1. \( j \leftarrow 0; \)
2. repeat
3. \( j \leftarrow j + 1; \)
4. generate independent \( \text{Bernoulli}(1/2) \) variables \( R_{-jT+1}, \ldots, R_{-(j-1)T} \);
5. until not \( \text{Detect}(\Phi, R_{-jT+1}, \ldots, R_{-(j-1)T}) \);
6. return \( F(V; R_{-jT+1}, \ldots, R_0) \).

**Algorithm 2** The Detect subroutine.

**Input:** A hypergraph \( \Phi = (V, C) \) and a sequence \( R_1, \ldots, R_T \).

**Output:** Whether \( B_L(R_1, \ldots, R_T) \) happens.

1. foreach \( C \in \mathcal{C} \) do
2. foreach \( t \in [T] : v_t \in C \) do
3. foreach \( v \in C \setminus \{ v_t \} \) do
4. foreach \( C' \in \mathcal{C} : v \in C' \neq C \) do
5. \( t' \leftarrow \text{UpdTime}(v, t); \)
6. connect \( e_{C,t} \rightarrow e_{C',t'} \) if both of them are open with regard to \( R_1, \ldots, R_T \);
7. breadth-first search from all \( e_{C,t} : C \in \mathcal{C}, t \in (T-n, T] \) until the current vertex intersects \([1, n] \);
8. return if the path has length \( \geq L \).

**Theorem 8.** Suppose there exist constant \( \varepsilon \in (0, 1) \) and \( \beta \) such that \( \Pr[B_L] \leq \beta \cdot (1 - \varepsilon)^L \) for all \( L \). With the concrete choice \( L := \left\lceil \frac{\log(2\beta)}{\log(1 - \varepsilon)} \right\rceil \), Algorithm 1 terminates with probability 1 and has expected running time \( O \left( \frac{\log(2\beta)}{\log(1 - \varepsilon)} \cdot \sum_{C \in \mathcal{C}} \sum_{v \in C} d_v |C| \right) \). Moreover, its output distribution \( \nu \) is exactly \( \mu \) upon termination.

**Proof.** The Detect subroutine (Algorithm 2) essentially constructs the witness graph \( H_T \) and decides if \( B_L(R_1, \ldots, R_T) \) happens. Since the inputs fed by Algorithm 1 are always independent \( \text{Bernoulli}(1/2) \) variables as required, we have

\[
p := \Pr[\text{Detect}(\Phi, R_{-jT+1}, \ldots, R_{-(j-1)T})] = \Pr[B_L] \leq \beta \cdot (1 - \varepsilon)^L = \frac{1}{2}.
\]

where the inequality follows from the assumption of the theorem.

Note that Algorithm 1 feeds disjoint (hence independent) sequences into Algorithm 2 in different rounds, so the corresponding return values are independent. Therefore, the total number of rounds \( J \) follows geometric distribution \( \text{Geom}(1 - p) \). In particular \( \Pr[J < \infty] = 1 \) and \( E[J] = \frac{1}{1-p} \leq 2 \). So the algorithm terminates with probability 1.
Next we analyze the running time of a single call to Algorithm 2. Each iteration of the nested loop can be implemented in constant time: We index each vertex \( e \in C, t \) by the pair \((C, t)\) to allow random access; the test of openness can be computed and stored beforehand to facilitate fast lookup. So the nested loop takes

\[
O \left( \sum_{C \in \mathcal{C}} |C| \cdot L \cdot \sum_{v \in C} d_v \right)
\]

time to finish. The standard breadth-first search consumes time proportional to the number of edges in the witness graph, which is bounded by the same quantity.

Finally, recall that Algorithm 1 calls Algorithm 2 \( J \) times. But \( E[J] \leq 2 \), so the expected running time of our sampler is

\[
O \left( L \cdot \sum_{C \in \mathcal{C}} \sum_{v \in C} d_v |C| \right).
\]

We could now turn to show \( \nu = \mu \) as advertised. Upon termination, the percolation event \( B_L(R_{JT+1}, \ldots, R_{(J-1)T}) \) does not happen and hence \( |F(\Omega; R_{JT+1}, \ldots, R_{(J-1)T})| = 1 \) by Lemma 5. But this in particular means \( |F(\Omega; R_{JT+1}, \ldots, R_0)| = 1 \).

For each integer \( s \geq 0 \), we define a (virtual) copy \((Z_s^t)_{-sn \leq t \leq 0}\) of systematic scan which starts from \( \{0\}^V \) at time \(-sn\):

\[
Z_s^t := F \left( \{0\}^V; R_{-sn+1}, \ldots, R_t \right).
\]

Let \( \nu^s \) denote the distribution of \( Z_s^0 \). Then \( \lim_{s \to \infty} \nu^s = \mu \) by convergence theorem. On the other hand,

\[
\| \nu^s - \nu \| \leq \Pr \left[ F \left( \{0\}^V; R_{-sn+1}, \ldots, R_0 \right) \neq F \left( \{0\}^V; R_{JT+1}, \ldots, R_0 \right) \right] \quad \text{(coupling)}
\]

\[
\leq \Pr \left[ sn < JT \right] \quad \text{(by (1))}
\]

\[
\to 0 \quad \text{(as } s \to \infty)\]

Hence \( \nu = \lim_{s \to \infty} \nu^s = \mu \). \hfill \blacksquare

### 3.4 Proofs of Theorem 1 and Theorem 2

Armed with Theorem 8, we only need to bound \( \Pr[B_L] \) under various conditions. We bound the quantity in Section 4. As a result, Theorem 1 follows from Lemma 9 and Theorem 2 follows from Lemma 13.

### 4 Percolation on Witness Graphs

Let \( \Phi = (V, \mathcal{C}) \) be a hypergraph with \(|V| = n\) and let \( H_T \) be its witness graph for any \( T = (L + 1)n, L \in \mathbb{N} \). We will analyze the probability \( \Pr[B_L] \) in the abstract probability space, where each time point \( t \in [T] \) is associated with an independent \( \text{Bernoulli}(1/2) \) variable.

#### 4.1 General hypergraphs

\textbf{Lemma 9.} If there exists a constant \( \varepsilon \in (0, 1) \) and a function \( x : \mathcal{C} \to (0, 1) \) satisfying

\[
\forall C \in \mathcal{C} : 2|C| \cdot 2^{-|C|} \leq (1 - \varepsilon) \cdot x(C) \cdot \prod_{C' \in \mathcal{P}_x(C)} (1 - x(C'))
\]

then \( \Pr[B_L] \leq \sum_{C \in \mathcal{E}} \frac{x(C)}{1 - x(C)} \cdot (1 - \varepsilon)^{\frac{L + 1}{2}} \) for all \( L \in \mathbb{N} \).
For every $C \in \mathcal{C}$, we use $P_{C,L}$ to denote the collection of induced paths $(e_1, e_2, \ldots, e_L)$ in $H_T$ where $e_1 \cap (T - u, T] \neq \emptyset$ and the label of $e_1$ is $C$ (namely $C(e_1) = C$). Let

$$P_{C,[L+1]}^2 := \left\{(e_1, e_3, e_5, \ldots, e_{2L+1}) : (e_1, e_2, e_3, \ldots, e_L) \in P_{C}\right\}. $$

Then by the union bound, we have

$$\Pr[B_L] \leq \sum_{C \in \mathcal{C}} \sum_{P \in P_{C,L}} \Pr[P \text{ is open}] \leq \sum_{C \in \mathcal{C}} \sum_{P \in P_{C,[L+1]}^2} \Pr[P \text{ is open}]$$

$$= \sum_{C \in \mathcal{C}} \sum_{e_1, \ldots, e_{L+1} \in P_{C,[L+1]}^2} \prod_{i=1}^{L+1} 2^{-|C(e_i)|}, \tag{2}$$

where the last equality follows from the fact that $e_i \cap e_{i+1} = \emptyset$ holds for every pair of consecutive vertices of a path in $P_{C,[L+1]}^2$.

**Lemma 10.** If there exists a function $x : \mathcal{C} \to (0, 1)$ satisfying

$$\forall C \in \mathcal{C} : 2|C| \cdot 2^{-|C|} \leq (1 - \varepsilon) \cdot x(C) \cdot \prod_{C' \subseteq \Gamma_{\leq 2}(C)} (1 - x(C')),$$

then

$$\sum_{C \in \mathcal{C}} \sum_{e_1, \ldots, e_{L+1} \in P_{C,[L+1]}^2} \prod_{i=1}^{L+1} 2^{-|C(e_i)|} \leq \sum_{C \in \mathcal{C}} \frac{x(C)}{1 - x(C)} \cdot (1 - \varepsilon)^{L+1}.$$
Proof. For a vertex \( v \in V_\tau \) we use \( W_v \subseteq \Gamma_{\Delta^+}(\mathcal{L}(v)) \) to denote the set of labels that do not occur as a label of some child nodes of \( v \). Then clearly, the probability that the Galton-Watson branching process produces exactly the tree \( \tau \) is given by

\[
\mu_{\mathcal{T}_C}(\tau) = \frac{1}{x(C)} \prod_{v \in V_\tau} x(\mathcal{L}(v)) \prod_{C' \in W_v} (1 - x(C')).
\]

In order to get rid of the \( W_v \), we can rewrite the expression as

\[
\mu_{\mathcal{T}_C}(\tau) = \frac{1 - x(C)}{x(C)} \prod_{v \in V_\tau} x(\mathcal{L}(v)) \prod_{C' \in \Gamma_{\Delta^+}(\mathcal{L}(v))} (1 - x(C'))
\]

\[
= \frac{1 - x(C)}{x(C)} \prod_{v \in V_\tau} x(\mathcal{L}(v)) \prod_{C' \in \Gamma_{\Delta^+}(\mathcal{L}(v))} (1 - x(C')). \quad \blacktriangleleft
\]

For every \( \ell \geq 1 \), we also use \( \mathcal{T}_{C,\ell} \) to denote the set of labelled trees in \( \mathcal{T}_C \) containing exactly \( \ell \) vertices. In the following, we slightly abuse notation and say that a labelled tree \( \tau = (V_\tau, E_\tau, \mathcal{L}) \in \mathcal{T}_C \) (or \( \mathcal{T}_{C,\ell} \)) if there exists some \( \tau' \in \mathcal{T}_C \) (or \( \mathcal{T}_{C,\ell} \)) such that \( \tau \) and \( \tau' \) are isomorphic.

### 4.1.2 Proof of Lemma 10

Let \( \ell = \lceil \frac{L+1}{2} \rceil \). It follows from the assumption that for every \( C \in \mathcal{C} \),

\[
\sum_{P=(e_1,e_2,\ldots,e_\ell) \in \mathcal{P}_{\mathcal{T}_C,\ell}} \prod_{i=1}^{\ell} 2^{-|C(e_i)|} \leq (1 - \epsilon)^{\ell} \sum_{P=(e_1,e_2,\ldots,e_\ell) \in \mathcal{P}_{\mathcal{T}_C,\ell}} \prod_{i=1}^{\ell} \frac{1}{2|C(e_i)|} \cdot x(C(e_i)) \prod_{C' \in \Gamma_{\Delta^+}(C(e_i))} (1 - x(C')). \quad (3)
\]

We now define a mapping \( \Psi : \mathcal{P}_{\mathcal{T}_C,\ell} \rightarrow \mathcal{T}_{C,\ell} \) that maps each \( P = (e_1,e_2,\ldots,e_\ell) \) to a labelled directed path \( \tau = (V_\tau, E_\tau, \mathcal{L}) \) where

- \( V_\tau = \{u_1,u_2,\ldots,u_\ell\} \);
- \( E_\tau = \{(u_i,u_{i+1}) : i \in [\ell - 1]\} \); and
- for every \( i \in [\ell] \), \( \mathcal{L}(u_i) = C(e_i) \).

The mapping \( \Psi \) is not necessarily an injection, but we can bound its multiplicity.

**Lemma 12.** For every labelled tree \( \tau = (V_\tau, E_\tau, \mathcal{L}) \in \mathcal{T}_{C,\ell} \) with \( V_\tau = \{u_1,\ldots,u_\ell\} \) and \( E_\tau = \{(u_i,u_{i+1}) : i \in [\ell - 1]\} \), it holds that

\[
|\Psi^{-1}(\tau)| \leq 2^{\ell-1} \cdot \prod_{i=1}^{\ell} |\mathcal{L}(u_i)|.
\]

**Proof.** We prove it by induction:

1. When \( \ell = 1 \), the lemma holds because there are at most \(|C|\) timestamps \( e \) with label \( C \) such that \( e \cap (T - n + 1, T] \neq \emptyset \);
2. We assume the statement holds for \( \ell \geq 1 \). Let \( V' = \{u_1,u_2,\ldots,u_\ell,u_{\ell+1}\} \) and \( V'' = \{u_1,u_2,\ldots,u_\ell\} \). According to the definition of \( \Psi \), we know that for all \( P \in \Psi^{-1}(\tau) \), it was extended from some \( P' \in \Psi^{-1}(\tau') \). Therefore, it suffices to analyze the possible extension from \( P' \) to \( P \in \Psi^{-1}(\tau) \). Assuming \( P' = (e_1,e_2,\ldots,e_\ell) \) and \( P = (e_1,e_2,\ldots,e_\ell,e_{\ell+1}) \), if \( P \)}
was extended from \( P' \), then \( \text{dist}(e_{i}, e_{i+1}) = 2 \) and \( C(e_{i+1}) = \mathcal{L}(u_{i+1}) \). Therefore, there are at most \( 2|\mathcal{L}(u_{i+1})| \) timestamps \( t \) satisfying \( P \in \Psi^{-1}(t) \) according to Lemma 4. Combining with the induction hypothesis, the statement holds for \( \ell + 1 \).

Clearly for those \( t \in \mathcal{T}_{C,\ell} \) which are not directed paths, its pre-image \( \Psi^{-1}(t) = \emptyset \). Therefore, it follows from Lemma 11 and Lemma 12 that

\[
\sum_{P=(e_{1}, e_{2}, \ldots, e_{\ell}) \in \mathcal{P}_{C,\ell}} \prod_{i=1}^{\ell} \frac{1}{2|\mathcal{C}(e_{i})|} \cdot x(\mathcal{C}(e_{i})) \prod_{C' \in \Gamma_{\mathcal{C}}(\mathcal{C}(e_{i}))} (1 - x(C'))
\]

\[
\leq \sum_{\tau=(V, E, L) \in \mathcal{T}_{C,\ell}} \prod_{i=1}^{\ell} x(\mathcal{L}(u_{i})) \prod_{C' \in \Gamma_{\mathcal{C}}(\mathcal{L}(u_{i}))} (1 - x(C'))
\]

\[
= \frac{x(C)}{1 - x(C)} \sum_{\tau \in \mathcal{T}_{C,\ell}} \mu_{\mathcal{T}_{C}}(\tau)
\]

\[
\leq \frac{x(C)}{1 - x(C)}.
\]

### 4.2 Refined analysis for uniform hypergraphs

If the instance \( \Phi = (V, \mathcal{C}) \) is a \( d \)-regular \( k \)-uniform hypergraph, we can choose \( x(C) = \frac{1}{x^{2}k^{2}} \) in Lemma 9 and the condition becomes \( d \leq \frac{c2^{k/2}}{x^{2}k^{2}} \) for some constant \( c > 0 \). In this section, we present a refined analysis for this case which removes the denominator \( k^{1.5} \).

**Lemma 13.** For all \( \varepsilon \in (0, 1) \) and \( k \geq 2 \), if

\[
d \leq \left( \frac{1}{4} \right)^{\frac{9 - \varepsilon}{2} - \frac{1}{2}} \cdot 2^{k/2}
\]

then \( \Pr [B_{L}] \leq \frac{k}{2^{r}} \cdot m \cdot (1 - \varepsilon)^{\frac{1.5}{2}} \) for all \( L \in \mathbb{N} \).

For every induced path \((u_{1}, \ldots, u_{k})\) in \( H_{T} \) and \( L > 0 \), we use \( \mathcal{P}_{(u_{1}, \ldots, u_{k}), L} \) to denote the collection of induced paths in \( H_{T} \) of length \( L \) with prefix \((u_{1}, \ldots, u_{k})^{2}\).

**Lemma 14.** For all \( \varepsilon \in (0, 1) \) and \( k \geq 2 \), if

\[
d \leq \left( \frac{1}{4} \right)^{\frac{9 - \varepsilon}{2} - \frac{1}{2}} \cdot 2^{k/2}
\]

then for every \( u \in V_{H_{T}} \) and every \( L \in \mathbb{N} \),

\[
\Pr \left[ \exists \text{ open } P \in \mathcal{P}_{(u), L} \mid u \text{ is open} \right] \leq (1 - \varepsilon)^{\frac{1.5}{2}}.
\]

We first show that Lemma 13 follows from Lemma 14.

**Proof of Lemma 13.** By the union bound, we have

\[
\Pr [B_{L}] \leq \sum_{C \in \mathcal{C}} \sum_{i \in (T-n, T)} \Pr \left[ \exists \text{ open } P \in \mathcal{P}_{(\varepsilon_{C}, i), L} \right]
\]

\[
= \sum_{C \in \mathcal{C}} \sum_{i \in (T-n, T)} \Pr \left[ \exists \text{ open } P \in \mathcal{P}_{(\varepsilon_{C}, i), L} \mid \varepsilon_{C, i} \text{ is open} \right] \cdot \Pr [\varepsilon_{C, i} \text{ is open}]
\]

\[
\leq km(1 - \varepsilon)^{\frac{1.5}{2}} \cdot 2^{-k}.
\]

\[2 \text{ Unlike the notation } \mathcal{P}_{C, L} \text{ defined in Section 4.1, we do not require } u_{1} \cap \{T - n + 1, \ldots, T\} = \emptyset \text{ here.}\]
The remaining part of the section is devoted to a proof of Lemma 14. We apply induction on $L$. The base case is that $L = 1$ and $L = 2$, in which the lemma trivially holds. For larger $L$ and every path $P = (u_1, u_2, u_3, \ldots, u_L) \in \mathcal{P}(u_1), L$, we discuss how the vertices $u_1$, $u_2$ and $u_3$ overlap.

Recall for every $u = e_{C,t} \in V_T$, $C(u) = C$ is its label. Similar to [10], we classify the tuple $(u_1, u_2, u_3)$ into three categories.

- We say $u_2$ is good if $|C(u_2) \cap C(u_1)| \leq \alpha \cdot k$ where $\alpha \in [0, 1]$ is a parameter to be set.
- If $u_2$ is not good, we say $(u_2, u_3)$ is of type I if $C(u_3) \cap C(u_1) \neq \emptyset$;
- If $u_2$ is not good, we say $(u_2, u_3)$ is of type II if $C(u_3) \cap C(u_1) = \emptyset$.

Then we can write

$$
\Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1), L \mid u_1 \text{ is open} \right] 
\leq \Pr \left[ \exists \text{ open } P = (u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1), L \mid u_2 \text{ is good } \mid u_1 \text{ is open} \right] 
+ \Pr \left[ \exists \text{ open } P = (u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1), L \mid (u_2, u_3) \text{ is of type I } \mid u_1 \text{ is open} \right] 
+ \Pr \left[ \exists \text{ open } P = (u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1), L \mid (u_2, u_3) \text{ is of type II } \mid u_1 \text{ is open} \right]
$$

In the following, we bound the probabilities in the three cases respectively.

$u_2$ is good

In this case, we have $|C(u_2) \cap C(u_1)| \leq \alpha \cdot k$. Therefore, conditioned on $u_1$ being open, at least $(1 - \alpha) \cdot k$ variables in $u_2$ are free (i.e. independent of those variables related to $u_1$). The number of choices of $C(u_2)$ is at most $k \cdot d$ and the number of choices of $u_2$ with fixed $C = C(u_2)$ is at most $k$. Combining this with the induction hypothesis, we have

$$
\Pr \left[ \exists \text{ open } P = (u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1), L \mid u_2 \text{ is good } \mid u_1 \text{ is open} \right] 
= \Pr \left[ \bigcup_{\text{good } u_2^*} \left[ \exists \text{ open } P \in \mathcal{P}(u_1, u_2^*), L \mid u_1 \text{ is open} \right] \right] 
\leq \sum_{\text{good } u_2^*} \Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1, u_2^*), L \mid u_1 \text{ is open} \right] 
= \sum_{\text{good } u_2^*} \Pr [u_2^* \text{ is open } \mid u_1 \text{ is open}] \cdot \Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1, u_2^*), L \mid (u_1, u_2^*) \text{ is open} \right] 
= \sum_{\text{good } u_2^*} \Pr [u_2^* \text{ is open } \mid u_1 \text{ is open}] \cdot \Pr \left[ \exists \text{ open } P' \in \mathcal{P}(u_1^*), L-1 \mid u_2^* \text{ is open} \right] 
\leq k^2 d \cdot 2^{-(1 - \alpha) k \cdot (1 - \epsilon)} \cdot \frac{k^{3 - 3}}{2^k}.
$$

($u_2, u_3$) is of type I

In this case, we know $C(u_3)$ is adjacent to $C(u_1)$ in $\Phi$ and therefore the number of the choices of $C(u_3)$ is at most $kd$. The choice of $u_3$ with fixed $C = C(u_3)$ is at most $2k$ by Lemma 4. On the other hand, since $P$ is an induced path, $u_1 \cap u_3 = \emptyset$ and $\Pr [u_3 \text{ is open } \mid u_1 \text{ is open} ] = 2^{-k}$. Combining these facts with the induction hypothesis, we have
To ease the notation, we let 

\[ C \]

with 

\[ C \]

To this end, we first enumerate all pairs 

This case is more complicated. We will enumerate all type II pairs and bound the sum of probabilities that some path in \( P_{(u_2, u_3)} \) is open conditioned on \( u_1 \) being open. To do this, we first consider all pairs \( (u_2, u_3) \) with \( C(u_2) = C_2 \) and \( C(u_3) = C_3 \).

For a fixed pair \((C_2, C_3)\), we aim to bound the probability

\[
\sum_{(u_2,u_3) \text{ is of type II,} \quad C(u_2)=C_2, C(u_3)=C_3} \Pr \left[ \exists \text{ open } P \in P_{(u_1, u_2, u_3),L} \mid u_1 \text{ is open} \right].
\]  

(6)

\((u_2, u_3)\) is of type II

This case is more complicated. We will enumerate all type II pairs \((u_2, u_3)\) and bound the sum of probabilities that some path in \( P_{(u_2, u_3)} \) is open conditioned on \( u_1 \) being open.

To do this, we first enumerate all pairs \((C_2, C_3)\) and then consider all pairs \((u_2, u_3)\) with \( C(u_2) = C_2 \) and \( C(u_3) = C_3 \).

For a fixed pair \((C_2, C_3)\), we aim to bound the probability

\[
\sum_{(u_2,u_3) \text{ is of type II,} \quad C(u_2)=C_2, C(u_3)=C_3} \Pr \left[ \exists \text{ open } P \in P_{(u_1, u_2, u_3),L} \mid u_1 \text{ is open} \right].
\]  

(7)

To ease the notation, we let \( C_1 := C(u_1) \). In order for the above sum to be nonzero, we need have \( |C_2 \cap C_1| > \alpha \cdot k \) and \( C_3 \cap C_1 = \emptyset \). Let \( a := |C_2 \cap C_1| \) and \( b := |C_3 \cap C_1| \). Recall the notations in Section 3.1. For those \((u_2, u_3)\) with \( C(u_2) = C_2 \) and \( C(u_3) = C_3 \), we write them as \( u_2 = e_{C_2, t_2} \) and \( u_3 = e_{C_3, t_3} \) respectively. In the following, all pairs \((u_2, u_3)\) discussed are of type II.

Note that \( e_{C_2, t_2} =: u_2 \neq u_2' := e_{C_2, t_2}' \) if and only if \( t_2 \neq t_2' \). Similarly \( e_{C_2, t_2} =: u_3' \neq u_3 := e_{C_2, t_2} \) if and only if \( t_3 \neq t_3' \). Moreover, if \( u_2 \neq u_2' \), then \( |u_2 \cap u_1| \neq |u_2' \cap u_1| \) and similarly if \( u_3 \neq u_3' \), then \( |u_3 \cap u_2| \neq |u_3' \cap u_2| \). As a result, we can enumerate type II pairs \((u_2, u_3)\) by enumerating the sizes of \( u_2 \cap u_1 \) and \( u_3 \cap u_2 \) respectively. On the other hand, if we know that \( |u_2 \cap u_1| = i \) and \( |u_3 \cap u_2| = j \), then

\[
\Pr \left[ \exists \text{ open } P \in P_{(u_1, u_2, u_3),L} \mid u_1 \text{ is open} \right] = \Pr [u_2 \text{ is open} \mid u_1 \text{ is open}] \cdot \Pr [u_3 \text{ is open} \mid (u_1, u_2) \text{ is open}] \leq \Pr [u_2 \text{ is open} \mid u_1 \text{ is open}] \cdot \Pr [u_3 \text{ is open} \mid u_2 \text{ is open}] \leq 2^{-k-i} \cdot 2^{-k-j} \cdot \Pr [\exists \text{ open } P' \in P_{(u_3),L-2} \mid u_3 \text{ is open}].
\]
To see this, consider the following way to enumerate all 3 hyperedges in total, and every hyperedge is at most \( \frac{4}{\alpha} \cdot \frac{1}{\epsilon^4} \).

It remains to enumerate \((C_2, C_3)\) pairs. Since we know that \( |C_2 \cap C_1| > \alpha \cdot k \), the number of choices of \( C_2 \) is at most \( \frac{\alpha}{\alpha} \cdot \frac{k}{\alpha} = \frac{d}{\alpha} \). For every fixed \( C_2 \), we let \( \mathcal{C} = \{C_3^{(1)}, C_3^{(2)}, \ldots, C_3^{(m)}\} \) be the collection of all possible \( C_3 \). Denote \( a := |C_2 \cap C_1| \) as before and for every \( i \in [m] \) let \( b_i := |C_3^{(i)} \cap C_2| \). The probability of interest can therefore be written as

\[
\Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1, L) \text{ where } (u_2, u_3) \text{ is of type II and } C(u_2) = C_2 \mid u_1 \text{ is open} \right] 
\leq \sum_{(u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1, L)} \Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1, u_2, u_3), L \mid u_1 \text{ is open} \right] 
\leq \left( \sum_{i=1}^{m} 2^{a+b_i-2k} \right) \cdot 4(1-\epsilon)\frac{1}{\epsilon^4}.
\]

To bound the term \( \sum_{i=1}^{m} 2^{a+b_i-2k} \), let us list some properties of the numbers involved:

- \( a \leq k \) and \( 1 \leq m \leq (k-a)d \);
- \( 1 \leq b_i \leq k-a \) for all \( i \in [m] \);
- \( \sum_{i=1}^{m} b_i \leq (k-a)d \). (This can be seen by an argument similar to the last footnote and the fact that \( C_3 \cap C_1 = \emptyset \).)

Roughly the numbers are acting against each other: If \( a \) and \( m \) are large then the \( b_i \)'s are small. So it is possible to control \( \sum_{i=1}^{m} 2^{a+b_i-2k} \) reasonably:

\[\text{Lemma 15. Suppose } a \leq k \text{ and } 1 \leq m \leq (k-a)d. \text{ Then for any integers } b_1, \ldots, b_m \in [k-a] \text{ such that } \sum_{i=1}^{m} b_i \leq (k-a)d, \text{ we have the inequality } \sum_{i=1}^{m} 2^{a+b_i-2k} \leq d \cdot 2^{-k}.\]

We arrange the proof in Appendix A. Returning to previous discussion, we derive

\[\Pr \left[ \exists \text{ open } P = (u_1, u_2, u_3, \ldots) \in \mathcal{P}(u_1, L) \text{ where } (u_2, u_3) \text{ is of type II} \mid u_1 \text{ is open} \right] \leq \frac{4a^2}{\alpha} \cdot 2^{-k} \cdot (1-\epsilon)^\frac{1}{\epsilon^4}. \quad (8)\]

Putting all together

Plugging Equations (5), (6) and (8) into Equation (4), we have

\[\Pr \left[ \exists \text{ open } P \in \mathcal{P}(u_1, L) \mid u_1 \text{ is open} \right] \leq \left( k^2d \cdot 2^{-1-\alpha}k + 2k^2d \cdot 2^{-k} + \frac{4a^2}{\alpha} \cdot 2^{-k} \right) \cdot (1-\epsilon)^\frac{1}{\epsilon^4} \]

\[< \left( 2^{a+k+2}k^2d + \frac{4}{\alpha}d^2 \right) 2^{-k} \cdot (1-\epsilon)^\frac{1}{\epsilon^4} \]

To see this, consider the following way to enumerate all \( C_2 \) incident to \( C_1 \) in \( \Phi \): First pick a vertex in \( C_1 \) and then pick one of its incident hyperedges. This way we enumerated (with repetitions) \( k \cdot d \) hyperedges in total, and every hyperedge \( C_2 : |C_2 \cap C_1| > \alpha \cdot k \) is enumerated at least \( \alpha \cdot k \) times.
where we used a very crude inequality $2^{\alpha_k} + 2 < 2^{\alpha_k+2}$. Let us take $\alpha = \alpha(k) := \frac{1}{2} + \frac{3-2\log k}{k} \in [0,1]$ for any $k \geq 2$. Note that $2^{\alpha_k+2}k^2 = 2^{\alpha_k+5}+3\epsilon$ by definition. On the other hand, from basic calculus one may find that $\alpha(k) : k \in \mathbb{N}$ attains minimum value $1/8$ at $k = 8$, thus $4/\alpha \leq 32 = 2^5$. Hence the above bound continues as:

$$
\cdots \leq \left(2^{k/2}d + d^2 \right) \cdot 2^{-k} \cdot (1 - \epsilon) \cdot \left(\frac{1}{2} + \frac{1}{2^5} \right)
\leq \left(1 - \epsilon \right) \cdot (1 - \epsilon) \cdot \left(\frac{1}{2} + \frac{1}{2^5} \right)
\leq \left(1 - \epsilon \right) \cdot \left(\frac{1}{2} + \frac{1}{2^5} \right)
$$

where the second line is due to our assumption $d \leq \left(\frac{1}{2} \sqrt{\frac{2 - \epsilon}{2}} - \frac{1}{2} \right) \cdot 2^{k/2} =: d^*$. To see this, observe that the function $g(d) := 2^{k/2}d + d^2$ is monotonically increasing when $d \geq 0$, and that $g(d^*) = (1 - \epsilon) \cdot 2^{k-5}$.

\textbf{Remark 16.} When $\epsilon := 0.1$, say, the condition becomes (approximately) $d \leq 0.027 \cdot 2^{k/2}$. Taking smaller $\epsilon$ will allow larger applicable range of $d$, but it comes at the price of slowing down the sampler (by a multiplicative log-factor; see Theorem 8).

If $k$ is sufficiently large we can further improve our bound to $d \lesssim \sqrt{\frac{1-\epsilon}{10}} \cdot 2^{k/2}$.

\textbf{References}

A Perfect Sampler for Hypergraph Independent Sets


A Proof of Lemma 15

For each \( j \in [k-a] \) we introduce a counter \( x_j := \left| \{ i : b_i = j \} \right| \) that counts the number of \( b_i \)'s taking a specific value \( j \). Then the constraint of the lemma translates to

\[
\sum_{j=1}^{k-a} j \cdot x_j \leq (k-a)d,
\]

and we want to show

\[
\sum_{j=1}^{k-a} 2^{a+j-2k} \cdot x_j \leq d \cdot 2^{-k}.
\]

To this end, we consider a (relaxed) linear program with variables \( x_1, \ldots, x_{k-a} \):

\[
\begin{align*}
\max & \quad \sum_{j=1}^{k-a} 2^{a+j-2k} \cdot x_j \\
\text{s.t} & \quad \sum_{j=1}^{k-a} j \cdot x_j \leq (k-a)d \\
& \quad x_j \geq 0 \quad (\forall j \in [k-a]).
\end{align*}
\]

By the strong duality theorem of linear programming, its maximum value equals the minimum value of the dual program

\[
\begin{align*}
\min & \quad (k-a)d \cdot y \\
\text{s.t} & \quad j \cdot y \geq 2^{a+j-2k} \quad (\forall j \in [k-a]) \\
& \quad y \geq 0.
\end{align*}
\]

Clearly the minimum value is obtained when \( y \) is as small as possible. It is clear that the sequence \( h_j := \frac{2^j}{j} \) is monotonically increasing for integers \( j \geq 1 \), so the minimum possible \( y \) is given by the \((k-a)\)-th constraint, namely

\[
y^* := \frac{2^{a+(k-a)-2k}}{k-a} = \frac{2^{-k}}{k-a}.
\]

Therefore, the minimum value of the dual – also the maximum value of the primal – is exactly \( d \cdot 2^{-k} \), as desired.