The Complexity of SPEs in Mean-Payoff Games

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Abstract
We establish that the subgame perfect equilibrium (SPE) threshold problem for mean-payoff games is NP-complete. While the SPE threshold problem was recently shown to be decidable (in doubly exponential time) and NP-hard, its exact worst case complexity was left open.

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1 Introduction

Nash equilibria (NEs), a fundamental solution concept in game theory, are defined as strategy profiles such that no player can improve their payoff by changing unilaterally their strategy. So NEs can be interpreted as self-enforcing contracts from which there is no incentive to deviate unilaterally. Unfortunately, NEs are known to suffer, in sequential games like infinite duration games played on graphs, from the issue of non-credible threats: to enforce a NE, some players may threaten other players to play irrationally in order to punish deviations. This is allowed by the definition of NEs, as in case of deviation from one player, the other players are not bound to rational behaviors anymore and they can therefore play irrationally w.r.t. their own objectives in order to sanction the deviating player. This drawback of NEs has triggered the introduction of the notion of subgame-perfect equilibria (SPEs) [22], a more complex but more natural solution concept for sequential games. A strategy profile is an SPE if after every history, i.e. in every subgame, the strategies of the players still form a NE. Thus, SPEs impose rationality even after a deviation and only rational behaviors can be used to coerce the behavior of other players.

In this paper, we study the complexity of SPE problems in infinite-duration sequential games played on graphs with mean-payoff objectives. While NEs always exist in those games, as proved in [8], SPEs do not always exist as shown in [23, 7]. The SPE threshold problem, i.e. the problem of deciding whether a given mean-payoff game admits an SPE satisfying some constraints on the payoffs it grants to the players, has recently been proved to be decidable.
in [3]. However, its worst-case computational complexity is open: [3] provides only an NP lower bound and a 2ExpTime upper bound. In this paper, we close this complexity gap and prove that the problem is actually NP-complete.

Contributions. The starting point of our algorithm is the characterization of SPEs recently presented in [3], based on the notions of requirement and negotiation function. A requirement on a game $G$ is a function $\lambda : V \rightarrow \mathbb{R} \cup \{\pm\infty\}$, where $V$ is the state space of $G$. For a given state $v$, the value $\lambda(v)$ should be understood as the minimal payoff that the player controlling the state $v$ will require in a play traversing $v$ in order to avoid deviating. A requirement captures, therefore, some level of rationality of the players. The negotiation function transforms each requirement $\lambda$ into a (possibly stronger) requirement $\text{nego}(\lambda)$, such that $\text{nego}(\lambda)(v)$ is the best payoff that the player controlling $v$ can ensure, while playing against a coalition of the other players that play rationally with regards to the requirement $\lambda$. A play is the outcome of an SPE if and only if it satisfies the requirement $\lambda^*$, the least fixed point of the negotiation function – or equivalently, one of its fixed points. We recall that result in Lemma 29. In order to obtain our nondeterministic polynomial time algorithm, the rest of the paper constructs a notion of witness recognizing the positive instances of the SPE threshold problem. Such witnesses admit three pieces. First, we show that the size of $\lambda^*$ can be bounded by a polynomial function of the size of the game (Theorem 37). This result is obtained by showing that the set of fixed points of the negotiation function can be characterized by a finite union of polyhedra that in turn can be represented by linear inequations. While the number of inequations that are needed for that characterization may be large (it cannot be bounded polynomially), we show that each of those inequations have coefficients and constants whose binary representations can be bounded polynomially. As the least fixed point is the minimal value in this set, it is represented by a vertex of one of those polyhedra. Then this guarantees, using results that bounds the solutions of linear equalities, that the least fixed point has a binary representation that is polynomial and so it can be guessed and verified in polynomial time by a nondeterministic algorithm: it will be the first piece of our notion of witness, in the non-deterministic algorithm we design to solve the SPE threshold problem. Second, we define a witness of polynomial size for the existence of a play, consistent with a given requirement, which generates a payoff vector between the desired thresholds (Theorem 38). This play is not guaranteed to be regular. Third, we define a witness of polynomial size to prove that a requirement is indeed a fixed point of the negotiation function. This notion of certificate relies on a new and more compact game characterization of the negotiation function called the reduced negotiation game (Definition 41, Theorem 44). These results are far from trivial as we also show that SPEs may rely on strategy profiles that are not regular and require infinite memory. As both the least fixed point and its two certificates can be guessed and verified in polynomial time, we obtain NP membership for the threshold problem, closing the complexity gap left open in [3] (Theorem 48).

Additionally, all the previous results do also apply to $\varepsilon$-SPEs, a quantitative relaxation of SPEs. In particular, Theorem 48 does also apply to the $\varepsilon$-SPE threshold problem.

Related works. Non-zero sum infinite duration games have attracted a large attention in recent years, with applications targeting reactive synthesis problems, see e.g. [1, 9, 10, 14, 19] and their references. We now detail other works more closely related to our contributions.

In [7], Brihaye et al. introduce and study the notion of weak SPE, which is a weakening of the classical notion of SPE. This weakening is equivalent to the original SPE concept on reward functions that are continuous. This is the case for example for the quantitative
reachability reward function, on which Brihaye et al. solve the SPE threshold problem in [6]. The mean-payoff cost function is not continuous and the techniques used in [7], and generalized in [11], cannot be used to characterize SPEs for the mean-payoff reward function.

In [21], Meunier develops a method based on Prover-Challenger games to solve the problem of the existence of SPEs on games with a finite number of possible payoffs. In mean-payoff games, the number of possible payoffs is uncountably infinite.

In [15], Flesch and Predtetchinski present another characterization of SPEs on games with finitely many possible payoffs, based on a game structure with infinite state space. In [3], Brice et al. define the notions of requirements and negotiation function. They prove that the negotiation function is characterized by a zero-sum two-player game called abstract negotiation game, which is similar to the game introduced in the characterization of Flesch and Predtetchinski. As a starting point for algorithms, they also provide an effective representation of this game, called concrete negotiation game, which turns out to be a zero-sum finite state multi-mean-payoff games [24]. Finally, they use those tools to prove that the SPE threshold problem is decidable for mean-payoff games. They left open the question of its precise complexity: they provide a \( \text{NP} \) lower bound and a \( 2\text{ExpTime} \) upper bound. In [5], the same authors use those tools to close the complexity gap for the SPE threshold problem in parity games, which had been proved to be \( \text{ExpTime} \)-easy and \( \text{NP} \)-hard by Ummels and Grädel in [16]. They prove that the problem is actually \( \text{NP} \)-complete. The techniques used in that paper heavily rely on the fact that parity objectives are \( \omega \)-regular, which is not the case of mean-payoff games in general.

In [13], Chatterjee et al. study mean-payoff automata, and give a result that can be translated into an expression of all the possible payoff vectors in a mean-payoff game. In [2], Brenguier and Raskin give an algorithm to build the Pareto curve of a multi-dimensional two-player zero-sum mean-payoff game. To do so, they study systems of equations and of inequations, and they prove that they always admit simple solutions (with polynomial size). Those technical results will be used along this paper.

Structure of the paper. In Section 2, we introduce the necessary background. Section 3 recalls the notions of requirement and negotiation function, and link them to NEs and SPEs. Section 4 recalls results about the size of solutions of systems of equations or inequations, and use them to bound the size of the least fixed point of the negotiation function. Section 5 defines a witness for the existence of a \( \lambda \)-consistent play between two given thresholds. Section 6 introduces the reduced negotiation game that is a new compact characterization of the negotiation function. Finally, Section 7 applies those results to prove the \( \text{NP} \)-completeness of the SPE threshold problem on mean-payoff games. The proofs that are not presented here can be found in appendices of [4], the full version of this paper.

2 Background

Games, strategies, equilibria. In all what follows, we study infinite duration turn-based quantitative games on finite graphs with complete information.

- **Definition 1 (Game).** A non-initialized game is a tuple \( G = (\Pi, V, (V_i)_{i \in \Pi}, E, \mu) \), where:
  - \( \Pi \) is a finite set of players;
  - \( (V, E) \) is a directed graph, called the underlying graph of \( G \), whose vertices are sometimes called states and whose edges are sometimes called transitions, and in which every state has at least one outgoing transition. For the simplicity of writing, a transition \( (v, w) \in E \) will often be written \( vw \);
\[ (V_i)_{i \in \mathbb{N}} \text{ is a partition of } V, \text{ in which } V_i \text{ is the set of states controlled by player } i; \]
\[ \mu : V^\omega \to \mathbb{R}^\Pi \text{ is an payoff function, that maps each infinite word } \rho \text{ to the tuple } (\mu_i(\rho))_{i \in \mathbb{N}} \text{ of the players’ payoffs.} \]

An initialized game is a tuple \((G, v_0)\), often written \(G[v_0]\), where \(G\) is a non-initialized game and \(v_0 \in V\) is a state called \textit{initialized state}. We often use the word \textit{game}, alone, for both initialized and non-initialized games.

\begin{definition} [Play, history.] A \textit{play} (resp. \textit{history}) in the game \(G\) is an infinite (resp. finite) path in the graph \((V, E)\). It is also a play (resp. history) in the initialized game \(G[v_0]\), when \(v_0\) is its first vertex. The set of plays (resp. histories) in the game \(G\) (resp. the initialized game \(G[v_0]\)) is denoted by \textit{Plays} \(G\) (resp. \textit{Plays} \(G[v_0]\), \textit{Hist} \(G\), \textit{Hist} \(G[v_0]\)). We write \(\text{Hist}_i G\) (resp. \(\text{Hist}_i G[v_0]\)) for the set of histories in \(G\) (resp. \(G[v_0]\)) of the form \(hv\), where \(v\) is a vertex controlled by player \(i\).

Given a play \(\rho\) (resp. a history \(h\)), we write \(\text{Occ}(\rho)\) (resp. \(\text{Occ}(h)\)) the set of vertices that appear in \(\rho\) (resp. \(h\)), and \(\text{Inf}(\rho)\) the set of vertices that appear infinitely often in \(\rho\). For a given index \(k\), we write \(\rho_{\leq k}\) (resp. \(h_{\leq k}\)), or \(\rho_{< k+1}\) (resp. \(h_{< k+1}\)), the finite prefix \(\rho_0 \ldots \rho_k\) (resp. \(h_0 \ldots h_k\)), and \(\rho_{\geq k}\) (resp. \(h_{\geq k}\)), or \(\rho_{> k}\) (resp. \(h_{> k}\)), the infinite (resp. finite) suffix \(\rho_k \rho_{k+1} \ldots\) (resp. \(h_k h_{k+1} \ldots h_{|h|-1}\)). Finally, we write first(\(\rho\)) (resp. first(\(h\))) the first vertex of \(\rho\) (and last(\(h\)) the last vertex of \(h\)).

\begin{definition} [Strategy, strategy profile.] A \textit{strategy} for player \(i\) in the initialized game \(G[v_0]\) is a function \(\sigma_i : \text{Hist}_i G[v_0] \to V\), such that \(v \sigma_i(hv)\) is an edge of \((V, E)\) for every \(hv\). A history \(h\) is \textit{compatible} with a strategy \(\sigma_i\) if and only if \(h_{k+1} = \sigma_i(h_0 \ldots h_k)\) for all \(k\) such that \(h_k \in V_i\). A play \(\rho\) is compatible with \(\sigma_i\) if all its prefixes are.

A \textit{strategy profile} for \(P \subseteq \Pi\) is a tuple \(\overline{\sigma}_P = (\sigma_i)_{i \in P}\), where each \(\sigma_i\) is a strategy for player \(i\) in \(G[v_0]\). A play or a history is \textit{compatible} with \(\overline{\sigma}_P\) if it is compatible with every \(\sigma_i\) for \(i \in P\). A \textit{complete strategy profile}, usually written \(\overline{\sigma}\), is a strategy profile for \(\Pi\). Exactly one play is compatible with a complete strategy profile: we write it \(\langle \overline{\sigma} \rangle\), and call it the \textit{outcome} of \(\overline{\sigma}\).

When \(i\) is a player and when the context is clear, we will often write \(-i\) for the set \(\Pi \setminus \{i\}\). When \(\bar{\tau}_P\) and \(\bar{\tau}_Q\) are two strategy profiles with \(P \cap Q = \emptyset\), we write \((\bar{\tau}_P, \bar{\tau}_Q)\) the strategy profile \(\bar{\sigma}_{P \cup Q}\) such that \(\sigma_i = \tau_i\) for \(i \in P\), and \(\sigma_i = \tau_i'\) for \(i \in Q\).

Before moving on to SPEs, let us recall that an NE is a strategy profile such that no player has an incentive to deviate unilaterally.

\begin{definition} [Nash equilibrium.] Let \(G[v_0]\) be a game. The strategy profile \(\overline{\sigma}\) is a Nash equilibrium – or \textit{NE} for short – in \(G[v_0]\) if and only if for each player \(i\) and for every strategy \(\sigma_i'\), called deviation of \(\sigma_i\), we have the inequality \(\mu_i(\langle \sigma_i', \sigma_{-i} \rangle) \leq \mu_i(\langle \overline{\sigma} \rangle)\).

An SPE is a strategy profile whose all substrategy profiles are NEs.

\begin{definition} [Subgame, substrategy.] Let \(hv\) be a history in the game \(G\). The \textit{subgame} of \(G\) after \(hv\) is the game \((\Pi, V, (V_i)_{i \in \mathbb{N}}, E, \mu_{|hv})\), where \(\mu_{|hv}\) maps each play to its payoff in \(G\), assuming that the history \(hv\) has already been played: formally, for every \(\rho \in \text{Plays}_{|hv} G\), we have \(\mu_{|hv}(\rho) = \mu(h \rho)\). If \(\sigma_i\) is a strategy in \(G[v_0]\), its \textit{substrategy} after \(hv\) is the strategy \(\sigma_{i|hv}\) in \(G_{|hv}\), defined by \(\sigma_{i|hv}(h') = \sigma_i(hh')\) for every \(h' \in \text{Hist}_{i} G_{|hv}\).

\begin{remark} The initialized game \(G[v_0]\) is also the subgame of \(G\) after the one-state history \(v_0\).

\begin{definition} [Subgame-perfect equilibrium.] Let \(G[v_0]\) be a game. The strategy profile \(\overline{\sigma}\) is a subgame-perfect equilibrium – or \textit{SPE} for short – in \(G[v_0]\) if and only if for every history \(h\) in \(G[v_0]\), the strategy profile \(\overline{\sigma}|_h\) is a Nash equilibrium in the subgame \(G|h\).

The notion of subgame-perfect equilibrium refines the notion of Nash equilibrium and excludes coercion by non-credible threats.
Two Nash equilibria.

(b) An example for the operator \(\mathcal{L}\).

Figure 1 Illustration for preliminary notions.

Example 7. Consider the game pictured in Figure 1a. It is initialized with initial state \(a\), and has two players, player \(\square\) and player \(\diamond\), who own respectively the circle and the square vertices. The payoff function assigns to each player a payoff of 1 for the play \(abd\), and 0 for all the other plays. Two different strategy profiles are represented here, one by the blue colored transitions, which has outcome \(abd\), one by the red colored ones, which has outcome \(acg\). Both are NEs: clearly, no player can increase their payoff by deviating from the blue choices, and in the case of the red profile, a deviation of player \(\#\) can only lead to the play \(acf\), and a deviation of player \(\#\) to \(abe\) – both plays give to both player the payoff 0.

However, for player \(\#\), going from \(b\) to \(e\) is not a rational choice, hence the red profile is not an SPE, while the blue profile is one.

An \(\varepsilon\)-SPE is a strategy profile which is almost an SPE: if a player deviates after some history, they will not be able to improve their payoff by more than a quantity \(\varepsilon \geq 0\). Note that a 0-SPE is an SPE, and conversely.

Definition 8 (\(\varepsilon\)-SPE). Let \(G_{|v_0}\) be a game, and \(\varepsilon \geq 0\). A strategy profile \(\bar{\sigma}\) from \(v_0\) is an \(\varepsilon\)-SPE if and only if for every history \(h\), for every player \(i\) and every strategy \(\sigma'_i\), we have

\[
\mu_i(\langle \bar{\sigma} - i | h, \sigma'_i | h \rangle) \leq \mu_i(\langle \bar{\sigma} | h \rangle) + \varepsilon.
\]

Mean-payoff games. We now turn to the definition of mean-payoff objectives.

Definition 9 (Mean-payoff, mean-payoff game). In a graph \((V, E)\), we associate to each mapping \(r: E \rightarrow \mathbb{Q}\) the mean-payoff function:

\[
\text{MP}_r : h_0 \ldots h_n \mapsto \frac{1}{n} \sum_{k=0}^{n-1} r_i(h_k h_{k+1}).
\]

A game \(G = (\Pi, V, (V_i), E, \mu)\) is a mean-payoff game if its underlying graph is finite, and if there exists a tuple \((r_i)_{i \in \Pi}\) of reward functions, such that for each player \(i\) and every play \(p\):

\[
\mu_i(p) = \lim \inf_{n \to \infty} \text{MP}_{r_i}(p_{\leq n}).
\]

The mapping \(r_i\) is called reward function of player \(i\): it represents the immediate reward that each action grants to player \(i\). The final payoff of player \(i\) is their average payoff along the play, classically defined as the limit inferior\(^1\) over \(n\) (since the limit may not be defined) of the average payoff after \(n\) steps. When the context is clear, we liberally write \(\text{MP}_{r_i}(h)\) for \(\text{MP}_{r_i}(h)\), and \(\text{MP}(h)\) for the tuple \((\text{MP}_i(h))_i\), as well as \(r(\mathcal{L})\) for the tuple \((r_i(\mathcal{L}))_i\).

\(^1\) An alternative definition of mean-payoff games exists, with a limit superior instead of inferior. While in zero-sum one dimensional games, the two definitions lead to the same notion of optimality, this is not the case when considering multiple dimensions, see e.g. [24]. All the results presented in this paper apply only on mean-payoff games defined with a limit inferior.
In the sequel, we develop a worst-case optimal algorithm to solve the \( \varepsilon \)-SPE threshold problem, which is a generalization of the SPE threshold problem, defined as follows.

\begin{definition} \text{(\( \varepsilon \)-SPE threshold problem)} \end{definition}

Given a rational number \( \varepsilon \geq 0 \), a mean-payoff game \( G_{v_0} \) and two thresholds \( \bar{x}, \bar{y} \in \mathbb{Q}^\Pi \), does there exist an \( \varepsilon \)-SPE \( \bar{\sigma} \) in \( G_{v_0} \) such that \( \bar{x} \leq \mu(\langle \bar{\sigma} \rangle) \leq \bar{y} \)?

That problem is already known, by [3], to be \( 2\text{ExpTime} \) and \( \text{NP} \)-hard. The proof given in that paper does also show that \( \text{NP} \)-hardness still holds when \( \varepsilon \) is fixed to 0. Let us also add that the existence of an SPE in a given mean-payoff game, i.e. the same problem with no thresholds and with \( \varepsilon = 0 \), is itself \( \text{NP} \)-hard.

\begin{definition} \text{(SPE existence problem)} \end{definition}

Given a mean-payoff game \( G_{v_0} \), does there exist an SPE in \( G_{v_0} \)?

\begin{lemma} \text{The SPE existence problem is \( \text{NP} \)-hard.} \end{lemma}

\section*{Set of possible payoffs.} A first important result that we need is the characterization of the set of possible payoffs in a mean-payoff game, which has been introduced in [13]. Given a graph \((V,E)\), we write \( \text{SC}(V,E) \) the set of simple cycles it contains. Given a finite set \( D \) of dimensions and a set \( X \subseteq \mathbb{R}^D \), we write \( \text{Conv} X \) the convex hull of \( X \). We will often use the subscript notation \( \text{Conv}_{x \in X} f(x) \) for the set \( \text{Conv} f(X) \).

\begin{definition} \text{(Downward sealing)} \end{definition}

Given a set \( Y \subseteq \mathbb{R}^D \), the downward sealing of \( Y \) is the set \( \uparrow Y = \{ (\min_{d \in D} z_d) : Z \text{ is a finite subset of } Y \} \).

\begin{example} \text{In \( \mathbb{R}^2 \), if \( Y \) is the blue area in Figure 1b, then \( \uparrow Y \) is the union of the blue area and the gray area.} \end{example}

\begin{lemma} [13] \text{Let } G \text{ be a mean-payoff game, whose underlying graph is strongly connected. The set of the payoffs } \mu(\rho), \text{ where } \rho \text{ is a play in } G, \text{ is exactly the set:} \end{lemma}

\begin{equation*}
\uparrow \left( \text{Conv}_{c \in \text{SC}(V,E)} \text{MP}(c) \right).
\end{equation*}

\section*{Two-player zero-sum games.} We now recall several definitions and two classical results about two-player zero-sum games.

\begin{definition} \text{(Two-player zero-sum game)} \end{definition}

A two-player zero sum game is a game \( G \) with \( \Pi = \{1,2\} \) and \( \mu_2 = -\mu_1 \).

\begin{definition} \text{(Borel game)} \end{definition}

A game \( G \) is Borel if the function \( \mu \), from the set \( V^\omega \) equipped with the product topology to the Euclidian space \( \mathbb{R}^\Pi \), is Borel, i.e. if, for every Borel set \( B \subseteq \mathbb{R}^\Pi \), the set \( \mu^{-1}(B) \) is Borel.

\begin{remark} \text{Mean-payoff games are Borel (see [12]).} \end{remark}

\begin{lemma} [Determinacy of Borel games, [20]] \text{Let } G_{v_0} \text{ be a zero-sum Borel game, with } \Pi = \{1,2\}. \text{ Then, we have the following equality:} \end{lemma}

\begin{equation*}
\sup_{\sigma_1} \inf_{\sigma_2} \mu_1(\langle \bar{\sigma} \rangle) = \inf_{\sigma_2} \sup_{\sigma_1} \mu_1(\langle \bar{\sigma} \rangle).
\end{equation*}

That quantity is called value of \( G_{v_0} \), denoted by \( \text{val}_1(G_{v_0}) \).
Definition 19 (Optimal strategy). Let $G_{[v_0]}$ be a zero-sum Borel game, with $\Pi = \{1, 2\}$. The strategy $\sigma_1$ is optimal in $G_{[v_0]}$ if $\inf_{\sigma_2} \mu_1((\sigma_1, \sigma_2)) = \text{val}_1(G_{[v_0]})$.

Let us now define memoryless strategies, and a condition under which they can be optimal.

Definition 20 (Memoryless strategy). A strategy $\sigma_i$ in a game $G_{[v_0]}$ is memoryless if for every vertex $v \in V$, and for all histories $h$ and $h'$, we have $\sigma_i(hv) = \sigma_i(h'v)$.

We usually write $\sigma_i(hv)$ for the state $\sigma_i(hv)$ for every $h$. For every game $G_{[v_0]}$, we write $\text{ML}(G_{[v_0]})$ for the set of memoryless strategies in $G_{[v_0]}$.

Definition 21 (Shuffling). Let $\rho, \eta$ and $\theta$ be three plays in a game $G$. The play $\theta$ is a shuffling of $\rho$ and $\eta$ if there exist two sequences of indices $k_0 < k_1 < \ldots$ and $l_0 < l_1 < \ldots$ such that $\eta_0 = \rho_{k_0} = \eta_{l_0} = \rho_{k_1} = \eta_{l_1} = \ldots$, and:

\[ \theta = \rho_0 \ldots \rho_{k_0}^{-1} \eta_0 \ldots \rho_{k_1}^{-1} \eta{l_0} \ldots \rho_{k_1}^{-1} \eta{l_1}^{-1} \ldots \]

Definition 22 (Convexity, concavity). A function $f : \text{Plays}G \to \mathbb{R}$ is convex if every shuffling $\theta$ of two plays $\rho$ and $\eta$ satisfies $f(\theta) \geq \min\{f(\rho), f(\eta)\}$. It is concave if $-f$ is convex.

Remark. Mean-payoff functions, defined with a limit inferior, are convex.

Lemma 23. In a two-player zero-sum game played on a finite graph, every player whose payoff function is concave has an optimal strategy that is memoryless.

Proof. According to [18], this result is true for qualitative objectives, i.e. when $\mu$ can only take the values $0$ and $1$. It follows that for every $\alpha \in \mathbb{R}$, if a player $i$, whose payoff function is concave, has a strategy that ensures $\mu_i(\rho) \geq \alpha$ (understood as a qualitative objective), then they have a memoryless one. Hence the equality:

\[ \text{val}_1(G_{[v_0]}) = \sup_{\sigma_1 \in \text{ML}(G_{[v_0]})} \inf_{\sigma_2} \mu_1((\bar{\sigma})) \]

Since the underlying graph $(V, E)$ is finite, memoryless strategies exist in finite number, hence the supremum above is realized by a memoryless strategy $\sigma_1$ that is, therefore, optimal.

3 Requirements and negotiation

We now recall some notions and results from [3], which are the starting point of our algorithm.

Requirements. In the sequel, we write $\bar{\mathbb{R}}$ the set $\mathbb{R} \cup \{\pm\infty\}$.

Definition 24 (Requirement). A requirement on the game $G$ is a mapping $\lambda : V \to \bar{\mathbb{R}}$.

For a given state $v$, the quantity $\lambda(v)$ represents the minimal payoff that the player controlling $v$ will require in a play traversing the state $v$.

Definition 25 ($\lambda$-consistency). Let $\lambda$ be a requirement on a game $G$. A play $\rho$ in $G$ is $\lambda$-consistent if and only if, for all $i \in \Pi$ and $n \in \mathbb{N}$ with $\rho_n \in V_i$, we have $\mu_i(\rho_{\geq n}) \geq \lambda(\rho_n)$. The set of $\lambda$-consistent plays from a state $v$ is denoted by $\lambda\text{Cons}(v)$.

Remark. The set $\lambda\text{Cons}(v)$ can be empty, and is not regular in general.

Definition 26 ($\lambda$-rationality). Let $\lambda$ be a requirement on a mean-payoff game G. Let $i \in \Pi$. A strategy profile $\bar{\sigma}_{-i}$ is $\lambda$-rational if and only if there exists a strategy $\sigma_i$ such that, for every history $hv$ compatible with $\bar{\sigma}_{-i}$, the play $\langle \bar{\sigma}_{hv} \rangle$ is $\lambda$-consistent. We then say that the strategy profile $\bar{\sigma}_{-i}$ is $\lambda$-rational assuming $\sigma_i$. The set of $\lambda$-rational strategy profiles in $G_{[v]}$ is denoted by $\lambda\text{Rat}(v)$.
**Negotiation.** In mean-payoff games, as well as in a wider class of games (see [3] and [15]), SPEs are characterized by the fixed points of the negotiation function, a function from the set of requirements into itself. We always use the convention $\inf \emptyset = +\infty$.

**Definition 27 (Negotiation function).** Let $G$ be a game. The negotiation function is the function that transforms each requirement $\lambda$ on $G$ into a requirement $\text{nego}(\lambda)$ on $G$ defined, for each $i \in I$ and $v \in V_i$, by:

$$\text{nego}(\lambda)(v) = \inf_{\bar{\sigma} \in \lambda \text{Rat}(v)} \sup_{\sigma_i} \mu_i(\langle \bar{\sigma} \rangle).$$

The quantity $\text{nego}(\lambda)(v)$ is the best payoff the player controlling the state $v$ can enforce if the other players play rationally with regards to the requirement $\lambda$.

**Remark.** The negotiation function satisfies the following properties.

1. It is monotone: if $\lambda \leq \lambda'$ (for the pointwise order), then $\text{nego}(\lambda) \leq \text{nego}(\lambda')$.
2. It is also non-decreasing: for every $\lambda$, we have $\lambda \leq \text{nego}(\lambda)$.
3. There exists a $\lambda$-rational strategy profile from $v$ against the player controlling $v$ if and only if $\text{nego}(\lambda)(v) \neq +\infty$.

**Link with SPEs.** The SPE outcomes in a mean-payoff game are characterized by the fixed points of the negotiation function, or equivalently by its least fixed point. That result can be extended to $\epsilon$-SPEs. To that end, we recall the notion of $\epsilon$-fixed points of a function.

**Definition 28 ($\epsilon$-fixed point).** Let $\epsilon \geq 0$, let $D$ be a finite set and let $f : \mathbb{R}^D \to \mathbb{R}^D$ be a mapping. A tuple $\bar{x} \in \mathbb{R}^D$ is an $\epsilon$-fixed point of $f$ if for each $d \in D$, for $y = f(\bar{x})$, we have $y_d \in [x_d - \epsilon, x_d + \epsilon]$.

**Remark.** A 0-fixed point is a fixed point, and conversely.

**Lemma 29 ([3]).** Let $G_{1\nu}$ be a mean-payoff game, and let $\epsilon \geq 0$. The negotiation function on $G$ has a least $\epsilon$-fixed point $\lambda^*$, and given a play $\rho$ in $G_{1\nu}$, the three following assertions are equivalent: (1) the play $\rho$ is an $\epsilon$-SPE outcome; (2) the play $\rho$ is $\lambda$-consistent for some $\epsilon$-fixed point $\lambda$ of the negotiation function; (3) the play $\rho$ is $\lambda^*$-consistent.

**The abstract negotiation game.** Given $\lambda$ and $u \in V$, the quantity $\text{nego}(\lambda)(u)$ can be characterized as the value of a negotiation game, a two-player zero-sum game opposing the player Prover, who simulates a $\lambda$-rational strategy profile and wants to minimize player $i$'s payoff, and the player Challenger, who simulates player $i$'s reaction by accepting or refusing Prover’s proposals. Two negotiation games were defined in [3]. Conceptually simpler, the abstract negotiation game $\text{Abs}_{\mu_i}(G)_{1\nu}$ unfolds as follows:

1. From the state $v$, Prover chooses a $\lambda$-consistent play $\rho$ from the state $v$ and proposes it to Challenger. If Prover has no play to propose, the game is over and Challenger gets the payoff $+\infty$.
2. Once a play $\rho$ has been proposed, Challenger can accept it. Or he can deviate, and choose a prefix $\rho_{\leq k}$ with $\rho_k \in V_i$ and a new transition $\rho_{k+1}w \in E$.
3. In the former case, the game is over. In the latter, it starts again from the state $w$.

If Challenger finally accepts a proposal $\rho$, then his payoff is $\mu_i(\rho)$. If he deviates infinitely often, then Prover’s proposals and his deviations construct a play $\bar{\pi} = \rho_{\leq k_0} \rho_{\leq k_1} \rho_{\leq k_2} \ldots$. Then, Challenger’s payoff is $\mu_i(\bar{\pi})$. It has been proved in [3] that the equality $\text{nego}(\lambda)(u) = \text{val}_c(\text{Abs}_{\mu_i}(G)_{1\nu})$ holds. Thus, the abstract negotiation game captures a first intuition on how the negotiation function can be computed.
Then, Challenger’s payoff in the play \( \pi \) where suffix \( \pi \) is either \( +\infty \) if there exists an index \( k \) such that the suffix \( \pi_{\geq 2k} \) contains no deviation and \( \hat{\pi}_{\geq k} \) is not \( \lambda \)-consistent, and \( \mu_i(\hat{\pi}) \) otherwise.

The concrete negotiation game. The abstract negotiation game cannot be directly used for an algorithmic purpose, since it has an infinite state space. However, it can be turned into a game on a finite graph if Prover does not propose plays as a whole, but edge by edge. In the concrete negotiation game \( \text{Conc}_\lambda(G) \), the states controlled by Prover have the form \( (v, M) \), where \( M \subseteq V \) memorizes the states seen since the last time Challenger deviated, in order to control that the play Prover is constructing since that moment is \( \lambda \)-consistent: for each \( u \in M \), Prover has to give to the player controlling \( u \) at least the payoff \( \lambda(u) \). Similarly, the states controlled by Challenger are of the form \( (v v', M) \), where \( v v' \in E \) is an edge proposed by Prover. The game unfolds as follows:

\begin{itemize}
    \item From the state \((v, M)\), Prover chooses a transition \( v v' \) and proposes it to Challenger.
    \item Once a transition \( v v' \) has been proposed, Challenger can accept it. Or, if \( v \in V_i \), he can deviate, and choose a new transition \( v w \).
    \item If the former case, the game starts again from the state \((v', M \cup \{v'\})\). In the latter, it starts from the state \((w, \{w\})\).
\end{itemize}

For every play \( \pi = (p_0, M_0)(p_0p_1, M_0)(p_1, M_1)(p_1p_1, M_1)\ldots \) in \( \text{Conc}_\lambda(G) \), we write \( \hat{\pi} = p_0p_1 \ldots \) the play in \( G \) constructed by Prover’s proposals and Challenger’s deviations.

\begin{itemize}
    \item Then, Challenger’s payoff in the play \( \pi \) is either \( +\infty \) if there exists an index \( k \) such that the suffix \( \pi_{\geq 2k} \) contains no deviation and \( \hat{\pi}_{\geq k} \) is not \( \lambda \)-consistent, and \( \mu_i(\hat{\pi}) \) otherwise.
\end{itemize}
In [3], a first algorithm was proposed to solve the $\varepsilon$-SPE threshold problem, using the fact that in the concrete negotiation game, Challenger has a memoryless optimal strategy, to design a complete representation of the negotiation function, and to compute its least $\varepsilon$-fixed point. However, that algorithm requires doubly exponential time, because it needs to enumerate all the memoryless strategies available for Challenger, whose number is exponential in the size of the concrete game, itself exponential in the size of $G$. Here, we make use of the concrete negotiation game only to bound the size of the least $\varepsilon$-fixed point: our algorithm will use a third negotiation game, the reduced negotiation game.

### 4 Size of the least $\varepsilon$-fixed point

In this section, after having recalled some results about the sizes of solutions to linear equations and inequations, we prove that the least $\varepsilon$-fixed point of the negotiation function in a game $G$ has a size that is polynomial in the size of $G$ and $\varepsilon$. The first piece of the witnesses identifying positive instances of the $\varepsilon$-SPE threshold problem will then be an $\varepsilon$-fixed point of the negotiation function of polynomial size.

#### About size, equations and inequations

We define here the notion of size that we use.

- **Definition 31** (Size). The size of a rational number $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ are co-prime, is the quantity $\|r\| = 1 + \lceil \log_q(|p| + 1) \rceil + \lceil \log_2(|q| + 1) \rceil$. The size of an irrational number is $+\infty$. The size of the infinite numbers is $\|+\infty\| = \|-\infty\| = 1$. The size of a tuple $\bar{x} \in O^D$, where $D$ is a set and $O$ is a set of objects for which the notion of size has been defined, is the quantity $\text{card}(D) + \sum_{d \in D} \|x_d\|$. Similarly, the size of a function $f : D \to X$ is the quantity $\text{card}(D) + \sum_{d \in D} \|f(d)\|$, and the size of a set $X \subseteq O$ is the quantity $\text{card}(X) + \sum_{x \in X} \|x\|$. The proof of Theorem 37 below requires the manipulation of polytopes, e.g. downward scalings of convex hulls (from Lemma 15), expressed as solution sets of systems of linear inequations.

- **Definition 32** (Linear equations, inequations, systems). Let $D$ be a finite set. A linear equation in $\mathbb{R}^D$ is a pair $(\bar{a}, b) \in (\mathbb{R}^D \setminus \{\emptyset\}) \times \mathbb{R}$. The solution set of the equation $(\bar{a}, b)$ is the set $\text{Sol}_{=}(\bar{a}, b) = \{\bar{x} \in \mathbb{R}^D \mid \bar{a} \cdot \bar{x} = b\}$, where $\cdot$ denotes the canonical scalar product on the euclidian space $\mathbb{R}^D$. A set $X \subseteq \mathbb{R}^D$ is a **hyperplane** of $\mathbb{R}^D$ if it is the solution set of some linear equation. A **system of linear equations** is a finite set $\Sigma$ of linear equations. The solution set of the system $\Sigma$ is the set $\text{Sol}_{=}(\Sigma) = \cap_{(\bar{a}, b) \in \Sigma} \text{Sol}_{=}(\bar{a}, b)$. A set $X \subseteq \mathbb{R}^D$ is a **linear subspace** of $\mathbb{R}^D$ if it is the solution set of some system of linear equations.

  A linear inequation in $\mathbb{R}^D$ is a pair $(\bar{a}, b) \in (\mathbb{R}^D \setminus \{\emptyset\}) \times \mathbb{R}$. The solution set of the inequation $(\bar{a}, b)$ is the set $\text{Sol}_{\geq}(\bar{a}, b) = \{\bar{x} \in \mathbb{R}^D \mid \bar{a} \cdot \bar{x} \geq b\}$. A set $X \subseteq \mathbb{R}^D$ is a **half-space** of $\mathbb{R}^D$ if it is the solution set of some linear inequation. A system of linear inequations is a finite set $\Sigma$ of linear inequations. The solution set of the system $\Sigma$ is the set $\text{Sol}_{\geq}(\Sigma) = \cap_{(\bar{a}, b) \in \Sigma} \text{Sol}_{\geq}(\bar{a}, b)$. A set $X \subseteq \mathbb{R}^D$ is a **polyhedron** of $\mathbb{R}^D$ if it is the solution set of some system of linear inequations $\Sigma$. A vertex of $X$ is a point $\bar{x} \in \mathbb{R}^D$ such that $\{\bar{x}\} = \text{Sol}_{=}(\Sigma')$ for some subset $\Sigma' \subseteq \Sigma$. A **polytope** is a bounded polyhedron.

- **Remark**. Polyhedra are closed sets. The polytopes of $\mathbb{R}^D$ are exactly the sets of the form $\text{Conv}(S)$, where $S$ is a finite subset of $\mathbb{R}^D$.

- **Lemma 33** ([13]). Let $\Sigma$ be a system of inequations, and let $X = \text{Sol}_{\geq}(\Sigma)$. The set $\mathbf{\Sigma}$ is itself a polyhedron, and there exists a system of inequations $\Sigma'$ such that $\mathbf{\Sigma} = \text{Sol}_{\geq}(\Sigma')$ and that for every $(\bar{a}', b') \in \Sigma'$, there exists $(\bar{a}, b) \in \Sigma$ with $\|\bar{a}'\| \leq \|(\bar{a}, b)\|$.
Lemma 34 ([2], Theorem 1). There exists a polynomial $P_1$ such that, for every system of equations $\Sigma$, there exists a point $\bar{x} \in \text{Sol}_\Sigma$, such that $\|\bar{x}\| \leq P_1(\max_{(\bar{a},b) \in \Sigma}(\|\bar{a},b\|))$.

Corollary 35. For every system of inequations $\Sigma$, each vertex $\bar{x}$ of the polyhedron $\text{Sol}_\Sigma(\Sigma)$ has size $\|\bar{x}\| \leq P_1(\max_{(\bar{a},b) \in \Sigma}(\|\bar{a},b\|))$.

Note that in Lemma 33, in Lemma 34 and in Corollary 35, the number of equations or inequations has no influence. A consequence of Lemma 34 is the following result.

Lemma 36. There exists a polynomial $P_2$ such that, for each finite set $D$ and every finite subset $X \subseteq \mathbb{R}^P$, there exists a system of linear inequations $\Sigma$, such that $\text{Sol}_\Sigma(\Sigma) = \text{Conv}(X)$ and $\|\bar{a},b\| \leq P_2(\|X\|)$ for every $(\bar{a},b) \in \Sigma$.

Size of the least $\varepsilon$-fixed point. The following theorem bounds the size of the least fixed point of the negotiation function. As we used the notation $\bar{x}$ for tuples so far, we use the notation $\bar{x}$ for tuples of tuples.

Theorem 37. There exists a polynomial $P_3$ such that for every mean-payoff game $G$, the least $\varepsilon$-fixed point $\lambda^*$ of the negotiation function has size $\|\lambda^*\| \leq P_3(\|G\| + \|\varepsilon\|)$.

Proof sketch. It has been proved in [3] that Challenger has a memoryless optimal strategy in every concrete negotiation game. Given a requirement $\lambda$, a player $i$, a state $v \in V_i$ and a memoryless strategy $\tau_C$, we can construct the set of payoff vectors $\mu(\bar{x})$, where $\bar{x}$ is a play in $\text{Conv}_{\tau_i}(G)$ compatible with $\tau_C$, as a union of polytopes defined using Lemma 15. If we intersect the upward closures of those sets, then $\text{nego}(\lambda)(v)$ is equal to the least value $x_i$, where $\bar{x}$ belongs to that intersection. Therefore, if $X_\lambda \subseteq \mathbb{R}^V$ is the product of those intersections, then for each $i$ and $v \in V_i$, we have $\text{nego}(\lambda) = \inf\{x_{vi} \mid \bar{x} \in X_\lambda\}$.

To each tuple of tuples $\bar{x} \in \mathbb{R}^V \times \Pi$, we associate the requirement $\lambda_\bar{x}$ defined by $\lambda_\bar{x}(v) = x_{vi} - \varepsilon$ for each $i \in I$ and $v \in V_i$. Then, we define $X = \{\bar{x} \mid \bar{x} \in X_\lambda\}$, and we show that any requirement $\lambda$ is an $\varepsilon$-fixed point of the negotiation function if and only if $\lambda = \lambda_{\bar{x}}$ for some $\bar{x} \in X$. Then, the set $X$ is itself a union of polyhedra, hence the linear mapping $\bar{x} \mapsto \sum_i \lambda_\bar{x}(v)$ has its minimum over $X$ on some vertex $\bar{x}$ of one of those polyhedra. The requirement $\lambda^*$ is equal to $\lambda_{\bar{x}}$, hence its size can be bounded using Corollary 35.

Constrained existence of a $\lambda$-consistent play

We claim that a non-deterministic algorithm can recognize the positive instances of the $\varepsilon$-SPE threshold problem by guessing an $\varepsilon$-fixed point $\lambda$ of the negotiation function. Once $\lambda$ has been guessed, according to Lemma 29, two assertions must be proved: on the one hand, that there exists a $\lambda$-consistent play between the two desired thresholds, and on the other hand, that $\lambda$ is actually an $\varepsilon$-fixed point of the negotiation function. The latter will be handled later through the concept of reduced negotiation game. Now, we tackle the former, and provide the second piece of our notion of witness: to prove the existence of a $\lambda$-consistent play $\rho$ with $\bar{x} \leq \mu(\rho) \leq \bar{y}$, we need to guess the sets $W = \inf(\rho)$ and $W' = \occ(\rho)$, and a tuple of tuples $\bar{a} \in [0,1]^\Pi \times \text{SC}(W)$ indicating how $\rho$ combines the cycles of $W$, i.e. such that:

$$
\mu(\rho) = \left( \min_{\bar{y} \in \Pi} \sum_{c \in \text{SC}(W)} \alpha_c \text{MP}(c) \right).$

Theorem 38. There exists a polynomial $P_4$ such that for every mean-payoff game $G_{\bar{v}0}$, for every $\bar{x}, \bar{y} \in \mathbb{R}^V$, and for every requirement $\lambda$ on $G$, there exists a $\lambda$-consistent play $\rho$ in $G_{\bar{v}0}$ satisfying $\bar{x} \leq \mu(\rho) \leq \bar{y}$ if and only if there exist two sets $W \subseteq W' \subseteq V$ and a tuple of tuples $\bar{a} \in [0,1]^\Pi \times \text{SC}(W)$ such that:

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The set $W$ is strongly connected in $(V, E)$, and accessible from the state $v_0$ using only and all the states of $W'$.

For each player $i$, we have $\sum c \alpha_{ic} = 1$, and:

$$x_i \leq \min_{j \in I} \sum_{c \in SC(W)} \alpha_{jc} \cdot MP_i(c) \leq y_i;$$

For each player $i$ and $v \in W \cap V_i$, we have:

$$\min_{j \in I} \sum_{c \in SC(W)} \alpha_{jc} \cdot MP_i(c) \geq \lambda(v);$$

And $||\delta|| \leq P_2(||G, \bar{x}, \bar{y}, \lambda||)$.

**Proof.** Let us first notice that given a set $X \subseteq \mathbb{R}^I$, the elements of the set $^\updownarrow (\text{Conv}X)$ are exactly the tuples of the form:

$$\left( \min_{j \in I} \sum_{x \in X} \alpha_{jx} x \right)_{i \in I}$$

For some tuple $\bar{\alpha} \in \mathbb{R}^{I \times X}$ satisfying $\sum_x \alpha_{ix} = 1$ for each $x$.

Now, let us assume that $W$, $W'$ and $\bar{\alpha}$ exist. Then, there exists a play $\eta$ with $\text{Occ}(\eta) = \text{Inf}(\eta) = W$ with payoff vector:

$$\mu(\eta) = \left( \min_{j \in I} \sum_{x \in X} \alpha_{jx} x \right)_{i \in I}.$$  

Moreover, since $W$ is accessible from $v_0$ using all and only the vertices of $W'$, there exists a history $h\eta_0$ from $v_0$ to $\eta_0$ with $\text{Occ}(h) = W'$. Then, the play $\rho = h\eta$ is $\lambda$-consistent and satisfies $\bar{x} \leq \mu(\rho) \leq \bar{y}$.

Conversely, if the play $\rho$ exists: let $W = \text{Inf}(\rho)$ and $W' = \text{Occ}(\rho)$. The polytope:

$$Z = \left\{ \mu(\eta) \mid \eta \in \lambda \text{Cons}(G_{\text{Inf}}), \text{Inf}(\eta) = W, \text{Occ}(\eta) = W', \text{Occ}(\eta), \bar{x} \leq \bar{\alpha} \leq \bar{y}, \text{ and } \bar{x} \leq \mu(\eta) \leq \bar{y} \right\}$$

(The equality holds by Lemma 15) is nonempty (it contains at least $\mu(\rho)$). By Lemma 36, the set $\text{Conv}_{\epsilon \in SC(W)} MP(c)$ is defined by a system of inequations which all have size $||\epsilon|| \leq P_2(||\epsilon||)$. Since by Lemma 33, the inequations defining $^\updownarrow (\text{Conv}_{\epsilon \in SC(W)} MP(c))$ are not larger, there exists a polynomial $P_6$, independent of $G, \bar{x}, \bar{y}$ and $\lambda$, such that $Z$ is defined by a system of inequations $\Sigma$ such that for every $(\alpha, b) \in \Sigma$, we have $||\epsilon|| \leq P_6(||\epsilon||)$.

Therefore, by Corollary 35, the polytope $Z$ admits a vertex $\bar{z}$ of size $||\bar{z}|| \leq P_4(||G, \bar{x}, \bar{y}, \lambda||)$.

Then, since we have $\bar{z} \in ^\updownarrow (\text{Conv}_{\epsilon \in SC(W)} MP(c))$, that vertex is, according to Definition 13, of the form:

$$\bar{z} = \left( \min_{j \in I} \sum_{c} \alpha_{jc} \cdot MP_j(c) \right)_i$$

For some tuple of tuples $\bar{\alpha} \in [0, 1]^{I \times SC(W)}$ with $\sum_c \alpha_{ic} = 1$ and having, by Corollary 35 again, size $||\bar{\alpha}|| \leq P_4 \left( \max_{c \in I} \sum_{\epsilon \in SC(W)} ||MP(c)|| \right)$, i.e. $||\bar{\alpha}|| \leq P_4(||G, \bar{x}, \bar{y}, \lambda||)$ for some polynomial $P_4$ independent of $G, \bar{x}, \bar{y}$ and $\lambda$.

Now, we need the third piece of our witness, which will be evidence of the fact that the requirement $\lambda$ is an $\epsilon$-fixed point of the negotiation function.
6 The reduced negotiation game

The abstract negotiation game has an infinite (and uncountable) state space in general, and the concrete negotiation game has an exponential one. In [5], the infiniteness of the abstract negotiation game has been handled in the case of parity games, by proving that Prover has an optimal strategy that is memoryless, and that proposes only simple plays with a finite representation. Unfortunately, this result does not apply to mean-payoff games, where Prover needs infinite memory in general. That fact is illustrated in the next example.

Example 39. In the game of Figure 3a, the requirement \( \lambda \) defined by \( \lambda(a) = \lambda(b) = 1 \) is a fixed point of the negotiation function (it is actually the least fixed point). Indeed, from the state \( a \) (the situation is symmetrical from the state \( b \)), consider the strategy for Prover that proposes always, from the state \( v \), the play \( eb^hR(\alpha^3b^3)\omega \), where \( h \) is the history that has already been constructed by her proposals and Challenger’s deviations. If Challenger accepts such a play, then he gets the payoff 1. If he deviates infinitely often, then Prover loops longer and longer on the state \( b \), and he also gets the payoff 1. The loop on \( b \) corresponds to what we will call later a *punishing cycle*. Now, if Prover uses only finite memory, Challenger can get a payoff better than 1 by always deviating and go to \( b \) as soon as he can: then, edges giving to player \( \bigcirc \) the reward 2 will occur with a nonzero frequency.

However, the plays proposed by Prover in the previous example are very similar: only the number of repetitions of the loop \( b \) does increase. More generally, one observes that Prover can play optimally while always proposing a play of the form \( hc^n\rho \), where \( h, c \) and \( \rho \) are constant, and only the number \( n \) increases, quadratically with the time – so that Challenger’s payoff is dominated by the mean-payoff \( MP_i(c) \) if he deviates infinitely often.

Definition 40 (Punishment family). A *punishment family* is a set of plays of the form:

\[ \{ hc^n\rho \mid n > 0, \mu(\rho) = \bar{x}, Occ(\rho) = W \} \]

where \( h \) is a simple history, \( c \) is a nonempty simple cycle, and where \( W \subseteq V \) and \( \bar{x} \in \mathbb{R}^\Pi \).

The cycle \( c \) is called its *punishing cycle*. For every \( \beta \in \mathbb{N} \), a \( \beta \)-punishment family is a punishment family with \( \| \bar{x} \| \leq \beta \). A \( \beta \)-punishment family is represented by the data \( h, c, \mu(\rho) \) and \( Occ(\rho) \), and that representation has a size smaller than or equal to the quantity \( 3\text{card}V[\log_2(\text{card}V + 1)] + \beta \).

We write \( hc^\infty\rho \) for the punishment family \( \{ hc^n\rho' \mid n > 0, \mu(\rho') = \mu(\rho), Occ(\rho') = Occ(\rho) \} \).

Beware that the play \( \rho \) matters only for its payoff vector and the vertices it traverses: if \( Occ(\rho) = Occ(\rho') \) and \( \mu(\rho) = \mu(\rho') \), then \( hc^\infty\rho = hc^\infty\rho' \). We write \( \mu(hc^\infty\rho) \) for the common payoff vector of all elements of \( hc^\infty\rho \), and we will say that \( hc^\infty\rho \) is \( \lambda \)-consistent if all its elements are (or equivalently, if one of its elements is). Let us clarify that a punishment family is not an equivalence class: for example, in the game of Figure 3a, the play \( ab^\omega \) belongs to both \( a^{\infty}b^\omega \) and \( ab^\infty b^\omega \), which are distinct. We can now define the reduced negotiation game, where Prover proposes \( \beta \)-punishment families instead of plays.

Definition 41 (Reduced negotiation game). Let \( G \) be a mean-payoff game, let \( \lambda \) be a requirement, let \( i \) be a player, let \( v_0 \in V_i \) and let \( \beta \) be a natural integer. The corresponding reduced negotiation game is the game \( \text{Red}_{\lambda,i}(G)|_{v_0} = (\{P, C\}, S, (S_P, S_C), \Delta, \nu)|_{v_0} \), where:

- the player \( P \) is called *Prover*, and the player \( C \) *Challenger*;
- the states controlled by Prover are the states of \( G \), i.e. \( S_P = V \);
- the states controlled by Challenger are the states of the form \([hc^\infty\rho], (c, u)\) or \([h'v] \), where:
Example 42. Figure 3b illustrates a (small) part of the game \( \text{Red}^2_{\lambda(G)} \), where \( G \) is the game of Figure 3a, and \( \lambda(a) = \lambda(b) = 1 \). Blue states are owned by Prover, orange ones by Challenger. When Prover proposes the punishment family \( ab^\omega (a^3b^3)^\omega \), the function \( \nu_C \) interprets it as the play \( ab^{\nu(C)} (a^3b^3)^\omega \), where \( b \) is the history that has already been constructed so far.

Remark. Reduced negotiation games are Borel, and are played on a finite graph.
Link with the negotiation function. We will now prove that the reduced negotiation game captures the negotiation function, as do the abstract and concrete ones. For that purpose, we first need the following key result.

▶ **Lemma 43.** In a reduced negotiation game, Prover has a memoryless optimal strategy.

**Proof.** This lemma is a consequence of Lemma 23: the payoff function \( \nu_C \) is concave. Indeed, let \( \xi \) be a shuffling of two plays \( \pi \) and \( \chi \). If either \( \pi \) or \( \chi \) reaches the state \( \bot \) (in which case both do), then we immediately have \( \nu_C(\xi) \leq \max\{\nu_C(\pi), \nu_C(\chi)\} = +\infty \). Otherwise, the play \( \xi \) is a shuffling of \( \pi \) and \( \chi \), and since mean-payoff objectives defined with a limit inferior are convex, we have \( \nu_C(\xi) \leq \max\{\nu_C(\pi), \nu_C(\chi)\} \).

This lemma enables us to prove that the reduced negotiation game is equivalent to the other negotiation games.

▶ **Theorem 44.** There exists a polynomial \( P_4 \) such that for every mean-payoff game \( G \), every requirement \( \lambda \) with rational values, each player \( i \) and each \( v_0 \in V_i \), for every \( \beta \geq P_4(\|G\| + \|\lambda\|) \), we have \( \text{nego}(\lambda)(v_0) = \text{val}_C\left(\text{Red}_{\lambda_i}(G)\right)\).

**Proof.** For every mean-payoff game \( G \) and every requirement \( \lambda \), we assume:

\[
\beta \geq P_4(P_2(\|[\text{MP}(c) \mid c \in \text{SC}(G)]\|))
\]

and for each \( v \in V \):

\[
\beta \geq \|\lambda(v)\| + 3,
\]

which are indeed quantities that are bounded by a polynomial of \( \|G\| + \|\lambda\| \).

First direction: \( \text{nego}(\lambda)(v_0) \geq \text{val}_C\left(\text{Red}_{\lambda_i}(G)\right) \).

Let \( \bar{\sigma}_{-i} \) be a strategy profile in \( G \) that is \( \lambda \)-rational assuming a strategy \( \sigma_i \), and let \( x = \sup_{\sigma_i} \mu_i((\bar{\sigma}_{-i}, \sigma_i)) \). We wish to prove that there exists a strategy \( \tau_\varphi \) in the reduced negotiation game such that \( \sup_{\sigma_i} \nu_G((\bar{\sigma}_{-i})) \leq x \). Thus, we will have proved that the quantity \( \text{val}_C\left(\text{Red}_{\lambda_i}(G)\right) \) is smaller than or equal to every such \( x \), and therefore smaller than or equal to \( \text{nego}(\lambda)(v_0) \).

Let us define simultaneously the strategy \( \tau_\varphi \) and a mapping \( \varphi : \text{Hist}_G \rightarrow \text{Hist}_{\text{Red}(G)} \), such that for each history \( H \), the punishment family \( \tau_\varphi(H) \) will be defined from the play \( \langle \bar{\sigma}_{\nu_G(H)} \rangle \). We guarantee inductively that if \( H \in \text{Hist}_G \), then \( \varphi(H) \in \text{Hist}_{\text{Red}(G)} \) is compatible with \( \bar{\sigma}_{-i} \). First, let us define \( \varphi(v_0) = v_0 \).

Let \( \overparen{H} \in \text{Hist}_{\text{Red}(G)} \) be a history compatible with \( \tau_\varphi \) as it has been defined so far, and such that \( \varphi(H) \) has already been defined. Let \( \eta^0 = \langle \bar{\sigma}_{\nu_G(H)} \rangle \). By induction hypothesis, the history \( \nu_G(H) \) is compatible with \( \bar{\sigma}_{-i} \), hence the play \( \eta^0 \) is \( \lambda \)-consistent, and satisfies \( \mu_i(\eta^0) \leq x \).

Let \( \eta^0_{\leq \ell} \) be the shortest prefix of \( \eta^0 \) that is not simple, i.e. such that there exists \( k < \ell \) with \( \eta^0_k = \eta^0_\ell \). If \( MP(\eta^0) \leq x \), then we define \( \tau_\varphi(H) = [\eta^0_{\leq k} \eta^0_{k+1} \ldots \eta^0_\ell] \), where \( \rho \) is a play such that \( \text{Occ}(\rho) = \text{Occ}(\eta^0_{\leq \ell}) \), that \( \mu_i(\rho) \leq x \), and that \( \|\mu(\rho)\| \leq \beta \).

Such a play exists, by the polytope:

\[
Z = \left\{ \mu(\rho) \mid \forall j, \forall v \in V_j \cap \text{Occ}(\eta^0_{\leq \ell}), \mu_j(\rho) \geq \nu_G(v), \text{ and } \text{Occ}(\rho) = \text{Occ}(\eta^0_{\leq \ell}) \right\}
\]

is nonempty (it contains \( \eta^0_{\leq \ell} \)), and has at least one vertex \( \bar{v} \) with \( z_i \leq x \) (because \( \mu_i(\eta^0_{\leq \ell}) \leq x \)), which by Lemma 36 and Corollary 35 has size \( \|\bar{v}\| \leq \beta \).
The Complexity of SPEs in Mean-Payoff Games

We have defined the strategy profile can be defined arbitrarily. To that end, we define

\[ \eta^j = \eta^0 \land \eta^0, \] for some \( n \). Then, for each prefix \( h, v \), we define \( \varphi(H[n^0 \land \eta^0]) \) \( (c, v) = \varphi(H[n^0 \land \eta^0]) \) \( (c, v) = \varphi(H[n^0 \land \eta^0]) \), where \( \eta^0 \land \eta^0 \) is the to obtain the prefix \( h \) of \( \eta^0 \); and similarly, for each pair \( (c, v) \), we define \( \varphi(H[n^0 \land \eta^0]) \) \( (c, v) = \varphi(H[n^0 \land \eta^0]) \), where \( \eta^0 \land \eta^0 \) is the to obtain the shortest prefix \( \eta^0 \land \eta^0 \) such that \( \eta^0 \land \eta^0 \in V_0 \) and \( \eta^0 \land \eta^0 \in E \).

Thus, the mapping \( \varphi \) is defined on every history compatible with \( \sigma \), and the image of such a history is always a history compatible with \( \sigma \). We define it arbitrarily on other histories. Note that for each history \( H \), the history \( H \) can be obtained from \( \varphi(H) \) by pulling out cycles \( c \) satisfying \( \varphi(H) \) \( (c) > x \), and adding cycles \( d \) with \( \varphi(H) \) \( (d) \leq x \). As a consequence, if \( \varphi(H) \) \( (d) \leq x \), then \( \varphi(H) \) \( (d) \leq x \) and the same result is true when we naturally extend the mapping \( \varphi \) to plays.

Let us now prove that \( \sup_{\sigma} \nu_c((\bar{\tau})) = \sup_{\sigma'} \mu_i((\bar{\sigma}-i, \sigma'_i)) \). Let \( \pi \) be a play compatible with \( \tau \).

- the state \( \perp \) does not appear in \( \pi \), because Prover’s strategy does never use a transition to it.
- If \( \pi \) has the form \( \pi = H[he^\infty] \) \( \perp \) : then, we have \( \nu_c(\pi) = \mu_i(\rho) \leq x \).
- If \( \pi \) is made of infinitely many deviations: the play \( \varphi(\pi) \) is compatible with \( \sigma \), hence \( \mu_i(\varphi(\pi)) \leq x \) which implies \( \mu_i(\pi) \leq x \), i.e. \( \nu_c(\pi) \leq x \).

Second direction: \( \text{nego}(\lambda)(v_0) \leq \text{val}_c(\text{Red}_A^\lambda(G)\{v_0\}) \).

Let \( \tau_0 \) be a memoryless strategy for Prover in the reduced negotiation game, and let \( y = \sup_{\sigma} \nu_c((\bar{\tau})) \). We want to show that \( \text{nego}(\lambda)(v_0) \leq y \): by Lemma 43, it will be enough to conclude. If \( y = +\infty \), it is clear. Let us assume that \( y \neq +\infty \). Then, we will define a strategy profile \( \bar{\sigma} \), where \( \bar{\sigma} \) is \( \lambda \)-rational assuming \( \sigma \), such that \( \sup_{\sigma'} \mu_i((\bar{\sigma}-i, \sigma'_i)) \leq y \); we proceed inductively by defining the play \( \langle \bar{\sigma}_{hv} \rangle \) for each history \( hv \) compatible with \( \bar{\sigma} \), such that \( h \) is empty, or last \( (h) \in V_0 \) and \( v \neq \sigma_i(h) \). Such a history is called a \( \text{bud} \) history. After other histories, the strategy profile can be defined arbitrarily. To that end, we construct a mapping \( \psi \) which maps each bud history to a history \( \psi(hv) \) \( \in \text{Hist}_{\text{Red}_A^\lambda(G)}\{v_0\} \) that is compatible with \( \tau_0 \). This mapping will induce a definition of \( \bar{\sigma} \): since \( y \neq +\infty \), we have \( \tau_0(\psi(hv)) \neq \perp \) : let then \( [he^\infty] \) \( (h) \in \tau_0(\psi(hv)) \). We then define \( \tau_0(\psi(hv)) = [he^\infty] \). \( (h) \), which is a \( \lambda \)-consistent play since \( he^\infty \) \( \rho \) is a \( \lambda \)-consistent play from \( hv \) \( \rho \) is a \( \lambda \)-consistent punishment family, by definition of the reduced negotiation game.

Let now \( hv_0v \) be a bud history: we assume that \( \bar{\sigma} \) has been defined on every prefix of \( h_0 \), but not on \( h_0v \) itself. If \( h_0 \) is empty, that is if \( h_0v = v_0 \), then we define \( \psi(h_0v) = v_0 \). Otherwise, let us write \( h_0 = h_1w_2 \), where \( h_1w_2 \) is the longest prefix of \( h_0 \) that is a bud history — that is, its longest prefix such that \( \psi(h_1w) \) has been defined, or its shortest prefix such that \( w_2h_2 \) is compatible with \( \bar{\sigma}_{hv} \). Let \( H = \psi(h_1w) \), and let \( [he^\infty] \) \( (h) = \tau_0(\psi(hv)) \). We have defined \( \langle \bar{\sigma}_{hv} \rangle = [he^\infty] \) \( (h) \), and consequently, the history \( wh_2 \) is a prefix of that play. If it is a prefix of the history \( he \), then we define \( \psi(h_0v) = [he^\infty] \) \( (h) = \tau_0(\psi(hv)) \). Otherwise, we define \( \psi(h_0v) = H[he^\infty] \) \( (h) \).\( (v) \).

Now, the strategy profile \( \bar{\sigma} \) has been defined, and since all the punishment families proposed by Prover are \( \lambda \)-consistent, the strategy profile \( \bar{\sigma} \) is \( \lambda \)-rational assuming \( \sigma \). Let \( \eta \) be a play compatible with \( \bar{\sigma} \), and let us prove that \( \mu_i(\eta) \leq y \). If \( \eta \) has finitely many prefixes that are bud histories, then let \( \eta_{<n} \) be the longest one: we have \( \eta_{<n} = he^\nu_{+\|h|} \) \( (h) \), where \( [he^\infty] \) \( (h) = \tau_0(\psi(\eta_{<n})) \). Then, we have \( \mu_i(\eta) = \mu_i(\rho) \leq y \).
Now, if $\eta$ has infinitely many such prefixes, then there exists a unique play $\pi$ in the reduced negotiation game such that for any prefix $\eta \leq n$ of $\eta$ that is a bud history, the history $\psi(\eta \leq n)$ is a prefix of $\pi$. Then, if $\pi$ contains finitely many post-cycle deviations, then there exist two indices $m$ and $n$ such that $\eta \geq m = \bar\pi \geq n$, hence $\mu_i(\eta) = \mu_i(\bar\pi) \leq y$.

Finally, if $\pi$ contains infinitely many post-cycle deviations, i.e., infinitely many occurrences of a state of the form $(c, v)$, then let us choose such state that minimizes the quantity $\text{MP}_i(c)$. The play $\eta$ has the form:

$$\eta = h_0e^{k_0}h_1e^{k_1}h_2 \ldots,$$

where for each $n$, we have $k_n = \left| h_0e^{k_0} \ldots e^{k_{n-1}}h_n \right|$. Then, if we write $M = \max r_i$, we have:

$$\text{MP}_i \left( h_0e^{k_0} \ldots h_ne^{k_n} \right) \leq \frac{1}{k_n + k_n^2|c|} - 1 \left( k_nM + (k_n^2|c| - 1) \text{MP}_i(c) \right),$$

which converges to $\text{MP}_i(c)$ when $n$ tends to $+\infty$, hence $\mu_i(\eta) \leq \text{MP}_i(c)$. Now, since $\tau_\beta$ is memoryless, there exists a play of the form $HC^\omega$ that is compatible with it, and such that $(c, v) \in \text{Occ}(C) \subseteq \text{Inf}(\pi)$; and by definition of $y$, we have $\text{MP}_i(C) = \nu(C)(HC^\omega) \leq y$.

By minimality of $\text{MP}_i(c)$, we have $\text{MP}_i(C) = \text{MP}_i(c)$, hence $\text{MP}_i(c) \leq y$, and therefore $\mu_i(\eta) \leq y$.

Thus, a given requirement $\lambda$ is an $\varepsilon$-fixed point of $\text{nego}$ if and only if for each $i$ and $v \in V_i$, there exists a memoryless strategy $\tau_\beta$ in the game $\text{Red}^\beta_{\lambda_\nu}$, with $\beta = P_i(\|G\| + \|\lambda\|)$, such that $\sup_{\tau_\beta} \nu_{\text{C}}((\bar\pi)) \leq \lambda(v) + \varepsilon$. The reduced negotiation game has an exponential size, but it contains only $\text{card}^V$ states that are controlled by Prover: memoryless strategies for Prover are therefore objects of polynomial size. Thus, such memoryless strategies constitute the third and last piece of our notion of witness.

### 7 Algorithm and complexity

We are now in a position to define formally our notion of witness.

**Definition 45 (Witness).** Let $I = (G_{|v_0}, \tilde{x}, \tilde{y}, \varepsilon)$ be an instance of the $\varepsilon$-SPE threshold problem. A **witness** for $I$ is a tuple $(W, W', \tilde{\alpha}, \tilde{\lambda}, (\tau_\beta)^\nu)$, where $W \subseteq W' \subseteq V_i; \tilde{\alpha} \in [0, 1]^{\Pi^\text{SC}(W)}$; $\lambda$ is a requirement; and each $\tau_\beta^\nu$ is a memoryless strategy in the game $\text{Red}^\beta_{\lambda_\nu}(G)_{|v_\beta}$, where $\beta = P_i(\|G\| + \|\lambda\|)$. A witness is **valid** if:

- each strategy $\tau_\beta^\nu$ satisfies the inequality $\sup_{\tau_\beta} \nu_{\text{C}}((\tau_\beta^\nu, \tau_\text{C})) \leq \lambda(v) + \varepsilon$;
- the sets $W$ and $W'$ and the tuple of triples $\tilde{\alpha}$ satisfy the hypotheses of Theorem 38.

**Remark.** The sets $W$ and $W'$, as well as the tuple of strategies $(\tau_\beta^\nu)_\nu$, have polynomial size. In order to bound the size of witnesses by a polynomial, we only have to bound $\|\lambda\|$ and $\|\tilde{\alpha}\|$.

The $\varepsilon$-SPE threshold problem will be $\text{NP}$-easy if we show, first, that there exists a valid witness of polynomial size if and only if the instance is positive, and second, that the validity of a witness can be decided in polynomial time. The former is a consequence of Lemma 29, Theorem 37, Theorem 38, Theorem 44, and Lemma 43:

**Lemma 46.** There exists a polynomial $P_3$ such that an instance $I$ of the $\varepsilon$-SPE threshold problem admits a valid witness of size $P_3(\|I\|)$ if and only if it is a positive instance.

Let us now tackle the latter.
Lemma 47. Given an instance of the $\varepsilon$-SPE threshold problem and a witness for it, deciding whether that witness is valid is $\textbf{P}$-easy.

Proof. The validity of a witness is defined by two conditions. As regards the second one, all the hypotheses of Theorem 38 can be checked in polynomial time with classical algorithms. Let us now show how the first condition can also be checked in polynomial time.

Let $n = \text{card}V$. Given a memoryless strategy $\tau^*_P$ of Prover in a reduced negotiation game, one can construct in a time polynomial in $||\tau^*_P||$ the graph $\text{Red}_{\lambda}^{\gamma}(G)[\tau^*_P]$, defined as the underlying graph of $\text{Red}_{\lambda}^{\gamma}(G)$ where all the transitions that are not compatible with $\tau^*_P$ have been omitted, as well as all the states that are, then, no longer accessible from the state $v$.

That graph has indeed a polynomial size, because it is composed only of:
- at most $n$ vertices of the form $w \in V$;
- at most $n$ vertices of the form $\tau_P(w)$ (either equal to $\bot$ or of the form $[hc^\infty \rho]$);
- at most $n^2$ vertices of the form $(c, w)$;
- at most $2n^2$ vertices of the form $[h'w']$, where $h'$ is a prefix of the history $hc$ for some punishment family $[hc^\infty \rho] = \tau_P(-w)$;
- possibly the state $\top$.

We call this connected graph the deviation graph. Note that if among those vertices, there is the vertex $\bot$, then since the vertices that are not accessible have been removed, we have $\sup_c \nu_C(\langle \bar{\tau} \rangle) = +\infty$ and the problem can be solved immediately. In what follows, we assume that it is not the case, i.e. that for each $w$, the state $\tau_P(w)$ has the form $[hc^\infty \rho]$. Deciding whether $\sup_c \nu_C(\langle \bar{\tau} \rangle) \leq \varepsilon$ is then equivalent to deciding whether there exists a path $\pi$, in that graph, such that $\nu_C(\pi) > \varepsilon$. Such a play can have three forms.

- It can end in the state $\top$, i.e. with Challenger accepting Prover’s proposal. The existence of such a play can be decided immediately, by checking whether in the deviation graph, there exists a vertex of the form $[hc^\infty \rho]$ with $\mu_C(hc^\infty \rho) > \varepsilon$.
- It can avoid the state $\top$, and comprise finitely many post-cycle deviations. This is the case if and only if there exists a cycle $C$ in the deviation graph, without post-cycle deviations, such that $\text{MP}_C(C) > \varepsilon$. The existence of such a cycle can be decided in polynomial time with Karp’s algorithm (see [17]).
- It can avoid the state $\top$, and comprise infinitely many post-cycle deviations. In that case, we have $\nu_C(\pi) \leq \text{MP}_C(c)$ for each state of the form $(c, w)$ appearing infinitely often along $\pi$; then, there exists a cycle $C$ in the deviation graph, such that every state of the form $(c, w)$ along $C$ satisfies $\text{MP}_C(c) > \varepsilon$. Conversely, if such a cycle exists, then $\pi$ exists. The existence of such a cycle can be decided in polynomial time with Karp’s algorithm.

Therefore, the existence of such a play is decidable in polynomial time. ▶

Thus, given an instance of the $\varepsilon$-SPE threshold problem, a valid witness can be guessed and checked in polynomial time. Since the $\varepsilon$-SPE threshold problem has been proved to be $\textbf{NP}$-hard, we finally obtain the following theorem:

Theorem 48. The $\varepsilon$-SPE threshold problem in mean-payoff games is $\textbf{NP}$-complete.

References

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