



# Weighted Model Counting with Twin-Width

Robert Ganian  

Algorithms and Complexity Group, TU Wien, Austria

Filip Pokrývka  

Masaryk University, Brno, Czech Republic

André Schidler 

Algorithms and Complexity Group, TU Wien, Austria

Kirill Simonov 

Algorithms and Complexity Group, TU Wien, Austria

Stefan Szeider  

Algorithms and Complexity Group, TU Wien, Austria

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## Abstract

Bonnet et al. (FOCS 2020) introduced the graph invariant twin-width and showed that many NP-hard problems are tractable for graphs of bounded twin-width, generalizing similar results for other width measures, including treewidth and clique-width. In this paper, we investigate the use of twin-width for solving the propositional satisfiability problem (SAT) and propositional model counting. We particularly focus on Bounded-ones Weighted Model Counting (BWMC), which takes as input a CNF formula  $F$  along with a bound  $k$  and asks for the weighted sum of all models with at most  $k$  positive literals. BWMC generalizes not only SAT but also (weighted) model counting.

We develop the notion of “signed” twin-width of CNF formulas and establish that BWMC is fixed-parameter tractable when parameterized by the certified signed twin-width of  $F$  plus  $k$ . We show that this result is tight: it is neither possible to drop the bound  $k$  nor use the vanilla twin-width instead if one wishes to retain fixed-parameter tractability, even for the easier problem SAT. Our theoretical results are complemented with an empirical evaluation and comparison of signed twin-width on various classes of CNF formulas.

**2012 ACM Subject Classification** Theory of computation → Parameterized complexity and exact algorithms

**Keywords and phrases** Weighted model counting, twin-width, parameterized complexity, SAT

**Digital Object Identifier** 10.4230/LIPIcs.SAT.2022.15

**Related Version** *Full Version*: <https://arxiv.org/abs/2206.01706>

**Supplementary Material** *Dataset*: <https://doi.org/10.5281/zenodo.6610819>

**Funding** *Robert Ganian*: support by the Austrian Science Fund (FWF, project Y1329).

*Filip Pokrývka*: supported by the Czech Science Foundation, project no. 20-04567S.

*Kirill Simonov*: supported by the Austrian Science Fund (FWF, projects Y1329 and P31336).

*Stefan Szeider*: supported by the Austrian Science Fund (FWF, project P32441) and by the Vienna Science and Technology Fund (WWTF, project ICT19-065).

**Acknowledgements** The authors thank Édouard Bonnet for his helpful feedback regarding the relationship between signed twin-width and planar graphs.

## 1 Introduction

In many cases, it is not sufficient to determine whether a propositional formula is satisfiable, but we also need to determine the number of models. This MODEL COUNTING problem arises in several areas of artificial intelligence, among others in the context of probabilistic



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25th International Conference on Theory and Applications of Satisfiability Testing (SAT 2022).

Editors: Kuldeep S. Meel and Ofer Strichman; Article No. 15; pp. 15:1–15:17

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

reasoning [1, 14, 25], and is often studied in a weighted setting, where each literal has a weight and each model contributes a weight that is equal to the product of the weights of its literals. Here, we will go a step further and consider a natural generalization of these problems called *Bounded-ones Weighted Model Counting* (BWMC), where we are additionally provided a bound  $k$  on the input and are only asked to count models with at most  $k$  literals sets to true.

MODEL COUNTING is known to be  $\#P$ -complete already when all variables have the weight 1 [27] and remains  $\#P$ -hard even for monotone 2CNF formulas and Horn 2CNF formulas [20]. Hence standard *syntactical restrictions* do not suffice to achieve tractability, not even for this restricted case of BWMC.

A more successful approach for tackling model counting problems is based on *structural restrictions*, which focus on exploiting the interactions between variables and/or clauses by considering suitable graph representations of the input formula. The two most popular graph representations used in this context are the primal graph and the incidence graph<sup>1</sup> (sometimes called the variable interaction and variable-clause interaction graphs, respectively) [23]. Typically, one aims at identifying structural properties of these graphs – measured by an integer *parameter*  $k$  – which can be exploited to obtain so-called *fixed-parameter* algorithms for a considered problem, which are algorithms whose worst-case running time is upper-bounded by  $f(k) \cdot n^{\mathcal{O}(1)}$  for some computable function  $f$  and inputs of size  $n$ . Within the broader context of *parameterized complexity theory* [6, 8], we then say that the problem is *fixed-parameter tractable* w.r.t. the considered parameter(s).

The arguably most classical results that arise from this “parameterized” approach to propositional satisfiability and MODEL COUNTING are the fixed-parameter algorithms w.r.t. the treewidth of the primal and incidence graphs [15, 22, 23]. These were then followed by the fixed-parameter tractability of the problem w.r.t. the signed clique-width [10] and signed rank-width [12], as well as other results which combine structural restrictions with syntactic ones [13]; all of these results can be seen as a push in the overarching aim of identifying the “broadest,” i.e., most general properties of graphs that suffice for fixed-parameter tractability. It is worth noting that all of these algorithmic results can be adapted to also solve BWMC.

Recently, Bonnet et al. [5] discovered a fundamental graph parameter called *twin-width* that is based on a novel type of graph decomposition called a *contraction sequence*. They showed that bounded twin-width generalizes many previously known graph classes for which first-order model checking (an important meta-problem in computational logic) is tractable; most notably, it is upper-bounded on graphs of bounded clique-width as well as on planar graphs [5]. In this sense, it provides a “common generalization” of both of these (otherwise very diverse) notions. In spite of its recent introduction, twin-width has already become the topic of extensive research [2–4, 9].

**Contributions.** With this paper, we embark on investigating the utilization of twin-width for the propositional satisfiability problem (SAT) and, more generally, BWMC. We begin by noting that, similarly to the case of clique-width, it is impossible to exploit the “vanilla” notion of twin-width, even for SAT. Indeed, as will become clear in Section 5, neither BWMC nor SAT is fixed-parameter tractable when parameterized by the vanilla twin-width of their primal or incidence graph representations. Hence, inspired by previous work on SAT using clique-width [10] and rank-width [12], we develop a notion of *signed twin-width* of CNF formulas that is based on the incidence graph representation along with new *bipartite contraction sequences*.

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<sup>1</sup> Definitions are provided in Section 2.

As our main algorithmic result, we establish the following:

► **Theorem 1.** *BWMC is fixed-parameter tractable when parameterized by  $k$  plus the twin-width of a signed contraction sequence provided on the input.*

We show that this result is essentially tight in the sense that to retain fixed-parameter tractability, it is neither possible to replace the use of “signed” twin-width with vanilla twin-width as introduced by Bonnet et al. [5], nor use the primal graph representation, nor drop  $k$  as a parameter. Our results are summarized in Table 1.

■ **Table 1** Overview of results. The fixed-parameter tractability applies to BWMC, while all lower bounds hold already for the problem of deciding whether a model exists.

|                     | Signed twin-width    | Vanilla twin-width   | Primal twin-width    |
|---------------------|----------------------|----------------------|----------------------|
| $k$ is parameter    | FPT (Theorem 1)      | W[1]-hard (Pr. 13)   | W[2]-hard (Pr. 10)   |
| $k$ is unrestricted | paraNP-hard (Pr. 11) | paraNP-hard (Pr. 11) | paraNP-hard (Pr. 11) |

Apart from establishing our main algorithmic result and the accompanying lower bounds highlighted above, we also prove that incidence graphs of bounded signed twin-width are a strictly more general class than both planar incidence graphs and incidence graphs of bounded signed clique-width. In fact, for the latter case, our proof also yields an improved bound on the vanilla twin-width for graphs of bounded clique-width compared to that of Bonnet et al. [5].

We complement our theoretical findings with a brief experimental evaluation, where we compare the signed clique-width, signed twin-width, and treewidth of CNF formulas. For this comparison, we utilize SAT encodings that determine the exact value of these parameters [16, 18, 24, 26]. For signed twin-width, we had to adapt the respective encoding to signed graphs and used the bipartiteness of the incidence graph to compute larger instances than would be possible with the plain encoding. Surprisingly, even though the signed twin-width could, in theory, be (at most a constant factor) larger than both treewidth and signed clique-width, in all of the experiments, it turned out to be smaller than these two other parameters.

*Statements where proofs are provided in the full version are marked with  $\star$ .*

## 2 Preliminaries

We assume a basic knowledge of common notions used in graph theory [7]. For an integer  $n > 0$  we denote the set  $\{1, \dots, n\}$  by  $[n]$ . All graphs considered in this paper are simple. For two vertices  $u, v \in V(G)$  we denote by  $uv$  the edge with endpoints  $u$  and  $v$ .

Following the terminology of [5], we call the following operation a *contraction* of two vertices  $u$  and  $v$ : introduce a new vertex  $w$  into the graph whose neighborhood consists of all the neighbors of  $u$  and  $v$ , and remove  $u$  and  $v$  from the graph. This definition of a contraction is distinguished from the more commonly used term “edge contraction” that corresponds to the same operation but requires  $u$  and  $v$  to be adjacent.

**Satisfiability and Weighted Model Counting.** We consider propositional formulas in conjunctive normal form (CNF), represented as sets of clauses over a variable set  $\text{var}$ . That is, a *literal* is a (propositional) variable  $x$  or a negated variable  $\bar{x}$ , a *clause* is a finite set of literals not containing a complementary pair  $x$  and  $\bar{x}$ , and a *formula* is a finite set of clauses. We use  $\text{var}(F)$  to denote the variables of  $F$ .

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A *truth assignment* (or *assignment*, for short) is a mapping  $\tau : X \rightarrow \{0, 1\}$  defined on some set  $X$  of variables in a formula  $F$ . We extend  $\tau$  to literals by setting  $\tau(\bar{x}) = 1 - \tau(x)$  for  $x \in X$ . An assignment  $\tau : X \rightarrow \{0, 1\}$  *satisfies* a formula  $F$  if every clause of  $F$  contains a literal  $z$  such that  $\tau(z) = 1$ . A truth assignment  $\tau : \text{var}(F) \rightarrow \{0, 1\}$  that satisfies  $F$  is a *model* of  $F$ , and let  $M(F)$  be the set of all models of  $F$ .

Let  $w$  be a *weight function* that maps each literal of  $F$  to a real. The weight of an assignment  $\tau$  is the product over the weights of its literals, i.e.,  $w(\tau) = (\prod_{v \in \tau^{-1}(1)} (w(v)) \cdot (\prod_{v \in \tau^{-1}(0)} (w(\bar{v}))$ ). Another property of assignments that will be useful for our considerations is the number of variables set to 1 by the assignment, which we define as  $\text{ones}(\tau) = |\tau^{-1}(1)|$ . Our main problem of interest is defined as follows:

|  |   |
|--|---|
| <b>BOUNDED-ONES WEIGHTED MODEL COUNTING (BWMC)</b> |   |
| Input:   | A formula $F$ with a weight function $w$ and an integer $k$ .         |
| Task:  | Compute $\sum_{\pi \in M(F) \wedge \text{ones}(\pi) \leq k} w(\pi)$ . |

We note that the WEIGHTED MODEL COUNTING problem precisely corresponds to BWMC when we set  $k = |\text{var}(F)|$  and to MODEL COUNTING when we additionally also set each literal to a weight of one. When describing our lower bounds, it will also be useful to consider a simpler decision problem called BOUNDED-ONES SAT (BSAT), which takes as input a formula  $F$  and integer  $k$  and asks whether the formula admits a satisfying assignment with at most  $k$  variables set to 1. In other words, BSAT is the restriction of BWMC to the case where each literal has a weight of one and where we only need to decide whether the output is  $\geq 1$  or 0; this problem precisely corresponds to SAT when we set  $k = |\text{var}(F)|$ .

The (*signed*) *incidence graph* of  $F$ , denoted here by  $G_F$ , is an edge-labeled graph defined as follows:  $V(G_F) = F \cup \text{var}(F)$  and  $E(G_F) = \{ab \mid a \in F \wedge b \in \text{var}(F) \wedge \{b, \bar{b}\} \cap a \neq \emptyset\}$ . The edge set is partitioned into the set of *positive* edges  $E^+(G_F) = \{ab \mid a \in F \wedge b \in \text{var}(F) \wedge b \in a\}$  and the set of *negative* edges  $E^-(G_F) = \{ab \mid a \in F \wedge b \in \text{var}(F) \wedge \bar{b} \in a\}$ . We will later also use the *primal graph* of  $F$  for comparison; the vertex set of the primal graph is the set of all variables, and two variables  $a, b$  are connected by an edge if and only if there is at least one clause containing literals of both variables.

**Twin-width.** Twin-width was introduced by Bonnet, Kim, Thomassé, and Watrigant [5]; in what follows, we recall the basic concepts and notations introduced there. A *trigraph* is defined by a triple  $(V(G), E(G), R(G))$  where  $E(G)$  and  $R(G)$  are both sets of edges with endpoints in  $V(G)$ , called (usual) black edges and *red edges*. We call  $G$  a *d-trigraph* if the maximum degree in a subgraph induced by  $R(G)$  is at most  $d$ . A trigraph is *red-connected* if it is connected and remains connected after removing all black edges.

For a trigraph  $G$  and vertices  $u, v \in V(G)$ , we define the trigraph  $G/u, v$  obtained by *contracting*  $u, v$  into a single vertex  $w$ . Specifically,  $V(G/u, v) = V(G) \setminus \{u, v\} \cup \{w\}$ ,  $G - \{u, v\} = G/u, v - \{w\}$ , and the edges incident to  $w$  are as follows: for a vertex  $x \in V(G/u, v) \setminus \{w\}$ ,

- $wx \in E(G/u, v)$  if and only if  $ux \in E(G)$  and  $vx \in E(G)$ ,
- $wx \notin E(G/u, v) \cup R(G/u, v)$  if and only if  $ux \notin E(G) \cup R(G)$  and  $vx \notin E(G) \cup R(G)$ ,
- $wx \in R(G/u, v)$  otherwise.

We say that  $G/u, v$  is a *contraction* of  $G$ , and if both are  $d$ -trigraphs,  $G/u, v$  is a *d-contraction* of  $G$ . A graph  $G$  is *d-collapsible* if there exists a sequence of  $d$ -contractions that contracts  $G$  to a single vertex. The minimum  $d$  for which  $G$  is  $d$ -collapsible is the *twin-width* of  $G$ , denoted  $\text{tw}(G)$ .

**Signed Clique-Width and Clique-Width Expressions.** In Section 3 we draw connections between twin-width and signed clique-width, and here we introduce the basic definitions for the latter. For a positive integer  $k$ , we let a  $k$ -graph be a graph whose vertices are labeled by  $[k]$ . For convenience, we consider a graph to be a  $k$ -graph with all vertices labeled by 1. We call the  $k$ -graph consisting of exactly one vertex  $v$  (say, labeled by  $i$ ) an initial  $k$ -graph and denote it by  $i(v)$ .

The (*signed*) *clique-width* of a signed graph  $G$  is the smallest integer  $k$  such that  $G$  can be constructed from initial  $k$ -graphs by means of iterative application of the following four operations:

1. Disjoint union (denoted by  $\oplus$ );
2. Relabeling: changing all labels  $i$  to  $j$  (denoted by  $\rho_{i \rightarrow j}$ );
3. Positive edge insertion: adding positive edges from each vertex labeled by  $i$  to each vertex labeled by  $j$  ( $i \neq j$ ; denoted by  $\eta_{i,j}^+$ );
4. Negative edge insertion: adding negative edges from each vertex labeled by  $i$  to each vertex labeled by  $j$  ( $i \neq j$ ; denoted by  $\eta_{i,j}^-$ ).

A construction of a  $k$ -graph  $G$  using the above operations can be represented by an algebraic term composed of  $\oplus$ ,  $\rho_{i \rightarrow j}$ ,  $\eta_{i,j}^+$  and  $\eta_{i,j}^-$  (where  $i \neq j$  and  $i, j \in [k]$ ). Such a term is called a  $k$ -expression defining  $G$ , and we often view it as a tree with each node labeled with the appropriate operation. Conversely, we call the  $k$ -graph that arises from a  $k$ -expression its *evaluation*. The *clique-width* of a signed graph  $G$  is the smallest integer  $k$  such that  $G$  can be defined by a  $k$ -expression, which we then also call a *clique-width expression* of  $G$ .

**Parameterized Complexity.** Next, we give a brief and rather informal review of the most important concepts of parameterized complexity. For an in-depth treatment of the subject, we refer the reader to other sources [6, 8, 11, 17, 23]. ( $\star$ )

The instances of a parameterized problem can be considered as pairs  $(I, k)$  where  $I$  is the *main part* of the instance and  $k$  is the *parameter* of the instance; the latter is usually a non-negative integer. A parameterized problem is *fixed-parameter tractable* (FPT) if instances  $(I, k)$  of size  $n$  (with respect to some reasonable encoding) can be solved in time  $f(k)n^c$  where  $f$  is a computable function and  $c$  is a constant independent of  $k$ . Such algorithms are called *fixed-parameter algorithms*.

All our lower bounds will be obtained already for the simpler BSAT problem, allowing us to restrict our attention to decision problems. Since this is NP-complete, the strongest form of parameterized intractability one can establish for a parameterization of BSAT is *paraNP-hardness*, which means that the parameterized problem remains NP-hard even for a fixed constant value of the considered parameter. A weaker notion of intractability is provided by the complexity classes W[1] or W[2]; while W[2] is believed to be a superclass of W[1], both W[1]-hardness and W[2]-hardness exclude fixed-parameter tractability under well-established complexity assumptions [6]. It is perhaps worth noting that the whole W-hierarchy (which contains the complexity classes W[1], W[2],  $\dots$ , W[P]) is itself defined through a weighted satisfiability problem that is similar in spirit to BSAT [8].

### 3 The Twin-Width of Signed Graphs

Motivated by the incidence graph of a formula, we define a *signed graph* to be a graph  $G$  distinguishing two sets of edges  $E^+(G)$  and  $E^-(G)$  (positive and negative edges) over the same set of vertices  $V(G)$ . We assume these two sets of edges to be disjoint, i.e.,  $E^+(G) \cap E^-(G) = \emptyset$ , so each pair of vertices  $u, v \in V(G)$  is either positively-adjacent, negatively-adjacent or non-adjacent. A signed graph  $G$  is bipartite if the set of vertices  $V(G)$  can be partitioned into two independent sets.

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We now proceed to define the twin-width for signed graphs based on contractions, analogously to how twin-width is defined for graphs [5] defines. First, we have to incorporate red edges in a signed graph, representing “errors” that were created during contractions. Let  $G$  be a *signed trigraph* which contains, in addition to  $E^+(G)$  and  $E^-(G)$ , an additional set  $R(G)$  of edges called the *red edges*. A signed trigraph  $G'$  is a contraction of  $G$  if there exist vertices  $u, v \in V(G)$  and a vertex  $w \in V(G')$  such that  $G \setminus \{u, v\} = G' \setminus \{w\}$  and all vertices  $x \in V(G') \setminus \{w\}$  satisfy the following:

- $wx$  is a positive edge if both  $ux$  and  $vx$  are positive,
- $wx$  is a negative edge if both  $ux$  and  $vx$  are negative,
- $w$  and  $x$  are non-adjacent if  $u$  and  $x$  are non-adjacent and  $v$  and  $x$  are non-adjacent,
- otherwise,  $wx$  is a red edge.

A contraction sequence of a signed graph  $G$  is a sequence of signed trigraphs  $G = G_n, G_{n-1}, \dots, G_1 = K_1$ , where  $G_i$  has  $i$  vertices and  $G_i$  is a contraction of  $G_{i+1}$ . A contraction sequence of a signed graph  $G$  is called a *d-sequence* if each vertex in each signed graph of the sequence has red degree at most  $d$ . The twin-width of a signed graph is the minimal such  $d$  over all contraction sequences of  $G$ ; we call this the *signed twin-width* of  $G$ . Later on, it will also be useful to compare this signed twin-width to the twin-width of the underlying unsigned graph for  $G$ ; to avoid any ambiguities, we refer to this parameter as the *unsigned twin-width* of  $G$ .

One useful way of viewing the contraction sequence is through vertex partitions. Let  $G$  be a signed graph, and consider a partition  $V'$  of  $V(G)$ . Intuitively, each set of vertices  $X \in V'$  corresponds to a vertex in the contracted graph that represents vertices in  $V(G)$  that were contracted into  $X$ . The set of positive, negative, and red edges over  $V'$  is then defined as follows.

- $X, Y \in V'$  are positively-adjacent, if each vertex  $x \in X$  is positively-adjacent to each vertex  $y \in Y$ .
- $X, Y \in V'$  are negatively-adjacent, if each vertex  $x \in X$  is negatively-adjacent to each vertex  $y \in Y$ .
- $X, Y \in V'$  are non-adjacent, if each vertex  $x \in X$  is non-adjacent to each vertex  $y \in Y$ .
- $X, Y \in V'$  are red-adjacent otherwise.

In this way, each signed graph  $G_i$  in a contraction sequence of  $G$  can be treated as a vertex partition of  $V(G)$ , where each vertex  $v \in V(G_i)$  is a subset of vertices of  $V(G)$  that were contracted into  $v$ .

**Bipartite contraction sequences.** In this section we define bipartite sequences of contractions, which are a core feature in the definition of the signed twin-width of formulas. In this context, they are helpful as they allow us to keep variable and clause vertices separate.

► **Definition 2.** *Let  $G$  be a bipartite graph with vertex partition  $A, B \subseteq V(G)$ ,  $A \cap B = \emptyset$ ,  $A \cup B = V(G)$ . A  $d$ -sequence of  $G$  is a bipartite  $d$ -sequence of  $G$ , if each contraction in the sequence contracts two vertices from the same part (either  $A$  or  $B$ ).*

Observe that each graph in a bipartite  $d$ -sequence is bipartite and a bipartite  $d$ -sequence of maximal length ends with  $G_2$ , where we can no longer contract. At this point, we can formalize our parameter of interest:

► **Definition 3.** *For a formula  $F$  with signed incidence graph  $G$ , let the signed twin-width of  $F$  be the minimal  $d$  such that  $G$  has a bipartite  $d$ -sequence ending in a two-vertex graph.*

Below, we show that the restriction to bipartite contraction sequences is not only natural in the context of incidence graph representations of formulas but also “essentially harmless” in terms of the parameter’s behavior.

► **Lemma 4.** *Let  $G$  be a bipartite signed graph  $G$  and  $G = G_n, \dots, G_1$  be a  $d$ -sequence. Then a bipartite  $(d + 2)$ -sequence  $G' = G'_n, \dots, G'_2$  can be computed in linear time.*

**Proof.** Suppose  $G = G_n, G_{n-1}, \dots, G_1 = K_1$  is a  $d$ -sequence of  $n$ -vertex bipartite graph  $G$ , with vertex partition  $A, B$ . Each vertex  $u$  of  $G_i$  is a set of vertices  $u(G)$  of  $G_n$  that contracted into  $u$ . Consider the following algorithm which constructs a bipartite  $(d + 2)$ -sequence  $G'_i$  along with a mapping  $m$  of indices between these sequences, which is necessary due to the fact that there will not be a one-to-one correspondence between trigraphs in sequence  $G'$  and in  $G$ ;  $m$  will map the index  $j$  of the trigraph we are currently processing in  $G'$  to a corresponding index  $i$  in  $G$ .

Initially, we set  $G'_n = G_n$ . For each  $G_i$  starting with  $i := n - 1$  and ending with  $i := 1$ , denote  $w$  the newly created vertex in  $G_i$  and  $u, v$  the vertices of  $G_{i+1}$  that contracted to  $w$ . The set of vertices  $w(G)$  can be split into  $w_A = w(G) \cap A$  and  $w_B = w(G) \cap B$ , and the sets  $u_A, u_B, v_A, v_B$  are defined analogously. We can proceed depending on the number of empty sets among  $u_A, u_B, v_A, v_B$ .

- Two sets are empty. Then both  $u$  and  $v$  belong to one part. If they are in the same part, it is safe to contract them as  $w$  is also contained in one part. We append  $G'_j$  to the contraction sequence, set  $m(j) = i$ , and proceed to the next trigraph by decrementing both  $i$  and  $j$ . Otherwise a contraction is not possible; we do not change the sequence of  $G'$  but continue processing the next graph in  $G$ , decrement  $i$ .
- One set is empty. Without loss of generality, assume  $v_B$  is the empty set (if not, swap the labels between  $u, v$  or  $A, B$ ). In this case, we construct the graph  $G'_j$  where  $u_A, v_A$  have been contracted and append it to  $G'$ , set  $m(j) = i$ , and proceed to the next trigraph by decrementing both  $i$  and  $j$ .
- All four sets are non-empty. We call this operation a *double step*, since two contractions are required. Firstly, we contract the  $A$ -parts ( $u_A$  and  $v_A$ ) to construct a first trigraph  $G'_j$ . We add this trigraph to the sequence  $G'$ , set  $m(j) = i$  and decrement  $j$ . Next, we proceed analogously for the new value of  $j$ : we contract the  $B$ -parts ( $u_B$  and  $v_B$ ), add this new trigraph to the sequence  $G'$ , and once again set  $m(j) = i$ . Finally, we proceed to the next trigraph by decrementing both  $i$  and  $j$ .

Obviously, the time complexity of the algorithm described above is linear; we merely process the input sequence and at each graph in the sequence we add either zero, one or two graphs into the output sequence.

What remains to be shown is that  $G'$  is a bipartite  $(d + 2)$ -sequence. First, it is obviously a bipartite sequence, since at each step we contracted only  $A$ -parts and  $B$ -parts of vertices being contracted in sequence of  $G$ , and since  $G_1$  is single vertex  $x$ ,  $G'_2$  contains two vertices  $x_A$  and  $x_B$ . Next, we turn our attention to the red degrees.

▷ **Claim 5.** Let  $j \in [n]$  such that  $m(j) \neq m(j - 1)$  (i.e.,  $G'_j$  is not an intermediate graph in a double step). Let  $x \in V(G_{m(j)})$  be a vertex in a trigraph in the original sequence. Then  $x_A$  has red degree at most  $d$  in  $G'_j \setminus x_B$ , and conversely  $x_B$  has red degree at most  $d$  in  $G'_j \setminus x_A$ .

**Proof.** We prove the claim for  $x_A$  only, since the situation for  $x_B$  is fully symmetrical. Suppose  $x_A$  has red degree more than  $d$  in  $G'_j \setminus x_B$ . This means that there are at least  $d + 1$  vertices in the  $B$ -part of  $G'_j \setminus x_B$  that are red-adjacent to  $x_A$  (there are no red edges

inside the  $A$ -part and inside the  $B$ -part). Each vertex  $b_i$  in the red neighborhood of  $x_A$  is then a  $B$ -part of some vertex of  $G_{m(j)}$ ; let us denote these vertices as  $u_i$ . Now, each  $u_i$  is red-adjacent to  $x$  in  $G_{m(j)}$ , because  $b_i$  is a red neighbor of  $x_A$ , and by contracting more vertices with  $x_A$  and  $u_i$ , the red edge does not disappear. That makes the red-degree of  $x$  at least  $d + 1$  in  $G_{m(j)}$ , which contradicts the input sequence of  $G$  being a  $d$ -sequence.  $\triangleleft$

As an immediate corollary of Claim 5, we obtain unless  $G'_j$  is an intermediate graph in a double step, all vertices in  $G'_j$  have red degree at most  $d + 1$ .

It remains to check intermediate graphs produced in the middle of double steps. In this case, there are two contractions in the sequence  $G'$ , while there is a single corresponding contraction in  $G$ . Thus, it might happen that after the first contraction in the  $A$ -part, we introduce a new red edge to a vertex that will be contracted in the second contraction in the  $B$ -part. For all other vertices, Claim 5 applies analogously. Furthermore, the same argument also upper-bounds the red degree of the vertices in the  $A$ -part that are being contracted, but here we must consider the graph  $G'_j \setminus \{u_B, v_B\}$  which yields a total upper-bound of  $d + 2$  on all red degrees in  $G'$ .  $\blacktriangleleft$

We remark that twin-width also admits a more general definition over matrices, where its definition on graphs coincides with the use of the matrix definition on the adjacency matrix of the graph [5]. In this context, the signed twin-width of a formula  $F$  could equivalently be stated as the twin-width of a “signed” bipartite adjacency matrix of the (signed) incidence graph, where the edge labels in the graph are represented as different values in the matrix.

**Comparison to Existing Measures.** Bonnet et al. [5] showed that any proper minor-closed graph class has bounded twin-width. Therefore, in particular, planar graphs and graphs of bounded genus have bounded twin-width. While, strictly speaking, this result considers only graphs and not signed graphs, the subsequent work [2] provides a way to lift these bounds to signed graphs. ( $\star$ )

► **Proposition 6** (Corollary of Theorem 10 [5] and Theorem 12 [2]). *Let  $G$  be a signed graph and  $G'$  be the corresponding unsigned graph. If  $G'$  has twin-width  $d$  and does not contain  $K_{t,t}$  as a subgraph then the signed twin-width of  $G$  is at most  $2^{2^{\mathcal{O}(dt)}}$ .*

► **Corollary 7.** *The class of all planar signed graphs has bounded signed twin-width.*

However, in the above it is crucial that the graph class in question is sparse. For arbitrary graphs one should not expect to derive a signed twin-width bound from a twin-width bound of the unsigned graph. For example, cliques have bounded twin-width, but by assigning signs to edges of the clique one may obtain an arbitrary binary structure. On the other hand, the bounds on twin-width based on other width measures can be extended to bounds on signed twin-width based on a *signed* version of the corresponding width measure. In what follows we show that bounded signed clique-width implies bounded signed twin-width.

► **Proposition 8** ( $\star$ ). *Let a graph  $G$  have a clique-width of  $d$ . Then  $G$  has a twin-width of at most  $2d$ . Moreover, if  $G$  is a signed graph with a signed clique-width of  $d$ , the signed twin-width of  $G$  is at most  $2d$ .*

**Proof Sketch.** We process the clique-width expression in a leaves-to-root fashion, during which we maintain the following invariant: all the vertices in the subexpression that have the same label are contracted into a single vertex. Thus, after processing each operation, the subexpression corresponds to at most  $d$  vertices in the current graph in the contraction



sequence, and no edges that go from the vertices of the subexpression to the outside, are red. Originally, no vertices are contracted, and by the time the whole expression is processed all vertices are contracted into one. It now remains to consider the three types of operations.

In an edge insertion node, no contractions occur as no vertices change labels. Moreover, the edges introduced correspond to exactly one edge in the current contraction sequence graph, and that edge is black since only vertices that have the same label were contracted together. In a relabeling node  $i \rightarrow j$ , we contract the vertices that represent labels  $i$  and  $j$  in the current subexpression. No outside edges become red as afterwards in the expression these vertices are treated identically, and inside the subexpression there are at most  $d$  vertices, thus the red degree in the subgraph corresponding to the subexpression does not exceed  $d$ . Finally, in a disjoint union node, we consecutively contract the vertices corresponding to the same label in the two disjoint parts. Analogously to the previous case, no outside edges become red. Since the whole subexpression corresponds to at most  $2d$  vertices at this point, the red degree is at most  $2d$  in each of the intermediate graphs. ◀

Moreover, twin-width strictly dominates clique-width: for example, the  $n \times n$  grid has constant twin-width, but unbounded clique-width as  $n$  grows [5]. The same holds for signed versions: signed clique-width clearly remains unbounded while the twin-width bound of Bonnet et al. [5] holds irrespectively of edge signs.

► **Proposition 9** (Theorem 4, [5]). *For every positive integers  $d$  and  $n$ , the  $d$ -dimensional  $n$ -grid has twin-width at most  $3d$ . The same holds for the signed twin-width of any orientation of the grid.*

**Proof.** It suffices to observe that [5] shows the bound for the case where all edges of the grid are red, thus the bound immediately transfers to grids with black edges where signs are arbitrary. ◀

In fact, even mim-width – a more general parameter than clique-width – is unbounded on  $n \times n$  grids. In general, however, twin-width and mim-width are incomparable as, e.g., interval graphs have mim-width one and unbounded twin-width [2]. While mim-width has also been used to solve variants of SAT [21], it does not yield fixed-parameter algorithms for these problems.

#### 4 Fixed-Parameter Algorithm for BWMC Parameterized by Twin-Width

For a formula  $F$  with signed incidence graph  $G$ , recall that the *signed twin-width* of  $F$  be the minimal red degree over all bipartite contraction sequences of  $G$ . This section is dedicated to our main technical contribution, which is a fixed-parameter algorithm for WMC parameterized by  $k$  plus the signed twin-width of the input formula. We note that since a fixed-parameter algorithm for computing contraction sequences is not yet known, we will adopt the assumption that such a sequence is provided in advance.

► **Theorem 1.** *BWMC is fixed-parameter tractable when parameterized by  $k$  plus the twin-width of a signed contraction sequence provided on the input.*

**Proof Sketch.** We begin by invoking Lemma 4 on the input sequence, which constructs a signed bipartite  $(\text{tw}(G) + 2)$ -sequence  $G_n, \dots, G_2$  of  $G$  in linear time. The core of the proof will be a dynamic programming procedure which will proceed along this sequence where, on a high level, for each graph  $G_i$  we will compute a record that will allow us to provide an

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output for the WMC problem once we reach  $G_2$ . To this end, we will dynamically compute records for  $G_n, G_{n-1}, \dots, G_2$ . Our records will consist of a mapping from so-called *profiles* to reals, where each profile will correspond to a set of assignments of  $\text{var}(F)$  which “behave the same” on the level of  $G_i$ .

Formally, a profile for a signed trigraph  $G_i$  is a tuple of the form  $(T, P, M, \ell, Q)$ , where:

- $T \subseteq V(G_i)$  that induce a red-connected subgraph of  $G_i$  of size at most  $k(d^2 + 1)$ , partitioned into  $T_{\text{cla}} \subseteq V_{\text{cla}}(G_i)$  and  $T_{\text{var}} \subseteq V_{\text{var}}(G_i)$ ,
- $P \subseteq T_{\text{var}}$  where  $|P| \leq k$ ,
- $M \subseteq P$ ,
- $\ell \leq k$ , and
- $Q \subseteq T_{\text{cla}}$ .

Intuitively, our records will only need to store information about parts of  $G_i$  which are red-connected subgraphs, since – as we will see later – the uniformity of black edges allows us to deal with them without dynamic programming. Furthermore, since the number of “ones” is bounded by  $k$ , it will suffice to store information only about local parts of each red-connected subgraph, and for each such subgraph we will consider a separate  $T$ . For each fixed  $T$ , we will then keep track of all possible variable-vertices in  $T_{\text{var}}$  where positive assignments occur (via  $P$ ), which of these also contain negative assignments (via  $M$ ), how many ones we have used up so far in this choice of  $T$  (via  $\ell$ ), and which clause-vertices in  $T_{\text{cla}}$  are already satisfied by variables in  $T_{\text{var}}$  (via  $Q$ ).

To formalize this intuition, we will use the notion of *realizability*. Consider a profile  $(T, P, M, \ell, Q)$  of  $G_i$ , and let  $S_{\text{var}}^T$  be the set of all variables in  $V_{\text{var}}(G_n)$  which are contracted to  $T_{\text{var}}$ , similarly let  $S_{\text{cla}}^T$  be the set of all clauses in  $V_{\text{cla}}(G_n)$  which are contracted to  $T_{\text{cla}}$ . We say that  $(T, P, M, \ell, Q)$  is *realizable* by an assignment  $\nu$  of  $S_{\text{var}}^T$  if and only if:

- R1** for each vertex  $u \in P$ , at least one variable  $v \in V(G_n)$  collapses to  $u$  and  $\nu(v) = 1$ ,
- R2** for each vertex  $u \in M$ , at least one variable  $v \in V(G_n)$  collapses to  $u$  and  $\nu(v) = 0$ ,
- R3** for each vertex  $u \in P \setminus M$  all variables  $v \in V(G_n)$  collapsing to  $u$  satisfy  $\nu(v) = 1$ ,
- R4** exactly  $\ell$  variables  $v \in S_{\text{var}}^T$  satisfy  $\nu(v) = 1$ .
- R5** if  $c \in Q$ , then every clause of  $S_{\text{cla}}^T$  collapsing to clause-vertex  $c$  is satisfied by  $\nu$ .
- R6** if  $c \notin Q$ , then at least one clause of  $S_{\text{cla}}^T$  collapsing to  $c$  is unsatisfied by  $\nu$ .

For each  $G_i$ , let  $\text{Profiles}(G_i)$  be the set of all profiles of  $G_i$ . For each profile  $\tau = (T, P, M, \ell, Q) \in \text{Profiles}(G_i)$ , let  $\alpha(\tau) = \{\nu : S_{\text{var}}^T \rightarrow \{0, 1\} \mid \tau \text{ is realizable by } \nu\}$  be the set of all assignments which are, intuitively, captured by this profile. Next, we will use  $w(\tau)$  to denote the sum of the weights of all these assignments; formally,  $w(\tau) = \sum_{\nu \in \alpha(\tau)} w(\nu)$ . Then the *record* for  $G_i$ , denoted  $R_{G_i}$ , is the mapping  $\text{Profiles}(G_i) \rightarrow \mathbb{R}$  which maps each profile  $\tau \in \text{Profiles}(G_i)$  to  $w(\tau)$ .

Denote  $t = k(d^2 + 1)$  as an upper-bound of  $|T|$ . We start by upper-bounding the size of the records. Specifically, we show that  $|\text{Profiles}(G_i)| \leq s_i$  where  $s_i$  is defined as  $i(d^{2t-2} + 1) \binom{t}{k} 2^{k+t} (k+1)$ . Using [3, Lemma 8], the number of possible red-connected sets  $T$ , where  $T \subseteq V(G_i)$  and  $|T| \leq t$  is upper-bounded by  $i(d^{2t-2} + 1)$ . There are at most  $\binom{t}{k}$  possible choices for the set  $P$  since  $|P| \leq k$  and  $P \subseteq T_{\text{var}} \subseteq T$ . Since  $M \subseteq P$ , there are  $2^k$  possible choices of  $M$ . For  $\ell$ , we have  $k+1$  possibilities because  $0 \leq \ell \leq k$ . For the last component  $Q$ , there are at most  $2^t$  possible choices since  $Q \subseteq T_{\text{cla}} \subseteq T$ .

Moreover, for  $G_n$  we can construct  $R_{G_n}$  as follows: for each  $v \in V_{\text{var}}(G_n)$ , we will have two profiles  $(\{v\}, \{v\}, \emptyset, 1, \emptyset)$  and  $(\{v\}, \emptyset, \emptyset, 0, \emptyset)$ , while for each  $v \in V_{\text{cla}}(G_n)$  we will have a single profile  $(\{v\}, \emptyset, \emptyset, 0, \emptyset)$ . The first two profiles will be mapped to  $w(\{v \mapsto 1\})$  and  $w(\{v \mapsto 0\})$ , respectively, while the last profile will be mapped to 0.

Once we obtain the record for  $G_2$ , we can output the solution for the WMC instance as follows. First, we observe that if  $G_2$  does not contain a red edge, then *every* variable in the input formula  $F$  occurs in *every* clause of  $F$  in the same way, which implies that  $|\text{cla}(F)| \leq 1$  and the instance is trivial. So, without loss of generality we can assume that  $G_2$  contains a red edge, and let  $\text{Profiles}'(G_2)$  be the restriction of  $\text{Profiles}(G_2)$  to tuples where the first component  $T$  contains precisely two vertices. From the definition of the records, it now follows that  $\sum_{\pi \in M(F) \wedge \text{ones}(\pi) \leq k} w(\pi) = \sum_{\tau \in \text{Profiles}'(G_2)} R_{G_2}(\tau)$ .

It remains to show how to compute  $w(\tau)$  for each profile  $\tau = (T, P, M, \ell, Q)$  of  $G_i$  assuming that the record for profiles in  $G_{i+1}$  is already computed. There are two possible cases, depending whether  $G_i$  is the result of contracting clause vertices or variable vertices of  $G_{i+1}$ . In both cases, denote by  $x, y \in V(G_{i+1})$  the vertices that are contracted into  $z \in V(G_i)$ . Let  $\tau = (T, P, M, \ell, Q)$  be a profile for  $G_{i+1}$ . If  $z \notin T$  then  $\tau$  is also a valid profile for  $G_{i+1}$ , and we conclude by setting  $R_{G_i}(\tau) = R_{G_{i+1}}(\tau) = w(\tau)$ . Thus, in the following we assume that  $z \in T$ .

Let  $T_1, \dots, T_m$  be the red-connected components of  $(T \setminus \{z\}) \cup \{x, y\}$  in  $G_{i+1}$ . Note that  $m \leq d + 2$  since each component contains either  $x, y$ , or a red neighbor of  $z$ . First, let us observe that one of the following two cases must occur: either for every  $T_j$  obtained for the current choice of  $T$  it holds that  $|T_j| \leq k(d^2 + 1)$  (which we call the *Standard Case*), or  $|T| = k(d^2 + 1)$  and  $m = 1$  and  $T_1 = T \setminus \{z\} \cup \{x, y\}$  (referred to as the *Large-Profile Case*).

**The Standard Case.** Here, for each  $T_j$  there is at least one and at most  $\binom{|T_j|}{k} 2^{k+|T_j|} (k+1)$ -many profiles in  $\text{Profiles}(G_{i+1})$ , and we branch over all such profiles for each  $T_j$ . We obtain a set of profiles  $\tau_1, \dots, \tau_m$  where for each  $j \in [m]$ ,  $\tau_j = (T_j, P_j, M_j, \ell_j, Q_j)$  is a profile of  $G_{i+1}$  as  $|T_j| \leq k(d^2 + 1)$ . We only proceed if the following *consistency conditions* between  $\tau$  and  $\tau_1, \dots, \tau_m$  hold. First, we state conditions that are defined outside of  $x, y, z$ .

**CC1**  $P \setminus \{z\} = \bigcup_{i=1}^m P_j \setminus \{x, y\}$ .

**CC2**  $M \setminus \{z\} = \bigcup_{i=1}^m M_j \setminus \{x, y\}$ .

**CC3**  $\ell = \sum_{i=1}^m \ell_j$ .

**CC4**  $Q \setminus \{z\} = (\bigcup_{j=1}^m Q_j \setminus \{x, y\}) \cup B$ , where  $B$  is the set of clauses in  $T_{\text{cla}} \setminus \{z\}$  satisfied by  $T_{\text{var}} \setminus \{z\} \cup \{x, y\}$  via a black edge in  $G_{i+1}$ . Specifically, a clause  $c \in T_{\text{cla}} \setminus \{z\}$  belongs to  $B$  if there is a variable  $v \in \bigcup_{j=1}^m P_j$  such that  $vc \in E^+(G_{i+1})$ , or there is a variable  $v \in \bigcup_{j=1}^m (T_j)_{\text{var}} \setminus (P \setminus M)$  such that  $vc \in E^-(G_{i+1})$ .

Moreover, we put special conditions depending on the type of the contraction that obtained  $G_i$ . Recall that since we consider only bipartite contraction sequences,  $x, y, z$  are all vertices of the same type, either variables or clauses. Let  $a, b \in [m]$  be such that  $x \in T_a, y \in T_b$  ( $a$  and  $b$  are not necessarily distinct).

**CC5** If  $x, y, z$  are variable vertices,  $z \in P \setminus M$  if and only if  $x \in P_a \setminus M_a$  and  $y \in P_b \setminus M_b$ ;  $z \notin P$  if and only if  $x \notin P_a$  and  $y \notin P_b$ ; otherwise  $z \in M$ .

**CC6** If  $x, y, z$  are clause vertices,  $z \in Q$  if and only if  $x \in Q_a$  or is satisfied by  $T_{\text{var}}$  via a black edge in  $G_{i+1}$ , and  $y \in Q_b$  or is satisfied by  $T_{\text{var}}$  via a black edge in  $G_{i+1}$ .

We say that  $\tau_1, \dots, \tau_m$  are *consistent* with  $\tau$  if  $T_1, \dots, T_m$  are defined as above, and all of CC1–6 hold. Finally, we set

$$R_{G_i}(\tau) = \sum_{\substack{\tau_1, \dots, \tau_m \\ \text{consistent with } \tau}} \prod_{j=1}^m w(\tau_j). \quad (1)$$

Since for each  $i \in [m]$ ,  $w(\tau_j)$  is already computed in  $R_{G_{i+1}}(\tau_j)$ , we can indeed compute the sum above.

**The Large-Profile Case.** Here, we do not have profiles on  $T_1$  in  $\text{Profiles}(G_{i+1})$ , as  $|T_1| = |T| + 1 > k(d^2 + 1)$ . However, we can compute the record for  $\tau$  using suitable profiles for  $G_{i+1}$  of smaller size. We elaborate below.

We branch over the choice of a “profile”  $\tau_1 = (T_1, P_1, M_1, \ell_1, Q_1)$  consistent with  $\tau$ . (Strictly speaking, a profile must have smaller size, but the consistency conditions are defined irrespectively of that.) It is easy to see that there are only constantly many consistent profiles as it only remains to determine how the vertices  $x$  and  $y$  belong to the sets  $P_1$ ,  $M_1$  (if variables are contracted) or  $Q_1$  (if clauses are contracted). Let  $v$  be the vertex with the maximum distance over the red edges from the set  $P_1$  in  $T_1$ . By this choice,  $T' = T_1 \setminus \{v\}$  remains red-connected. We consider profiles of  $G_{i+1}$  of the form  $(T', P_1, M_1, \ell, Q')$ , where  $Q'$  is one of the suitable subsets of  $Q_1 \setminus \{v\}$  that will be specified later.

If  $v$  is a clause vertex, we only consider  $Q' = Q_1 \setminus \{v\}$ . We check whether  $v$  is satisfied by a black positive edge from  $P_1$ , or by a negative black edge from  $T'_{\text{var}} \setminus (P_1 \setminus M_1)$ , or by a red edge. In the latter case, we check explicitly whether variables that are contracted into the red neighborhood of  $v$  satisfy all clauses contracted to  $v$  when set to zero. Next, we check whether all clauses of  $v$  are satisfied if and only if  $v \in Q_1$ . If this equivalence holds, we add to  $R_{G_i}(\tau)$  the value  $R_{G_{i+1}}(\tau')$ , where  $\tau' = (T', P, M, \ell, Q \setminus \{v\})$ . If the checks above fail for all  $\tau_1$ , then the profile  $\tau$  is not realizable.

If  $v$  is a variable vertex, we go through all possible subsets  $Q' \subseteq Q_1$  and check whether the profile  $\tau' = (T', P, M, \ell, Q')$  is realizable. If so, then we also check whether all clauses in  $Q_1 \setminus Q'$  are satisfied and all clauses not in  $Q_1$  are not satisfied by setting the variables of  $v$  to 0. If all conditions hold, we add  $R_{G_{i+1}}(\tau')$  to  $R_{G_i}(\tau)$  computed so far.

At this point, it remains to argue the correctness of the computation in both cases  $(\star)$  and provide the claimed bound on the running time.

**Running time.** The number of profiles for  $G_i$  is bounded by  $s_i$ , and  $s_i \leq s_n$ . Let  $t = k(d^2 + 1)$  be the upper bound on the size of a profile. To compute the record for a single profile in  $G_i$ , we iterate through all possible tuples of consistent profiles. In the Large-Profile Case, we need to perform checks over all vertices contracted to a clause or a variable vertex, which can be done in time  $\mathcal{O}(n^2)$ . This could happen at most  $\mathcal{O}(2^t)$  times, since we might iterate through all possible subsets of  $Q$  in the profile. Thus we obtain the upper bound of  $\mathcal{O}(2^t n^2)$  for processing a single profile. In the Standard Case, the number of tuples of consistent profiles is upper-bounded by  $f(k, d) = \binom{t+1}{k} \cdot 2^{k+t+1} \cdot (k+1)^{d+2}$ . This holds since  $T_1, \dots, T_m$  is fixed for a particular profile  $\tau$  of  $G_i$ , and it only remains to decide the sets  $P_j, M_j, Q_j$ , and the values  $\ell_j$ . There are at most  $\binom{t+1}{k}$  choices for all sets  $P_j$  as all of the profiles are disjoint and contain at most  $t+1$  vertices, and at most  $k$  of them can belong to any of  $P_j$  in total. After fixing  $P_j$ , there are at most  $2^k$  choices for the sets  $M_j$  as for each  $j \in [m]$ ,  $M_j \subset P_j$ . At most  $2^{t+1}$  choices exist for deciding which clauses be part of the sets  $Q_j$ . Finally, for each  $j \in [m]$ ,  $0 \leq \ell_j \leq k$ , so the number of choices for  $\ell_j$  is upper-bounded by  $(k+1)^{d+2}$ .

Hence,  $R_{G_i}$  can be computed in time  $\mathcal{O}(s_i \cdot (f(k, d) + 2^t n^2)) = \mathcal{O}(s_n \cdot (f(k, d) + 2^t n^2))$ . Since the length of the contraction sequence is  $n - 2 \leq n$ , this results in the overall running time bound of  $\mathcal{O}(n \cdot s_n \cdot (f(k, d) + 2^t n^2)) = n^4 \cdot d^{\mathcal{O}(kd^2)}$ .  $\blacktriangleleft$

## 5 Tightness

In this section, we show that our main result (Theorem 1) is tight, in the sense that it is not possible to strengthen any of the parameterizations if one wishes to retain fixed-parameter tractability. Our hardness results even hold for BSAT, which merely asks whether a given CNF formula has a satisfying assignment that sets at most  $k$  variables to True.

1. BWMC parameterized by  $k$  plus the twin-width of the primal graph is  $W[2]$ -hard (Proposition 10).
2. BWMC parameterized by the signed twin-width alone is **paraNP**-hard (Proposition 11).
3. BWMC parameterized by  $k$  plus the unsigned twin-width is  $W[1]$ -hard (Proposition 13).

The proofs of these claims are provided in the full version.

► **Proposition 10** (★). *BSAT (and hence BWMC) parameterized by  $k$  is  $W[2]$ -hard, even when restricted to formulas whose primal graphs have twin-width zero and even assuming that an optimal contraction sequence is provided on the input.*

► **Proposition 11** (★). *BSAT (and hence BWMC) parameterized by the signed twin-width is **paraNP**-hard, even assuming that an optimal bipartite contraction sequence is provided on the input. The same also holds if we replace the signed twin-width with the vanilla twin-width of the incidence graph (i.e., where signs are ignored), or the twin-width of the primal graph.*

To establish the third claim, we first show that an edge subdivision of a clique has twin-width bounded by the size of the original clique. Note that we do not assume any additional properties of this subdivision – each edge could be subdivided an arbitrary number of times. We remark that this result provides an upper bound that complement the asymptotic lower bounds developed in earlier work on twin-width [2, Section 6].

► **Lemma 12** (★). *Let  $d \geq 2$  and  $G$  be a graph obtained by an arbitrary sequence of subdivisions of the edges in the complete graph  $K_d$ . Then the twin-width of  $G$  is at most  $d - 1$ .*

► **Proposition 13**. *BSAT (and hence BWMC) is  $W[1]$ -hard when parameterized by  $k$  plus the twin-width of the incidence graph, even assuming that an optimal contraction sequence for the incidence graph is provided on the input.*

**Proof.** A graph  $G$  is *balanced  $d$ -partite* if  $V(G)$  is the disjoint union of  $d$  sets  $V_1, \dots, V_d$ , each of the same size, such that no edge of  $G$  has both endpoints in the same set  $V_i$ . The following problem is well-known to be  $W[1]$ -complete when parameterized by  $d$  [19].

PARTITIONED CLIQUE

Input: A balanced  $d$ -partite graph  $G$ .  
 Question: Does  $G$  have a  $d$ -clique?

Let  $G$  be a balanced  $d$ -partite graph for  $d \geq 2$  with  $d$ -partition  $V_1, \dots, V_d$ . We write  $V_i = \{v_1^i, \dots, v_n^i\}$ , for  $1 \leq i \leq d$ . We construct a CNF formula  $F$  which has a satisfying assignment that sets  $\leq d$  variables to true if and only if  $G$  has a  $d$ -clique. As the variables of  $F$  we take the vertices of  $G$ .  $F$  consists of  $d + |E(G)|$  many clauses: For each  $1 \leq i \leq d$  we add the clause  $C_i = V_i$ , which asserts that at least one variable from each set  $V_i$  must be set to true. Each satisfying assignment that sets at most  $d$  variables to true will therefore set exactly one variable from each  $V_i$  to true,  $1 \leq i \leq d$ . For each pair of vertices  $u, v$  with  $u \in V_i$  and  $v \in V_j$  and  $uv \notin E(G)$  we add the clause  $C_{u,v} = (V_i \setminus \{u\}) \cup (V_j \setminus \{v\}) \cup \{-u, -v\}$ , which asserts that for each satisfying assignment that sets at most  $d$  variables to true, cannot set  $u$  and  $v$  to true if  $uv \notin E(G)$ . We conclude that, indeed  $F$  has a satisfying assignment that sets  $\leq d$  variables to true if and only if  $G$  has a  $d$ -clique.

It remains to show that the twin-width of the (unsigned) incidence graph  $I$  of  $F$  is bounded by a function of  $d$ . We observe that each set  $V_i$  and each set  $F_{i,j} = \{C_{u,v} \mid u \in V_i, v \in V_j, uv \notin E(G)\}$  is a module of  $I$ . Hence we can contract each of these modules into a single vertex, obtaining a new graph  $I'$  which is a subdivision of  $K_d$ . By Lemma 12,  $I'$  has (signed) twin-width at most  $d - 1$ . ◀

## 6 Experiments

In our experiments, we compute the signed twin-width, signed clique-width, and treewidth of several SAT instances generated with the tool CNFGen<sup>2</sup> and 100 instances from the uniform random benchmark set *uf20-91*<sup>3</sup>. We used the respective SAT-encodings [16, 18, 24, 26] to compute the widths for the instances. Signed clique-width and treewidth were chosen because of being prominent and general examples of structural parameters which yield the fixed-parameter tractability of SAT and model counting [23].

We adapted the encoding for twin-width to signed graphs of formulas and added an improvement that can compute the signed twin-width of formulas with larger incidence graphs. In particular, the SAT encoding uses variables and clauses to represent all possible contraction steps. With bipartite contraction sequences, it is possible to reduce the total number of admissible steps significantly. On a high level, we observed a significant performance improvement due to not needing to consider all possible contraction steps.

We used two types of instances in our experiments: hard instances from *proof complexity* and *random k-SAT* instances. We generated all families of formulas available with CNFGen such that the resulting signed graph had at most 125 vertices. Where available, we varied the parameters to see how the widths develop with the change in the parameter. We generated most of the random *k-SAT* instances with CNFGen; all of these instances use 15 variables and a varying number of literals per clause and clauses. For each choice of parameters, we generated 10 different instances and report the average. Apart from our generated instances, we use the publicized instance set *uf20-91*, which contains 1000 random 3-SAT instances with 20 variables and 91 clauses, of which we randomly selected 100 instances.

Table 2 shows the results of our experiments. For signed twin-width, we start from a greedy upper bound and can therefore provide an upper bound whenever we cannot compute or verify the exact value of signed twin-width.

The results show that the signed twin-width never exceeds the treewidth. Interestingly, it turned out that computing the signed clique-width using the best known SAT-encoding [18] is much harder than computing the signed twin-width. Therefore, we can compare the two widths only on very few instances, and the signed twin-width is smaller on all these instances.

## 7 Concluding Remarks

We have provided an exhaustive investigation of how twin-width can be used in SAT solving and model counting and have developed the notion of signed twin-width for formulas. Our complexity-theoretic results follow up on the classical line of research that investigates the complexity of SAT and its extensions from the viewpoint of variable-clause interactions. On the empirical side, we have computed the exact signed twin-width of several formulas and compared these values to those of treewidth and signed clique-width.

In future work, it would be interesting to investigate whether there is a structural parameter that generalizes signed clique-width and can yield fixed-parameter algorithms for SAT alone; in the case of twin-width, we show that this is not possible (Proposition 11). Moreover, while the fixed-parameter tractability established in Theorem 1 should be viewed as a classification result, it would be interesting to see whether the ideas developed there could be used to inspire improvements to existing heuristics for SAT.

<sup>2</sup> <https://massimolauria.net/cnfgen/>

<sup>3</sup> <https://www.cs.ubc.ca/~hoos/SATLIB/benchm.html>

■ **Table 2** Experimental results comparing signed twin-width (tww), signed clique-width (cw) and treewidth (tw). Random instances are named such that the first number is the number of variables, the second the number of literals per clause, and the third the number of clauses. For random instances, each row represents a set of 10 generated instances. Signed twin-width values marked with an \* are only an upper bound.

|                  | Instance     | $ V $  | $ E $ | Signed tww | Signed cw | tw   |
|------------------|--------------|--------|-------|------------|-----------|------|
| Proof Complexity | cliquecolor5 | 130    | 225   | *6.0       | -         | -    |
|                  | count4       | 22     | 36    | 3.0        | 7.0       | 4.0  |
|                  | count5       | 45     | 80    | 4.0        | -         | 7.0  |
|                  | count6       | 81     | 150   | *5.0       | -         | 10.0 |
|                  | matching4    | 22     | 36    | 3.0        | 7.0       | 4.0  |
|                  | matching5    | 45     | 80    | 4.0        | -         | 7.0  |
|                  | matching6    | 81     | 150   | *5.0       | -         | 10.0 |
|                  | order4       | 46     | 96    | 5.0        | -         | 9.0  |
|                  | order5       | 95     | 220   | *7.0       | -         | -    |
|                  | parity5      | 45     | 80    | 4.0        | -         | 7.0  |
|                  | parity6      | 81     | 150   | *5.0       | -         | 10.0 |
|                  | parity7      | 133    | 252   | *6.0       | -         | -    |
|                  | pidgeon4     | 34     | 48    | 4.0        | 6.0       | 4.0  |
|                  | pidgeon5     | 65     | 100   | *4.0       | -         | 5.0  |
|                  | subsetcard9  | 121    | 252   | *6.0       | -         | -    |
|                  | tsetin9      | 90     | 288   | 4.0        | -         | 7.0  |
|                  | tsetin10     | 100    | 320   | 4.0        | -         | 9.0  |
| tsetin11         | 110          | 352    | 4.0   | -          | 7.0       |      |
| $k$ -Random      | uf15_2_15    | 28.0   | 30.0  | *3.0       | 4.0       | 2.0  |
|                  | uf15_2_30    | 45.0   | 60.0  | *3.8       | 6.2       | 4.0  |
|                  | uf15_2_45    | 60.0   | 90.0  | *4.0       | -         | 6.3  |
|                  | uf15_2_65    | 80.0   | 130.0 | *4.2       | -         | 7.6  |
|                  | uf15_2_80    | 95.0   | 160.0 | *5.0       | -         | 8.5  |
|                  | uf15_2_95    | 110.0  | 190.0 | *5.0       | -         | 9.4  |
|                  | uf15_2_110   | 125.0  | 220.0 | *5.0       | -         | 9.8  |
|                  | uf15_3_15    | 30.0   | 45.0  | 3.4        | 7.1       | 4.9  |
|                  | uf15_3_30    | 45.0   | 90.0  | *4.9       | -         | 8.0  |
|                  | uf15_3_45    | 60.0   | 135.0 | *5.0       | -         | 10.0 |
|                  | uf15_3_65    | 80.0   | 195.0 | *6.0       | -         | 11.5 |
|                  | uf15_3_80    | 95.0   | 240.0 | *7.0       | -         | 11.7 |
|                  | uf15_3_95    | 110.0  | 285.0 | *7.0       | -         | 12.5 |
|                  | uf15_3_110   | 125.0  | 330.0 | *7.2       | -         | 12.5 |
|                  | uf15_5_15    | 30.0   | 75.0  | 5.0        | 11.4      | 9.6  |
|                  | uf15_5_30    | 45.0   | 150.0 | 6.9        | -         | 12.3 |
|                  | uf15_5_45    | 60.0   | 225.0 | *7.9       | -         | 13.2 |
|                  | uf15_5_65    | 80.0   | 325.0 | *9.0       | -         | 13.8 |
|                  | uf15_5_80    | 95.0   | 400.0 | *9.3       | -         | 14.0 |
|                  | uf15_5_95    | 110.0  | 475.0 | *10.0      | -         | 14.0 |
|                  | uf15_5_110   | 125.0  | 550.0 | *10.0      | -         | 14.0 |
|                  | uf15_7_15    | 30.0   | 105.0 | 6.0        | 14.1      | 12.6 |
|                  | uf15_7_30    | 45.0   | 210.0 | 8.0        | -         | 13.8 |
|                  | uf15_7_45    | 60.0   | 315.0 | *9.0       | -         | 14.0 |
|                  | uf15_7_65    | 80.0   | 455.0 | *10.0      | -         | 14.0 |
|                  | uf15_7_80    | 95.0   | 560.0 | *11.0      | -         | 14.0 |
|                  | uf15_7_95    | 110.0  | 665.0 | *11.0      | -         | 14.0 |
|                  | uf15_7_110   | 125.0  | 770.0 | *11.5      | -         | 14.0 |
|                  | uf15_10_15   | 30.0   | 150.0 | 6.7        | 15.3      | 13.9 |
|                  | uf15_10_30   | 45.0   | 300.0 | 9.0        | -         | 14.0 |
|                  | uf15_10_45   | 60.0   | 450.0 | *10.0      | -         | 14.0 |
|                  | uf15_10_65   | 80.0   | 650.0 | *11.0      | -         | 14.0 |
|                  | uf15_10_80   | 95.0   | 800.0 | *12.0      | -         | 14.0 |
| uf15_10_95       | 110.0        | 950.0  | *12.0 | -          | 14.0      |      |
| uf15_10_110      | 125.0        | 1100.0 | *12.0 | -          | 14.0      |      |
| uf20_3_91        | 111.0        | 273.0  | *7.7  | -          | 14.5      |      |

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