QBF Merge Resolution Is Powerful but Unnatural

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Abstract

The Merge Resolution proof system (M-Res) for QBFs, proposed by Beyersdorff et al. in 2019, explicitly builds partial strategies inside refutations. The original motivation for this approach was to overcome the limitations encountered in long-distance Q-Resolution proof system (LD-Q-Res), where the syntactic side-conditions, while prohibiting all unsound resolutions, also end up prohibiting some sound resolutions. However, while the advantage of M-Res over many other resolution-based QBF proof systems was already demonstrated, a comparison with LD-Q-Res itself had remained open. In this paper, we settle this question. We show that M-Res has an exponential advantage over not only LD-Q-Res, but even over LQU^+-Res and IRM, the most powerful among currently known resolution-based QBF proof systems. Combining this with results from Beyersdorff et al. 2020, we conclude that M-Res is incomparable with LQU-Res and LQU^+-Res.

Our proof method reveals two additional and curious features about MRes: (i) M-Res is not closed under restrictions, and is hence not a natural proof system, and (ii) weakening axiom clauses with existential variables provably yields an exponential advantage over MRes without weakening. We further show that in the context of regular derivations, weakening axiom clauses with universal variables provably yields an exponential advantage over M-Res without weakening. These results suggest that M-Res is better used with weakening, though whether M-Res with weakening is closed under restrictions remains open. We note that even with weakening, M-Res continues to be simulated by eFrege + $\forall$red (the simulation of ordinary M-Res was shown recently by Chew and Slivovsky).

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1 Introduction

Testing satisfiability of CNF formulas (the propositional SAT problem) is NP-complete and is hence believed to be hard in the worst case. Despite this, modern SAT solvers routinely solve industrial SAT instances with hundreds of thousands or even millions of variables in close to linear time [31, 13, 24]. Recently some mathematics problems, some of which were open for almost a century, have been solved by employing SAT solvers (see [21] for a survey). This apparent disconnect between theory and practice has led to a more detailed study of the different solving techniques.

Most successful SAT solvers use a non-deterministic algorithm called conflict-driven clause learning (CDCL) [28, 25], which is inspired by and an improvement of the DPLL algorithm [18, 17]. The solvers use some heuristics to make deterministic or randomized choices for the non-deterministic steps of the CDCL algorithm. The CDCL algorithm (and the resulting solvers) can be studied by analysing a proof system called resolution. Resolution contains a single inference rule, which given clauses $x \vee A$ and $\overline{x} \vee B$, allows the derivation of clause $A \vee B$ [11, 26]. To be more precise, from a run of the CDCL algorithm (or a solver) on an unsatisfiable formula, resolution refutations of the same length (as the run of the solver) can be extracted. This means that refutation size lower bounds on resolution translate to runtime lower bounds for the CDCL algorithm and the solvers based on it. See [24] for more on CDCL based SAT solvers and [13] for their connection to resolution.

With SAT solvers performing so well, the community has set sights on solving Quantified Boolean formulas (QBFs). Some of the variables in QBFs are quantified universally, allowing a more natural and succinct encoding of many constraints. As a result, QBF solving has many more practical applications (see [27] for a survey). However, it is PSPACE-complete [29] and hence believed to be much harder than SAT.

The main way of tackling QBFs is by adapting resolution (and CDCL based solvers) to handle universal variables. There are two major ways of doing this, which have given rise to two orthogonal families of proof systems. Reduction-based systems allow dropping a universal variable from a clause if some conditions are met – proof systems Q-Res and QU-Res [23, 20] are of this type. In contrast, expansion-based systems eliminate universal variables at the outset by expanding the universal quantifiers into conjunctions, giving a purely propositional formula – proof systems $\forall\mathrm{Exp}+\mathrm{Res}$ and IR [22, 9] are of this type. It was soon observed that, under certain conditions, producing a clause containing a universal variable in both polarities (to be interpreted in a special way, not as a tautology) is not only sound but also very useful for making proofs shorter [33, 19]. This led to new proof systems of both types: reduction-based systems LD-Q-Res, LQU-Res and LQU$^+$-Res [2, 3], and expansion-based system IRM [9].

Since all these proof systems degenerate to resolution on propositional formulas, lower bounds for resolution continue to hold for these systems as well. However such lower bounds do not tell us much about the relative powers and weaknesses of these systems. QBF proof complexity aims to understand this. This is done by finding formula families which have polynomial-size refutations in one system but require super-polynomial size refutations in the other system. For example, among the reduction-based and expansion-based resolution systems, LQU$^+$-Res and IRM respectively are the most powerful and are known to be incomparable [3, 9].

In this paper, we study a proof system called Merge Resolution (M-Res). This system was proposed in [6] with the goal of circumventing a limitation of LD-Q-Res. The main feature of this system is that each line of the refutation contains information about partial strategies for the universal player in the standard two-player evaluation game associated with QBFs. These strategies are built up as the proof proceeds. The information about these partial strategies allows some resolution steps which are blocked in LD-Q-Res. This makes M-Res very powerful – it has short refutations for formula families requiring exponential-size refutations in Q-Res, QU-Res, $\forall\mathrm{Exp}+\mathrm{Res}$, and IR, and also in the system $\mathrm{CP}+\forall\mathrm{red}$ introduced in [10]. However, the authors of [6] did not show any advantage over LD-Q-Res – the system that M-Res was designed to improve. They only showed advantage over a restricted version of LD-Q-Res, the system reductionless LD-Q-Res. In a subsequent paper [7], limitations of M-Res were shown – there are formula families which have polynomial-size refutations in QU-Res, LQU-Res, LQU$^+$-Res and $\mathrm{CP}+\forall\mathrm{red}$, but require exponential size refutations in M-Res. This, combined with the results from [6], showed that M-Res is incomparable with QU-Res and $\mathrm{CP}+\forall\mathrm{red}$. More recently, it has been shown that $\mathrm{eFrege}+\forall\mathrm{red}$ proof system p-simulates M-Res [15]. On the solving side, M-Res has recently been used to build a solver, though with a different representation for strategies [12].

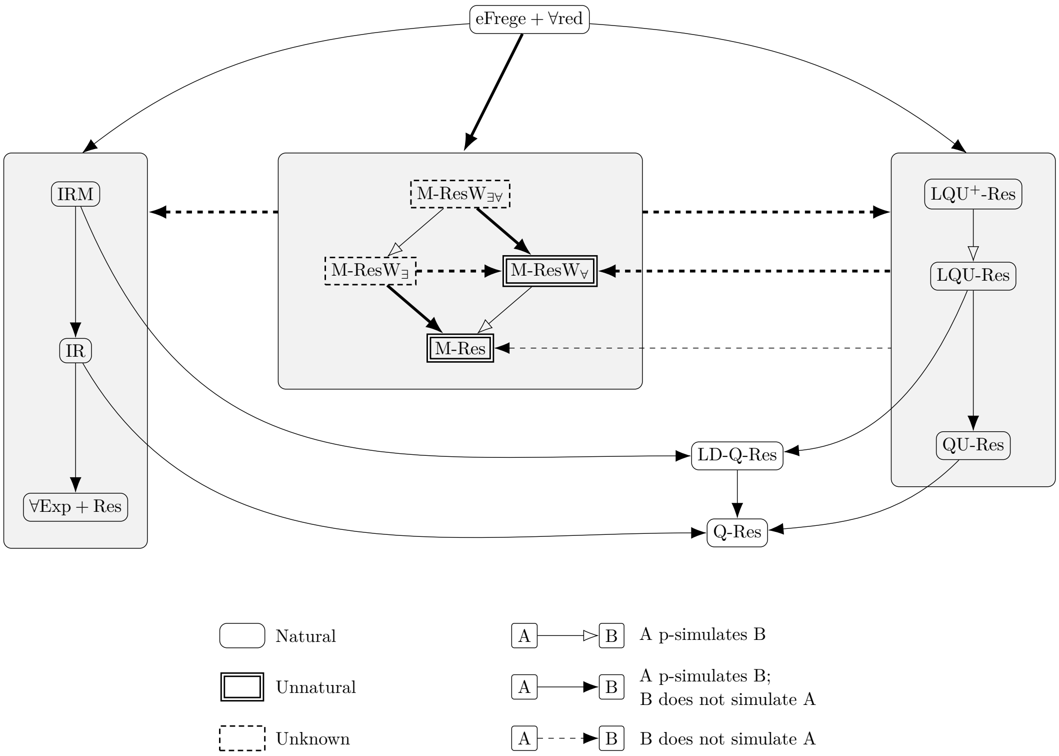


Figure 1 Relations among resolution-based QBF proof systems, with new results and observations highlighted using thicker lines. In addition, regular M-ResW$_\forall$ strictly p-simulates regular M-Res. (i) Lines from a big grey box mean that the line is from every proof system within the box. (ii) The missing relations follow from transitivity, otherwise the systems are incomparable.

In this paper, we show that M-Res is indeed quite powerful, answering one of the main questions left open in [6]. We show that there are formula families which have polynomial-size refutations in M-Res but require exponential-size refutations in LD-Q-Res. In fact, we show that there are formula families having short refutations in M-Res but requiring exponential-size refutations in LQU$^+$-Res and IRM – the most powerful resolution-based QBF proof systems. Combining this with the results in [7], we conclude that M-Res is incomparable with LQU-Res and LQU$^+$-Res; see Theorems 3.6 and 3.12.

The power of M-Res is shown using modifications of two well-known formula families: KBKF-lq [3] which is hard for M-Res [7], and QUParity [9] which we believe is also hard. The main observation is that the reason making these formulas hard for M-Res is the mismatch of partial strategies at some point in the refutation. This mismatch can be eliminated if the formulas are modified appropriately. The resultant formulas, called KBKF-lq-split and MParity, have polynomial-size refutations in M-Res but require exponential-size refutations in IRM and LQU$^+$-Res respectively.

We observe that the modification of KBKF-lq is actually a weakening of the clauses. This leads to an observation that weakening adds power to M-Res. Weakening is a rule that is sometimes augmented to resolution. This rule allows the derivation of $A \vee x$ from A, provided that A does not contain the literal $\overline{x}$. The weakening rule is mainly used to make resolution refutations more readable – it can not make them shorter [1]. The same holds for all the known resolution-based QBF proof systems with the exception of M-Res – allowing weakening can make M-Res refutations exponentially shorter. We distinguish between two types of weakenings, namely existential clause weakening and strategy weakening. Both these weakenings were defined in the original paper [6] in which M-Res was introduced. However, these weakenings were used only for Dependency-QBFs (DQBFs); in that setting they are necessary for completeness. The potential use of weakening for QBFs was not explicitly addressed. Here, we show that existential clause weakening adds exponential power to M-Res; see Theorem 4.1. We do not know whether strategy weakening adds power to M-Res. However, we show that it does add exponential power to regular M-Res; see Theorem 4.5. At the same time, weakening of any or both types does not make M-Res unduly powerful; we show in Theorem 4.13 that eFrege + ∀red polynomially simulates (p-simulates) M-Res even with both types of weakenings added. This is proven by observing that the p-simulation of M-Res in [15] can very easily be extended to handle weakenings.

Another observation from our main result is that M-Res is not closed under restrictions. Closure under restrictions is a very important property of proof systems. For a (QBF) proof system, it means that restricting a false formula by a partial assignment to some of the (existential) variables does not make the formula much harder to refute. Note that a refutation of satisfiability of a formula implicitly encodes a refutation of satisfiability of all its restrictions, and it is reasonable to expect that such refutations can be extracted without paying too large a price. This is indeed the case for virtually all known proof systems to date. Algorithmically, CDCL-based solvers work by setting some variables and simplifying the formula [24]. Without closure under restrictions, setting a bad variable may make the job of refuting the formula exponentially harder. Because of this reason, proofs systems which are closed under restrictions have been called *natural proof systems* [4]. We show in Theorem 4.14 that M-Res, with and without strategy weakening, is unnatural. We believe this would mean that it is hard to build QBF solvers based on it. On the other hand, we do not yet know whether it remains unnatural if existential clause weakening or both types of weakenings are added. We believe that this is the most important open question about M-Res – a negative answer can salvage it.

Our results are summarized in Figure 1.

2 Preliminaries

The sets $\{1, 2, \ldots, n\}$ and $\{m, m+1, \ldots, n\}$ are abbreviated as $[n]$ and $[m, n]$ respectively. A literal is a variable or its negation; a clause is a disjunction of literals. We will interchangeably denote clauses as disjunctions of literals as well as sets of literals.

A Quantified Boolean Formula (QBF) in prenex conjunctive normal form (p-cnf), denoted $\Phi = \mathcal{Q}.\phi$, consists of two parts: (i) a quantifier prefix $\mathcal{Q} = Q_1 Z_1, Q_2 Z_2, \ldots, Q_n Z_n$ where the Z_i are pairwise disjoint sets of variables, each $Q_i \in \{\exists, \forall\}$, and $Q_i \neq Q_{i+1}$; and (ii) a conjunction of clauses ϕ with variables in $Z = Z_1 \cup \cdots \cup Z_n$. In this paper, when we say QBF, we mean a p-cnf QBF. The set of existential (resp. universal) variables of Φ, denoted X (resp. U), is the union of Z_i for which $Q_i = \exists$ (resp. $Q_i = \forall$).

The semantics of a QBF is given by a two-player evaluation game played on the QBF. In a run of the game, the existential player and the universal player take turns setting the existential and the universal variables respectively in the order of the quantification prefix. The existential player wins the run of the game if every clause is set to true. Otherwise the universal player wins. The QBF is true (resp. false) if and only if the existential player (resp. universal player) has a strategy to win all potential runs, i.e. a winning strategy.

For a formula Φ and a partial assignment ρ to some of its variables, $\Phi\restriction_\rho$ denotes the restricted formula resulting from setting the specified variables according to ρ.

▶ **Definition 2.1.** *A QBF proof system P is* closed under restrictions *if for every false QBF Φ and every partial assignment ρ to some existential variables, the size of the smallest P-refutation of $\Phi\lceil_\rho$ is at most polynomial in the size of the smallest P-refutation of Φ.*

▶ Remark 2.2. Sometimes a stricter definition is used, requiring that a refutation of $\Phi\lceil_\rho$ be constructible in polynomial time from every refutation of Φ. We will prove that M-Res is not closed under restrictions for the weaker definition (and hence also for the stricter definition).

▶ **Definition 2.3** ([4])**.** *A proof system is* natural *if it is closed under restrictions.*

Merge Resolution

Merge Resolution (M-Res) is a proof system for refuting false QBFs. Its definition is a bit technical and can be found in [6]. We give an informal description, see [7] for a slightly longer but still informal description.

Each line of an M-Res refutation consists of an ordered pair – the first element of the pair is a clause C over the existential variables; and the second part is a set of functions $\{M^u \mid u \in U\}$, one for each universal variable. The function for a particular universal variable u takes as input an assignment to the existential variables to the left of u in the quantifier prefix and outputs an assignment for u. Each function is stored as some deterministic branching program [32] (also called merge-map here) computing the function.

In a refutation, each axiom A is converted into a clause, merge-map pair where C is the maximal sub-clause of A with existential variables, and the merge-maps falsify each universal literal in A (constant merge-maps) and leave other universal variables unassigned (trivial merge-maps). Given two lines $L_1 = (x \vee A, \{M_1^u \mid u \in U\})$ and $L_2 = (\overline{x} \vee B, \{M_2^u \mid u \in U\})$, we can derive $(A \vee B, \{M_3^u \mid u \in U\})$ provided the merge-maps M_1^u and M_2^u satisfy certain conditions. These conditions specify that for each u preceding x in the prefix, either at least one of M_1^u or M_2^u is trivial, or M_1^u and M_2^u are isomorphic (and hence compute the same function). If the rule is applicable, the resultant merge-map M_3^u is a combination of M_1^u and M_2^u. To be precise, for each $u \in U$,

- if M_1^u is trivial, then $M_3^u = M_2^u$;
- else if M_2^u is trivial, then $M_3^u = M_1^u$;
- else if M_1^u is isomorphic to M_2^u, then $M_3^u = M_1^u$;
- else if x precedes u in the quantifier prefix, then M_3^u is the following branching program: if $x = 0$, then go to M_1^u, otherwise go to M_2^u. (In this, if M_1^u and M_2^u have some nodes in common, these nodes are not duplicated in M_3^u. This makes M_3^u a branching program rather than a decision tree, and prevents an exponential blowup).

The following invariant is maintained at each line $L_i = (C_i, \{M_i^u \mid u \in U\})$ of the refutation: for every existential assignment α falsifying C_i, if $\beta = \{u = M_i^u(\alpha) \mid u \in U\}$, then the assignment $\alpha \cup \beta$ falsifies at least one axiom [6]. The end goal is to derive a line whose existential clause part is $\square$, i.e. the empty clause – implying that the given QBF is false.

For concreteness, we reproduce a simple example from [6]:

▶ **Example 2.4.** For the QBF family
$\exists x, \forall u, \exists t.\, (x \vee u \vee t) \wedge (\overline{x} \vee \overline{u} \vee t) \wedge \left(x \vee u \vee \overline{t}\right) \wedge \left(\overline{x} \vee \overline{u} \vee \overline{t}\right)$, here is an M-Res refutation.

$$\dfrac{\dfrac{x \vee t, u=0 \qquad \overline{x} \vee t, u=1}{t, u=x} \qquad \dfrac{x \vee \overline{t}, u=0 \qquad \overline{x} \vee \overline{t}, u=1}{\overline{t}, u=x}}{\square, u=x}$$

(To be pedantic, each line should contain the merge-map for u. For simplicity, we avoid it here, describing only the function computed by the merge-map.)

3 Power of Merge Resolution

In this section, we prove that neither IRM nor LQU$^+$-Res simulates M-Res.

3.1 Advantage over IRM

To show that M-Res is not simulated by IRM, we use a variant of the well-studied KBKF formula family. This family was first introduced in [23], and along with multiple variants, has been a very influential example in showing many separations. In particular, it was used to prove that LD-Q-Res is strictly stronger than Q-Res [19]. The variant KBKF-lq was defined in [3] and used to show that LD-Q-Res does not simulate QU-Res. In [7], KBKF-lq was also shown to require exponentially large M-Res refutations. We reproduce the definition of this family below and then define two further variants that will be useful for our purpose.

▶ **Definition 3.1** ([3]). *KBKF-lq$[n]$ is the QBF with the quantifier prefix $\exists d_1, e_1, \forall x_1, \ldots, \exists d_n, e_n, \forall x_n, \exists f_1, \ldots, f_n$ and with the following clauses:*

$$\begin{array}{lll}
A_0 = \{\overline{d_1}, \overline{e_1}, \overline{f_1}, \ldots, \overline{f_n}\} & & \\
A_i^d = \{d_i, x_i, \overline{d_{i+1}}, \overline{e_{i+1}}, \overline{f_1}, \ldots, \overline{f_n}\} & A_i^e = \{e_i, \overline{x_i}, \overline{d_{i+1}}, \overline{e_{i+1}}, \overline{f_1}, \ldots, \overline{f_n}\} & \forall i \in [n-1] \\
A_n^d = \{d_n, x_n, \overline{f_1}, \ldots, \overline{f_n}\} & A_n^e = \{e_n, \overline{x_n}, \overline{f_1}, \ldots, \overline{f_n}\} & \\
B_i^0 = \{x_i, f_i, \overline{f_{i+1}}, \ldots \overline{f_n}\} & B_i^1 = \{\overline{x_i}, f_i, \overline{f_{i+1}}, \ldots \overline{f_n}\} & \forall i \in [n-1] \\
B_n^0 = \{x_n, f_n\} & B_n^1 = \{\overline{x_n}, f_n\} &
\end{array}$$

We now define two new formula families: KBKF-lq-weak and KBKF-lq-split.

▶ **Definition 3.2.** *KBKF-lq-weak has the same quantifier prefix as KBKF, and all the A-clauses of KBKF-lq, but it has the following clauses instead of B_i^0 and B_i^1:*

$$\left.\begin{array}{ll}
\textit{weak-}B_i^0 & = d_i \vee B_i^0 \\
\textit{weak-}B_i^1 & = \overline{d_i} \vee B_i^1
\end{array}\right\} \qquad \forall i \in [n]$$

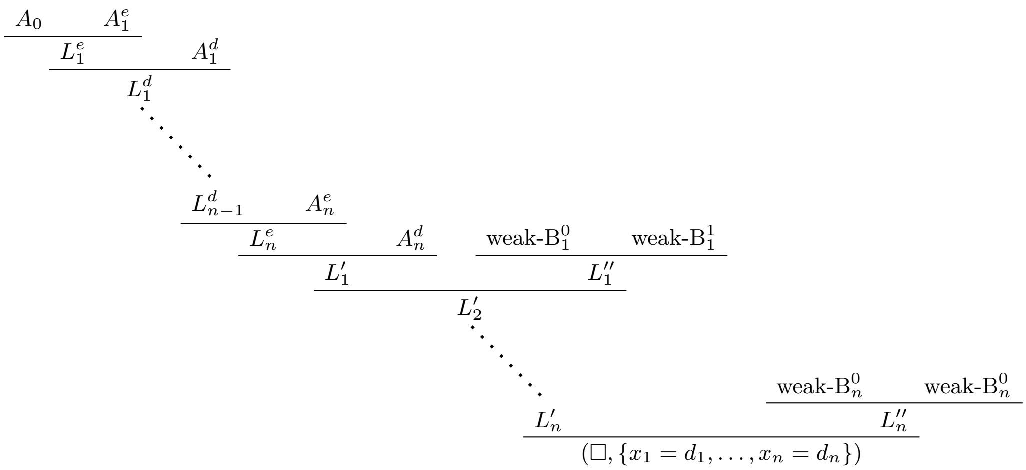
▶ **Definition 3.3.** *KBKF-lq-split has all variables of KBKF-lq and one new variable t quantified existentially in the first block, so its quantifier prefix is $\exists t, \exists d_1, e_1, \forall x_1, \ldots, \exists d_n, e_n, \forall x_n, \exists f_1, \ldots, f_n$. It has all the A-clauses of KBKF-lq, but the following clauses instead of B_i^0 and B_i^1:*

$$\left.\begin{array}{ll}
\textit{split-}B_i^0 & = t \vee B_i^0 \\
\textit{split-}B_i^1 & = t \vee B_i^1 \\
T_i^0 & = \{\bar{t}, d_i\} \\
T_i^1 & = \{\bar{t}, \overline{d_i}\}
\end{array}\right\} \qquad \forall i \in [n]$$

▶ **Lemma 3.4.** *KBKF-lq-weak has polynomial-size M-Res refutations.*

Proof. Let L_i'' denote the M-Res-resolvent of weak-B$_i^0$ and weak-B$_i^1$. It has only one non-trivial merge-map, setting $x_i = d_i$. Starting with A_0, resolve in sequence with A_1^e, A_1^d, A_2^e, A_2^d, and so on up to A_n^e, A_n^d to derive the line with all negated f literals and merge-maps computing $x_i = d_i$ for each i. Now sequentially resolve this with L_1'', L_2'', up to L_n'' to obtain the empty clause. It can be verified that none of these resolutions are blocked, and the final merge-maps compute the winning strategy $x_i = d_i$ for each i.

The refutation is pictorially depicted below. The abbreviations A_0, A_i^d, weak-B$_i^0$ etc. will denote the clause, merge-map pair corresponding to the respective axioms.



Here $L_i^e, L_i^d, L_i', L_i''$, for all $i \in [n]$, are the following lines, with only non-trivial merge-maps written explicitly:

- $L_1^e = \left(\left\{\overline{d_1}, \overline{d_2}, \overline{e_2}, \overline{f_1}, \ldots, \overline{f_n}\right\}, \{x_1 = 1\}\right)$
- $L_i^e = \left(\left\{\overline{d_i}, \overline{d_{i+1}}, \overline{e_{i+1}}, \overline{f_1}, \ldots, \overline{f_n}\right\}, \{x_1 = d_1, \ldots, x_{i-1} = d_{i-1}, x_i = 1\}\right)$ for all $i \in [2, n-1]$
- $L_n^e = \left(\left\{\overline{d_n}, \overline{f_1}, \ldots, \overline{f_n}\right\}, \{x_1 = d_1, \ldots, x_{n-1} = d_{n-1}, x_n = 1\}\right)$
- $L_i^d = \left(\left\{\overline{d_{i+1}}, \overline{e_{i+1}}, \overline{f_1}, \ldots, \overline{f_n}\right\}, \{x_1 = d_1, \ldots, x_i = d_i\}\right)$ for all $i \in [n-1]$
- $L_i' = \left(\left\{\overline{f_i}, \ldots, \overline{f_n}\right\}, \{x_1 = d_1, \ldots, x_n = d_n\}\right)$ for all $i \in [n]$
- $L_i'' = \left(\left\{f_i, \overline{f_{i+1}}, \ldots \overline{f_n}\right\}, \{x_i = d_i\}\right)$ for all $i \in [n-1]$
- $L_n'' = (\{f_n\}, \{x_n = d_n\})$ ◀

Observe that the extra d_i variable in weak-B_i^0 and weak-B_i^1 (in contrast to B_i^0 and B_i^1 in KBKF-lq) allows us to resolve these two lines. This gives the clause L_i'' whose merge-map computes $x_i = d_i$. This merge-map is isomorphic to the merge-map for x_i in the line derived by resolving the A_i lines. This isomorphism allows the polynomial-size refutation.

▶ **Lemma 3.5.** *KBKF-lq-split has polynomial-size M-Res refutations.*

Proof. For each $i \in [n]$ and $k \in \{0, 1\}$, resolving split-B_i^k and T_i^k yields weak-B_i^k. This gives us the KBKF-lq-weak formula family which, as shown in Lemma 3.4, has polynomial-size M-Res refutations. ◀

▶ **Theorem 3.6.** *IRM does not simulate M-Res.*

Proof. The KBKF-lq-split formula family witnesses the separation. By Lemma 3.5, it has polynomial size M-Res refutations. Restricting it by setting $t = 0$ gives the family KBKF-lq, which requires exponential size to refute in IRM, [9]. Since IRM is closed under restrictions (Lemma 11 in [9]), KBKF-lq-split also requires exponential size to refute in IRM. ◀

3.2 Advantage over LD-Q-Res, LQU-Res and LQU$^+$-Res

To show that LQU$^+$-Res does not simulate M-Res, we define a new formula family called MParity, as a modification of the QParity formula family [9].

Let us first give a brief history of QParity and other formulas based on it. This was first defined in [9] and was used to show that Q-Res does not p-simulate $\forall$Exp + Res [9] and LD-Q-Res [16]. (A subsequent elegant argument in [5] reproved its hardness for QU-Res and CP + $\forall$red.) The variant LQParity, also defined in [9], was used to show that LD-Q-Res does not p-simulate $\forall$Exp + Res. Finally, the variant QUParity, built by duplicating the universal variable of LQParity, was used to show that LQU$^+$-Res does not p-simulate $\forall$Exp + Res.

We give the definition of QParity, informally describe the variants LQParity and QUParity, and then define our new variant MParity. We use $\text{parity}^c(y_1, y_2, \ldots, y_k)$ as a shorthand for the following conjunction of clauses: $\bigwedge_{S\subseteq[k],\,|S|\equiv 1\,(\text{mod }2)}((\vee_{i\in S}\overline{y_i}) \vee (\vee_{i\notin S} y_i))$. Thus $\text{parity}^c(y_1, y_2, \ldots, y_k)$ is satisfied by assignment $a_1, \ldots, a_k$ iff $a_1 + a_2 + \cdots + a_k \equiv 0 \pmod 2$.

▶ **Definition 3.7.** *$QParity_n$ is the following QBF:*

$$\exists x_1, \ldots, x_n, \forall z, \exists t_1, \ldots, t_n.\ \left(\bigwedge_{i\in[n+1]} \zeta_i\right)$$

where each ζ_i contains the following clauses:

- *For $i = 1$, each $C \in \text{parity}^c(x_1, t_1)$.*
- *For all $i \in [2, n]$, each $C \in \text{parity}^c(t_{i-1}, x_i, t_i)$.*
- *For $i = n + 1$, the clauses $\{t_n, z\}$ and $\{\overline{t_n}, \overline{z}\}$.*

With the same quantifier prefix, replacing each clause C of QParity that does not contain z with the two clauses $C \vee z$ and $C \vee \overline{z}$ gives the family LQParity.

To obtain QUParity, the universal variable is duplicated. That is, the block $\forall z$ is replaced with the block $\forall z_1, z_2$. Each clause of the form $C \cup \{z\}$ in LQParity is replaced with the clause $C \cup \{z_1, z_2\}$, and each clause of the form $C \cup \{\overline{z}\}$ is replaced with the clause $C \cup \{\overline{z_1}, \overline{z_2}\}$.

The short LD-Q-Res refutation of QParity (from [16, p. 54]) relies on the fact that most axioms do not have universal variable z. This enables steps in which a merged literal z^* is present in one antecedent but there is no literal over z in the other antecedent. LQParity is created from QParity by replacing each clause C not containing z by two clauses $C \vee z$ and $C \vee \overline{z}$. Since, every axiom of LQParity (and hence also each derived clause) now has a literal over z, we can no longer resolve clauses containing the merged literal z^* with any other clause. This forbids the creation of merged literals, which in turn, forbids all possible short refutations. The same problem seems to occur in M-Res also – though M-Res allows resolution steps if the merge-maps are isomorphic, we do not know of any way of making them isomorphic. This leads us to define the new variant MParity. We notice that if the formula family is modified appropriately, we can indeed make the merge-maps isomorphic, and additionally throwing in the modifications of LQParity and QUParity does not destroy this feature. This leads us to define the modified family MParity.

▶ **Definition 3.8.** *$MParity_n$ is the following QBF:*

$$\underset{i,j\in[n]}{\exists} a_{i,j}, \exists x_1, \ldots, x_n, \forall z_1, z_2, \exists t_1, \ldots, t_n.\ \left(\bigwedge_{i\in[n+1]} \psi_i\right)$$

where each ψ_i contains the following clauses:

- *For $i = 1$, for all $C \in \text{parity}^c(x_1, t_1)$, the clauses $A^0_{1,C} = C \cup \{z_1, z_2, a_{1,n}\}$ and $A^1_{1,C} = C \cup \{\overline{z_1}, \overline{z_2}, a_{1,n}\}$*
- *For all $i \in [2, n-1]$, for all $C \in \text{parity}^c(t_{i-1}, x_i, t_i)$, the clauses $A^0_{i,C} = C \cup \{z_1, z_2, a_{i,n}\}$ and $A^1_{i,C} = C \cup \{\overline{z_1}, \overline{z_2}, a_{i,n}\}$.*
- *For $i = n$, for all $C \in \text{parity}^c(t_{n-1}, x_n, t_n)$, the clauses $A^0_{i,C} = C \cup \{z_1, z_2\}$ and $A^1_{i,C} = C \cup \{\overline{z_1}, \overline{z_2}\}$.*
- *For $i = n + 1$, the clauses $\{t_n, z_1, z_2\}$ and $\{\overline{t_n}, \overline{z_1}, \overline{z_2}\}$.*
- *For all $i \in [n-1]$, the following clauses:*

$$B^0_{i,j} = \{\overline{a_{i,j}}, x_j, a_{i,j-1}\}, \quad B^1_{i,j} = \{\overline{a_{i,j}}, \overline{x_j}, a_{i,j-1}\} \quad \forall j \in \{n, n-1, \ldots, i+2\}$$
$$B^0_{i,i+1} = \{\overline{a_{i,i+1}}, x_{i+1}\}, \quad B^1_{i,i+1} = \{\overline{a_{i,i+1}}, \overline{x_{i+1}}\}$$

We can adapt the LD-Q-Res refutation of QParity to an M-Res refutation of MParity. We describe below exactly how this is achieved. The proof has two stages. In the first stage, the a variables are eliminated. The role of these $a_{i,j}$ variables and the B-clauses is to build up complex merge-maps meeting the isomorphism condition, so that subsequent resolution steps are enabled. In the second phase, the LD-Q-Res refutation of QParity is mimicked, eliminating the t variables.

(In the proofs below, notice that each line contains a single merge-map. This is done because the merge-maps for z_1 and z_2 in every line are same. So, we write them only once to save space.)

For $i \in [n+1]$, let g_i be the function $\oplus_{j\geq i} x_j$, and let h_i denote its complement. (The parity of an empty set of variables is 0; thus $g_{n+1} = 0$ and $h_{n+1} = 1$.) Let M_i^1 (resp. M_i^0) be the smallest merge-map which queries variables in the order $x_i, \ldots, x_n$ and computes the function g_i (resp. h_i). Note that both these merge-maps have $2(n-i)+1$ internal nodes and two leaf nodes labelled 0 and 1.

The main idea is to replace the constant merge-maps in the axioms of $A^0_{i,C}$ and $A^1_{i,C}$ by the merge-maps M^0_{i+1} and M^1_{i+1} – the clause, merge-map pairs so generated will be denoted by $\widetilde{\psi_i}$ (and are defined below). These merge-maps will allow us to pass the isomorphism checks later in the proofs.

For $i \in [n]$, let $\widetilde{\psi_i}$ be the following sets of clause, merge-map pairs:

$$\widetilde{\psi_i} = \left\{ \left(C, M^b_{i+1}\right) \mid C \in \text{parity}^c(t_{i-1}, x_i, t_i), b \in \{0,1\} \right\} \qquad \forall i \in [2,n]$$
$$\widetilde{\psi_1} = \left\{ \left(C, M^b_2\right) \mid C \in \text{parity}^c(x_1, t_1), b \in \{0,1\} \right\}$$

▶ **Lemma 3.9.** *For all $i \in [n]$, $\psi_i \vdash_{M\text{-}Res} \widetilde{\psi_i}$. Moreover the size of these derivations is polynomial in n.*

Proof. At $i = n$, $\widetilde{\psi_n}$ is the same as ψ_n so there is nothing to prove.

Consider now an $i \in [n-1]$. For each $b \in \{0,1\}$ and each $C \in \text{parity}^c(t_{i-1}, x_i, t_i)$ (if $i = 1$, omit t_{i-1}), the clause $A^b_{i,C} \in \psi_i$ yields the line $(C \cup \{a_{i,n}\}, M^{1-b}_{n+1})$. Resolving each of these with each of $B^d_{i,n}$ for $d \in \{0,1\}$, we obtain four clauses that can be resolved in two pairs to produce the lines $(C \cup \{a_{i,n-1}\}, M^b_n)$. (See the derivation at the end of this proof.) Repeating this process successively for $j = n, n-1, \ldots, i+2$, using the clause pairs $B^d_{i,j}$ with the previously derived clauses, we can obtain each $(C \cup \{a_{i,j}\}, M^b_{j+1})$. In each stage, the index j of the variable $a_{i,j}$ present in the clause decreases, while the merge-map accounts for one more variable. Finally, when we use the clause pairs $B^d_{i,i+1}$, the $a_{i,i+1}$ variable is eliminated, variables $x_{i+1}, \ldots, x_n$ are accounted for in the merge-map, and we obtain the lines (C, M^b_{i+1}), corresponding to the clauses in $\widetilde{\psi_i}$.

The derivation at one stage is as shown below.

$$\dfrac{\dfrac{\left(C \cup \{a_{i,j}\}, M^1_{j+1}\right) \qquad \overbrace{\left(\{\overline{a_{i,j}}, x_j, a_{i,j-1}\}, *\right)}^{B^0_{i,j}}}{\left(C \cup \{x_j, a_{i,j-1}\}, M^1_{j+1}\right)} \qquad \dfrac{\left(C \cup \{a_{i,j}\}, M^0_{j+1}\right) \qquad \overbrace{\left(\{\overline{a_{i,j}}, \overline{x_j}, a_{i,j-1}\}, *\right)}^{B^1_{i,j}}}{\left(C \cup \{\overline{x_j}, a_{i,j-1}\}, M^0_{j+1}\right)}}{\left(C \cup \{a_{i,j-1}\}, M^1_j\right)}$$

$$\dfrac{\dfrac{\left(C \cup \{a_{i,j}\}, M^1_{j+1}\right) \qquad \overbrace{\left(\{\overline{a_{i,j}}, \overline{x_j}, a_{i,j-1}\}, *\right)}^{B^1_{i,j}}}{\left(C \cup \{\overline{x_j}, a_{i,j-1}\}, M^1_{j+1}\right)} \qquad \dfrac{\left(C \cup \{a_{i,j}\}, M^0_{j+1}\right) \qquad \overbrace{\left(\{\overline{a_{i,j}}, x_j, a_{i,j-1}\}, *\right)}^{B^0_{i,j}}}{\left(C \cup \{x_j, a_{i,j-1}\}, M^0_{j+1}\right)}}{\left(C \cup \{a_{i,j-1}\}, M^0_j\right)}$$ ◀

In the second phase, we successively eliminate the t variables in stages.

▶ **Lemma 3.10.** *The following derivations can be done in M-Res in size polynomial in n:*

1. *For $i = n, n-1, \ldots, 2$,* $\left(\{t_i\}, M^1_{i+1}\right), \left(\{\overline{t_i}\}, M^0_{i+1}\right), \widetilde{\psi_i} \vdash \left(\{t_{i-1}\}, M^1_i\right), \left(\{\overline{t_{i-1}}\}, M^0_i\right)$.
2. $\left(\{t_1\}, M^1_2\right), \left(\{\overline{t_1}\}, M^0_2\right), \widetilde{\psi_1} \vdash \left(\Box, M^1_1\right)$.

Proof. For $i \geq 2$, the derivation is as follows:

$$\dfrac{\dfrac{\left(\{t_{i-1}, x_i, \overline{t_i}\}, M^1_{i+1}\right) \qquad \left(\{t_i\}, M^1_{i+1}\right)}{\left(\{t_{i-1}, x_i\}, M^1_{i+1}\right)} \qquad \dfrac{\left(\{t_{i-1}, \overline{x_i}, t_i\}, M^0_{i+1}\right) \qquad \left(\{\overline{t_i}\}, M^0_{i+1}\right)}{\left(\{t_{i-1}, \overline{x_i}\}, M^0_{i+1}\right)}}{\left(\{t_{i-1}\}, M^1_i\right)}$$

$$\dfrac{\dfrac{\left(\{\overline{t_{i-1}}, \overline{x_i}, \overline{t_i}\}, M^1_{i+1}\right) \qquad \left(\{t_i\}, M^1_{i+1}\right)}{\left(\{\overline{t_{i-1}}, \overline{x_i}\}, M^1_{i+1}\right)} \qquad \dfrac{\left(\{\overline{t_{i-1}}, x_i, t_i\}, M^0_{i+1}\right) \qquad \left(\{\overline{t_i}\}, M^0_{i+1}\right)}{\left(\{\overline{t_{i-1}}, x_i\}, M^0_{i+1}\right)}}{\left(\{\overline{t_{i-1}}\}, M^0_i\right)}$$

The derivation at the last stage is as follows:

$$\dfrac{\dfrac{\left(\{x_1, \overline{t_1}\}, M^1_2\right) \qquad \left(\{t_1\}, M^1_2\right)}{\left(\{x_1\}, M^1_2\right)} \qquad \dfrac{\left(\{\overline{x_1}, t_1\}, M^0_2\right) \qquad \left(\{\overline{t_1}\}, M^0_2\right)}{\left(\{\overline{x_1}\}, M^0_2\right)}}{\left(\Box, M^1_1\right)}$$

◀

We can now conclude the following:

▶ **Lemma 3.11.** *MParity has polynomial size M-Res refutations.*

Proof. We first use Lemma 3.9 to derive all the $\widetilde{\psi_i}$. Next, we start with $\left(\{t_n\}, M^1_{n+1}\right)$ and $\left(\{\overline{t_n}\}, M^0_{n+1}\right)$, the lines corresponding to the clauses in ψ_{n+1}. From these lines and $\widetilde{\psi_n}$, we derive $\left(\{t_{n-1}\}, M^1_n\right)$ and $\left(\{\overline{t_{n-1}}\}, M^0_n\right)$, using Lemma 3.10. We continue in this manner deriving $\left(\{t_i\}, M^1_{i+1}\right)$ and $\left(\{\overline{t_i}\}, M^0_{i+1}\right)$ for $i = n-2, n-3, \ldots, 1$. From $\left(\{t_1\}, M^1_2\right)$ and $\left(\{\overline{t_1}\}, M^0_2\right)$, we derive $\left(\Box, M^1_1\right)$ using $\widetilde{\psi_1}$ using Lemma 3.10. ◀

▶ **Theorem 3.12.** *LD-Q-Res does not p-simulate M-Res; and LQU-Res and LQU$^+$-Res are incomparable with M-Res.*

Proof. We showed in Lemma 3.11 that the MParity formulas have polynomial size M-Res refutations. We will now show that MParity requires exponential size LQU$^+$-Res refutations. We first note that QUParity requires exponential size LQU$^+$-Res refutations [9]. We further note that LQU$^+$-Res is closed under restrictions (Proposition 2 in [3]). Since restricting the MParity formulas by setting $a_{i,j} = 0$, for all $i, j \in [n]$, gives the QUParity formulas, we conclude that MParity requires exponential size LQU$^+$-Res refutations. Therefore LQU$^+$-Res does not simulate M-Res. Since LQU$^+$-Res p-simulates LD-Q-Res and LQU-Res, these two systems also do not simulate M-Res.

In [7] it is shown that M-Res does not simulate QU-Res. (The separating formula is in fact KBKF-lq.) Since LQU-Res and LQU$^+$-Res p-simulate QU-Res [3] and the simulation order is transitive, it follows that M-Res does not simulate LQU-Res and LQU$^+$-Res.

Hence LQU-Res and LQU$^+$-Res are incomparable with M-Res. ◀

▶ Remark 3.13. In these proofs, note that the hardness for LQU$^+$-Res and IRM was proven using restrictions. But the same did not apply to M-Res – a restricted formula being hard for M-Res does not mean that the original formula is also hard. This means that M-Res is not closed under restrictions, and is hence unnatural.

▶ Remark 3.14. Another observation is that the clauses of the KBKF-lq-weak formula family are weakenings of the clauses of KBKF-lq. Since KBKF-lq requires exponential-size M-Res refutations but KBKF-lq-weak has polynomial-size M-Res refutations, we conclude that weakening adds power to M-Res.

4 Role of weakenings, and unnaturalness

4.1 Weakenings

Let $(C, \{M^u \mid u \in U\})$ be a line. Then it can be weakened in two different ways [6]:

- Existential clause weakening: $C \vee x$ can be derived from C, provided it does not contain the literal $\overline{x}$. The merge-maps remain the same. Similarly, $C \vee \overline{x}$ can be derived if $x \notin C$.
- Strategy weakening: A trivial merge-map $(*)$ can be replaced by a constant merge-map (0 or 1). The existential clause remains the same.

Adding these weakenings to M-Res gives the following three proof systems:

- M-Res with existential clause weakening (M-ResW$_\exists$),
- M-Res with strategy weakening (M-ResW$_\forall$), and
- M-Res with both existential clause and strategy weakening (M-ResW$_{\exists\forall}$).

In the remainder of this subsection, we will study the relation among these systems. First, we note that existential clause weakening adds exponential power.

▶ **Theorem 4.1.** *M-ResW$_\exists$ is strictly stronger than M-Res.*

Proof. Since M-ResW$_\exists$ is a generalization of M-Res, M-ResW$_\exists$ p-simulates M-Res.

The KBKF-lq formulas can be transformed into the KBKF-lq-weak formulas in M-ResW$_\exists$ using a linear number of applications of the existential weakening rule. The transformed KBKF-lq-weak formulas have polynomial size M-Res (and hence M-ResW$_\exists$) refutations, Lemma 3.4. Thus the KBKF-lq formulas have polynomial size M-ResW$_\exists$ refutations. Since the KBKF-lq formulas require exponential size M-Res refutations [7], we get the desired separation. ◀

Next we observe that a lower bound for M-Res from [7] can be lifted to M-ResW$_\forall$.

▶ **Lemma 4.2.** *KBKF-lq requires exponential size refutations in M-ResW$_\forall$.*

Proof. We observe that the M-Res lower bound for KBKF-lq in [7] works with a minor modification. In [7, Lemma 21], item 3 says that $M^{x_i} = *$. However a weaker condition $M^{x_i} \in \{*, 0, 1\}$ is sufficient for the lower bound. With this modification, we observe that the remaining argument carries over, and hence the lower bound also works for M-ResW$_\forall$. ◀

This tells us that strategy weakening is not as powerful as existential weakening.

▶ **Theorem 4.3.** *M-ResW$_\forall$ does not simulate M-ResW$_\exists$; and M-ResW$_{\exists\forall}$ is strictly stronger than M-ResW$_\forall$.*

Proof. We showed that the KBKF-lq formulas require exponential size refutations in M-ResW$_\forall$ (Lemma 4.2) but have polynomial size refutations in M-ResW$_\exists$ and M-ResW$_{\exists\forall}$ (proof of Theorem 4.1). Therefore M-ResW$_\forall$ does not simulate M-ResW$_\exists$ and M-ResW$_{\exists\forall}$. Since M-ResW$_{\exists\forall}$ p-simulates M-ResW$_\forall$, M-ResW$_{\exists\forall}$ is strictly stronger than M-ResW$_\forall$. ◀

The next logical question is whether strategy weakening adds power to M-Res. We do not know the answer. However, we can answer this for the regular versions of these systems.

▶ **Definition 4.4.** *A refutation (in M-Res, M-ResW$_\exists$, M-ResW$_\forall$ or M-ResW$_{\exists\forall}$) is called regular if each variable is resolved at most once along every path.*

▶ **Theorem 4.5.** *Regular M-ResW$_\forall$ is strictly stronger than regular M-Res.*

To prove this theorem, we will use a variant of the Squared-Equality (Eq^2) formula family, called Squared-Equality-with-Holes (H-Eq$^2(n)$). Squared-Equality, defined in [6], is a two-dimensional version of the Equality formula family [5], and has short regular tree-like M-Res refutations. It was used to show that the systems Q-Res, QU-Res, reductionless LD-Q-Res, $\forall$Exp + Res, IR and CP + $\forall$red do not p-simulate M-Res. We recall its definition below:

▶ **Definition 4.6.** Squared-Equality *(Eq$^2(n)$) is the following QBF family:*

$$\underset{i\in[n]}{\exists} x_i, y_i, \underset{j\in[n]}{\forall} u_j, v_j, \underset{i,j\in[n]}{\exists} t_{i,j}. \left(\bigwedge_{i,j\in[n]} A_{i,j}\right) \wedge B$$

where

- $B = \vee_{i,j\in[n]}\overline{t_{i,j}}$,
- *For* $i, j \in [n]$, $A_{i,j}$ *contains the following four clauses:*

$$x_i \vee y_j \vee u_i \vee v_j \vee t_{i,j}, \qquad x_i \vee \overline{y_j} \vee u_i \vee \overline{v_j} \vee t_{i,j},$$
$$\overline{x_i} \vee y_j \vee \overline{u_i} \vee v_j \vee t_{i,j}, \qquad \overline{x_i} \vee \overline{y_j} \vee \overline{u_i} \vee \overline{v_j} \vee t_{i,j}$$

We observe that the short M-Res refutation of Eq$^2(n)$ crucially uses the isomorphism of merge-maps. For each $i, j \in [n]$, the four clauses in $A_{i,j}$ are resolved to derive the line $(t_{i,j}, \{u_i = x_i, v_j = y_j\})$. These lines are then resolved with the line $\left(\vee_{i,j\in[n]}\overline{t_{i,j}}, \{*, \cdots, *\}\right)$ to derive the line $(\Box, \{u_i = x_i, v_j = y_j \mid \forall i, j \in [n]\})$. The resolutions over the $t_{i,j}$ variables are possible only because the merge-maps are isomorphic. If we modify the clauses of Eq^2 such that the merge-maps produced from different $A_{i,j}$ are non-isomorphic, then the refutation described above is forbidden. This is the motivation behind the Squared-Equality-with-Holes (H-Eq^2) formula family defined below. It is constructed from Eq^2 by removing some of the universal variables from the $A_{i,j}$ clauses. The resulting QBF family remains false but different $A_{i,j}$ lead to different merge-maps. We believe that this QBF family is hard for M-Res, but we have only been able to prove the hardness for regular M-Res, and hence the separation is between the regular versions of M-Res and M-ResW$_\forall$.

The variant identifies regions in the $[n] \times [n]$ grid, and changes the clause sets $A_{i,j}$ depending on the region that (i, j) belongs to. We can use any partition of $[n] \times [n]$ into two regions R_0, R_1 such that each region has at least one position in each row and at least one position in each column; call such a partition a *covering partition*. One possible choice for R_0 and R_1 is the following: $R_0 = ([1, n/2] \times [1, n/2]) \cup ([n/2 + 1, n] \times [n/2 + 1, n])$ and $R_1 = ([1, n/2] \times [n/2 + 1, n]) \cup ([n/2 + 1, n] \times [1, n/2])$. We will call R_0 and R_1 two regions of the matrix.

▶ **Definition 4.7.** *Let R_0, R_1 be a covering partition of $[n] \times [n]$.*
Squared-Equality-with-Holes (H-Eq$^2(n)(R_0, R_1)$) is the following QBF family:

$$\underset{i\in[n]}{\exists} x_i, y_i, \underset{j\in[n]}{\forall} u_j, v_j, \underset{i,j\in[n]}{\exists} t_{i,j}. \left(\bigwedge_{i,j\in[n]} A_{i,j}\right) \wedge B$$

where

- $B = \vee_{i,j\in[n]}\overline{t_{i,j}}$,
- *For* $(i,j) \in R_0$*,* $A_{i,j}$ *contains the following four clauses:*

$$x_i \vee y_j \vee u_i \vee v_j \vee t_{i,j}, \qquad x_i \vee \overline{y_j} \vee u_i \vee t_{i,j},$$
$$\overline{x_i} \vee y_j \vee v_j \vee t_{i,j}, \qquad \overline{x_i} \vee \overline{y_j} \vee t_{i,j}$$

- *For* $(i,j) \in R_1$*,* $A_{i,j}$ *contains the following four clauses:*

$$x_i \vee y_j \vee t_{i,j}, \qquad x_i \vee \overline{y_j} \vee \overline{v_j} \vee t_{i,j},$$
$$\overline{x_i} \vee y_j \vee \overline{u_i} \vee t_{i,j}, \qquad \overline{x_i} \vee \overline{y_j} \vee \overline{u_i} \vee \overline{v_j} \vee t_{i,j}$$

(We do not always specify the regions explicitly but merely say H-Eq^2*.)*

▶ **Lemma 4.8.** *H-Eq*$^2(n)$ *requires exponential size refutations in regular M-Res.*

Before proving this, we show how to obtain Theorem 4.5.

Proof of Theorem 4.5. Since regular M-ResW$_\forall$ is a generalization of regular M-Res, it p-simulates regular M-Res.

Using strategy weakening, we can get Eq^2 from H-Eq^2 in a linear number of steps. Since Eq^2 has polynomial-size refutations in regular M-Res, we get polynomial-size refutations for H-Eq^2 in regular M-ResW$_\forall$. On the other hand, Lemma 4.8 gives an exponential lower bound for H-Eq^2 in regular M-Res. Therefore regular M-ResW$_\forall$ is strictly stronger than regular M-Res. ◀

It remains to prove Lemma 4.8. This is a fairly involved proof, but in broad outline and in many details it is similar to the lower bound for Eq^2 in reductionless LD-Q-Res ([6]).

The size bound is trivially true for $n = 1$, so we assume that $n > 1$. Let Π be a Regular M-Res refutation of H-Eq$^2(n)$. Since a tautological clause cannot occur in a regular M-Res refutation, we assume that Π does not have a line whose clause part is tautological.

Let us first fix some notation. Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$, $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$, and $T = \{t_{i,j} \mid i,j \in [n]\}$. For lines L_1, L_2, etc., the respective clauses and merge-maps will be denoted by C_1, C_2 and M_1, M_2 etc. For a line L in Π, Π_L denotes the sub-derivation of Π ending in L. Viewing Π as a directed acyclic graph, we can talk of leaves and paths in Π. For a line L of Π, let $\textsc{Uci}(L) = \{(i,j) \mid A_{i,j} \cap \text{leaves}(\Pi_L) \neq \emptyset\}$.

We first show some structural properties about Π. The first property excludes using many axioms in certain derivations.

▶ **Lemma 4.9.** *For line* $L = (C, M)$ *of* Π*, and* $i,j \in [n]$*, if* $t_{i,j} \in C$*, then* $\textsc{Uci}(L) = \{(i,j)\}$*.*

Proof. Since the literal $t_{i,j}$ only occurs in clauses in $A_{i,j}$, so leaves$(L) \cap A_{i,j} \neq \emptyset$, hence $\textsc{Uci}(L) \supseteq \{(i,j)\}$.

Now suppose $\textsc{Uci}(L) > 1$. Let (i',j') be an arbitrary element of $\textsc{Uci}(L)$ distinct from (i,j). Pick a leaf of Π_L using a clause in $A_{i',j'}$, and let p be a path from this leaf to L and then to the final line of Π. Both $t_{i,j}$ and $t_{i',j'}$ are necessarily used as pivots on this path. Assume that $t_{i,j}$ is used as a pivot later (closer to the final line) than $t_{i',j'}$; the other case is symmetric. Let $L_c = \text{res}(L_a, L_b, t_{i',j'})$ and $L_f = \text{res}(L_d, L_e, t_{i,j})$ respectively be the positions where $t_{i',j'}$ and $t_{i,j}$ are used as resolution pivots on this path (here L_a and L_d are the lines of path p, hence $t_{i',j'} \in C_a$ and $t_{i,j} \in C_d$). Then C_b has the negated literal $\overline{t_{i',j'}}$; hence $B \in \text{leaves}(L_b)$. Since $\overline{t_{i,j}} \in B$ but $\overline{t_{i,j}} \notin L_d$, $t_{i,j}$ is used as a resolution pivot in the derivation Π_{L_d}. This contradicts the fact that Π is regular. ◀

The next property is the heart of the proof, and shows that paths with B at the leaf must have a suitable wide clause.

▶ **Lemma 4.10.** *On every path from* $\left(\vee_{i,j\in[n]}\overline{t_{i,j}}, \{*, \cdots, *\}\right)$ *(the line for axiom clause* B*) to the final line, there exists a line* $L = (C, M)$ *such that either* $X \subseteq var(C)$ *or* $Y \subseteq var(C)$.

Proof. With each line $L_l = (C_l, M_l)$ in Π, we associate an $n \times n$ matrix N_l in which $N_l[i, j] = 1$ if $\overline{t_{i,j}} \in C_l$ and $N_l[i, j] = 0$ otherwise.

Let $p = L_1, \ldots, L_k$ be a path from $\left(\vee_{i,j\in[n]}\overline{t_{i,j}}, \{*, \cdots, *\}\right)$ to the final line in Π. Since Π is regular, each $\overline{t_{i,j}}$ is resolved away exactly once, so no clause on p has any positive $t_{i,j}$ literal. Let l be the least integer such that N_l has a 0 in each row or a 0 in each column. Note that $l \geq 2$ since N_1 has no zeros. Consider the case that N_l has a 0 in each row; the argument for the other case is identical. We will show in this case that $X \subseteq \mathrm{var}(C_l)$. We will use the following claim:

▷ Claim 4.11. In each row of N_l, there is a 0 and a 1 such that the 0 and 1 are in different regions (i.e. one is in R_0 and the other in R_1).

We proceed assuming the claim. We want to prove that $X \subseteq var(C_l)$. Suppose, to the contrary, there exists $i \in [n]$ such that $x_i \notin \mathrm{var}(C_l)$. We know that there exist $j_1, j_2 \in [n]$ such that $N_l[i, j_1] = 0$ and $N_l[i, j_2] = 1$; and either $(i, j_1) \in R_0$ and $(i, j_2) \in R_1$, or $(i, j_1) \in R_1$ and $(i, j_2) \in R_0$. Without loss of generality, we may assume that $(i, j_1) \in R_0$ and $(i, j_2) \in R_1$.

We know that on path p, there is a resolution with pivot t_{i,j_1} before L_l and a resolution with pivot t_{i,j_2} after L_l. Let the former resolution be $L_c = \mathrm{res}(L_a, L_b, t_{i,j_1})$ where L_b is on path p, and let the latter resolution be $L_f = \mathrm{res}(L_d, L_e, t_{i,j_2})$ where L_e is on path p. Since Π is a regular refutation, $t_{i,j_1} \in C_a$, $\overline{t_{i,j_1}} \in C_b$ and $t_{i,j_2} \in C_d$, $\overline{t_{i,j_2}} \in C_e$. Thus along path p these lines appear in the relative order $B, L_b, L_c, L_l, L_e, L_f, \Box$.

▷ Claim 4.12. $\overline{x_i} \in C_c$.

Proof. By Lemma 4.9, $\textsc{Uci}(L_d) = \{(i, j_2)\}$, or equivalently leaves$(L_d) \subseteq A_{i,j_2}$. Since $(i, j_2) \in R_1$, no clause in A_{i,j_2} has literal u_i. Hence $M_d^{u_i} \in \{*, 1\}$. Furthermore, if $M_d^{u_i} = *$, then $x_i \in C_d$. Since the pivot for resolving L_d and L_e is t_{i,j_2}, this would imply that $x_i \in C_f$.

By a similar argument, we can conclude that (i) leaves$(L_a) \subseteq A_{i,j_1}$, (ii) $M_a^{u_i} \in \{*, 0\}$, and (iii) if $M_a^{u_i} = *$, then $\overline{x_i} \in C_c$.

If $M_d^{u_i} = *$ and $M_a^{u_i} = *$, then $x_i \in C_f$ and $\overline{x_i} \in C_c$. So x_i must be used twice as pivot, contradicting regularity.

If $M_d^{u_i} = *$ and $M_a^{u_i} = 0$, then $x_i \in C_f$ and Π_{L_a} uses some clause containing x_i to make the merge-map for u_i non-trivial. Thus $x_i \in \Pi_{L_a}$, $x_i \notin L_l$ by assumption, $x_i \in L_f$. Hence x_i is used twice as pivot, contradicting regularity.

Hence $M_d^{u_i} = 1$. Since the resolution at line L_f is not blocked, $M_e^{u_i} \in \{*, 1\}$. But L_e is derived after, and using, L_a. Since merge-maps don't get simpler along a path, $M_a^{u_i} \in \{*, 1\}$. It follows that $M_a^{u_i} = *$. Hence $\overline{x_i} \in C_c$. ◁

Since $\overline{x_i} \notin C_l$, x_i has been used as a resolution pivot between L_c and L_l on path p. Let $L_w = \mathrm{res}(L_u, L_v, x_i)$ be the position on path p where x_i is used as pivot (since the refutation is regular, such a position is unique). Let L_v be the line on path p. By regularity of the refutation, $x_i \in L_u$ and $\overline{x_i} \in L_v$.

As observed at the outset, L_w is on path p and so does not contain a positive t literal. Since C_w is obtained via pivot x_i, this implies that C_u also does not contain a positive t literal. Since all axioms contain at least one t variable but only B contains negated t literals, so $B \in$ leaves(L_u).

Let q be a path that starts from a leaf using B, passes through L_u to L_w, and then continues along path p to the final clause. Since the refutation is regular, $N_v = N_u = N_w$. Hence $N_v[i, j_1] = 0$ i.e. $\overline{t_{i,j_1}} \notin C_v$. This implies that t_{i,j_1} is used as resolution pivot before L_v on path q.

We already know that t_{i,j_2} is used as a pivot after line L_l on path p, and hence on path q. Arguing analogous to Claim 4.12 for path p but with respect to path q, we observe that $\overline{x_i}$ belongs to at least one leaf of L_u. Since $x_i \in C_u$ and since the refutation is regular, x_i is not used as a resolution pivot before C_u on path q. This implies that $\overline{x_u} \in C_u$. We already know that $x_i \in C_u$, since it contributed the pivot at L_w. This means that C_u is a tautological clause, a contradiction.

It remains to prove Claim 4.11.

Proof of Claim 4.11. We already know that N_l has a 0 in each row. We will first prove that N_l also has a 1 in each row. Aiming for contradiction, suppose that N_l has a full 0 row r. Since $l \geq 2$, N_{l-1} exists. Note that, by definition of resolution, there can be at most one element that changes from 1 in N_{l-1} to 0 in N_l. Since N_{l-1} does not have a 0 in every column, it does not contain a full 0 row. Hence, the unique element that changed from 1 in N_{l-1} to 0 in N_l must be in row r. Thus all other rows of N_{l-1} already contain the one 0 of that row in N_l. Since $n \geq 2$, N_{l-1} also has at least one 0 in row r; thus N_{l-1} has a 0 in each row, contradicting the minimality of l.

Since R_0 and R_1 form a covering partition, it cannot be the case that all the 0s and 1s of any row are in the same region R_b; that would imply that R_{1-b} does not cover the row. ◁

With the claim proven, the proof of Lemma 4.10 is now complete. ◀

We can finally prove Lemma 4.8. This part is identical to the corresponding part of the proof of Theorem 28 in [6]; we include it here for completeness.

Proof of Lemma 4.8. For each $a = (a_1, \ldots, a_n) \in \{0,1\}^n$, consider the assignment σ_a to the existential variables which sets $x_i = y_i = a_i$ for all $i \in [n]$, and $t_{i,j} = 1$ for all $i, j \in [n]$. Call such an assignment a symmetric assignment. Given a symmetric assignment σ_a, walk from the final line of Π towards the leaves maintaining the following invariant: for each line $L = (C, \{M^u \mid u \in U \cup V\})$, σ_a falsifies C. Let p_a be the path followed. By Lemma 4.10, this path will contain a line $L = (C, \{M^u \mid u \in U \cup V\})$ such that either $X \subseteq var(C)$ or $Y \subseteq var(C)$. Let us define a function f from symmetric assignments to the lines of Π as follows: $f(a) = (C, \{M^u \mid u \in U \cup V\})$ is the last line (i.e. nearest to the leaves) on p_a such that either $X \subseteq var(C)$ or $Y \subseteq var(C)$. Note that, for any line L of Π, there can be at most one symmetric assignment a such that $f(a) = L$. This means that there are at least 2^n lines in Π. This gives the desired lower bound. ◀

4.2 Simulation by eFrege + ∀red

It was recently shown that eFrege + ∀red p-simulates all known resolution-based QBF proof systems; in particular, it p-simulates M-Res [15]. We observe that this p-simulation can be extended in a straightforward manner to handle both the weakenings in M-Res. Hence we obtain a p-simulation of M-ResW$_\exists$, M-ResW$_\forall$ and M-ResW$_{\exists\forall}$ by eFrege + ∀red.

▶ **Theorem 4.13.** *eFrege + ∀red strictly p-simulates M-ResW$_\exists$, M-ResW$_\forall$ and M-ResW$_{\exists\forall}$.*

Proof. The separation follows from the separation of the propositional proof systems resolution and eFrege [30]. We prove the p-simulation below.

It suffices to prove that eFrege + $\forall$red p-simulates M-ResW$_{\exists\forall}$. The proof is essentially same as that of the p-simulation of M-Res in [15], but with two additional cases for the two weakenings. So, we will briefly describe that proof and then describe the required modifications.

Let Π be an M-ResW$_{\exists\forall}$ refutation Π of a QBF Φ. The last line of this refutation gives a winning strategy for the universal player; let us call this strategy S. We will first prove that there is a short eFrege derivation $\Phi \vdash \neg S$. Then, as mentioned in [15], the technique of [8, 14] can be used to derive the empty clause from $\neg S$ using universal reduction.

We will now describe an eFrege derivation $\Phi \vdash \neg S$. Let $L_i = (C_i, \{M_i^u \mid u \in U\})$ be the i$^{\text{th}}$ line of Π. We create new extension variables: $s_{i,j}^u$ is the variable for the j$^{\text{th}}$ node of M_i^u. If node j is a leaf of M_i^u labeled by constant c, then $s_{i,j}^u$ is defined to be c. Otherwise, if $M_i^u(j) = (x, a, b)$, then $s_{i,j}^u$ is defined as $s_{i,j}^u \triangleq \left(x \wedge s_{i,a}^u\right) \vee \left(\overline{x} \wedge s_{i,b}^u\right)$. The extension variables for u will be to its left in the quantifier prefix.

We will prove that for each line L_i of Π, we can derive the formula $F_i \triangleq \wedge_{u\in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \to C_i$; where $r(u,i)$ is the index of the root of merge-map M_i^u, and U_i is the set of universal variables for which M_i^u is non-trivial.

Our proof will proceed by induction on the lines of the refutation.

The base case is when L_i is an axiom; and the inductive step will have three cases depending on which rule is used to derive L_i: (i) resolution, (ii) existential clause weakening, or (iii) strategy weakening. The proof for the base case and the resolution step case is as given in [15]. We give proofs for the other two cases below:

- Existential clause weakening: Let line $L_b = (C_b, \{M_b^u \mid u \in U\})$ be derived from line $L_a = (C_a, \{M_a^u \mid u \in U\})$ using existential clause weakening. Then $C_b = C_a \vee x$ for some existential literal x such that $\overline{x} \notin C_a$, and $M_b^u = M_a^u$ for all $u \in U$. By the induction hypothesis, we have derived the formula $F_a \triangleq \wedge_{u\in U_a}(u \leftrightarrow s_{a,r(u,a)}^u) \to C_a$. We have to derive the formula $F_b \triangleq \wedge_{u\in U_b}(u \leftrightarrow s_{b,r(u,b)}^u) \to C_b = \wedge_{u\in U_b}(u \leftrightarrow s_{b,r(u,b)}^u) \to C_a \vee x$. Since $M_b^u = M_a^u$ for each u, there is a short eFrege + $\forall$red derivation of the formula $s_{a,j}^u \leftrightarrow s_{b,j}^u$ for each $u \in U_i$, and each node j of M_a^u. This allows us to replace variable $s_{a,j}^u$ by $s_{b,j}^u$ in F_a. As a result, we get the formula $F_b' \triangleq \wedge_{u\in U_b}(u \leftrightarrow s_{b,r(u,b)}^u) \to C_a$. Now, using an inference of the form $p \to q \vdash p \to q \vee r$, we obtain the formula F_b.
- Strategy weakening: Let line $L_b = (C_b, \{M_b^u \mid u \in U\})$ be derived from line $L_a = (C_a, \{M_a^u \mid u \in U\})$ using strategy weakening for a variable v. Then $C_b = C_a$, $M_b^u = M_a^u$ for all $u \in U \setminus \{v\}$, and $M_a^v = *$, M_b^v is a constant, say d. Similar to the above case, we start with the inductively obtained F_a and replace each $s_{a,j}^u$ with $s_{b,j}^u$ to obtain a formula $F_b' \triangleq \wedge_{u\in U_b\setminus\{v\}}(u \leftrightarrow s_{b,r(u,b)}^u) \to C_b$. With a final inference of the form $p \to q \vdash p \wedge r \to q$, we can then add $(v \leftrightarrow s_{b,r(v,b)}^v)$ to the conjunction to obtain F_b. ◀

4.3 Unnaturalness

In this section, we observe that M-Res and M-ResW$_\forall$ are unnatural proof systems, i.e. they are not closed under restrictions.

▶ **Theorem 4.14.** *M-Res and M-ResW$_\forall$ are unnatural proof systems.*

Proof. The KBKF-lq-split formula family has polynomial-size refutations in M-Res (and M-ResW$_\forall$), as seen in Lemma 3.5. The restriction of this family obtained by setting $t = 0$ is exactly the KBKF-lq formula family, which, as shown in Lemma 4.2, is exponentially hard for M-ResW$_\forall$ and hence also for M-Res. ◀

5 Conclusion and future work

M-Res was introduced in [6] to overcome the weakness of LD-Q-Res. It was shown that M-Res has advantages over many proof systems, but the advantage over LD-Q-Res was not demonstrated. In this paper, we have filled this gap. We have shown that M-Res has advantages over not only LD-Q-Res, but also over more powerful systems, LQU^{+}-Res and IRM. We have also looked at the role of weakening – that it adds power to M-Res. On the negative side, we have shown that M-Res with and without strategy weakening is unnatural – which we believe makes it useless in practice.

For the system to still be useful in practice, one will have to prove that it can be made natural by adding existential weakening or both weakenings. This, in our opinion, is the most important open question about M-Res.

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