Fixed-Point Cycles and Approximate EFX Allocations

Benjamin Aram Berendsohn ✉
Institut für Informatik, Freie Universität Berlin, Germany

Simona Boyadzhiyska ✉
Institut für Mathematik, Freie Universität Berlin, Germany

László Kozma ✉
Institut für Informatik, Freie Universität Berlin, Germany

─────── **Abstract** ───────

We study edge-labelings of the complete bidirected graph $\overleftrightarrow{K}_n$ with *functions* that map the set $[d] = \{1, \ldots, d\}$ to itself. We call a directed cycle in $\overleftrightarrow{K}_n$ a *fixed-point cycle* if composing the labels of its edges in order results in a map that has a fixed point, and we say that a labeling is *fixed-point-free* if no fixed-point cycle exists. For a given d, we ask for the largest value of n, denoted $R_f(d)$, for which there exists a fixed-point-free labeling of $\overleftrightarrow{K}_n$. Determining $R_f(d)$ for all $d > 0$ is a natural Ramsey-type question, generalizing some well-studied zero-sum problems in extremal combinatorics. The problem was recently introduced by Chaudhury, Garg, Mehlhorn, Mehta, and Misra [EC 2021], who proved that $d \leq R_f(d) \leq d^4 + d$ and showed that the problem has close connections to *EFX allocations*, a central problem of fair allocation in social choice theory.

In this paper we show the improved bound $R_f(d) \leq d^{2+o(1)}$, yielding an efficient $(1 - \varepsilon)$-EFX allocation with n agents and $O((n/\varepsilon)^{0.67})$ unallocated goods; this improves the bound of $O((n/\varepsilon)^{0.8})$ of Chaudhury, Garg, Mehlhorn, Mehta, and Misra.

Additionally, we prove the stronger upper bound $2d - 2$, in the case where all edge-labels are *permutations*. A very special case of this problem, that of finding *zero-sum cycles* in digraphs whose edges are labeled with elements of $\mathbb{Z}_d$, was recently considered by Alon and Krivelevich [JGT 2021] and by Mészáros and Steiner [EJC 2021]. Our result improves the bounds obtained by these authors and extends them to labelings with elements of an arbitrary (not necessarily commutative) group, while also simplifying the proof.

1 Introduction

Let $\overleftrightarrow{K}_n$ denote the complete bidirected graph on n vertices, that is, $\overleftrightarrow{K}_n$ has directed edges (u, v) and (v, u) for every pair u, v of distinct vertices. For an integer $d > 0$, a *d-labeling*, or simply *labeling*, ℓ of $\overleftrightarrow{K}_n$ is an assignment of a function $\ell_e \colon [d] \to [d]$ to each edge e of $\overleftrightarrow{K}_n$, where $[d] = \{1, \ldots, d\}$.

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Let C be a simple cycle of $\overset{\leftrightarrow}{K}_n$ with edges $e_1, \ldots, e_k$ (appearing in this order), where each edge e_i is labeled with a function f_i, for $i \in [k]$. We say that C is a *fixed-point cycle* if the function $f = f_1 \circ f_2 \circ \cdots \circ f_k = f_k(f_{k-1}(\cdots f_1(x) \cdots))$ has a fixed point, i.e., $f(x) = x$ for some $x \in [d]$. Observe that cyclically permuting the edges along C does not affect the existence of a fixed point.

Let $R_f(d)$ be the largest value n such that a d-labeling of $\overset{\leftrightarrow}{K}_n$ with no fixed-point cycle exists. Although not obvious a priori, the finiteness of $R_f(d)$ for all d can be seen through a simple reduction to Ramsey's theorem. In light of the known exponential bounds on multicolor Ramsey numbers [13, 20, 23], the bound obtained in this way is *doubly exponential* in d (see [12] for details).

A lower bound of $R_f(d) \geq d$ can be obtained through the following construction (found independently by various authors in different contexts). Say $V(\overset{\leftrightarrow}{K}_d) = [d]$, and label all edges (i, j) such that $i < j$ with the function $x \mapsto x$ and all other edges with the function $x \mapsto x + 1 \pmod{d}$. The avoidance of fixed-point cycles follows from the observation that every cycle contains at least one and at most $d - 1$ edges of the form (i, j) with $i > j$.

The question of determining $R_f(d)$ was recently raised by Chaudhury, Garg, Mehlhorn, Mehta, and Misra [12] in a slightly different, but equivalent form they call the *rainbow cycle problem*. They proved the upper bound $R_f(d) \leq d^4 + d$.

The motivation in [12] for introducing this problem comes from an application to *discrete fair division*. In particular, through an elegant and surprising connection, they showed that *polynomial* upper bounds on $R_f(d)$ yield nontrivial guarantees for the quality of allocations in a certain natural setting with a strong fairness condition.

EFX allocations

In economics and computational social choice theory, the *discrete fair division* problem asks to distribute a set of m indivisible goods among n agents in a *fair* way. The problem has a long history (see for example [22]) and different notions of fairness have been extensively studied, leading to a rich set of algorithmic and hardness results (e.g., see [8] or [9] and the references therein for an overview and precise definitions).

As simple examples show, complete *envy-freeness* cannot, in general, be achieved. One of the most compelling relaxations of this notion is *envy-freeness up to any good* (EFX). Informally, EFX means that no agent should prefer the goods received by any other agent to their own, if an arbitrary single good is removed from the other's set. The existence of EFX allocations is considered one of the central open questions of contemporary social choice theory (e.g., see [6, 9, 10, 12, 19]), and thus, various relaxations of it have been proposed. In particular, it is desirable to show that an EFX allocation exists (and can be efficiently computed) when (i) a certain global number t of goods are left unallocated, and (ii) envy-freeness is required to hold when every agent scales the values of others' goods by a factor of $1 - \varepsilon$ for some $0 \leq \varepsilon < 1$. Such an allocation is called $(1-\varepsilon)$-*EFX with t unallocated goods*. It is desirable to minimize both t and ε, with the original EFX question requiring $t = \varepsilon = 0$.

Very recently, Chaudhury, Garg, Mehlhorn, Mehta, and Misra [12] introduced a correspondence between approximate EFX allocations and the fixed-point cycle problem described above (in their terminology, the *rainbow cycle problem*). Suppressing constant factors, the connection can be summarized as follows.

▶ **Theorem 1** (Theorem 4 in [12]). *For all $\varepsilon \in (0, 1/2]$, if $R_f(d) \in O(d^c)$ for some $c \geq 1$, then there exists a $(1 - \varepsilon)$-EFX allocation with $O((n/\varepsilon)^{\frac{c}{1+c}})$ unallocated goods, where n is the number of agents.*

Moreover, if the upper bound on $R_f(d)$ is *constructive* (i.e., a fixed-point cycle can be found in time polynomial in the number n of vertices whenever n exceeds the given bound on $R_f(d)$), then the claimed allocation can also be found in polynomial time. The result of $R_f(d) \le d^4 + d$ from [12] is constructive; together with Theorem 1 it thus implies an efficient algorithm for computing an approximate EFX allocation with $O((n/\varepsilon)^{0.8})$ unallocated goods, the first guarantee of this kind with a *sublinear* (in n) number of unallocated goods.

Rainbow cycles

As remarked in [12], the question of determining $R_f(d)$ is in itself a natural question of extremal graph theory, independently of its application to EFX allocations.

Chaudhury, Garg, Mehlhorn, Mehta, and Misra formulated the problem in a slightly different, but essentially equivalent, way as follows. Given an n-partite digraph with each part having *at most d* vertices, a *rainbow cycle* is a cycle that visits each part *at most once*. The task is to determine the largest $n = n(d)$ for which such a digraph can avoid rainbow cycles, with the requirement that each vertex in part i has an incoming edge from at least one vertex in part j, for all distinct $i, j \in [n]$.

Note that we may assume that there is an extremal example containing *exactly d* vertices in each part. Indeed, if $\vec{G}$ is an extremal example and some part has fewer than d vertices, let v be any vertex from that part; then add the necessary number of new vertices, making them all "clones" of v, that is, every new vertex has the same in- and out-neighborhood as v. Then the newly obtained digraph has a rainbow cycle if and only if $\vec{G}$ does. Further, we may assume that each vertex v of $\vec{G}$ has *exactly* one incoming edge from every part other than its own. With these observations, the equivalence between the rainbow cycle problem and the fixed-point cycle problem is evident: we can view the bipartite digraph containing the edges from a part V_i to another part V_j as the mapping given by $y \mapsto x$, where $(x, y) \in E(V_i, V_j)$.

Our main result improves the upper bound of Chaudhury, Garg, Mehlhorn, Mehta, and Misra.

▶ **Theorem 2.** *For all $d > 0$ we have $R_f(d) \le d^{2+o(1)}$.*

Similarly to the previous result, our bound is constructive. Combining Theorem 2 with Theorem 1 thus yields the following result about EFX allocations.

▶ **Corollary 3.** *For all $\varepsilon \in (0, 1/2]$, there exists an efficient $(1 - \varepsilon)$-EFX allocation with $O((n/\varepsilon)^{0.67})$ unallocated goods.*

Zero-sum problems in extremal combinatorics

The problem of determining $R_f(d)$ can be cast as a generalization of some classical zero-sum problems in extremal combinatorics (this is somewhat less apparent in the original multi-partite formulation of the problem). Zero-sum problems have received substantial attention and form a well-defined subfield of combinatorics, with an algebraic flavour. Perhaps the earliest result in this area is the Erdős-Ginzburg-Ziv theorem [14] which states that every collection of $2m - 1$ integers contains m integers whose sum modulo m is zero (see [3] for multiple proofs and extensions). Zero-sum problems *in graphs* typically ask, given an edge- or vertex-weighted graph, whether a certain substructure exists with zero total weight (modulo some fixed integer). Well-studied cases include complete graphs, cycles, stars, and trees (e.g., see [2, 5, 7, 11, 15, 21] for surveys and representative results).

More recently, for a given positive integer d, Alon and Krivelevich [4] asked for the maximum integer n such that the edges of the complete bidirected graph $\overleftrightarrow{K}_n$ can be labeled with integers so that there is no zero-sum cycle modulo d. In the following, we denote

this quantity by $n = R_i(d)$. Alon and Krivelevich showed through an elegant probabilistic argument that $R_i(d) \in O(d \log d)$, with an improvement to $R_i(d) \in O(d)$ when d is prime. The application considered in [4] is finding cycles of length divisible by d in minors of complete graphs. It is easy to see this question as a special case of our fixed-point cycle problem: simply replace every edge-label k by the function $x \mapsto x + k \pmod d$. Zero-sum cycles in the original labeling are then in bijection with fixed-point cycles in our new labeling, and thus $R_i(d) \le R_f(d)$.

Recently, Mészáros and Steiner [17] improved the result of Alon and Krivelevich, showing $R_i(d) \le 8d - 1$, with further improvements for prime d. In fact, Mészáros and Steiner generalized the result, allowing the labels to come from an arbitrary commutative group of order d. The proof of the main result in [17] can be seen as an extension of an incremental construction of [4], combined with an intricate inductive argument that makes use of group-theoretic results.

We improve these results and show an upper bound of $R_i(d) \le 2d - 2$ through somewhat similar, but arguably simpler arguments. Our result extends to arbitrary groups (not necessarily commutative). In fact, we prove the result in the more general setting of fixed-point cycles, when the edge-labels are restricted to *permutations* of $[d]$. The permutation case subsumes the integer case, since the functions $x \mapsto x + k \pmod d$ in the above reduction are permutations. Denoting the corresponding quantity by $R_p(d)$, we thus have $R_i(d) \le R_p(d) \le R_f(d)$, and (omitting the easy case $R_i(1) = R_p(1) = R_f(1) = 1$), prove the following.

▶ **Theorem 4.** *For all $d \ge 2$, we have $R_p(d) \le 2d - 2$.*

By the above discussion, Theorem 4 implies that $R_i(d) \le 2d - 2$. In fact, the following more general result holds.

▶ **Corollary 5.** *Let ℓ be a labeling of $\overleftrightarrow{K}_{2d-1}$ with elements of a (not necessarily abelian) group $(G, \cdot)$ of order d. Then there is a cycle whose labels multiply to the identity $1 \in G$.*

To our knowledge, the question of permutation labels has not been considered before. We see it as a natural problem of intermediate generality; it facilitates a simple proof for the integer case, allowing to sidestep the group-theoretic tools used in previous proofs.

In the case of permutation labels, it is natural to ask whether we can guarantee, instead of a fixed-point cycle, the existence of an *identity-cycle*. Indeed, Corollary 5 implies, as a very special case, that if $n \ge 2d! - 1$ and the edges of $\overleftrightarrow{K}_n$ are labeled with permutations of $[d]$, then there is a cycle whose labels compose to $\mathbf{1}_d$. The bound of $2d! - 1$ may appear rather loose, and one may wonder if a condition similar to that of Theorem 4, or at least one with polynomial dependence on d, would be sufficient. The following lower bound rules out this possibility.

▶ **Lemma 6.** *There exists a fixed $d_0 > 0$ such that for all $d \ge d_0$ and some $n \ge e^{\sqrt{d \ln d}}$, there is a labeling of $\overleftrightarrow{K}_n$ with permutations of $[d]$ such that there is no cycle whose labels compose to $\mathbf{1}_d$.*

Open questions

Closing the gap between the lower and upper bounds remains an interesting challenge for all three considered quantities ($R_f(d)$, $R_p(d)$, and $R_i(d)$). The lower bound construction with d vertices discussed earlier can be adapted to all three settings, and it remains a plausible conjecture that $R_i(d) = R_p(d) = R_f(d) = d$. This is easily verified [12] for $d \le 3$ in the case

of $R_f(d)$ (and hence for $R_p(d)$), and was verified via SAT-solvers [17] for $d \le 6$ in the case of $R_i(d)$. Showing $R_f(d) \in O(d)$ would yield, via Theorem 1, an approximate EFX allocation with $O(\sqrt{n/\varepsilon})$ unallocated goods.

A further natural question is whether other classical zero-sum results from extremal combinatorics can be extended to the fixed-point setting.

Organization of the paper

In §2 we introduce some useful terminology. In §3 we prove Theorem 4, Corollary 5, and Lemma 6 and in §4 we prove Theorem 2.

Independent work

Independently and concurrently to our work, Akrami, Chaudhury, Garg, Mehlhorn, and Mehta [1] also obtained the upper bound $R_p(d) \le 2d - 2$, and an improved upper bound $R_f(d) \in O(d^2)$.

2 Preliminaries

Let $\overleftrightarrow{K}_n$ be given together with a labeling ℓ that assigns functions $[d] \to [d]$ to the edges. By *path* or *cycle* we always mean a *simple* path or cycle, and for a path P, we denote by $V(P)$ its set of vertices. All *edges* in this paper are directed, and an edge (u, v) is alternatively denoted by $u \to v$.

When we say that the edge $u \to v$ *maps* x to y, we mean that the function ℓ_{uv} assigned to $u \to v$ maps x to y. Similarly, a path (or cycle) $u_1 \to \cdots \to u_k$ maps x to y if, given the label f_i on $u_i \to u_{i+1}$ for each $i \in [k-1]$, we have $(f_1 \circ \cdots \circ f_{k-1})(x) = y$. For consistency, a path with only one vertex is said to map every value to itself.

We write $v{:}i$ for a pair of a vertex $v \in V(\overleftrightarrow{K}_n)$ and a value $i \in [d]$. By $u{:}x \xrightarrow{\ell} v{:}y$ (or simply $u{:}x \to v{:}y$ when the labeling ℓ is clear from the context) we denote the fact that the edge $u \to v$ maps x to y.

Let $W = (v_1{:}i_1, v_2{:}i_2, \ldots, v_k{:}i_k)$ be a sequence of vertex-value pairs with $v_j{:}i_j \to v_{j+1}{:}i_{j+1}$ for each $j \in [k-1]$. We call W a *valued walk* in $(\overleftrightarrow{K}_n, \ell)$. If $v_1 = v_k$, then W is a *valued circuit*. If all vertices in W are distinct, then W is a *valued path*. A valued circuit in which all vertices but the last are distinct is a *valued cycle*.

Finally, let $u, v, w \in V(\overleftrightarrow{K}_n)$ and $x, y \in [d]$. We say that w *routes* $u{:}x$ to $v{:}y$ if $u{:}x \to w{:}z \to v{:}y$ for some $z \in [d]$.
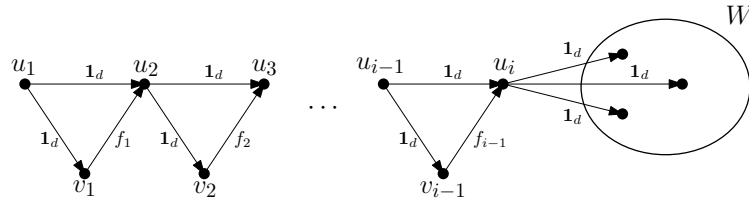
3 Permutation labels

In this section we prove our main result for permutation labels.

▶ **Theorem 4.** *For all $d \ge 2$, we have $R_p(d) \le 2d - 2$.*

First, we introduce a useful tool, generalizing a technique used in [4, 17] to the fixed-point cycle setting.

Let ℓ be a labeling of $\overleftrightarrow{K}_n$ that assigns to every edge e a permutation $\ell_e \colon [d] \to [d]$. Consider an edge $u \to v$ of $\overleftrightarrow{K}_n$ with label ℓ_{uv}. Let $(g_i)_i$ and $(h_i)_i$ denote the labels of the incoming and outgoing edges at v, respectively. A *shifting at $u \to v$* changes the labeling ℓ by replacing each g_i by $g_i \circ \ell_{uv}{}^{-1}$ and each h_i by $\ell_{uv} \circ h_i$. In particular, the label of $u \to v$ becomes the identity permutation $\mathbf{1}_d$. Let ℓ' be the resulting labeling. Observe that mappings along cycles remain unchanged, in particular $(\overleftrightarrow{K}_n, \ell')$ has a fixed-point cycle if and only if $(\overleftrightarrow{K}_n, \ell)$ does.

Figure 1 Illustration of the proof of Theorem 4.

Proof of Theorem 4. We start with the case $d = 2$. Suppose there is a permutation 2-labeling ℓ of $\overleftrightarrow{K}_3$ without a fixed-point cycle. The two possible labels are $\mathbf{1}_2$ and the function $\bar{\mathbf{1}}_2$ that maps $1 \mapsto 2$ and $2 \mapsto 1$. For each pair of vertices u, v the labels ℓ_{uv} and ℓ_{vu} must be distinct, otherwise we have a fixed-point cycle $u \to v \to u$. This means that the number of edges labeled $\bar{\mathbf{1}}_2$ in ℓ is precisely 3. Now consider two directed 3-cycles in $\overleftrightarrow{K}_3$ that form a partition of the edges. One of these cycles has to contain an even number of edges labeled $\bar{\mathbf{1}}_2$. Clearly, that cycle maps each value to itself, a contradiction. Thus, $R_p(2) \leq 2$.

Consider now the case $d \geq 3$; let $n \geq 2d - 1$, and let ℓ be an arbitrary permutation d-labeling of $\overleftrightarrow{K}_n$. We show that a fixed-point cycle exists.

We construct a "chain" consisting of vertices $u_1, \ldots, u_j$ and $v_1, \ldots, v_{j-1}$ (for some $j \leq d$) and transform the labeling so that each edge of the form $u_i \to u_{i+1}$ or $u_i \to v_i$ is labeled $\mathbf{1}_d$. By *step i* we mean either the edge $u_i \to u_{i+1}$ or the path $u_i \to v_i \to u_{i+1}$. Let $S_i \subseteq [d]$ denote the set of possible values to which 1 can be mapped along some path that concatenates steps $1, \ldots, i-1$. Observe that if $S_i = [d]$ then we are done, since $u_i \to u_1$ maps some $x \in [d]$ to 1, and adding this edge to the path from u_1 to u_i that maps 1 to x yields a fixed-point cycle.

We construct the chain step-by-step (see Figure 1), ensuring that $|S_i| \geq i$ for all i. If we reach $|S_i| = d$, then we are done, having used at most $2d - 1$ vertices.

Pick vertex $u_1 \in V(\overleftrightarrow{K}_n)$ arbitrarily. The condition is then trivially satisfied.

Assume now that we have identified vertices $u_1, \ldots, u_i$ and $v_1, \ldots, v_{i-1}$ of the chain. Let $W \subseteq V(\overleftrightarrow{K}_n)$ denote the set of vertices *not used* yet, and shift at all edges $u_i \to u$ for $u \in W$. Observe that no label along the chain is affected. Now consider all edges *between* vertices in W. There are two possible cases.

Case 1: If some edge $v \to u$ (for $u, v \in W$) maps some element $x \in S_i$ to some element $y \notin S_i$, then extend the chain with $u_{i+1} = u$ and $v_i = v$. The set of reachable values becomes $S_{i+1} \supseteq S_i \cup \{y\}$, establishing the claim for the next step.

Case 2: All edges within W map S_i to S_i, and consequently $[d] \setminus S_i$ to $[d] \setminus S_i$. Since the chain has used up $2i - 1 \leq 2|S_i| - 1$ vertices, the digraph induced by W has at least $2d - 1 - (2|S_i| - 1) \geq 2\,|[d] \setminus S_i|$ vertices. If $i \leq d - 2$, then $|[d] \setminus S_i| \geq 2$, and we can argue inductively that the digraph induced by W has a fixed-point cycle. If $i = d - 1$, then $|[d] \setminus S_i| = 1$ and $|W| \geq 2$, so we trivially have a fixed-point cycle. ◄

We next show the extension of the result to the case of labels from an arbitrary (not necessarily abelian) group.

▶ **Corollary 5.** *Let ℓ be a labeling of $\overleftrightarrow{K}_{2d-1}$ with elements of a (not necessarily abelian) group $(G, \cdot)$ of order d. Then there is a cycle whose labels multiply to the identity $1 \in G$.*

Proof. Let ℓ be a labeling that assigns elements of the group $(G, \cdot)$ of order d to edges of $\overleftrightarrow{K}_{2d-1}$. Construct a labeling ℓ' of $\overleftrightarrow{K}_{2d-1}$, assigning the function $x \mapsto x \cdot k$ to every edge with label k in ℓ. By Theorem 4, ℓ' has a fixed-point cycle. Suppose its labels are $f_1, \ldots, f_t$, where $f_i(x) = x \cdot k_i$ for all $i \in [t]$. Then $(f_1 \circ \cdots \circ f_t)(x) = x$ for some $x \in G$, which implies that $k_1 \cdot \cdots \cdot k_t = 1 \in G$. ◄

Finally, we prove that to guarantee a cycle whose labels compose to the identity permutation, we need a *super-polynomial* number of vertices.

▶ **Lemma 6.** *There exists a fixed $d_0 > 0$ such that for all $d \geq d_0$ and some $n \geq e^{\sqrt{d \ln d}}$, there is a labeling of $\overleftrightarrow{K}_n$ with permutations of $[d]$ such that there is no cycle whose labels compose to $\mathbf{1}_d$.*

Proof. For given $d > 0$, let $n = n(d)$ denote the *maximum order* of a permutation of $[d]$, that is, n is the largest integer for which there exists a permutation $\pi \colon [d] \to [d]$ such that $\pi^n = \mathbf{1}_d$ but $\pi^k \neq \mathbf{1}_d$ for all $1 \leq k < n$. The function $n(d)$ is *Landau's function*, and it is well known [16, 18] that $n(d) = e^{(1+o(1))\sqrt{d \ln d}}$. (One can obtain high-order permutations by combining cycles of different prime lengths.)

The construction is now similar to the one used earlier. Let $\pi \colon [d] \to [d]$ be a permutation of order n, let $V(\overleftrightarrow{K}_n) = [n]$, and label all edges (i, j) such that $i < j$ with $\mathbf{1}_d$ and all other edges with π. Since every cycle contains at least one and at most $n - 1$ edges of the form (i, j) with $i > j$, composing the labels along a cycle cannot yield $\mathbf{1}_d$. ◀

4   General function labels

Before proving our main theorem, we first present a weaker result, in order to introduce some techniques that will be used later. The result already improves the bound from [12], while using a similar argument.

▶ **Lemma 7.** *For all $d > 0$ we have $R_f(d) \leq d^3 - d^2 + d$.*

Proof. Suppose that $n \geq d^3 - d^2 + d + 1$, and let ℓ be an arbitrary d-labeling of $\overleftrightarrow{K}_n$. We show that a fixed-point cycle exists. For that, we proceed algorithmically.

Partition $V(\overleftrightarrow{K}_n)$ arbitrarily into two parts $V = \{v_1, \ldots, v_d\}$ and $U = \{u_1, \ldots, u_{d^3-d^2+1}\}$, and consider all vertices of U initially *unmarked*.

In the preprocessing phase, for each triplet $(i, x, y) \in [d-1] \times [d]^2$ in turn, check whether there exists an unmarked vertex $u \in U$ that routes $v_i{:}x$ to $v_{i+1}{:}y$. If yes, then *mark u*, and say that u is *responsible* for the triplet (i, x, y).

Observe that as we mark at most $(d-1)d^2$ vertices of U, we have at least one unmarked vertex remaining in U after the preprocessing phase. Let $c \in U$ be such a vertex. Consider now the walk that alternates between visiting c and visiting the vertices of V in order, giving rise to the valued walk $W = (c{:}c_0, v_1{:}x_1, c{:}c_1, v_2{:}x_2, \ldots, v_d{:}x_d, c{:}c_d)$, where $x_i, c_i \in [d]$ for all $i \in [d]$ and $c_0 = 1$. Since $|\{c_0, \ldots, c_d\}| \leq d$, there must be some i, j such that $0 \leq i < j \leq d$ and $c_i = c_j$, yielding the valued fixed-point *circuit* $C = (c{:}c_i, v_{i+1}{:}x_{i+1}, \ldots, v_j{:}x_j, c{:}c_j)$.

It remains to transform C into a cycle. For all k such that $i < k < j$ replace the subpath $v_k \to c \to v_{k+1}$ in C by $v_k \to c' \to v_{k+1}$, where $c' \in U$ is the *unique* marked vertex that is responsible for the triplet (k, x_k, x_{k+1}). Such a vertex must exist in U, for otherwise c itself would have been chosen as responsible for this triplet in the preprocessing phase. We have removed all occurrences of c in C but the first and last, and we have not changed the mappings of values; thus we obtain a fixed-point cycle. An efficient algorithm for finding this cycle is implicit in the proof. ◀

Next, we introduce a transformation that will be useful in the main proof.

Compression

Given a d-labeling ℓ of $\overleftrightarrow{K}_n$, define the *imageset* of a vertex v as $\mathrm{im}(v) = \{y \mid u{:}x \xrightarrow{\ell} v{:}y, \text{ for some } u, x\}$. In words, $\mathrm{im}(v)$ is the subset of values in $[d]$ that can be mapped to by edges to v.

We say that v is k-*compressed* if $|\mathrm{im}(v)| \leq k$. Observe that all vertices are trivially d-compressed, and the existence of a 1-compressed vertex would immediately yield a fixed-point cycle. Indeed, if v is 1-compressed, then the cycle $v \to u \to v$ maps the unique element in $\mathrm{im}(v)$ to itself for any $u \in V(\overleftrightarrow{K}_n)$.

We now describe the compression operation, illustrated in Figure 2. Let w be a k-compressed vertex for some $k \geq 2$ and $w' \in V(\overleftrightarrow{K}_n) \setminus \{w\}$. Suppose there exist two paths P_1 and P_2 from w to w', together with two distinct values $i_1, i_2 \in \mathrm{im}(w)$ and $j \in [d]$ such that P_1 maps i_1 to j and P_2 maps i_2 to j. Note that the sets of interior vertices of P_1 and P_2 do not need to be disjoint and that either of the paths may consist of a single edge.

Define the function $f\colon [d] \to [d]$ as follows: let $f(i_2) = j$; for all $x \in \mathrm{im}(w) \setminus \{i_2\}$, let $f(x) = y$, where P_1 maps x to y; for all $x \in [d] \setminus \mathrm{im}(w)$, choose $f(x)$ arbitrarily. Now, delete all vertices of P_1 and P_2, and add a new vertex $w^\star$ with edges to and from all remaining vertices. For all $v \in V(\overleftrightarrow{K}_n) \setminus (V(P_1) \cup V(P_2))$, if the edge $v \to w$ had the label g, then the edge $v \to w^\star$ gets the label $g \circ f$, and if an edge $w' \to u$ had the label h, then the edge $w^\star \to u$ gets the label h. The labels of edges not involving $w^\star$ remain unchanged.

We refer to this operation as *compressing P_1 and P_2 to $w^\star$*. Suppose we are left with n' vertices and observe that $n - n' \leq |V(P_1)| + |V(P_2)| - 3$, since P_1 and P_2 have common endpoints. Let ℓ' denote the resulting labeling of $\overleftrightarrow{K}_{n'}$. We prove two crucial properties.

▶ **Lemma 8.** *Suppose that P_1 and P_2 (with starting vertex w) are compressed to $w^\star$ in the way described above, resulting in a labeling ℓ' of $\overleftrightarrow{K}_{n'}$.*
  (i) *If w is k-compressed in $(\overleftrightarrow{K}_n, \ell)$, then $w^\star$ is $(k-1)$-compressed in $(\overleftrightarrow{K}_{n'}, \ell')$.*
  (ii) *If $(\overleftrightarrow{K}_{n'}, \ell')$ has a fixed-point cycle, then $(\overleftrightarrow{K}_n, \ell)$ has a fixed-point cycle.*

Proof.
  (i) Let $S \subseteq [d]$ denote the set of values to which values in $\mathrm{im}(w) \setminus \{i_2\}$ are mapped by P_1. Clearly $|S| \leq |\mathrm{im}(w)| - 1 \leq k - 1$. Since P_2 maps i_2 to j and $j \in S$ by construction (as P_1 maps i_1 to j), every edge $v \to w^\star$ maps all values in $[d]$ to S and thus $|\mathrm{im}(w^\star)| \leq |\mathrm{im}(w)| - 1$.
  (ii) If a fixed-point cycle in $(\overleftrightarrow{K}_{n'}, \ell')$ avoids $w^\star$, then it also exists in $(\overleftrightarrow{K}_n, \ell)$. Otherwise, suppose a cycle in $(\overleftrightarrow{K}_{n'}, \ell')$ contains the segment $v{:}x \to w^\star{:}y \to u{:}z$. Then, in $(\overleftrightarrow{K}_n, \ell)$, the edge $v \to w$ maps x to some value in $\mathrm{im}(w)$ that is mapped by P_1 or P_2 to $w'{:}y$, and $w' \to u$ maps y to z. Replacing $w^\star$ by P_1 or P_2 thus gives a fixed-point cycle in $(\overleftrightarrow{K}_n, \ell)$. Given a fixed-point cycle in $(\overleftrightarrow{K}_{n'}, \ell')$, a fixed-point cycle in $(\overleftrightarrow{K}_n, \ell)$ can be reconstructed (i.e., the compression can be undone) efficiently, with minor bookkeeping.

In the remainder of the section we prove our main theorem.

▶ **Theorem 2.** *For all $d > 0$ we have $R_f(d) \leq d^{2+o(1)}$.*

The high-level strategy is similar to the one used in the proof of Lemma 7. The main difference is that, instead of trying to build a cycle using a sequence of d designated vertices $(v_1, \ldots, v_d)$ and a well-chosen center-vertex (c), we pick a sequence of *far fewer* designated vertices, and use the structure imposed upon them by a special compressed vertex. We may fail to find a fixed-point cycle (or even a circuit) with any candidate center. In that case, however, we make progress by *compressing* the special vertex further. After repeating the
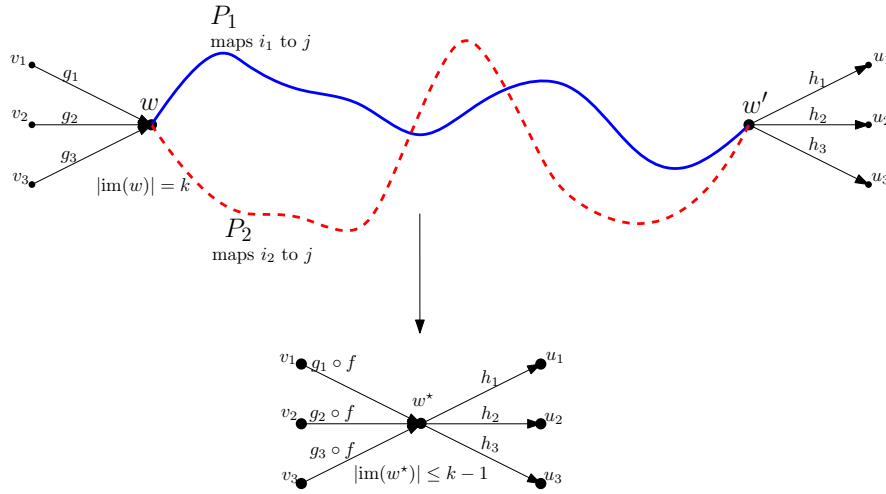
Figure 2 Compressing P_1 and P_2 to $w^\star$.

process sufficiently many times, we will have created a large number of highly compressed vertices. The mappings between these vertices are sufficiently restricted that we can find a fixed-point cycle in the digraph induced by them, through a *recursive application* of the same procedure.

The following lemma describes the key *win-win* step of the procedure: we either find a fixed-point cycle, or identify two paths that allow the compression of a vertex.

▶ **Lemma 9.** *Let $k \geq 2$, $n \geq 4d\lceil\frac{d}{k}\rceil^2 + 2\lceil\frac{d}{k}\rceil + 2$, and $w \in V(\overleftrightarrow{K}_n)$ be a k-compressed vertex. Then either $(\overleftrightarrow{K}_n, \ell)$ has a fixed-point cycle, or there exist a vertex $u \in V(\overleftrightarrow{K}_n) \setminus \{w\}$, values $i_1, i_2 \in \text{im}(w)$ with $i_1 \neq i_2$ and $x \in [d]$, and two paths on at most $4\lceil\frac{d}{k}\rceil + 2$ vertices each from $w{:}i_1$ and $w{:}i_2$ to $u{:}x$.*

Proof. Let $q = 2\lceil\frac{d}{k}\rceil + 1$ and arbitrarily fix a q-subset $V \subseteq V(\overleftrightarrow{K}_n) \setminus \{w\}$. Write $V = \{v_1, v_2, \ldots, v_q\}$ and $U = V(\overleftrightarrow{K}_n) \setminus (V \cup \{w\})$. Notice that $|U| \geq 4d\lceil\frac{d}{k}\rceil^2$.

Call a vertex $c \in U$ *valid* for (i, x, y), where $i \in [q-1]$ and $x, y \in [d]$, if c routes $v_i{:}x$ to $v_{i+1}{:}y$ and there are at least $q - 1$ vertices in U that route $v_i{:}x$ to $v_{i+1}{:}y$ (including c). Let k_c denote the number of triples (i, x, y) for which c is *not* valid. Double-counting yields

$$\sum_{c \in U} k_c \leq (q-1) \cdot d^2 \cdot (q-2) < 4d^2 \left\lceil\frac{d}{k}\right\rceil^2 .$$

Thus, by the pigeonhole principle, there exists a $c \in U$ with $k_c < \frac{4d^2\lceil\frac{d}{k}\rceil^2}{|U|} \leq d$. Fix such a c.

Recall that $|\text{im}(w)| = k$ and assume without loss of generality that $\text{im}(w) = [k]$. For each $i \in [k]$, we construct a (valued) walk W_i that starts with $v_1{:}\ell_{wv_1}(i)$ and visits $v_2, \ldots, v_q$ in that order, possibly taking a detour through c at every step.

Fix $i \in [k]$. We describe the construction of W_i by iteratively constructing prefixes W_i^j that start with v_1 and end with v_j. First, let $W_i^1 = v_1{:}x_1$, where $x_1 = \ell_{wv_1}(i)$.

For $1 < j \leq q$, suppose that W_i^{j-1} ends with $v_{j-1}{:}x_{j-1}$, and consider the values y_j, x_j reached in the valued path $v_{j-1}{:}x_{j-1} \to c{:}y_j \to v_j{:}x_j$. If c is valid for $(j-1, x_{j-1}, x_j)$, then let $W_i^j = W_i^{j-1} \to c{:}y_j \to v_j{:}x_j$. Otherwise, let $W_i^j = W_i^{j-1} \to v_j{:}x_j'$ for the appropriate x_j'. Finally, let $W_i = W_i^q$. Note that W_i contains at most $2q - 1$ vertices (including up to $q - 1$ occurrences of c).

Observe that in the process of constructing $W_1, \ldots, W_k$, we make $(q-1) \cdot k \geq 2d$ steps in total, where each step either adds c to the current walk, or finds that c is not valid for some triple $(j-1, x_{j-1}, x_j)$.

We claim that some vertex-value pair occurs at least twice in all the valued walks $W_1, W_2, \ldots, W_k$. Suppose not. Then, in particular, c occurs at most d times. Moreover, the triples $(j-1, x_{j-1}, x_j)$ for which c is not valid that we encounter in the construction of the walks must be pairwise distinct (if $(j-1, x_{j-1}, x_j)$ occurs twice in this way, then two distinct paths must contain $v_{j-1}{:}x_{j-1}$). This means that we encounter such triples at most k_c times in total. The total number of steps is then at most $d + k_c < 2d$, contradicting our earlier observation. This proves the claim.

First, consider the case where some $v{:}x$ occurs in *two different walks* W_{i_1}, W_{i_2}. Define W'_{i_1} and W'_{i_2} by removing all vertices after v from W_{i_1} and W_{i_2}, respectively. We turn W'_{i_1} and W'_{i_2} into (simple) paths as follows. Suppose c occurs more than once in W'_{i_1}. Suppose W'_{i_1} contains the segment $v_i{:}x_i \to c{:}y_i \to v_{i+1}{:}x_{i+1}$. If this is not the first occurrence of c, then we simply replace c by some vertex $c' \in U \setminus \{c\}$ that routes $v_i{:}x_i$ to $v_{i+1}{:}x_{i+1}$ and has not been used in this way before. Note that we do this at most $q-2$ times (once for each occurrence of c in W'_{i_1} except for the first one), and there are at least $q-2$ such vertices in $U \setminus \{c\}$ by construction, so the operation is well-defined. We proceed the same way for W'_{i_2} (recall that the paths needed for the compression operation need not be internally disjoint). Call the resulting (simple) paths W''_{i_1} and W''_{i_2}, respectively. Then $w{:}i_1 \to W''_{i_1}$ and $w{:}i_2 \to W''_{i_2}$ are the desired paths of at most $2q \leq 4\lceil \frac{d}{k} \rceil + 2$ vertices, and we are done.

Second, suppose that some $v{:}x$ occurs twice *in a single walk* $W = W_i$. Then $v = c$, since each v_i occurs only once in W. We now find a sub-walk of W that has a fixed point. Remove all vertices before the first occurrence of $c{:}x$ and all vertices after the second occurrence of $c{:}x$, and call the resulting valued walk W'. Clearly, W' is a fixed-point walk and contains at most $q-1$ occurrences of c (counting both start and end individually). Then, similarly as in the first case, transform W' into a simple cycle by replacing all occurrences of c, except at the start/end, by suitable vertices (at most $q-3$ of them) in $U \setminus \{c\}$. The result is a fixed-point cycle. ◀

We prove Theorem 2 via the following recurrence.

▶ **Lemma 10.** *For all $d \geq 2^9$, we have $R_f(d) < 5d^2 + 9d \log_2 d \cdot (R_f(\lfloor \sqrt{d} \rfloor) + 1)$.*

Proof. Set $n = 5d^2 + 9d \log_2 d \cdot (R_f(\lfloor \sqrt{d} \rfloor) + 1)$ and consider an arbitrary labeling ℓ of $\overleftrightarrow{K}_n$. We show that $(\overleftrightarrow{K}_n, \ell)$ contains a fixed-point cycle. We again proceed algorithmically. Our strategy is to perform transformations on $\overleftrightarrow{K}_n$ and ℓ, such as to compress vertices using Lemma 9. We say that a $\lfloor \sqrt{d} \rfloor$-compressed vertex is *fully compressed*.

We first observe that if we have more than $R_f(\lfloor \sqrt{d} \rfloor)$ fully compressed vertices, then we are done. Indeed, consider the subdigraph induced by the set F of vertices that are fully compressed. For all $u, v \in F$, restrict each function ℓ_{uv} to an arbitrary subset of $[d]$ of size $\lfloor \sqrt{d} \rfloor$ containing $\text{im}(u)$. Applying an arbitrary bijection from each such set to $[\lfloor \sqrt{d} \rfloor]$, transform the restricted labeling into a valid $\lfloor \sqrt{d} \rfloor$-labeling on the subdigraph induced by F without creating new fixed-point cycles. Then, if $|F| > R_f(\lfloor \sqrt{d} \rfloor)$, we can recursively find a fixed-point cycle in the induced subdigraph, from which we can recover a fixed-point cycle in $(\overleftrightarrow{K}_n, \ell)$.

Now we explain how we obtain the required number of fully compressed vertices. Let T be an arbitrary set of $R_f(\lfloor \sqrt{d} \rfloor) + 1$ vertices which are to be compressed and let S be the remaining set of vertices of $\overleftrightarrow{K}_n$. At each step, let $w \in T$ be any vertex that is *not* fully compressed; say w is k-compressed. Apply Lemma 9 to w and the subdigraph induced by $\{w\} \cup S$. If we find a fixed-point cycle, then we are immediately done by Lemma 8.

Otherwise, we find two paths P_1 and P_2 starting at w that we can compress into a single $(k-1)$-compressed vertex $w^\star$. Now set $T = T \setminus \{w\} \cup \{w^\star\}$ and $S = S \setminus (V(P_1) \cup V(P_2))$. Note that the size of S reduces by at most $8\lceil \frac{d}{k} \rceil + 1 \leq 8\frac{d}{k} + 9$.

We need to ensure that we always have enough vertices to apply Lemma 9. First we count the number of vertices that get removed from S throughout the process. Note that, for each k such that $\lfloor \sqrt{d} \rfloor + 1 \leq k \leq d$, we transform a k-compressed vertex of T into a $(k-1)$-compressed vertex of T at most $R_f(\lfloor \sqrt{d} \rfloor) + 1$ times, and every time we perform such an operation, we remove at most $8\frac{d}{k} + 9$ vertices from S. Thus, the number of vertices we remove throughout the process is at most

$$(R_f(\lfloor \sqrt{d} \rfloor) + 1) \sum_{k=\lfloor \sqrt{d} \rfloor + 1}^{d} \left(8\frac{d}{k} + 9 \right) \leq (R_f(\lfloor \sqrt{d} \rfloor) + 1)(8d \log_2 d + 9d - 1)$$

$$\leq (R_f(\lfloor \sqrt{d} \rfloor) + 1)(9d \log_2 d - 1),$$

where the first inequality uses the fact that $\sum_{k=1}^{d} 1/k \leq \log_2 d$, and the second inequality uses our assumption $\log_2 d \geq 9$.

Also accounting for the $R_f(\lfloor \sqrt{d} \rfloor) + 1$ vertices in T, it follows that the final set S has at least $5d^2$ remaining vertices, which is enough to ensure that we can apply Lemma 9. Indeed, the number of additional vertices needed to apply Lemma 9 is

$$4d\left\lceil \frac{d}{k} \right\rceil^2 + 2\left\lceil \frac{d}{k} \right\rceil + 1 \leq 4d(\sqrt{d} + 1)^2 + 2(\sqrt{d} + 1) + 1$$

$$\leq 5d^2,$$

since we always have $k \geq \lfloor \sqrt{d} \rfloor + 1 \geq \sqrt{d}$, and $d \geq 16$. Thus, the process terminates successfully, leaving us with the set T of $R_f(\lfloor \sqrt{d} \rfloor) + 1$ fully compressed vertices, using which we find a fixed-point cycle, as discussed above. ◄

Finally, we show that the above recurrence solves to $R_f(d) \in \mathcal{O}(d^2 \cdot 2^{(\log_2 \log_2 d)^2})$, thus implying Theorem 2.

▶ **Lemma 11.** *For all $d \geq 4$, we have $R_f(d) \leq 16 \cdot d^2 \cdot 2^{(\log_2 \log_2 d)^2}$.*

Proof. If $d < 2^{20}$, the claim follows by Lemma 7, since $16 \cdot d^2 \cdot 2^{(\log_2 \log_2 d)^2} \geq d^3 - d^2 + d$ for all $d < 2^{20}$. Assume therefore that $d \geq 2^{20}$. Denoting $C = 16$ we have:

$$
\begin{aligned}
R_f(d) &\leq 5d^2 + 9d\log_2 d \cdot (R_f(\lfloor \sqrt{d} \rfloor) + 1) & \text{(from Lemma 10)} \\
&\leq 5d^2 + 9d\log_2 d \cdot \left(C \cdot (\lfloor \sqrt{d} \rfloor)^2 \cdot 2^{(\log_2 \log_2 \lfloor \sqrt{d} \rfloor)^2} + 1 \right) & \text{(by induction)} \\
&\leq 5d^2 + 9d\log_2 d \cdot \left(C \cdot d \cdot 2^{(\log_2 \log_2 \sqrt{d})^2} + 1 \right) & \text{(using } \lfloor \sqrt{d} \rfloor \leq \sqrt{d}) \\
&\leq 6d^2 + 9C \cdot d^2 \log_2 d \cdot 2^{(\log_2 \log_2 \sqrt{d})^2} & \text{(using } 9\log_2 d \leq d, \text{ for } d \geq 52) \\
&\leq 10C \cdot d^2 \log_2 d \cdot 2^{(\log_2 \log_2 \sqrt{d})^2} & \text{(using } d \geq 2^6) \\
&= 10C \cdot d^2 \log_2 d \cdot 2^{((\log_2 \log_2 d) - 1)^2} \\
&= 10C \cdot d^2 \cdot 2^{((\log_2 \log_2 d) - 1)^2 + \log_2 \log_2 d} \\
&= C \cdot d^2 \cdot 2^{(\log_2 \log_2 d)^2} \cdot \left(10 \cdot 2^{1 - \log_2 \log_2 d} \right) \\
&= C \cdot d^2 \cdot 2^{(\log_2 \log_2 d)^2} \cdot (20/\log_2 d) \\
&\leq C \cdot d^2 \cdot 2^{(\log_2 \log_2 d)^2}. & \text{(using } d \geq 2^{20})
\end{aligned}
$$

This concludes the proof. ◄

Note that the constant factor 16 in Lemma 11 can be significantly reduced through a more careful optimization, but we have avoided this, preferring a simpler presentation.

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