Complete ZX-Calculi for the Stabiliser Fragment in Odd Prime Dimensions

Robert I. Booth
University of Edinburgh, Edinburgh, United Kingdom
Sorbonne Université, CNRS, LIP6, 4 place Jussieu, F-75005 Paris, France
LORIA CNRS, Inria Mocqua, Université de Lorraine, F-54000 Nancy, France

Titouan Carette
Université Paris-Saclay, Inria, CNRS, LMF, 91190, Gif-sur-Yvette, France

Abstract

We introduce a family of ZX-calculi which axiomatise the stabiliser fragment of quantum theory in odd prime dimensions. These calculi recover many of the nice features of the qubit ZX-calculus which were lost in previous proposals for higher-dimensional systems. We then prove that these calculi are complete, i.e. provide a set of rewrite rules which can be used to prove any equality of stabiliser quantum operations. Adding a discard construction, we obtain a calculus complete for mixed state stabiliser quantum mechanics in odd prime dimensions, and this furthermore gives a complete axiomatisation for the related diagrammatic language for affine co-isotropic relations.

2012 ACM Subject Classification Theory of computation → Quantum computation theory

Keywords and phrases ZX-calculus, completeness, quantum, stabiliser, qudits

Digital Object Identifier 10.4230/LIPIcs.MFCS.2022.24

Related Version *Full Version*: https://arxiv.org/abs/2204.12531 [14]

Funding *Robert I. Booth*: ANR VanQuTe project (ANR-17-CE24-0035) and Cisco University Research Program Fund.

Acknowledgements We thank Cole Comfort and Simon Perdrix for enlightening discussions.

1 Introduction

The ZX-calculus is a powerful yet intuitive graphical language for reasoning about quantum computing, or, more generally, about operations on quantum systems [23, 46]. It allows one to represent such quantum operations pictorially, and comes equipped with a set of rules which, in principle, make it possible to derive any equality between those pictures [3, 38, 42]. It has now had several applications in quantum information processing, from MBQC [32, 5] through quantum error correction codes [31, 29, 20, 33]. More recently, it has been used to obtain state-of-the-art optimisation techniques for quantum circuits [39, 28, 30] and faster classical simulation algorithms for general quantum computations [40].

Despite its origins in categorical quantum mechanics and the diagrammatic language for finite-dimensional linear spaces [1, 22, 23] the literature on the ZX-calculus has been concerned almost exclusively with small-dimensional quantum systems, and even then mostly with the case of two-dimensional quantum systems, or qubits. The qubit ZX-calculus is remarkable in its simple treatment of stabiliser quantum mechanics, along with the fact that any diagram can be treated purely graph-theoretically, without concern to its overall layout, and without losing its quantum-mechanical interpretation. Those proposals that go beyond qubits lose many of these nice features, and are significantly more complicated than the qubit case [43, 51, 9, 50, 49]. In particular, they eschew the prised "Only Connectivity Matters" (OCM) meta-rule, often cited as one of the key features in the qubit case. In these calculi, which can represent any linear map between the corresponding Hilbert spaces, it is also not necessarily obvious (at least, to us) how to pick out and work with the stabiliser fragment.

Stabiliser quantum mechanics is a simple yet particularly important fragment of quantum theory. While much less powerful than the full fragment – it can be efficiently classically simulated, even in odd prime dimensions [27] – it has seen significant study [34, 36] and forms the basis for a number of key methods in quantum information theory [35]. Operationally, it can be described as the fragment of quantum mechanics which is obtained if one allows only state preparation and measurement in the computational basis, and unitary operations from the *Clifford groups* [35]. In the qubit case, the stabiliser fragment of the ZX-calculus was proved complete in [3] while ignoring global scalars, and extended to include scalars in [4]. A simplified calculus further reducing the set of axioms of the calculus was presented in [6].

In this article, we present a simple family of ZX-calculi which are complete for stabiliser quantum mechanics in odd prime dimensions, and which recover as many of the nice features of the qubit calculus as possible. In odd prime dimensions, stabiliser quantum mechanics can be given a particularly nice graphical presentation, owing to the group-theory underlying the corresponding Clifford groups [41, 2, 27]. We then give this calculus a set of rewrite rules that is complete, i.e. rich enough to derive any equality of stabiliser quantum operations. In particular, it is a design priority to recover OCM, and to make explicit the stabiliser fragment and its group-theoretical underpinnings. Adding a discard construction [18], we obtain a universal and complete calculus for mixed state stabiliser quantum mechanics in odd prime dimensions. By previous work [26], this gives a complete axiomatisation for the related diagrammatic language for affine co-isotropic relations, while still maintaining OCM.

Although we do not do so here, these calculi can naturally be extended to represent much larger fragments of quantum theory, up to the entire theory in odd prime dimensions [51]. However, finding a complete axiomatisation for such calculi will presumably be a much more complicated task, and we leave this for future work.

All proofs and additional appendices are available in the full version of the paper [14].

2 Stabiliser quantum mechanics in odd prime dimensions

Throughout this paper, p denotes an arbitrary odd prime, and $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ the ring of integers with arithmetic modulo p. We also put $\omega := e^{i\frac{2\pi}{p}}$, and let $\mathbb{Z}_p^*$ be the group of units of $\mathbb{Z}_p$. Since p is prime, $\mathbb{Z}_p$ is a field and $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$ as a set. We also have need of the following definition:

$$\chi_p(x) = \begin{cases} 1 & \text{if} \quad \text{there is no } y \in \mathbb{Z}_p \text{ s.t. } x = y^2; \\ 0 & \text{otherwise;} \end{cases} \tag{1}$$

which is just the characteristic function of the complement of the set of squares in $\mathbb{Z}_p$.

The Hilbert space of a qupit [35, 52] is $\mathcal{H} = \text{span}\{|m\rangle \mid m \in \mathbb{Z}_p\} \cong \mathbb{C}^p$, and we write $U(\mathcal{H})$ the group of unitary operators acting on $\mathcal{H}$. We have the following standard operators on $\mathcal{H}$, also known as the *clock* and *shift* operators: $Z\,|m\rangle := \omega^m\,|m\rangle$ and $X\,|m\rangle := |m+1\rangle$ for any $m \in \mathbb{Z}_p$. In particular, note that $ZX = \omega XZ$.

We call any operator of the form $\omega^k X^a Z^b$ for $k, a, b \in \mathbb{Z}_p$ a *Pauli operator*. We say a Pauli operator is *trivial* if it is proportional to the identity. The collection of all Pauli operators is denoted $\mathscr{P}_1$ and called the *Pauli group*. For $n \in \mathbb{N}^*$, the *generalised Pauli group* is $\mathscr{P}_n := \bigotimes_{k=1}^n \mathscr{P}_1$.

Of particular importance to us are the *(generalised) Clifford groups*. These are defined, for each $n \in \mathbb{N}^*$, as the normaliser of $\mathscr{P}_n$ in $U(\mathcal{H}^{\otimes n})$: C is a Clifford operator if for any $P \in \mathscr{P}_n$, $CPC^\dagger \in \mathscr{P}_n$. It is clear that that every Pauli operator is Clifford, but there are non-Pauli Clifford operators. An important example is the *Fourier* gate: $F\,|m\rangle = \frac{1}{\sqrt{d}} \sum_{n\in\mathbb{Z}_p} \omega^{mn}\,|n\rangle$

which is such that $FXF^\dagger = Z \quad \text{and} \quad FZF^\dagger = X^{-1}$. We also need the *phase* gate: $S\,|m\rangle = \omega^{2^{-1}m(m+1)}\,|m\rangle$ such that $SXS^\dagger = \omega XZ \quad \text{and} \quad SZS^\dagger = Z$. Yet another important example is the *controlled-phase* gate, which acts on $\mathcal{H} \otimes \mathcal{H}$, $E\,|m\rangle\,|n\rangle \coloneqq \omega^{mn}\,|m\rangle\,|n\rangle$. It is important to emphasise a key difference between the qupit and the qubit case: when $p \neq 2$, none of these operators are self-inverse. In fact, if Q is a Pauli and I the identity operator on $\mathcal{H}$, we have $Q^p = I$, $E^p = I \otimes I$ and $F^4 = I$.

As a side note, these equations imply that both X and Z, and in fact every Pauli, have spectrum $\{\omega^k \mid k \in \mathbb{Z}_p\}$. As a result, we denote $|k : Q\rangle$ the eigenvector of a given Pauli Q associated with eigenvalue ω^k, and furthermore use the notation $|k, \dots, k : Q\rangle = \bigotimes_{k=1}^n |k : Q\rangle$. It follows from the definition of Z that we can identify $|k : Z\rangle = |k\rangle$.

Now, for any $\alpha \in [0, 2\pi)$, the operator $e^{i\alpha} I$ is Clifford. However, we want to construct calculi with a finite axiomatisation. As a result, the diagrams in the calculus are countable and this makes it impossible for us to represent all such phases $e^{i\alpha}$. Unfortunately, finding a group of phases that behaves well diagrammatically is somewhat inconvenient, and involves some elementary number theory. This is why, for an odd prime p, we consider the group of phases generated by the composition of the previously defined Clifford gates. Explicitly, it is given by:

if $p \equiv 1 \mod 4$,	if $p \equiv 3 \mod 4$,
$\mathbb{P}_p \coloneqq \left\{(-1)^s \omega^t \mid s, t \in \mathbb{Z}\right\}$	$\mathbb{P}_p \coloneqq \left\{i^s \omega^t \mid s, t \in \mathbb{Z}\right\}$

This of course covers all cases since p is odd. Then, we restrict our attention to the *reduced* Clifford group, $\mathscr{C}_n = \{\lambda C \mid \lambda \in \mathbb{P}_p, C \in U(\mathcal{H}^{\otimes n}) \text{ is Clifford and special unitary}\}$. We call $\mathscr{C}_1^{\otimes n}$ the *local Clifford group* on n qupits. It is clear from these examples that $\mathscr{C}_n$ is strictly larger than $\mathscr{C}_1^{\otimes n}$, but it turns out to not be that much larger:

▶ **Proposition 1** ([21, 41]). *The reduced Clifford group $\mathscr{C}_n$ is generated by the gate-set $\{F_j, S_j, E_{j,k} \mid j, k = 1, \dots, n\}$.*

Stabiliser quantum mechanics can be operationally described as the fragment of quantum mechanics in which the only operations allowed are initialisations and measurements in the eigenbases of Pauli operators, and unitary operations from the generalised Clifford groups. As before, we restrict our attention to the fragment of stabiliser quantum mechanics where only unitary operations from the reduced Clifford group are allowed. Scalars are then taken from the monoid $\mathbb{G}_p \coloneqq \{0, \sqrt{p^r}\lambda \mid r \in \mathbb{Z}, \lambda \in \mathbb{P}_p\}$. Little is lost for the description of quantum algorithms, since we can always simplify by a global phase to make the Clifford generators special unitary. Thus, we can embed any stabiliser circuit into the calculus, and then calculate the relative phases of different branches of a computation without restriction.

▶ **Definition 2.** *The symmetric monoidal category* Stab_p *has as objects $\mathbb{C}^{pn}$ for each $n \in \mathbb{N}$, and morphisms generated by:*

- $\mathbb{C} \to \mathbb{C}^p : \lambda \mapsto \lambda\,|0\rangle$;
- $\mathbb{C}^{pn} \to \mathbb{C}^{pn} : |\psi\rangle \mapsto U\,|\psi\rangle$ *for any* $U \in \mathscr{C}_n$;
- $\mathbb{C}^p \to \mathbb{C} : |\psi\rangle \mapsto \langle 0|\psi\rangle$.

The monoidal product is given by the usual tensor product of linear spaces.

It is clear that Stab_p is a subcategory of the category FLin of finite dimensional $\mathbb{C}$-linear spaces; it is also a PROP.

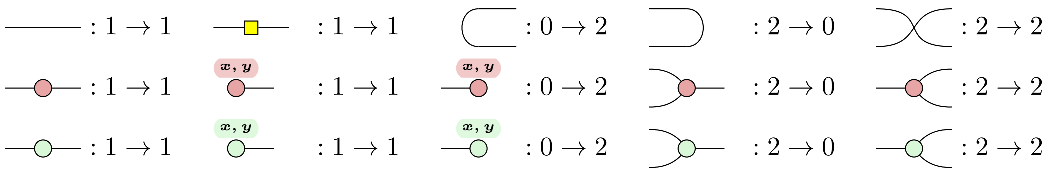
3 A ZX-calculus for odd prime dimensions

In this section, we present our family of ZX-calculi, with one for each odd prime. Relying on some of the group theoretical properties of the qupit Clifford groups, we can give a relatively simple presentation of the calculi, which avoids the need to explicitly consider rotations in p-dimensional space, significantly simplifing the presentation compared to previous work [43]. These calculi are also constructed in order to satisfy the property of *flexsymmetry*, proposed in [15, 16], and which allows one to recover the OCM meta-rule. OCM is an intuitively desirable feature for the design of a graphical language; anecdotally, it greatly simplifies the human manipulation of diagrams, including in the proofs of this paper. More formally, it means that the equational theory can be formalised in terms of double pushout rewriting over graphs rather than over hypergraphs as is necessary in the more general theory [11, 10, 12].

Another key concern is the issue of completeness, which we begin to address in this article. Outside of qubits [38, 42, 47], there has so far been a complete axiomatisation only for the stabiliser fragment in dimension $p = 3$ [48]. We present an axiomatisation which is complete for the stabiliser fragment for any odd prime p, and leave the general case for future work.

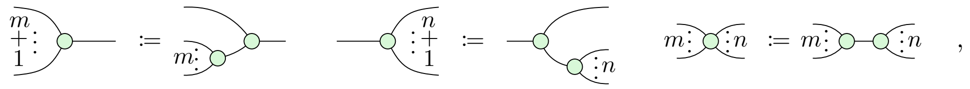
3.1 Generators

For any odd prime p, consider the symmetric monoidal category $\mathsf{ZX}_p^{\mathrm{Stab}}$ with objects $\mathbb{N}$ and morphisms generated by:



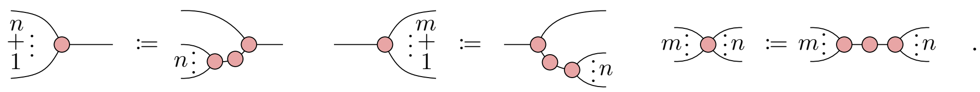
where $x, y \in \mathbb{Z}_p$. We also introduce a generator $\star : 0 \to 0$ to simplify the calculus; it will correspond to a scalar whose representation in terms of the other generators depends non-trivially on the dimension p. Morphisms are composed by connecting output wires to input wires, and the monoidal product is given on objects by $n \otimes m = n + m$ and on morphisms by vertical juxtaposition of diagrams.

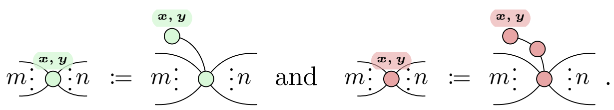
We extend this elementary notation with a first piece of syntactic sugar, which is standard for the ZX-calculus family: *green spiders* are defined inductively, for any $m, n \in \mathbb{N}$, by



and it is clear that these have types $m + 1 \to 1$, $1 \to n + 1$ and $m \to n$ respectively.

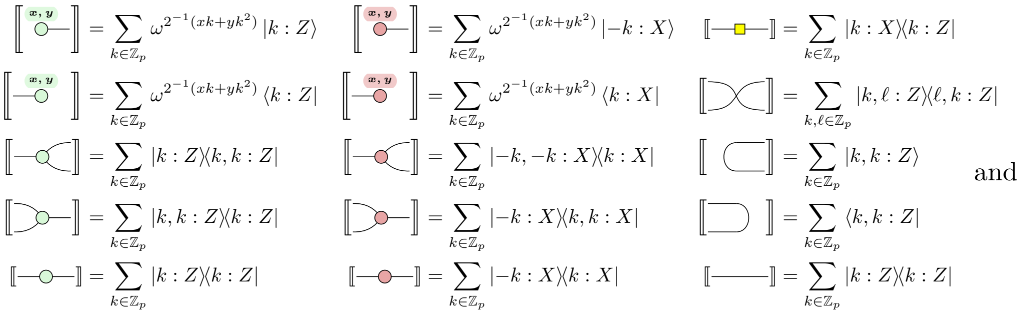
Red spiders are defined analogously to ZH-calculus harvestmen:



Labelled spiders are given by: 

3.2 Standard interpretation and universality

The standard interpretation of a $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram is a symmetric monoidal functor $[\![-]\!] : \mathsf{ZX}_p^{\mathrm{Stab}} \to \mathsf{FLin}$ (the category of finite-dimensional complex Hilbert spaces). It is defined on objects as $[\![m]\!] := \mathbb{C}^{p \times m}$, and on the generators of the morphisms as:

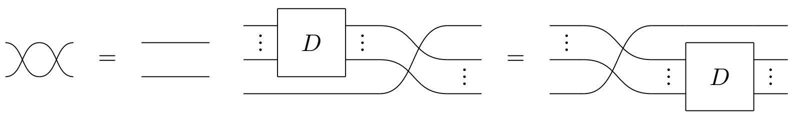


$[\![\star]\!] = -1$. Then, by the functoriality of the standard interpretation, we deduce that $[\![m : \overset{x,y}{\cdot} : n]\!] = \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |k:Z\rangle^{\otimes n} \langle k:Z|^{\otimes m}$ and $[\![m : \overset{x,y}{\cdot} : n]\!] = \sum_{k\in\mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |-k:X\rangle^{\otimes n} \langle k:X|^{\otimes m}$.

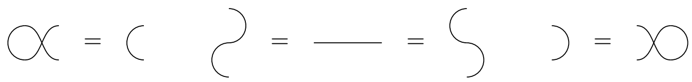
▶ **Theorem 3** (Universality). *The standard interpretation $[\![-]\!]$ is universal for the qupit stabiliser fragment, i.e. for any stabiliser operation $C : \mathbb{C}^{pm} \to \mathbb{C}^{pn}$ there is a diagram $D \in \mathsf{ZX}_p^{\mathrm{Stab}}$ such that $[\![D]\!] = C$. Put formally, the co-restriction of $[\![-]\!]$ to Stab_p is full.*

3.3 Axiomatisation

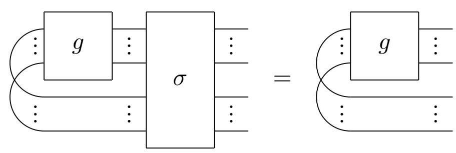
We now begin to introduce rewrite rules with which to perform a purely diagrammatic reasoning. By doing so we are in fact describing a PROP by generators and relations [7], thus the swap is required to verify the following properties:



Note that the last equation is required to hold for any diagram $D : n \to m$. This property states that our diagrams form a symmetric monoidal category. Furthermore, we want this category to be self-dual compact-closed, hence the cup and cap must verify:



Furthermore, as long as the connectivity of the diagram remains the same, vertices can be freely moved around without changing the standard interpretation of the diagram. This is a consequence of the fact that we require our generators to be *flexsymmetric*, as shown in [15]. This amounts to imposing that all generators except the swap verify:



where $\sigma : n + m \to n + m$ stands for any permutation of the wires involving swap maps. We will consider all the previous rules as being purely structural and will not explicitly state their use. Using these rules, we can in fact deduce that both the green and red spiders (and

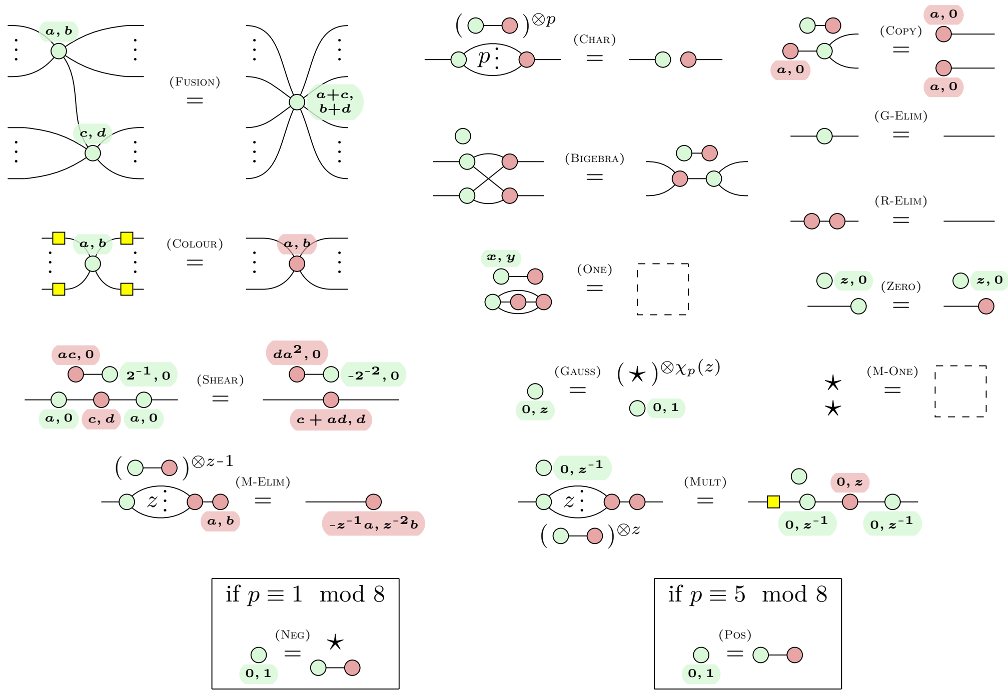


Figure 1 A presentation of the equational theory zx_p, which is sound and complete for the stabiliser fragment. The equations hold for any $a, b, c, d \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^*$. χ_p is the characteristic function of the complement of the set of squares in $\mathbb{Z}_p$, defined in equation (1).

their labelled varieties) are themselves flexsymmetric. This means that the language follows the OCM meta-rule, and we can formally treat any $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram as a graph[1] whose vertices are the spiders, and whose edges are labelled by the $1 \to 1$ generators of the language.[2]

Figure 1 presents the remainder of the equational theory zx_p, which as we shall see axiomatises the stabiliser fragment of quantum mechanics in the qupit ZX-calculus. Firstly though, we must be sure that all of these rules are sound for the standard interpretation, i.e. it should not be possible to derive an equality of diagrams whose quantum mechanical interpretations are different.

▶ **Theorem 4** (Soundness). *The equational theory zx_p is sound for $[\![-]\!]$, i.e., for any $A, B \in \mathsf{ZX}_p^{\mathrm{Stab}}$, $\mathrm{zx}_p \vdash A = B$ implies $[\![A]\!] = [\![B]\!]$. Put formally, $[\![-]\!]$ factors through the projection $\mathsf{ZX}_p^{\mathrm{Stab}} \to \mathsf{ZX}_p^{\mathrm{Stab}}/\mathrm{zx}_p$.*

This set of rewriting rule also turns out to be also complete:

▶ **Theorem 5** (Completeness). *The equational theory zx_p is complete for $[\![-]\!]$, i.e., for any $A, B \in \mathsf{ZX}_p^{\mathrm{Stab}}$, $[\![A]\!] = [\![B]\!]$ implies $\mathrm{zx}_p \vdash A = B$.*

The proof of Theorem 5 will be the object of the following sections.

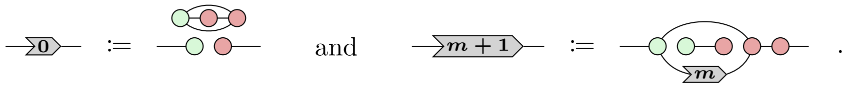
[1] Note that the graphs in question must be allowed to have loops and parallel edges, so are perhaps better called *pseudographs* or *multigraphs*.

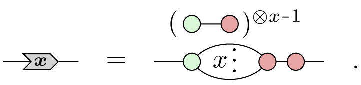
[2] There is a small ambiguity: $1 \to 1$ spiders can be treated as either edges or vertices. When considering diagrams, it matters little which, since any given graph is always to be understood as one of the many equivalent $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagrams constructed formally out of the generators. Any computer implementation of the calculus will have to carefully resolve this ambiguity in its internal representation.

3.4 Syntactic sugar for multi-edges

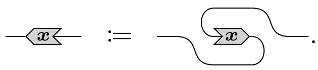
Before we move into the proof of completeness, we introduce some syntactic sugar to the language. Given axiom (CHAR), in $\mathsf{ZX}_p^{\mathrm{Stab}}$, unlike the qubit case, spiders of opposite colours can be connected by more that one edge, and these multi-edges cannot be simplified. We therefore add some syntactic sugar to represent such multi-edges. These constructions add no expressiveness to the language, and are simply used to reduce the size of some recurring diagrams. They are shamelessly stolen from previous work [13, 53, 17, 19], and we use them to obtain a particularly nice representation of qupit graph states [54]. Graph states a central role in our proof of completeness, as they have in previous completeness results of the stabiliser fragments for dimensions 2 and 3 [3, 30, 48]. In particular, these constructions permit a nice presentation of how graph states evolve under local Clifford operations.

Firstly, we extend $\mathsf{ZX}_p^{\mathrm{Stab}}$ by *multipliers*, which are defined inductively by:

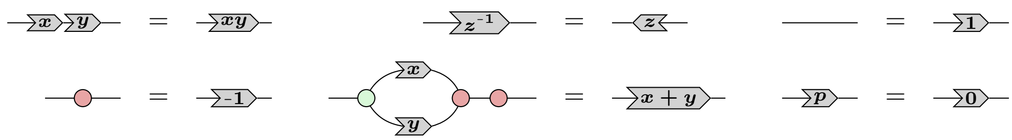


Explicitly, then, for $x \in \mathbb{Z}_p^*$, 

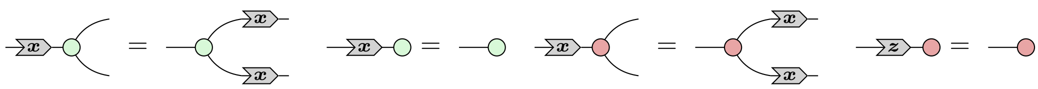
We also define inverted multipliers, using the standard notation for graphical languages based on symmetric monoidal categories:

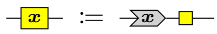
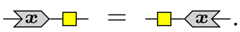


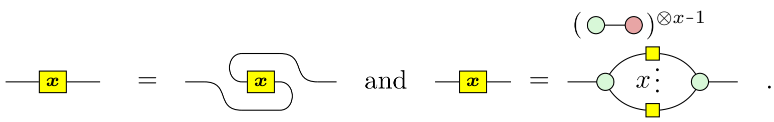
▶ **Proposition 6.** *Multipliers verify the following equations under* zx_p*: for any* $x, y \in \mathbb{Z}_p$ *and* $z \in \mathbb{Z}_p^*$,



which amounts to saying that the multipliers form a presentation of the field $\mathbb{Z}_p$*. They also verify the following useful copy and elimination identities:*

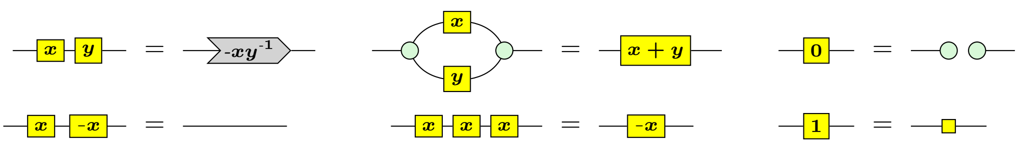


We can also unambiguously define *Fourier boxes*:  since . Fourier boxes exactly match the labelled "Hadamard" boxes introduced for the qutrit case [45] when $p = 3$. Owing to the flexsymmetry of the language, we have that for any $x \in \mathbb{Z}_p$,



(2)

▶ **Proposition 7.** zx_p *proves the following equations:*



(3)

4 Completeness

We now have all the necessary tools to show that our equational theory is complete. The structure of our proof is similar to the one used to show the completeness in the qubit case [3]. However, if the overall scheme is very similar, each step separately can involve different approaches more suited to the qupit situation. The plan is as follows. We identify a family of scalars, called *elementary scalars*, which correspond to those which appear when applying the rewrites of zx_p. We first show the completeness up to non-zero elementary scalars, which allows us to work with simpler diagrams without taking care of all the invertible scalars appearing along the way. Then, we show the completeness for elementary scalars independently, leading to a general proof of completeness. The proof of completeness up to non-zero elementary scalars goes something like this:

1. Take two diagrams with the same interpretation.
2. Put them into a simplified form, called the *GS+LC* form in [3].
3. Define the notion of simplified GS+LC form for a pair of diagrams in which some vertices are marked. Then show that in a simplified GS+LC pair if a vertex is marked on one side, it must also be marked on the other side, else the two diagrams cannot have the same interpretation.
4. Show that two diagram forming an rGS+LC pair such that their marked vertices matches and having the same interpretation are equal modulo the equational theory.

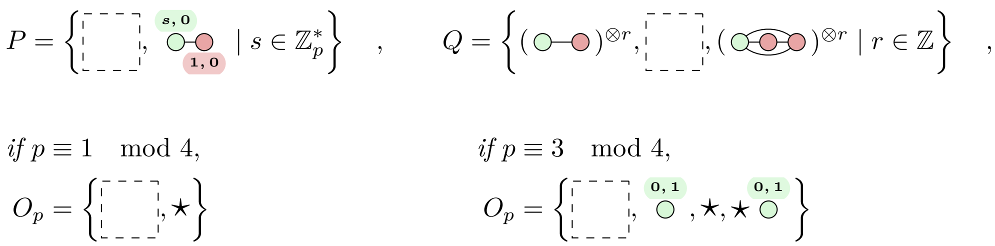
4.1 Elementary scalars

The following is standard from categorical quantum mechanics:

▶ **Lemma 8.** *If $A, B \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,0]$, then $[\![A \otimes B]\!] = [\![A]\!] \cdot [\![B]\!] = [\![A \circ B]\!]$, where $\cdot$ is the usual multiplication on $\mathbb{C}$ restricted to the monoid $\mathbb{G}$.*

Now, as when we were defining the group of phases $\mathbb{P}_p$, the set of normal forms for phases must depend on the prime p in question:

▶ **Definition 9.** *An* elementary scalar *is a diagram $A \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,0]$ which is a tensor product of diagrams from the collection $O_p \cup P \cup Q$: where*



If $A, B \in \mathsf{ZX}_p^{\mathrm{Stab}}$, we say that A and B are equal up to an elementary scalar *if there is an elementary scalar C such that $A = B \otimes C$. In that case, we write $A \simeq B$.*

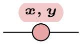
Comparing with the definition of $\mathbb{G}$, the interpretation of the elements of P correspond to powers of ω, the elements of Q to (possible negative) powers of $\sqrt{p}$, and the elements of O_p to powers of -1 or i depending on the value of p. This remark will naturally lead to the normal for for scalars in a few sections.

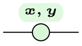
Now, as written, equality up to an elementary scalar might seem like a relation that is not symmetric and therefore not an equivalence relation.

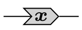
▶ **Proposition 10.** *Every elementary scalar $C \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,0]$ has a multiplicative inverse, i.e. an elementary scalar $C^{-1} \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,0]$ such that $C \otimes C^{-1} = C \circ C^{-1} =$ ⬚.*

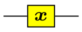
In light of this fact, if $A \simeq B$, there is an elementary scalar C such that $A = B \otimes C$, and then $B = B \otimes C \otimes C^{-1} = A \otimes C^{-1}$, so that $B \simeq A$.

▶ **Proposition 11.** *Every equation in zx_p can be loosened to equality up to an elementary scalar by erasing every part of the LHS and RHS diagrams which is disconnected from the inputs and outputs.*

Probably the most important case of equality up to elementary scalars is the completeness of the single-qupit Clifford groups, on which the entire proof of completeness of the calculus rests. The fragment of $\mathsf{ZX}_p^{\mathrm{Stab}}$ which corresponds to $\mathscr{C}_1$ is that generated by the $1 \to 1$ diagrams: 

, 

, 

and 

. We call any such diagram a *single-qupit Clifford diagram* or $\mathscr{C}_1$-diagram.

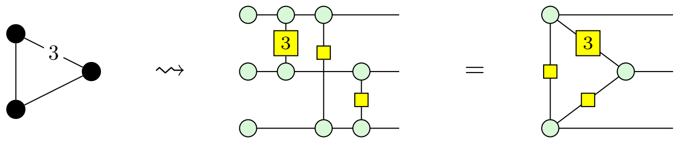
▶ **Proposition 12.** *If $A \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,0]$ is a single-qupit Clifford diagram, then zx_p proves that*

(4)

for some $s, t, u, v \in \mathbb{Z}_p$ and $w \in \mathbb{Z}_p^*$. Furthermore, this form is unique.

4.2 Relating stabiliser diagrams to graphs

The completeness proof begins by relating every diagram to simpler one in the form of a graph state diagram. Graph states are defined as usual – diagrammatically, we have an embedding of graphs given (informally) by:



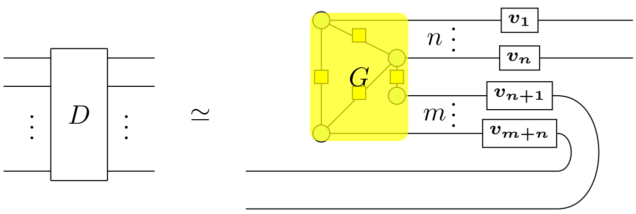
(5)

More specifically, we want to relate the diagram to the following form:

▶ **Definition 13** ([3, 48]). *A* GS+LC diagram *is a $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram which consists of a graph state diagram with arbitrary single-qupit Clifford operations applied to each output. These associated Clifford operations are called* vertex operators.

▶ **Proposition 14.** *Every $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram $0 \to n$ can be rewritten, up to elementary scalars, to GS+LC form under zx_p.*

In other words, zx_p proves that, for any stabiliser $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram $D : m \to n$, there is a graph G on $m + n$ vertices and a set $(v_k)_{k=1}^{m+n}$ of $\mathscr{C}_1$-diagrams such that



(6)

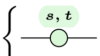
and we need only consider the question of whether zx_p can prove the equality of two GS+LC diagrams.

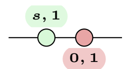
4.3 Completeness modulo elementary scalars

Now, as a first step, we show that zx_p can normalise any pair of diagrams with equal interpretations, up to elementary scalars. In particular, as was shown in the previous section, we can relax zx_p to reason about equality up to elementary scalars by simply ignoring the scalar part of each rule, and make free use of the "scalarless" equational theory. We will take care of the resulting scalars in the next section.

This part of the completeness proof follows the general ideas of [3]. The first step on the way to completeness is to note that, considering a diagram in GS+LC form, where the vertex operators have been normalised, we can obtain a yet more reduced diagram by absorbing as much as possible of the vertex operators into local scalings and local complementations. We then obtain the following form for the vertex operators:

▶ **Definition 15** ([3]). *A $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram is in* reduced *GS+LC (or rGS+LC) form if it is in GS+LC form, and furthermore:*

1. *All vertex operators belong to the following set:* $R = \Bigg\{$ 

 , 

 $\Bigg|\ s,t \in \mathbb{Z}_p \Bigg\}$.
2. *Two adjacent vertices do not have vertex operators that both include red spiders.*

▶ **Proposition 16.** *If $D \in \mathsf{ZX}_p^{\mathrm{Stab}}$ is a Clifford diagram, then there is a diagram $G \in \mathsf{ZX}_p^{\mathrm{Stab}}$ in rGS+LC form such that $\mathrm{zx}_p \vdash D \simeq G$.*

Then, given two diagrams with equal interpretations, taking them both to rGS+LC makes the task of comparing the diagrams considerably easier. In particular, we can guarantee that the corresponding vertex operators in each diagram always have matching forms:

▶ **Definition 17** ([3]). *A pair of rGS+LC of the same type (i.e. whose graphs have the same vertex set V) is said to be* simplified *if there is no pair of vertices $q, p \in V$ such that q has a red vertex operator in the first diagram but not the second, q has a red vertex operator in the second diagram but not the first, and q and p are adjacent in at least one of the diagrams.*

▶ **Proposition 18.** *Any pair A, B of rGS+LC diagrams of the same type (i.e. on the same vertex set) can be simplified.*

For the sake of clarity, we shall say that a vertex operator (or equivalently, the vertex itself) is *marked* if it contains a red spider (i.e. it belongs to the right-hand form of definition 15). Then, two diagrams with the same interpretation can always be rewriten so that the marked vertices match:

▶ **Proposition 19.** *Let $C, D \in \mathsf{ZX}_p^{\mathrm{Stab}}$ be a simplified pair in rGS+LC form, then $[\![C]\!] = [\![D]\!]$ only if the marked vertices in C and D are the same.*

We have enough control over the pair of diagrams to finish the completeness proof:

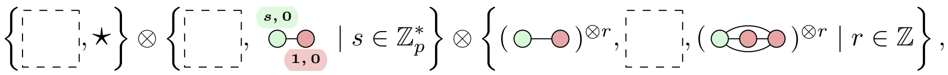
▶ **Theorem 20.** *zx_p is complete for $[\![-]\!]$, i.e. if for any pair of diagrams $A, B \in \mathsf{ZX}_p^{\mathrm{Stab}}[0,n]$ with $n \neq 0$, $[\![A]\!] = [\![B]\!]$, then $\mathrm{zx}_p \vdash A \simeq B$.*

4.4 Completeness of the scalar fragment

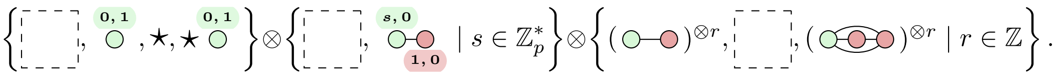
Finally, we are ready to leap-frog off of the previous section into the full completeness (including scalars). First, we need to find a normal form for diagrams which evaluate to 0. In fact, we need pick one normal form for each type $m \to n$:

▶ **Proposition 21.** *The zero scalar "destroys" diagrams: for any $m, n \in \mathbb{N}$ and $D \in \mathsf{ZX}_p^{\mathrm{Stab}}[m,n]$, ⊗ m D n = m n. We take the RHS diagram to be the "zero" diagram of type $m \to n$.*

A scalar diagram is in *normal form* if it is either the zero scalar, or it belongs to the set



when $p \equiv 1 \mod 4$, or to the set



when $p \equiv 3 \mod 4$. It is straightforward to see, by evaluating $[\![-]\!]$ on each element, that these sets contain exactly one diagram for each scalar in $\mathbb{G}_p \setminus \{0\}$ (and the zero scalar corresponds to $0 \in \mathbb{G}_p$).

▶ **Theorem 22.** zx_p *proves any scalar diagram equal to a scalar in normal form (depending on the congruence of p modulo 4), or to the zero scalar .*

The completeness for the whole stabiliser fragment follows immediately, combining theorems 20 and 22:

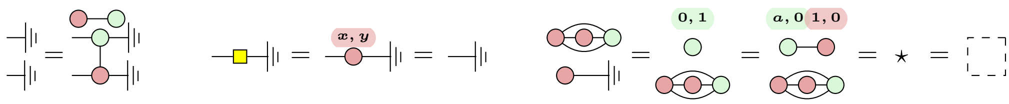
▶ **Theorem 23.** *The equational theory zx_p is complete for Stab_p, i.e. for any $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagrams A and B, if $[\![A]\!] = [\![B]\!]$, then $\mathrm{zx}_p \vdash A = B$.*

5 Mixed states and co-isotropic relations

In this last section we use the work of [18] to extend our completeness result to the mixed-state case. We then unravel the connection to the Lagrangian relation investigated in [26].

5.1 A complete graphical language for CPM(Stab_p)

We now extend our completeness result form Stab_p to CPM(Stab_p), the category of completely positive maps corresponding to mixed state stabiliser quantum mechanics, see [44, 23] for a formal definition. We will rely on the discard construction of [18] to define a graphical language $(\mathsf{ZX}_p^{\mathrm{Stab}})^{\ground}$. It consists in adding to the equational theory one generator, the discard $\ground : 1 \to 0$ and equations stating that this generator erases all isometries. In Stab_p, the isometries are generated by a small family of diagrams, the equations to add are then:



A new interpretation $[\![\_]\!] : \mathsf{ZX}_p^{\mathrm{Stab}\,\ground} \to$ CPM(Stab_p) is defined as $[\![\ground]\!] : \rho \mapsto \mathrm{Tr}(\rho)$ for the ground and for all $\mathsf{ZX}_p^{\mathrm{Stab}}$-diagram $D : n \to m$:

$$[\![D]\!]^{\ground} : \rho \mapsto [\![D]\!]^{\dagger} \rho [\![D]\!].$$

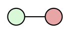
Corollary 22 of [18] provides a sufficient condition for the previous construction to extend to a universal and complete graphical language for Stab_p into a universal complete graphical language for $\mathrm{CPM}(\mathsf{Stab}_p)$. This condition is for Stab_p to have *enough isometries* in the sense of [18], By similar arguments as is the qubit case, we can show that Stab_2 indeed has enough isometries and then it follows that:

▶ **Theorem 24.** $(\mathsf{ZX}_p^{\mathrm{Stab}})^{\doteqdot}$ *is universal and complete for* $\mathrm{CPM}(\mathsf{Stab}_p)$.

5.2 Co-isotropic relations

It has been shown in [26, 25, 24] that $\mathrm{CPM}(\mathsf{Stab}_p)$ is equivalent to the category of affine co-isotropic relations up to scalars. More formally, we endow $\mathbb{Z}_p^2$ with the symplectic form: $\omega\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right) = ad - bc$, and $\mathbb{Z}_p^{2m} = \bigoplus_m \mathbb{Z}_p^2$ with the direct sum symplectic form.

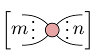
▶ **Definition 25.** *The symmetric monoidal category* $\mathsf{AffCoIsoRel}_{\mathbb{Z}_p}$ *has as objects* $\mathbb{N}$, *and as morphisms, relations* $R : \mathbb{Z}_p^{2m} \to \mathbb{Z}_p^{2n}$ *such that* R *viewed as a subset of* $\mathbb{Z}_p^m \times \mathbb{Z}_p^n$ *is an affine co-isotropic subspace thereof.*

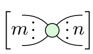
Since [26] works in the scalarless ZX-calculus, we need to add one extra axiom, which suffices to eliminate all remaining (non-zero) scalars in Stab_p: we impose the rule (Mod) that $p = 1$. Diagrammatically, this amounts to quotienting $(\mathsf{ZX}_p^{\mathrm{Stab}})^{\doteqdot}$ by the rule:  $=$  .

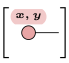
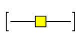
Then we can give an interpretation $[-]$ of $(\mathsf{ZX}_p^{\mathrm{Stab}})^{\doteqdot}$ making it universal and complete for $\mathsf{AffCoIsoRel}_{\mathbb{Z}_p}$, and which is defined uniquely by the commutative diagram:

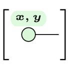
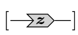
$$\begin{tikzcd}
(\mathsf{ZX}_p^{\mathrm{Stab}})^{\doteqdot} \arrow[r, dashed, "{[-]}"] \arrow[d, "{[\![-]\!]}"'] & \mathsf{AffCoIsoRel}_{\mathbb{Z}_p} \arrow[d, leftrightarrow] \\
\mathrm{CPM}(\mathsf{Stab}_p) \arrow[r, two heads] & \mathrm{CPM}(\mathsf{Stab}_p)/(\mathrm{Mod})
\end{tikzcd}$$

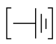
Explicitly, it is given by the identity on objects, $[m] = m$, and is defined on morphisms by: for $x, y \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^*$,

 $= \left\{ \left(\bigoplus_{k=1}^m \begin{bmatrix}a\\b_k\end{bmatrix}, \bigoplus_{k=1}^n \begin{bmatrix}-a\\-c_k\end{bmatrix} \right) \middle| \, a, b_k, c_k \in \mathbb{Z}_p \quad \text{and} \quad \sum_k b_k = \sum_k c_k \right\}$

 $= \left\{ \left(\bigoplus_{k=1}^m \begin{bmatrix}a_k\\c\end{bmatrix}, \bigoplus_{k=1}^n \begin{bmatrix}b_k\\c\end{bmatrix} \right) \middle| \, a, b_k, c_k \in \mathbb{Z}_p \quad \text{and} \quad \sum_k a_k = \sum_k b_k \right\}$

 $= \left\{ \left(\bullet, \begin{bmatrix}-1 & 0\\-y & -1\end{bmatrix}\begin{bmatrix}v\\0\end{bmatrix} + \begin{bmatrix}-x\\0\end{bmatrix} \right) \middle| \, v \in \mathbb{Z}_p \right\}$  $= \left\{ \left(\mathbf{v}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\mathbf{v} \right) \middle| \, \mathbf{v} \in \mathbb{Z}_p^2 \right\}$

 $= \left\{ \left(\bullet, \begin{bmatrix}1 & -y\\0 & 1\end{bmatrix}\begin{bmatrix}0\\v\end{bmatrix} + \begin{bmatrix}0\\x\end{bmatrix} \right) \middle| \, v \in \mathbb{Z}_p \right\}$  $= \left\{ \left(\mathbf{v}, \begin{bmatrix}z & 0\\0 & z^{-1}\end{bmatrix}\mathbf{v} \right) \middle| \, \mathbf{v} \in \mathbb{Z}_p^2 \right\}$

Note that all of these are actually affine *Lagrangian* relations. The only generator which has a co-isotropic but not Lagrangian semantics is the discard map:  $= \left\{ (\mathbf{v}, \bullet) \,\middle|\, \mathbf{v} \in \mathbb{Z}_p^2 \right\}$.

As pointed out in [7, 8, 26], the related category of affine Lagrangian relations over the field $\mathbb{R}[x,y]/(xy-1)$ can be used to represent a fragment of electrical circuits. We expect that the axiomatisation of figure 1 can be adapted to that setting, but leave this for a future article.

6 Conclusion

We have constructed a ZX-calculus which captures the stabiliser fragment in odd prime dimensions, whilst retaining many of the “nice” features of the qubit ZX-calculus. Of course, there are a few obvious questions that we leave for future work.

First amongst these is the question of whether a fully universal calculus can be obtained from the ideas we used here. The spiders we have used here labelled with elements $a, b \in \mathbb{Z}_p \times \mathbb{Z}_p$ and which can be interpreted as polynomials $x \mapsto ax + bx^2$ which parametrise the phases of the spider. Adding one additional term of degree 3 is already sufficient to obtain a universal calculus in prime dimensions (strictly) greater than 3 [37]. In fact, one might as well add all higher order of polynomials (mod p) since access to such higher degrees will hopefully prove useful in finding commutation relations for the resulting spiders.

Secondly, it remains to be seen how to formulate a universal ZX-calculus for non-prime dimensions, even for just the stabiliser fragment. For this, the methods in this article are clearly inadequate: for example local scaling is no longer an invertible operation and thus certainly not in the Clifford group.

Finally, the set of axioms we provide here is probably not minimal. It would be nice to see if a simplified version can be obtained, as was done in [6] for the qubit case.

References

1 Samson Abramsky and Bob Coecke. Categorical quantum mechanics, 2008. arXiv:0808.1023.

2 D. M. Appleby. Properties of the extended Clifford group with applications to SIC-POVMs and MUBs, 2009. arXiv:0909.5233.

3 Miriam Backens. The ZX-calculus is complete for stabilizer quantum mechanics. *New Journal of Physics*, 16(9):093021, 2014. doi:10.1088/1367-2630/16/9/093021.

4 Miriam Backens. Making the stabilizer ZX-calculus complete for scalars. *Electronic Proceedings in Theoretical Computer Science*, 195:17–32, 2015. doi:10.4204/EPTCS.195.2.

5 Miriam Backens, Hector Miller-Bakewell, Giovanni de Felice, Leo Lobski, and John van de Wetering. There and back again: A circuit extraction tale. *Quantum*, 5:421, 2021. doi:10.22331/q-2021-03-25-421.

6 Miriam Backens, Simon Perdrix, and Quanlong Wang. A Simplified Stabilizer ZX-calculus. *Electronic Proceedings in Theoretical Computer Science*, 236:1–20, 2017. doi:10.4204/EPTCS.236.1.

7 John C. Baez, Brandon Coya, and Franciscus Rebro. Props in Network Theory. *Theory and Application of Categories*, 33(25):727–783, 2018. arXiv:1707.08321.

8 John C. Baez and Brendan Fong. A Compositional Framework for Passive Linear Networks. *Theory and Application of Categories*, 33(38):1158–1222, 2018. arXiv:1504.05625.

9 Xiaoning Bian and Quanlong Wang. Graphical Calculus for Qutrit Systems. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, E98.A(1):391–399, 2015. doi:10.1587/transfun.E98.A.391.

10 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Pawel Sobocinski, and Fabio Zanasi. String Diagram Rewrite Theory II: Rewriting with Symmetric Monoidal Structure, 2021. doi:10.48550/arXiv.2104.14686.

11 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobocinski, and Fabio Zanasi. String Diagram Rewrite Theory I: Rewriting with Frobenius Structure. *Journal of the ACM (JACM)*, 2022. doi:10.1145/3502719.

12 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. String Diagram Rewrite Theory III: Confluence with and without Frobenius, 2022. doi:10.48550/arXiv.2109.06049.

13 Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. Interacting Hopf Algebras. *Journal of Pure and Applied Algebra*, 221(1):144–184, 2017. doi:10.1016/j.jpaa.2016.06.002.

14 Robert I. Booth and Titouan Carette. Complete ZX-calculi for the stabiliser fragment in odd prime dimensions, 2022. arXiv:2204.12531.

15 Titouan Carette. When Only Topology Matters, 2021. arXiv:2102.03178.

16 Titouan Carette. *Wielding the ZX-calculus, Flexsymmetry, Mixed States, and Scalable Notations*. Theses, Université de Lorraine, 2021. URL: https://hal.archives-ouvertes.fr/tel-03468027.

17 Titouan Carette, Dominic Horsman, and Simon Perdrix. SZX-calculus: Scalable Graphical Quantum Reasoning, 2019. arXiv:1905.00041.

18 Titouan Carette, Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Completeness of graphical languages for mixed states quantum mechanics. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece*, volume 132 of *LIPIcs*, pages 108:1–108:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.ICALP.2019.108.

19 Titouan Carette and Simon Perdrix. Colored props for large scale graphical reasoning, 2020. arXiv:2007.03564.

20 Nicholas Chancellor, Aleks Kissinger, Joschka Roffe, Stefan Zohren, and Dominic Horsman. Graphical Structures for Design and Verification of Quantum Error Correction, 2018. arXiv:1611.08012.

21 Sean Clark. Valence bond solid formalism for d-level one-way quantum computation. *Journal of Physics A: Mathematical and General*, 39(11):2701–2721, 2006. doi:10.1088/0305-4470/39/11/010.

22 Bob Coecke and Ross Duncan. Interacting Quantum Observables: Categorical Algebra and Diagrammatics. *New Journal of Physics*, 13(4):043016, 2011. doi:10.1088/1367-2630/13/4/043016.

23 Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2017. doi:10.1017/9781316219317.

24 Cole Comfort. Relational semantics for quantum algorithms.

25 Cole Comfort. A symplectic setting for mixed stabiliser circuits, 2021.

26 Cole Comfort and Aleks Kissinger. A Graphical Calculus for Lagrangian Relations, 2021. arXiv:2105.06244.

27 Niel de Beaudrap. A linearized stabilizer formalism for systems of finite dimension, 2012. arXiv:1102.3354.

28 Niel de Beaudrap, Xiaoning Bian, and Quanlong Wang. Fast and effective techniques for T-count reduction via spider nest identities, 2020. arXiv:2004.05164.

29 Niel de Beaudrap and Dominic Horsman. The ZX calculus is a language for surface code lattice surgery, 2017. arXiv:1704.08670.

30 Ross Duncan, Aleks Kissinger, Simon Perdrix, and John van de Wetering. Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus. *Quantum*, 4:279, 2020. doi:10.22331/q-2020-06-04-279.

31 Ross Duncan and Maxime Lucas. Verifying the Steane code with Quantomatic. *Electronic Proceedings in Theoretical Computer Science*, 171:33–49, 2014. doi:10.4204/EPTCS.171.4.

32 Ross Duncan and Simon Perdrix. Rewriting Measurement-Based Quantum Computations with Generalised Flow. In Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis, editors, *Automata, Languages and Programming*, volume 6199, pages 285–296. Springer Berlin Heidelberg, 2010. doi:10.1007/978-3-642-14162-1_24.

33 Liam Garvie and Ross Duncan. Verifying the Smallest Interesting Colour Code with Quantomatic. *Electronic Proceedings in Theoretical Computer Science*, 266:147–163, 2018. doi:10.4204/EPTCS.266.10.

34 Vlad Gheorghiu. Standard form of qudit stabilizer groups. *Physics Letters A*, page 5, 2014.

35 Daniel Gottesman. Fault-Tolerant Quantum Computation with Higher-Dimensional Systems. *Chaos, Solitons & Fractals*, 10(10):1749–1758, 1999. doi:10.1016/S0960-0779(98)00218-5.

36 Ladina Hausmann, Nuriya Nurgalieva, and Lídia del Rio. A consolidating review of Spekkens' toy theory. arXiv:2105.03277.

37 Mark Howard and Jiri Vala. Qudit versions of the qubit "pi-over-eight" gate. *Physical Review A*, 86(2):022316, 2012. doi:10.1103/PhysRevA.86.022316.

38 Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics, 2017. arXiv:1705.11151.

39 Aleks Kissinger and John van de Wetering. Reducing T-count with the ZX-calculus, 2020. arXiv:1903.10477.

40 Aleks Kissinger, John van de Wetering, and Renaud Vilmart. Classical simulation of quantum circuits with partial and graphical stabiliser decompositions, 2022. arXiv:2202.09202.

41 Markus Nenhauser. An Explicit Construction of the Metaplectic Representation over a Finite Field. *Journal of Lie Theory*, 12(15), 2002.

42 Kang Feng Ng and Quanlong Wang. A universal completion of the ZX-calculus, 2017. arXiv:1706.09877.

43 André Ranchin. Depicting qudit quantum mechanics and mutually unbiased qudit theories. *Electronic Proceedings in Theoretical Computer Science*, 172:68–91, 2014. doi:10.4204/EPTCS.172.6.

44 Peter Selinger. Dagger compact closed categories and completely positive maps: (extended abstract). *Electron. Notes Theor. Comput. Sci.*, 170:139–163, 2007. doi:10.1016/j.entcs.2006.12.018.

45 Alex Townsend-Teague and Konstantinos Meichanetzidis. Classifying Complexity with the ZX-Calculus: Jones Polynomials and Potts Partition Functions, 2021. arXiv:2103.06914.

46 John van de Wetering. ZX-calculus for the working quantum computer scientist, 2020. arXiv:2012.13966.

47 Renaud Vilmart. A Near-Optimal Axiomatisation of ZX-Calculus for Pure Qubit Quantum Mechanics, 2018. arXiv:1812.09114.

48 Quanlong Wang. Qutrit ZX-calculus is Complete for Stabilizer Quantum Mechanics. *Electronic Proceedings in Theoretical Computer Science*, 266:58–70, 2018. doi:10.4204/EPTCS.266.3.

49 Quanlong Wang. A non-anyonic qudit ZW-calculus, 2021. arXiv:2109.11285.

50 Quanlong Wang. Qufinite ZX-calculus: A unified framework of qudit ZX-calculi, 2021. arXiv:2104.06429.

51 Quanlong Wang and Xiaoning Bian. Qutrit Dichromatic Calculus and Its Universality. *Electronic Proceedings in Theoretical Computer Science*, 172:92–101, 2014. doi:10.4204/EPTCS.172.7.

52 Yuchen Wang, Zixuan Hu, Barry C. Sanders, and Sabre Kais. Qudits and High-Dimensional Quantum Computing. *Frontiers in Physics*, 8, 2020. doi:10.3389/fphy.2020.589504.

53 Fabio Zanasi. *Interacting Hopf Algebras: The Theory of Linear Systems.* phdthesis, Ecole Normale Superieure de Lyon, 2018. arXiv:1805.03032.

54 D. L. Zhou, B. Zeng, Z. Xu, and C. P. Sun. Quantum computation based on d-level cluster states. *Physical Review A*, 68(6):062303, 2003. doi:10.1103/PhysRevA.68.062303.