CNF Encodings of Parity

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Abstract

The minimum number of clauses in a CNF representation of the parity function $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is 2^{n-1}. One can obtain a more compact CNF encoding by using non-deterministic variables (also known as guess or auxiliary variables). In this paper, we prove the following lower bounds, that almost match known upper bounds, on the number m of clauses and the maximum width k of clauses: 1) if there are at most s auxiliary variables, then $m \geq \Omega\left(2^{n/(s+1)}/n\right)$ and $k \geq n/(s+1)$; 2) the minimum number of clauses is at least $3n$. We derive the first two bounds from the Satisfiability Coding Lemma due to Paturi, Pudlák, and Zane using a tight connection between CNF encodings and depth-3 circuits. In particular, we show that lower bounds on the size of a CNF encoding of a Boolean function imply depth-3 circuit lower bounds for this function.

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1 Overview

1.1 Motivation

A popular approach for solving a difficult combinatorial problem in practice is to encode it in conjunctive normal form (CNF) and to invoke a SAT-solver. There are two main reasons why this approach works well for many hard problems: the state-of-the-art SAT-solvers are extremely efficient and many combinatorial problems are expressed naturally in CNF. At the same time, a CNF encoding is not unique and one usually determines a good encoding empirically. Moreover, there is no such thing as the best encoding of a given problem as it also depends on a SAT-solver at hand. Prestwich [10] gives an overview of various ways to translate problems into CNF and discusses their desirable properties, both from theoretical and practical points of view.

Already for such simple functions as the parity function $x_1 \oplus x_2 \oplus \cdots \oplus x_n$, it is not immediate how to encode them in CNF (to make it efficiently handled by SAT-solvers). Parity function is used frequently in cryptography (hash functions, stream ciphers, etc.). It is known that the minimum number of clauses in a CNF computing parity is 2^{n-1}. This becomes impractical quickly as n grows. A standard way to reduce the size of an encoding is by using non-deterministic variables (also known as guess or auxiliary variables). Namely, one introduces s non-deterministic variables $y_1, \dots, y_s$ and partitions the set of input variables into $s+1$ blocks of size at most $\lceil n/(s+1) \rceil$: $\{x_1, x_2, \dots, x_n\} = X_1 \sqcup X_2 \sqcup \dots \sqcup X_{s+1}$. Then, one writes down the following $s+1$ parity functions in CNF:

$$\left(y_1 = \bigoplus_{x \in X_1} x\right), \left(y_2 = y_1 \oplus \bigoplus_{x \in X_2} x\right), \dots,$$
$$\left(y_s = y_{s-1} \oplus \bigoplus_{x \in X_s} x\right), \left(1 = y_s \oplus \bigoplus_{x \in X_{s+1}} x\right). \quad (1)$$

The value for the parameter s is usually determined experimentally. For example, Prestwich [9] reports that taking $s = 10$ gives the best results when solving the minimal disagreement parity learning problem using local search based SAT-solvers.

The simple construction above implies several upper bounds on the number m of clauses, the number s of non-deterministic variables, and the width k of clauses:

Limited non-determinism: using s non-deterministic variables, one can encode parity either as a CNF with at most

$$m \le (s+1)2^{\lceil n/(s+1) \rceil + 2 - 1} \le 4(s+1)2^{n/(s+1)}$$

clauses or as a k-CNF, where

$$k = 2 + \lceil n/(s+1) \rceil \le 3 + n/(s+1)\,.$$

Unlimited non-determinism: one can encode parity as a CNF with at most $4n$ clauses (to do this, use $s = n-1$ non-deterministic variables; then, each of n functions in (1) can be written in CNF using at most four clauses).

1.2 Results

In this paper, we show that the upper bounds mentioned above are essentially optimal.

▶ **Theorem 1.** *Let F be a CNF-encoding of PAR_n with m clauses, s non-deterministic variables, and maximum clause width k.*

1. *The parameters s and m cannot be too small simultaneously:*
$$m \ge \Omega\left(\frac{s+1}{n} \cdot 2^{n/(s+1)}\right). \quad (2)$$
2. *The parameters s and k cannot be too small simultaneously:*
$$k \ge n/(s+1)\,. \quad (3)$$
3. *The parameter m cannot be too small:*
$$m \ge 3n - 9\,. \quad (4)$$

1.3 Techniques

We derive a lower bound $m \ge \Omega((s+1)2^{n/(s+1)}/n)$ from the Satisfiability Coding Lemma due to Paturi, Pudlák, and Zane [8]. This lemma allows to prove a $2^{\sqrt{n}}$ lower bound on the size of depth-3 circuits computing the parity function. Interestingly, the lower bound $m \ge \Omega((s+1)2^{n/(s+1)}/n)$ implies a lower bound $2^{\Omega(\sqrt{n})}$ almost immediately, though it is not clear whether a converse implication can be easily proved.

To prove a lower bound $m \ge 3n-9$, we analyze carefully the structure of a CNF encoding.

1.4 Related work

Many results for various computational models with limited non-determinism are surveyed by Goldsmith, Levy, and Mundhenk [2]. An overview of known approaches for CNF encodings is given by Prestwich [10]. Two recent results that are close to the results of this paper are the following. Morizumi [7] proved that non-deterministic inputs do not help in the model of Boolean circuits over the U_2 basis (the set of all binary functions except for the binary parity and its complement) for computing the parity function: with and without non-deterministic inputs, the minimum size of a circuit computing parity is $3(n-1)$. Kucera, Savický, Vorel [5] prove almost tight bounds on the size of CNF encodings of the at-most-one Boolean function ($[x_1 + \cdots + x_n \leq 1]$). Sinz [11] proves a linear lower bound on the size of CNF encodings of the at-most-k Boolean function.

2 General setting

2.1 Computing Boolean functions by CNFs

For a Boolean function $f(x_1, \ldots, x_n)\colon \{0,1\}^n \to \{0,1\}$, we say that a CNF $F(x_1, \ldots, x_n)$ *computes* f if $f \equiv F$, that is, for all $x_1, \ldots, x_n \in \{0,1\}$, $f(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$. We treat a CNF as a set of clauses and by the *size* of a CNF we mean its number of clauses. It is well known that for every function f, there exists a CNF computing it. One way to construct such a CNF is the following: for every input $x \in \{0,1\}^n$ such that $f(x) = 0$, populate a CNF with a clause of length n that is falsified by x.

This method does not guarantee that the produced CNF has the minimal number of clauses: this would be too good to be true as the problem of finding a CNF of minimum size for a given Boolean function (specified by its truth table) is NP-complete as proved by Masek [6] (see also [1] and references herein). For example, for a function $f(x_1, x_2) = x_1$ the method produces a CNF $(\overline{x_1} \lor x_2) \land (\overline{x_1} \lor \overline{x_2})$ whereas the function x_1 is already in CNF format.

2.2 Parity

It is well known that for many functions, the minimum size of a CNF is exponential. The canonical example is the parity function $\mathrm{PAR}_n(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n$. The property of PAR_n that prevents it from being computable by short CNF's is its high *sensitivity*: by flipping *any* bit in *any* input $x \in \{0,1\}^n$, one flips the value of $\mathrm{PAR}_n(x)$.

▶ **Lemma 2.** *The minimum size of a CNF computing* PAR_n *has size* 2^{n-1}.

Proof. An upper bound follows from the method above by noting that $|\mathrm{PAR}_n^{-1}(0)| = 2^{n-1}$.

A lower bound is based on the fact that any clause of a CNF F computing PAR_n must contain all variables $x_1, \ldots, x_n$. Indeed, if a clause $C \in F$ did not depend on x_i, one could find an input $x \in \{0,1\}^n$ that falsifies C (hence, $F(x) = \mathrm{PAR}_n(x) = 0$) and remains to be falsifying even after flipping x_i. As any clause of F has exactly n variables, it rejects exactly one $x \in \{0,1\}^n$. Hence, F must contain at least $|\mathrm{PAR}_n^{-1}(0)| = 2^{n-1}$ clauses. ◀

2.3 Encoding Boolean functions by CNFs

We say that a CNF F *encodes* a Boolean function $f(x_1, \ldots, x_n)$ if the following two conditions hold.

1. In addition to the input bits $x_1, \ldots, x_n$, F also depends on s bits $y_1, \ldots, y_s$ called *guess inputs* or *non-deterministic inputs*.
2. For every $x \in \{0,1\}^n$, $f(x) = 1$ iff there exists $y \in \{0,1\}^s$ such that $F(x,y) = 1$. In other words, for every $x \in \{0,1\}^n$,

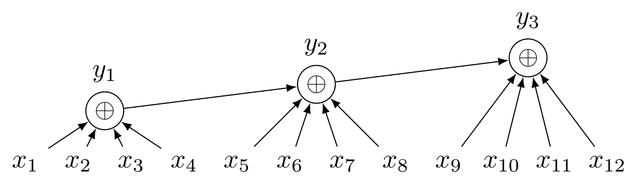
$$f(x) = \bigvee_{y \in \{0,1\}^s} F(x,y)\,. \tag{5}$$

Such representations of Boolean functions are widely used in practice when one translates a problem to SAT. For example, the following CNF encodes PAR_4:

$$(x_1 \vee x_2 \vee \overline{y_1}) \wedge (x_1 \vee \overline{x_2} \vee y_1) \wedge (\overline{x_1} \vee x_2 \vee y_1) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{y_1}) \wedge (y_1 \vee x_3 \vee \overline{y_2}) \wedge \\ (y_1 \vee \overline{x_3} \vee y_2) \wedge (\overline{y_1} \vee x_3 \vee y_2) \wedge (\overline{y_1} \vee \overline{x_3} \vee \overline{y_2}) \wedge (\overline{x_4} \vee y_2) \wedge (x_4 \vee \overline{y_2})\,. \tag{6}$$

2.4 Boolean Circuits and Tseitin Transformation

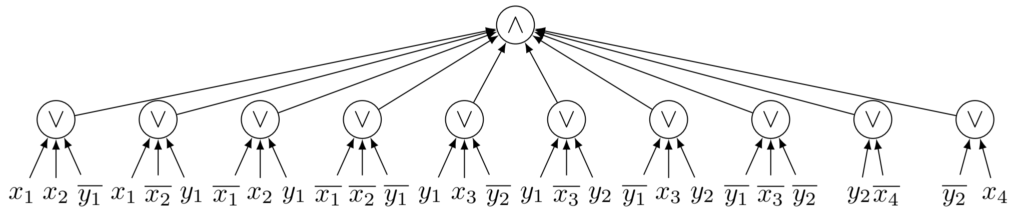
A natural way to get a CNF encoding of a Boolean function f is to take a circuit computing f and apply Tseitin transformation [12]. We describe this transformation using a toy example. The following circuit computes PAR_{12} with three gates. It has 12 inputs, 3 gates (one of which is an output gate), and has depth 3.


y_1
y_2
y_3
$\oplus$
$\oplus$
$\oplus$
x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}

$$y_1 = x_1 \oplus x_2 \oplus x_3 \oplus x_4$$
$$y_2 = y_1 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8$$
$$y_3 = y_2 \oplus x_9 \oplus x_{10} \oplus x_{11} \oplus x_{12}$$

To the right of the circuit, we show the functions computed by each gate. One can translate each line into CNF. Adding a clause (y_3) to the resulting CNF gives a CNF encoding of the function computed by the circuit. In fact, the CNF (1) can be obtained this way (after propagating the value of the output gate).

A CNF can be viewed as a depth-2 circuit where the output gate is an AND, all other gates are ORs, and the inputs are variables and their negations. For example, the following circuit corresponds to a CNF (6). Such depth-2 circuits are also denoted as AND ∘ OR circuits.


$\wedge$
$\vee$ $\vee$ $\vee$ $\vee$ $\vee$ $\vee$ $\vee$ $\vee$ $\vee$ $\vee$
x_1 x_2 $\overline{y_1}$ x_1 $\overline{x_2}$ y_1 $\overline{x_1}$ x_2 y_1 $\overline{x_1}$ $\overline{x_2}$ $\overline{y_1}$ y_1 x_3 $\overline{y_2}$ y_1 $\overline{x_3}$ y_2 $\overline{y_1}$ x_3 y_2 $\overline{y_1}$ $\overline{x_3}$ $\overline{y_2}$ y_2 $\overline{x_4}$ $\overline{y_2}$ x_4

2.5 Depth-3 circuits

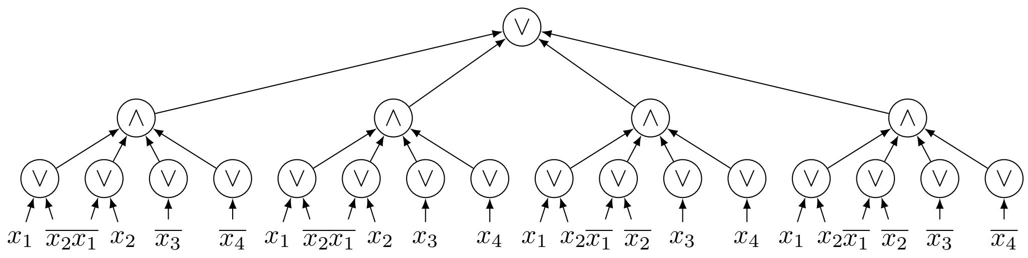
Depth-3 circuits is a natural generalization of CNFs: a Σ_3-*circuit* is simply an OR of CNFs. In a circuit, these CNFs are allowed to share clauses. A Σ_3-*formula* is a Σ_3-circuit whose CNFs do not share clauses (in other words, it is a circuit where the out-degree of every gate is equal to one).

On the one hand, this computation model is still simple enough. On the other hand, proving lower bounds against this model is much harder: getting a $2^{\omega(n)}$ lower bound for an explicit function (say, from NP or E^{NP}) is a major challenge. Proving a lower bound $2^{\omega(n/\log\log n)}$ would resolve another open question, through Valiant's depth reduction [13]: proving a superlinear lower bound on the size of logarithmic depth circuits. We refer the reader to Jukna's book [4, Chapter 11] for an exposition of known results for depth-3 circuits. For the parity function, the best known lower bound on depth-3 circuits is $\Omega(2^{\sqrt{n}})$ [8]. If one additionally requires that a circuit is a formula, i.e., that every gate has out-degree at most 1, then the best lower bound is $\Omega(2^{2\sqrt{n}})$ [3]. Both lower bounds are tight up to polynomial factors.

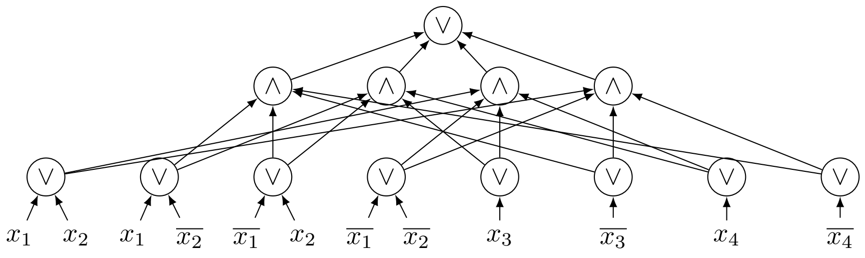
Equation (5) shows a tight connection between CNF encodings and depth-3 circuits of type OR $\circ$ AND $\circ$ OR. Namely, let $F(x_1,\dots,x_n,y_1,\dots,y_s)=\{C_1,\dots,C_m\}$ be a CNF encoding of a Boolean function $f\colon\{0,1\}^n\to\{0,1\}$. Then, $f(x)=\vee_{y\in\{0,1\}^s}F(x,y)$. By assigning y's in all 2^s ways, one gets an Σ_3-formula that computes f:

$$f(x)=\bigvee_{j\in[2^s]}F_j(x)\,, \tag{7}$$

where each F_j is a CNF. We call this an *expansion* of F. For example, an expansion of a CNF (6) looks as follows. It is an OR of four CNFs.


x_1 $\overline{x_2}$ $\overline{x_1}$ x_2 $\overline{x_3}$ $\overline{x_4}$
x_1 $\overline{x_2}$ $\overline{x_1}$ x_2 x_3 x_4
x_1 x_2 $\overline{x_1}$ $\overline{x_2}$ x_3 x_4
x_1 x_2 $\overline{x_1}$ $\overline{x_2}$ $\overline{x_3}$ $\overline{x_4}$

An expansion is a formula: it is an OR of CNFs, every gate has out-degree one. One can also get a *circuit-expansion*: in this case, gates are allowed to have out-degree more than one; alternatively, CNFs are allowed to share clauses. For example, this is a circuit-expansion of (6).


x_1 x_2 x_1 $\overline{x_2}$ $\overline{x_1}$ x_2 $\overline{x_1}$ $\overline{x_2}$ x_3 $\overline{x_3}$ x_4 $\overline{x_4}$

Below, we show that CNF encodings and depth-3 circuits can be easily transformed one into the other. It will prove convenient to define the size of a circuit as its number of gates *excluding* the output gate. This way, the size of a CNF formula equals its number of clauses (a CNF is a depth-2 formula). By a $\Sigma_3(t,r)$-circuit we denote a Σ_3-circuit having at most t ANDs on the second layer and at most r ORs on the third layer (hence, its size is at most $t+r$).

▶ **Lemma 3.** *Let $F(x_1, \dots, x_n, y_1, \dots, y_s)$ be a CNF encoding of size m of a function $f\colon \{0,1\}^n \to \{0,1\}$. Then, f can be computed by a $\Sigma_3(2^s, m \cdot 2^s)$-formula and by a $\Sigma_3(2^s, m)$-circuit.*

Proof. Let $F = \{C_1, \dots, C_m\}$. To expand F as $\bigvee_{j \in [2^s]} F_j$, we go through all 2^s assignments to non-deterministic variables $y_1, \dots, y_s$. Under any such assignment, each clause C_i is either satisfied or becomes a clause $C_i' \subseteq C_i$ resulting from C_i by removing all its non-deterministic variables. Thus, for each $j \in [2^s]$, $F_j \subseteq \{C_1', \dots, C_m'\}$. The corresponding Σ_3-formula contains at most $2^s + m2^s$ gates: there are 2^s gates for F_j's, each F_j contains no more than m clauses. The corresponding Σ_3-circuit contains no more than $2^s + m$ gates: there are 2^s gates for F_j's and m gates for $C_1', \dots, C_m'$ (each F_j selects which of these m clauses to contain). ◀

Interestingly, the upper bounds on depth-3 circuits resulting from this simple transformation cannot be substantially improved. Indeed, by plugging in a CNF encoding of PAR_n with $s = \sqrt{n}$ and $m = O(\sqrt{n}2^{\sqrt{n}})$ (see (1)), one gets a Σ_3-formula and a Σ_3-circuit of size $2^{2\sqrt{n}}$ and $2^{\sqrt{n}}$, respectively, up to polynomial factors. As discussed above, these bounds are known to be optimal.

Below, we show a converse transformation.

▶ **Lemma 4.** *Let C be a $\Sigma_3(t, r)$-formula (circuit) computing a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$. Then, f can be encoded as a CNF with $\lceil \log t \rceil$ non-deterministic variables of size r ($2rt$, respectively).*

Proof. Let $C = F_1 \vee \dots \vee F_t$ be a Σ_3-formula (hence, $r = \mathrm{size}(F_1) + \dots + \mathrm{size}(F_t)$). Introduce $s = \lceil \log t \rceil$ non-deterministic variables $y_1, \dots, y_s$. Then, for every assignment to $y_1, \dots, y_s$, take the corresponding CNF F_i ($1 \le i \le 2^s$ is the unique integer corresponding to this assignment) and add y_i's with the corresponding signs to every clause of F_i. Call the resulting CNF F_i'. Then, $F = F_1' \wedge \dots \wedge F_{2^s}'$ encodes f and F has at most r clauses.

If C is a Σ_3-circuit, we need to create a separate copy of every gate corresponding to a clause in each of 2^s CNFs. Hence, the size of the resulting CNF encoding is at most $r2^s \le 2rt$. ◀

Finally, we show that proving strong lower bounds on the size of CNF encodings is not easier than proving strong lower bounds on the size of depth-3 circuits. Let C be a $\Sigma_3(t, r)$-formula computing PAR_n. Lemma 4 guarantees that PAR_n can be encoded as a CNF of size r with $\lceil \log t \rceil$ non-deterministic variables. Then, by the inequality (2),

$$\mathrm{size}(C) = t + r \ge t + \Omega\left(\frac{1}{n} \cdot 2^{\frac{n}{\log t + 2}}\right) \ge \frac{1}{n}\left(t + \Omega\left(2^{\frac{n}{\log t + 2}}\right)\right) \ge \Omega\left(\frac{2^{\sqrt{n}}}{n}\right).$$

Similarly, if C is a $\Sigma_3(t, r)$-circuit, Lemma 4 guarantees that PAR_n can be encoded as a CNF of size $2rt$ with $\lceil \log t \rceil$ non-deterministic variables. Then,

$$\mathrm{size}(C) = t + r \ge t + \Omega\left(\frac{1}{2tn} \cdot 2^{\frac{n}{\log t + 2}}\right) \ge \Omega\left(\frac{2^{\sqrt{n/2}}}{n}\right).$$

3 Lower bounds for CNF encodings of parity

In this section, we prove Theorem 1. The essential property of the parity function used in the proof is its high sensitivity (every satisfying assignment is isolated): for any $i \in [n]$ and any $x, x' \in \{0,1\}^n$ that differ in the i-th position only, $\mathrm{PAR}(x) \neq \mathrm{PAR}(x')$. This means that if a CNF F computes PAR and $F(x) = 1$, then F must contain a clause that is satisfied by x_i only. Following [8], we call such a clause *critical* with respect to (x, i). This notion extends to CNF encodings in a natural way. Namely, let $F(x, y)$ be a CNF encoding of PAR. Then, for any (x, y) such that $F(x, y) = 1$ and any $i \in [n]$, F contains a clause that becomes falsified if one flips the bit x_i. We call it critical w.r.t. (x, y, i).

3.1 Limited non-determinism

To prove a lower bound $m \geq \Omega((s+1)2^{n/(s+1)}/n)$, we adapt a proof of the $\Omega(n^{1/4}2^{\sqrt{n}})$ lower bound for depth-3 circuits computing PAR_n by Paturi, Pudlák, and Zane [8]. Let $F(x_1, \ldots, x_n)$ be a CNF. For every isolated satisfying assignment $x \in \{0,1\}^n$ of F and every $i \in [n]$, fix a shortest critical clause w.r.t. (x, i) and denote it by $C_{F,x,i}$. Then, for an isolated satisfying assignment x, define its weight w.r.t. F as

$$w_F(x) = \sum_{i=1}^{n} \frac{1}{|C_{F,x,i}|} .$$

▶ **Lemma 5** (Lemma 5 in [8]). *For any μ, F has at most $2^{n-\mu}$ isolated satisfying assignments of weight at least μ.*

Proof of (2), $m \geq \Omega\left(\frac{s+1}{n} \cdot 2^{n/(s+1)}\right)$**.** Let $F(x_1, \ldots, x_n, y_1, \ldots, y_s)$ be a CNF encoding of size m of PAR_n. Consider its expansion:

$$\mathrm{PAR}_n(x) = \bigvee_{j \in [2^s]} F_j(x) .$$

We extend the definitions of $C_{F,x,i}$ and $w(x)$ to CNFs with non-deterministic variables as follows. Let $x \in \mathrm{PAR}_n^{-1}(1)$ and let $j \in [2^s]$ be the smallest index such that $F_j(x) = 1$. For $i \in [n]$, let $C'_{F,x,i} = C_{F_j,x,i}$ (that is, we simply take the first F_j that is satisfied by x and take its critical clause w.r.t. (x, i)). Then, the weight $w'_F(x)$ of x w.r.t. to F is defined simply as $w_{F_j}(x)$. Clearly,

$$w'_F(x) = \sum_{i \in [n]} \frac{1}{|C'_{(F,x,i)}|} .$$

For $l \in [n]$, let also $N_{l,F}(x) = |\{i \in [n] : |C'_{F,x,i}| = l\}|$ be the number of critical clauses (w.r.t. x) of length l. Clearly,

$$w'_F(x) = \sum_{l \in [n]} \frac{N_{l,F}(x)}{l} . \tag{8}$$

For a parameter $0 < \varepsilon < 1$ to be chosen later, split $\mathrm{PAR}_n^{-1}(1)$ into light and heavy parts:

$$H = \{x \in \mathrm{PAR}_n^{-1}(1) : w'_F(x) \geq s + 1 + \varepsilon\} ,$$
$$L = \{x \in \mathrm{PAR}_n^{-1}(1) : w'_F(x) < s + 1 + \varepsilon\} .$$

We claim that

$$|H| \leq 2^s \cdot 2^{n-s-1-\varepsilon} .$$

Indeed, for every $x \in H$, $w'_F(x) = w_{F_j}(x)$ for some $j \in [2^s]$, and by Lemma 5, F_j cannot accept more than $2^{n-s-1-\varepsilon}$ isolated solutions of weight at least $s + 1 + \varepsilon$. Since $|H| + |L| = |\mathrm{PAR}_n^{-1}(1)| = 2^{n-1}$, we conclude that

$$|L| = 2^{n-1} - |H| \geq (1 - 2^{-\varepsilon})2^{n-1} . \tag{9}$$

Let $F = \{C_1, \ldots, C_m\}$. For every $k \in [m]$, let $C'_k \subseteq C_k$ be the clause C_k with all non-deterministic variables removed. Hence, for every $j \in [2^s]$, $F_j \subseteq \{C'_1, \ldots, C'_m\}$. For $l \in [n]$, let $m_l = |\{k \in [m] \colon |C'_k| = l\}|$ be the number of such clauses of length l. Consider a clause C'_k and let $l = |C'_k|$. Then, there are at most $l2^{n-l}$ pairs (x, i), where $x \in \mathrm{PAR}^{-1}(1)$ and $i \in [n]$, such that $C'_{F,x,i} = C'_k$: there are at most l choices for i, fixing i fixes the values of all l literals in C'_k (all of them are equal to zero except for the i-th one), and there are no more than 2^{n-l} choices for the other bits of x. Recall that $N_{l,F}(x)$ is the number of critical clauses w.r.t. x of length l. Thus, we arrive at the following inequality:

$$m_l \cdot l \cdot 2^{n-l} \geq \sum_{x \in \mathrm{PAR}^{-1}(1)} N_{F,l}(x) \geq \sum_{x \in L} N_{F,l}(x)\,.$$

Then,

$$m = \sum_{l \in [n]} m_l \geq \sum_{l \in [n]} \frac{\sum_{x \in L} N_{F,l}(x)}{l2^{n-l}} = \sum_{x \in L} \sum_{l \in [n]} \frac{N_{F,l}(x)}{l2^{n-l}} = \sum_{x \in L} n2^{-n} \sum_{l \in [n]} \frac{N_{F,l}(x)}{n} \cdot \frac{2^l}{l}\,. \tag{10}$$

To estimate the last sum, let

$$T(x) = \sum_{l \in [n]} \frac{N_{F,l}(x)}{n} \cdot \frac{2^l}{l} = \sum_{l \in [n]} \frac{N_{F,l}(x)}{n} \cdot g(l)\,,$$

where $g(l) = \frac{2^l}{l}$. Since $g(l)$ is convex (for $l > 0$) and $\sum_{l \in [n]} \frac{N_{F,l}(x)}{n} = 1$, Jensen's inequality gives

$$T(x) \geq g\left(\sum_{l \in [n]} \frac{N_{F,l}(x)}{n} \cdot l\right)\,. \tag{11}$$

Further, Sedrakyan's inequality[1] (combined with (8) and $\sum_{l \in [n]} N_{F,l}(x) = n$) gives

$$\sum_{l \in [n]} l N_{F,l}(x) = \sum_{l \in [n]} \frac{N^2_{F,l}(x)}{N_{F,l}(x)/l} \geq \frac{\left(\sum_{l \in [n]} N_{F,l}(x)\right)^2}{\sum_{l \in [n]} N_{F,l}(x)/l} = \frac{n^2}{w'_F(x)}\,. \tag{12}$$

Since $g(l)$ is monotonically increasing for $l \geq 1/\ln 2$ and $w'_F(x) < s + 1 + \varepsilon$ for every $x \in L$, combining (11) and (12), we get

$$T(x) \geq g\left(\frac{n}{w'_F(x)}\right) \geq g\left(\frac{n}{s+1+\varepsilon}\right)\,, \tag{13}$$

for $s \leq n \ln 2 - 1 - \varepsilon$. (If $s > n \ln 2 - 1 - \varepsilon$, then the lower bound $m \geq \Omega(2^{n/(s+1)}/n)$ is trivial.)

[1] Sedrakyan's inequality is a special case of Cauchy–Schwarz inequality: for all $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}_{>0}$, $\sum_{i=1}^n a_i^2/b_i \geq \left(\sum_{i=1}^n a_i\right)^2 / \sum_{i=1}^n b_i$.

Thus,

$$\begin{aligned}
m &\geq \sum_{x \in L} n 2^{-n} T(x) \geq && \text{(10 and 13)}\\
&\geq \sum_{x \in L} n 2^{-n} g\left(\frac{n}{s+1+\varepsilon}\right) = && \text{(definition of } g)\\
&= |L| 2^{-n} 2^{\frac{n}{s+1+\varepsilon}} (s+1+\varepsilon) \geq && (9)\\
&\geq \left(\frac{1}{2} - \frac{1}{2^{\varepsilon+1}}\right)(s+1+\varepsilon) 2^{\frac{n}{s+1+\varepsilon}} = && \text{(rewriting)}\\
&= \left(\frac{1}{2} - \frac{1}{2^{\varepsilon+1}}\right)(s+1+\varepsilon) 2^{\frac{n}{s+1}} 2^{\frac{-n\varepsilon}{(s+1)(s+1+\varepsilon)}} .
\end{aligned}$$

Set $\varepsilon = 1/n$. Then,

$$\left(\frac{1}{2} - \frac{1}{2^{\frac{1}{n}+1}}\right) = \Theta\left(\frac{1}{n}\right) .$$

Also,

$$\frac{1}{2} \leq 2^{\frac{-1}{(s+1)(s+1+1/n)}} \leq 1 ,$$

as $2^{-1/x}$ is increasing for $x > 0$. This finally gives a lower bound

$$m \geq \Omega\left(\frac{s+1}{n} \cdot 2^{\frac{n}{s+1}}\right) . \qquad ◀$$

3.2 Width of clauses

To prove the lower bound $k \geq n/(s+1)$, we use the following corollary of the Satisfiability Coding Lemma.

▶ **Lemma 6** (Lemma 2 in [8]). *Any k-CNF $F(x_1, \dots, x_n)$ has at most $2^{n-n/k}$ isolated satisfying assignments.*

Proof of (3), $k \geq n/(s+1)$. Consider a k-CNF $F(x_1, \dots, x_n, y_1, \dots, y_s)$ that encodes PAR_n. Expand F to an OR of 2^s k-CNFs:

$$\mathrm{PAR}_n(x) = \bigvee_{j \in [2^s]} F_j(x) .$$

By Lemma 6, each F_j accepts at most $2^{n-n/k}$ isolated solutions. Hence,

$$2^s \geq \frac{2^{n-1}}{2^{n-n/k}} = 2^{n/k-1}$$

and thus, $k \geq n/(s+1)$. ◀

3.3 Unlimited non-determinism

In this section, we prove the lower bound $m \geq 3n - 9$.

Proof of (4), $m \geq 3n - 9$. We use induction on n. The base case $n \leq 3$ is clear. To prove the induction step, assume that $n > 3$ and consider a CNF encoding $F(x_1, \ldots, x_n, y_1, \ldots, y_s)$ of PAR_n with the minimum number of clauses. Below, we show that one can find k deterministic variables (where $k = 1$ or $k = 2$) such that assigning appropriately chosen constants to them reduces the number of clauses by at least $3k$, respectively. The resulting function computes PAR_{n-k} or its negation. It is not difficult to see that the minimum number of clauses in encodings of PAR and its negation are equal (by flipping the signs of all occurrences of any deterministic variable in a CNF encoding of PAR, one gets a CNF encoding of the negation of PAR, and vice versa). Hence, one can proceed by induction and conclude that F contains at least $3(n-k) - 9 + 3k = 3n - 9$ clauses.

To find the required k deterministic variables, we go through a number of cases. In the analysis below, by a d-literal we mean a literal that appears exactly d times in F, a d^+-literal appears at least d times. A (d_1, d_2)-literal occurs d_1 times positively and d_2 times negatively. Other types of literals are defined similarly. We treat a clause as a set of literals (that do not contain a literal together with its negation) and a CNF formula as a set of clauses.

Note that for all $i \in [s]$, y_i must be a $(2^+, 2^+)$-literal. Indeed, if y_i (or $\overline{y_i}$) is a 0-literal, one can assign $y_i \leftarrow 0$ ($y_1 \leftarrow 1$, respectively). It is not difficult to see that the resulting formula still encodes PAR. If y_i is a $(1, t)$-literal, one can eliminate it using resolution: for all pairs of clauses $C_0, C_1 \in F$ such that $\overline{y_i} \in C_0$ and $y_i \in C_1$, add a clause $C_0 \cup C_1 \setminus \{y_i, \overline{y_i}\}$ (if this clause contains a pair of complementary literals, ignore it); then, remove all clauses containing y_i or $\overline{y_i}$. The resulting formula still encodes PAR_n, but has a smaller number of clauses than F (we remove $1 + t$ clauses and add at most t clauses).

In the case analysis below, by l_i we denote a literal that corresponds to a deterministic variable x_i or its negation $\overline{x_i}$.

1. *F contains a 3^+-literal l_i.* Assigning $l_i \leftarrow 1$ eliminates at least three clauses from F.
2. *F contains a 1-literal l_i.* Let $l_i \in C \in F$ be a clause containing l_i. C cannot contain other deterministic variables: if $l_i, l_j \in C$ (for $i \neq j \in [n]$), consider $x \in \{0,1\}^n$ such that $\mathrm{PAR}_n(x) = 1$ and $l_i = l_j = 1$ (such x exists since $n > 3$), and its extension $y \in \{0,1\}^s$ such that $F(x, y) = 1$; then, F does not contain a critical clause w.r.t. (x, y, i). Clearly, C cannot be a unit clause, hence it must contain a non-deterministic variable y_j. Consider $x \in \{0,1\}^n$, such that $\mathrm{PAR}_n(x) = 1$ and $l_i = 1$, and its extension $y \in \{0,1\}^s$ such that $F(x, y) = 1$. If $y_j = 1$, then F does not contain a critical clause w.r.t. (x, y, i). Thus, for every $(x, y) \in \{0,1\}^{n+s}$ such that $F(x, y) = 1$ and $l_i = 1$, it holds that $y_j = 0$. This observation allows us to proceed as follows: first assign $l_i \leftarrow 1$, then assign $y_j \leftarrow 0$. The former assignment satisfies the clause C, the latter one satisfies all the clauses containing $\overline{y_j}$. Thus, at least three clauses are removed.
3. *For all $i \in [n]$, x_i is a $(2, 2)$-literal.* If there is no clause in F containing at least two deterministic variables, then F contains at least $4n$ clauses and there is nothing to prove. Let $l_i, l_j \in C_1 \in F$, where $i \neq j$, be a clause containing two deterministic variables and let $l_i \in C_2 \in F$ and $l_j \in C_3 \in F$ be the two clauses containing other occurrences of l_i and l_j ($C_1 \neq C_2$ and $C_1 \neq C_3$, but it can be the case that $C_2 = C_3$).
   Assume that C_2 contains another deterministic variable: $l_k \in C_2$, where $k \neq i, j$. Consider $x \in \{0,1\}^n$, such that $\mathrm{PAR}_n(x) = 1$ and $l_i = l_j = l_k = 1$ (such x exists since $n > 3$), and its extension $y \in \{0,1\}^s$ such that $F(x, y) = 1$. Then, F does not contain a critical clause w.r.t. (x, y, i): C_1 is satisfied by l_j, C_2 is satisfied by l_k. For the same reason, C_2 cannot contain the literal l_j. Similarly, C_3 cannot contain other deterministic variables and the literal l_i. (At the same time, it is not excluded that $\overline{l_j} \in C_2$ or $\overline{l_i} \in C_3$.) Hence, $C_2 \neq C_3$. Note that each of C_2 and C_3 must contain at least one non-deterministic variable: otherwise, it would be possible to falsify F by assigning l_i and l_j.

a. *At least one of C_2 and C_3 contains a single non-deterministic variable.* Assume that it is C_2:

$$\{l_i, y_1\} \subseteq C_2 \subseteq \{l_i, \overline{l_j}, y_1\}\,.$$

Assign $l_j \leftarrow 1$. This eliminates two clauses: C_1 and C_3 are satisfied. Also, under this substitution, $C_2 = \{l_i, y_1\}$ and l_i is a 1-literal. We claim that in any satisfying assignment of the resulting formula F', $l_i = \overline{y_1}$. Indeed, if (x, y) satisfies F' and $l_i = y_1$, then $l_i = y_1 = 1$ (otherwise C_2 is falsified). But then there is no critical clause in F' w.r.t. (x, y, i). Since in every satisfying assignment $l_i = \overline{y_1}$, we can replace every occurrence of y_1 ($\overline{y_1}$) by $\overline{l_i}$ (y_1, respectively). This, in particular, satisfies the clause C_2.

b. *Both C_2 and C_3 contain at least two non-deterministic variables:*

$$\{l_i,\ l_j\} \subseteq C_1, \quad \{l_i,\ y_1,\ y_2\} \subseteq C_2, \quad \{l_j,\ y_3,\ y_4\} \subseteq C_3\,.$$

Here, y_1 and y_2 are different variables, y_3 and y_4 are also different, though it is not excluded that some of y_1 and y_2 coincide with some of y_3 and y_4. Let $Y \subseteq \{y_1, \ldots, y_s\}$ be non-deterministic variables appearing in C_2 or C_3.
Recall that for every $(x, y) \in \{0,1\}^{n+s}$ such that $F(x, y) = 1$ and $l_i = l_j = 1$, it holds that $y = 0$ for all $y \in Y$. This means that if a variable $y \in Y$ appears in both C_2 and C_3, then it has the same sign in both clauses. Consider two subcases.

i. $Y = \{y_1, y_2\}$:

$$\{l_i,\ l_j\} \subseteq C_1, \quad \{l_i,\ y_1,\ y_2\} \subseteq C_2, \quad \{l_j,\ y_1,\ y_2\} \subseteq C_3\,.$$

Assume that $\overline{y_1} \notin C_1$. Assign $l_i \leftarrow 1$, $l_j \leftarrow 1$. Then, assigning $y_1 \leftarrow 0$ eliminates at least two clauses. Let us show that there remains a clause that contains $\overline{y_2}$. Consider $x \in \mathrm{PAR}_n^{-1}(1)$, such that $l_i = l_j = 1$, and its extension $y \in \{0,1\}^s$, such $F(x, y) = 1$. We know that y_1 and y_2 must be equal to 0. However, flipping the value of y_2 results in a satisfying assignment. Thus, it remains to analyze the following case:

$$\{l_i,\ l_j, \overline{y_1}, \overline{y_2}\} \subseteq C_1, \quad \{l_i,\ y_1,\ y_2\} \subseteq C_2, \quad \{l_j,\ y_1,\ y_2\} \subseteq C_3\,.$$

Assume that $\overline{l_j} \notin C_2$ and $\overline{l_i} \notin C_1$. Assign $l_i \leftarrow 1$, then assign $y_1 \leftarrow 0$ and $y_2 \leftarrow 0$. Under this assignment, $C_3 = \{l_j\}$ (recall that C_3 cannot contain other deterministic variables, see Case 3). This would mean that $l_j = 1$ in every satisfying assignment of the resulting CNF formula which cannot be the case for a CNF encoding of parity. Thus, we may assume that either $\overline{l_j} \in C_2$ or $\overline{l_i} \in C_1$. Without loss of generality, assume that $\overline{l_j} \in C_2$.
Let us show that for every $(x, y) \in \{0,1\}^{n+s}$, such that $F(x, y) = 1$ and $l_i = 1$, it holds that $l_j \neq y_1$ and $l_j \neq y_2$. Indeed, if there is $(x, y) \in \{0,1\}^{n+s}$ such that $F(x, y) = 1$ and $l_i = l_j = 1$, then y_1 and y_2 must be equal to 0. If there is $(x, y) \in \{0,1\}^{n+s}$, such that $F(x, y) = 1, l_i = 1, l_j = 0$, then y_1 and y_2 must be equal to 0, otherwise F does not contain a critical clause w.r.t. (x, y, i). Thus, assigning $l_i \leftarrow 1$ eliminates two clauses (C_1 and C_2). We then replace y_1 and y_2 with $\overline{l_j}$ and delete the clause C_3.

ii. $|Y| \geq 3, \{y_1, y_2, y_3\} \subseteq Y$:

$$\{l_i,\ l_j\} \subseteq C_1, \quad \{l_i,\ y_1,\ y_2\} \subseteq C_2, \quad \{l_j,\ y_1,\ y_3\} \subseteq C_3\,.$$

Assigning $l_i \leftarrow 1, l_j \leftarrow 1$ eliminates C_1, C_2, C_3. Assigning $y_1 \leftarrow 0$ eliminates at least one more clause (y_1 appears positively at least two times, but it may appear in C_1). There must be a clause with $\overline{y_2}$ (otherwise we could assign $y_2 \leftarrow 1$). Assigning $y_2 \leftarrow 0$ eliminates at least one more clause. Similarly, assigning $y_3 \leftarrow 1$ eliminates another clause. In total, we eliminate at least six clauses. ◀

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