A Systematic Study of Isomorphism Invariants of Finite Groups via the Weisfeiler-Leman Dimension

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Abstract

We investigate the relationship between various isomorphism invariants for finite groups. Specifically, we use the Weisfeiler-Leman dimension (WL) to characterize, compare and quantify the effectiveness and complexity of invariants for group isomorphism.

It turns out that a surprising number of invariants and characteristic subgroups that are classic to group theory can be detected and identified by a low dimensional Weisfeiler-Leman algorithm. These include the center, the inner automorphism group, the commutator subgroup and the derived series, the abelian radical, the solvable radical, the Fitting group and π-radicals. A low dimensional WL-algorithm additionally determines the isomorphism type of the socle as well as the factors in the derives series and the upper and lower central series.

We also analyze the behavior of the WL-algorithm for group extensions and prove that a low dimensional WL-algorithm determines the isomorphism types of the composition factors of a group.

Finally we develop a new tool to define a canonical maximal central decomposition for groups. This allows us to show that the Weisfeiler-Leman dimension of a group is at most one larger than the dimensions of its direct indecomposable factors. In other words the Weisfeiler-Leman dimension increases by at most 1 when taking direct products.

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1 Introduction

Tasks of classifying finite groups up to isomorphism and generating particular classes of finite groups are fundamental and recurring themes in computational group theory. Yet, in particular the computational complexity of such problems remains most illusive to date.

For example, for most orders up to 20.000 the number of non-isomorphic finite groups has been computed and the groups have been exhaustively generated [13]. But there are currently 38 notoriously difficult, exceptional cases, for which this information is beyond our current means (see [13]). The varying difficulty across different orders is in part caused by the erratic fluctuation of the number of isomorphism classes of finite groups as the order increases. This number appears to be closely linked to the multiplicities of the prime factors of the respective order, but even estimating the number of groups of a given order is non-trivial.

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Generation tasks for classes of groups have a long tradition dating back to Cayley [7]. Nowadays, there is extensive work on generating particular classes of groups. For example there are practically efficient algorithms for the generation of finite nilpotent or finite solvable groups [12]. However, the algorithms come without efficient running time guarantees.

One of the difficulties for a complexity analysis stems from the group isomorphism problem. Indeed, the group isomorphism problem for finite groups stays among the few standard tasks in computational group theory with uncertain complexity. In principle, we desire algorithms with an efficient worst case running time measured in the number of generators through which the groups are given. However, we do not even have algorithms with an efficient worst case running time when measured in the order of the group. In fact the only improvement for the worst case complexity over Tarjan's classic $n^{\log(n)+O(1)}$ algorithm are $n^{\frac{1}{c}\cdot\log(n)+O(1)}$ algorithms with a small constant c depending on the model of computation (randomization, quantum computing etc.) [21, 23, 24]. There is however a nearly-linear time algorithm that solves group isomorphism for most orders [11].

A closely related problem is that of computing isomorphism invariants to distinguish groups. Efficiently computable complete invariants are sufficient for general isomorphism testing. However, we do not know efficiently computable complete invariants even for very special cases, such as nilpotent p-groups of class 2. Partial invariants only give incomplete isomorphism tests, but they still find application in generation tasks allowing for heuristic fast pruning [13]. Given the long history of (algorithmic) group theory, there is an abundance of partial invariants.

Generally the techniques involved in generation and isomorphism computations exploit the existence of various characteristic subgroups classic to group theory. As outlined in [13], these include exploiting the Frattini subgroup $\Phi(G)$ [3], the exponent-p-central series [22], characteristic series [25] and similar.

Overall, many of the techniques currently in use are ad-hoc, focused on practical performance, and do not lead to efficient worst case upper bounds for the complexity of the algorithmic problems. As a consequence, the general picture for finite groups is somewhat chaotic. There is often no structured way of comparing or combining invariants for group isomorphism. E.g., two given invariants may be incomparable in their distinguishing power, making it unclear which invariant to use. Also the required time to evaluate an invariant may be difficult to estimate and can depend significantly on the input group. Even when we are given a class of efficiently computable invariants, it will generally be unclear which invariants to choose or how to efficiently combine their evaluation algorithmically.

In Summary, we lack the formal means to characterize, compare, or quantify the effectiveness and complexity of invariants for group isomorphism. We therefore propose a systematic study of computationally tractable invariants for finite groups.

For inspiration on how to systematize such a study, we turn to algorithmic finite model theory and specifically descriptive complexity theory. This allows us to characterize the complexity of an invariant by considering a formula within a logic that captures the invariant. A natural choice for a logic from which to choose the formulas is the powerful fixed point logic with counting. Not only can this logic express all polynomial time computable languages on ordered structures [17, 26], but in the context of graphs it has also proven to be an effective tool in comparing invariants (see [20]). As a measure for the complexity of an invariant we can then use the number of variables required to express the invariant in fixed point logic with counting. Crucially there is a corresponding algorithm, the k-dimensional Weisfeiler-Leman algorithm (WL-refinement, WL), that (implicitly) simultaneously evaluates all invariants that are expressible by formulas requiring at most $k+1$ variables in polynomial time[1].

[1] For groups there are actually two natural, closely related versions of the logic and of the algorithm,

Thus, to enable a quantification and comparison of the complexity of invariants we suggest the Weisfeiler-Leman algorithm. More specifically we suggest to use the Weisfeiler-Leman dimension, which determines how many variables are required to express a given invariant as a formula. This gives us a natural and robust framework for studying group invariants. In fact, the k-dimensional Weisfeiler-Leman algorithm is universal for all invariants of the corresponding dimension, resolving the issue of how to combine invariants. With this approach we also include an abundance of invariants that have not been considered before. However, it is a priori not clear at all that commonly used invariants can even be captured by the framework, i.e., that they even have bounded WL-dimension.

Contribution. The first contribution of this paper is to show that a surprising number of isomorphism invariants and subgroups that are classic to group theory can be detected and identified by a low dimensional Weisfeiler-Leman algorithm.

Specifically, we show first that for a small value of k, groups not distinguished by k-$\mathrm{WL}_{\mathrm{II}}$ have centers ($k \geq 2$), inner automorphism groups ($k \geq 4$), derived series ($k \geq 3$), abelian radicals ($k \geq 3$), solvable radicals ($k \geq 2$), Fitting groups ($k \geq 3$) and π-radicals ($k \geq 3$) that are indistinguishable by k-$\mathrm{WL}_{\mathrm{II}}$. They also have isomorphic socles ($k \geq 5$), stepwise isomorphic factors in the derives series ($k \geq 4$), upper central series ($k \geq 4$), and lower central series ($k \geq 4$). Our techniques regarding characteristic subgroups are fairly general. We thus expect them to be applicable to a large variety of other isomorphism invariants. In particular they should facilitate the analysis of combinations of invariants one might be interested in (such as the Fitting series or the hypercenter).

Beyond these characteristic subgroups, in our second contribution we show that composition factors are implicitly computed by a Weisfeiler-Leman algorithm of bounded dimension.

▶ **Theorem 1.1.** *If $k \geq 5$ and G is indistinguishable from H via k-WL_{I}, then G and H have the same (isomorphism types of) composition factors (with multiplicities).*

The theorem shows that the WL-algorithm, which is a purely combinatorial algorithm, can compute group theoretic invariants that do not even appear as a canonical subset of the group. In particular, the composition factors cannot be localized within the group, and at first sight it might not be clear that WL grasps quotient groups.

Our third contribution, having the most technical proof and building on our other results, regards direct products of groups. Here we consider the decomposition of a group into direct factors. We show that direct products indistinguishable by k-WL must arise from factors that are indistinguishable by $(k+1)$-WL.

▶ **Theorem 1.2.** *Let $G = G_1 \times \cdots \times G_d$ be a direct product and $k \geq 5$. If G and H are not distinguished by k-$\mathrm{WL}_{\mathrm{II}}$ then there are direct factors $H_i \leq H$ such that $H = H_1 \times \cdots \times H_d$ and such that for all i the groups G_i and H_i are not distinguished by $(k-1)$-$\mathrm{WL}_{\mathrm{II}}$.*

In other words, the Weisfeiler-Leman dimension increases by at most 1 when taking direct products. The main difficulty here is that decompositions into direct products are not unique, and thus not definable. These complications arise mainly due to central elements. However we manage to define a canonical maximal central decomposition, that is generally finer than a decomposition into direct factors. We then show that this canonical decomposition is implicitly computed by the WL-algorithm.

k-WL_{I} and k-$\mathrm{WL}_{\mathrm{II}}$, see Section 3.

One way of interpreting our results is that the Weisfeiler-Leman algorithm comprises a unified way of computing all the mentioned invariants and characteristics simultaneously. The dimension can therefore be used to compare the complexity of invariants.

Techniques. To show the various results on characteristic subgroups, we prove a general result on group expressions. It essentially shows that subsets that can be defined by equation systems can be detected by k-WL (see Lemma 4.3).

The result on composition factors involves a technique that relates k-WL distinguishability of groups to detectable normal subgroups and detectable quotients (Theorem 4.8).

To deal with direct products, we extend the technique to simultaneously relate chains of subgroups in two indistinguishable groups (Lemma 4.9). Here we exploit well-known connections of pebble games to Weisfeiler-Leman algorithms. However, the main difficulty regarding our result on decompositions into direct factors is that such decompositions are not unique. In fact in general, a group element cannot be assigned to a direct factor in a well defined sense, making it impossible for WL to detect direct factors. For this purpose we develop a new technical tool, component-wise filtrations (Definition 6.8), which compensate for the non-uniqueness to extract at least the isomorphism type of the direct factors (Lemma 6.10). We also exploit the non-commuting graph of the group and show that certain subsets, which we call non-abelian components, can be detected by k-WL (Lemma 6.14). These non-abelian components lead to a WL-definable maximal central decomposition of every finite group.

Outline. Section 2 provides preliminaries. Section 3 treats WL-refinement in the context of colored groups. In Section 4, we show that invariants generated via WL-refinement fulfill group theoretic closure properties. Section 5 is an extensive collection of specific structure properties and invariants which Weisfeiler-Leman algorithms detect in finite groups. Finally, in Section 6 we investigate the ability of WL-refinement to detect direct product decompositions, building on the results of the previous sections. Throughout the paper various lemmas have been condensed and proofs are omitted. Attached is a full version of the paper (sections agree but the sections are expanded and numbers may disagree.).

Further related work. We should point out that there are various results in the literature on decomposing groups into indecomposable direct factors for various input models of groups. For example there is a polynomial time algorithm to decompose permutation groups into direct products [28]. Finally, there is a recent algorithm that finds direct product decompositions of permutation groups with factors having disjoint support [8]. There is also a polynomial time algorithm that computes direct factors efficiently for groups given by multiplication table [19]. Aspects of this algorithm are related to arguments we use for studying the behavior of WL on direct products (see the beginning of Section 6 for a discussion).

Regarding group isomorphism problems, for isomorphism of Abelian groups a linear time algorithm is known [18] and there are near linear time algorithms for some classes of non-abelian groups (e.g, [10]). Recent directions relate group isomorphism to tensor problems [15]. The Weisfeiler-Leman algorithm has also been incorporated as a subroutine within other sophisticated group isomorphism algorithms [6].

Regarding Weisfeiler-Leman algorithms, the literature is somewhat limited when it comes to groups [4, 6] but quite extensive when it comes to graphs. In [2], for example the authors investigate some graph invariants that are captured by the Weisfeiler-Leman algorithm. We refer to [20] for an introduction and extensive overview over recent results for WL on graphs.

2 Preliminaries

Sets & Partitions. We denote multisets as $\{\{\dots\}\}$. Given disjoint sets M and N, their union is $M \uplus N$. The m-th Cartesian power of M is $M^{(m)}$.

Graphs. We use $V(\Gamma)$ and $E(\Gamma)$ to refer to the vertices or edges of a graph Γ. For a subset $S \subseteq V(\Gamma)$, let $\Gamma[S]$ denote the subgraph induced by the set S.

Groups. Groups are assumed to be finite. The symmetric group on m symbols is denoted by S_m. The order of a group element $g \in G$ is the order of the group generated by g, i.e., $|g| := |\langle g \rangle|$. Given a finite set of primes π, a *π-group* is a group whose order is only divisible by primes in π. A group element is called a *π-element* if it generates a π-group. For any $d \in \mathbb{Z}$, set $(G)^d := \langle \{g^d \mid g \in G\} \rangle$ in contrast to the d-fold direct power G^d. Given $g, h \in G$, we define the *commutator* $[g,h] := ghg^{-1}h^{-1}$. Conjugation is denoted by $g^h := hgh^{-1}$. If $M, N \subseteq G$ we set $[M,N] := \langle [m,n] \mid m \in M, n \in N \rangle$ and we write $G' := [G,G]$.

3 Colored Groups & Weisfeiler-Leman Algorithms

We recapitulate various notions regarding WL-algorithms on groups. For WL on graphs we refer to [20]. For uncolored groups, versions of WL were defined in [4]. For our purpose, we need to generalize the concepts to the setting of colored groups. Let us point out that in the case of graphs, colors can be replaced by gadget constructions to obtain uncolored graphs while maintaining the graph's combinatorial properties. However, for groups it is unclear how to do this. Nevertheless, we can still use colors to restrict the set of possible automorphisms.

3.1 Colorings on Finite Groups

Given a natural number k and a finite group G, a *(k-)coloring* (over G) is just a map $\gamma : G^{(k)} \to \mathcal{C}$ where $\mathcal{C}$ denotes some finite set of colors. A k-coloring γ partitions $G^{(k)}$ into *color classes*. We refer to 1-colorings as *element-colorings*.

The color set $\mathcal{C}$ is often omitted. Considering two natural numbers $m < k$, a *k-coloring* $\gamma : G^{(k)} \to \mathcal{C}$ induces an m-coloring $\gamma^{(m)} : G^{(m)} \to \mathcal{C}$ via $\gamma^{(m)}((g_1, \dots, g_m)) := \gamma((g_1, \dots, g_m, 1, \dots, 1))$. To simplify notation, we may write γ again instead of $\gamma^{(m)}$ and instead of $\gamma^{(1)}$ we use $\gamma^{(G)}$ to emphasize that the coloring is pulled back to group elements.

▶ **Definition 3.1.** *A* colored group *is a group G together with an element-coloring γ over G. We say $M \subseteq G$ is* γ-induced *if $\gamma(M) \cap \gamma(G \setminus M) = \emptyset$ holds, i.e., M is a union of γ-color classes. Colored groups (G, γ_G) and (H, γ_H) are* isomorphic *if there is a group isomorphism $\varphi : G \to H$ that respects colors, i.e., $\gamma_H \circ \varphi = \gamma_G$. We set $Aut_{\gamma_G}(G) := \{\varphi \in Aut(G) \mid \gamma_G \circ \varphi = \gamma_G\}$.*

3.2 Weisfeiler-Leman Refinement on Colored Groups

In [4], three versions of Weisfeiler-Leman algorithms on groups were defined. For us it is sufficient to consider two of these versions. The relevant definitions and results are discussed below. They are essentially taken from [4], but we added colorings.

For $k \geq 2$ we devise a *Weisfeiler-Leman algorithm of dimension k (k-WL)* that takes as input a colored group (G, γ) and computes an $\mathrm{Aut}_\gamma(G)$-invariant coloring on $G^{(k)}$. The algorithm computes an initial coloring from isomorphism invariant properties of k-tuples and then iteratively refines color classes until the process stabilizes. The *stable colorings* arising from k-WL provide (possibly incomplete) polynomial-time non-isomorphism tests.

Version I (k-WL$_\mathrm{I}$). The initial coloring $\chi^{\mathrm{I},k}_{\gamma,0}$ is defined via the group's multiplication relation while also taking into account element-colors. Two tuples $\bar{g} := (g_1, \ldots, g_k)$ and $\bar{h} := (h_1, \ldots, h_k)$ obtain the same initial color if and only if for all indices i, j, and m between 1 and k it holds $\gamma(g_i) = \gamma(h_i)$, $g_i = g_j \iff h_i = h_j$, and $g_i g_j = g_m \iff h_i h_j = h_m$. The subsequent refinements are defined iteratively via $\chi^{\mathrm{I},k}_{\gamma,i+1}(\bar{g}) := \Big(\chi^{\mathrm{I},k}_{\gamma,i}(\bar{g}), \mathcal{M}(\bar{g})\Big)$. Here, $\mathcal{M}(\bar{g})$ is the multiset of k-tuples of colors given by $\mathcal{M}(\bar{g}) := \{\{(\chi^{\mathrm{I},k}_{\gamma,i}(\bar{g}_{1\leftarrow x}), \ldots, \chi^{\mathrm{I},k}_{\gamma,i}(\bar{g}_{k\leftarrow x})) \mid x \in G\}\}$, where $\bar{g}_{j\leftarrow x}$ is obtained by replacing the j-th entry of $\bar{g}$ by x.

Version II (k-WL$_\mathrm{II}$). The initial coloring $\chi^{\mathrm{II},k}_{\gamma,0}$ is defined in terms of *colored, ordered isomorphism* of tuples. Thus, $\bar{g} = (g_1, \ldots, g_k)$ and $\bar{h} = (h_1, \ldots, h_k)$ obtain the same initial color if and only if there exists an isomorphism of colored subgroups $\varphi : \langle \bar{g} \rangle \to \langle \bar{h} \rangle$ such that $\varphi(g_i) = h_i$ for all i. The refinement step is unchanged from Version I.

For finite G there is a smallest i such that $\chi^{\mathrm{I},k}_{\gamma,i}$ and $\chi^{\mathrm{I},k}_{\gamma,i+1}$ induce the same color class partition on $G^{(k)}$. At this point color classes become stable and we obtain the *stable coloring* $\chi^{\mathrm{I},k}_{\gamma} := \chi^{\mathrm{I},k}_{\gamma,i}$. Define $\chi^{\mathrm{II},k}_{\gamma}$ analogously. For uncolored groups write $\chi^{\mathrm{I},k}_{G}$ and $\chi^{\mathrm{II},k}_{G}$, respectively.

By definition, the initial colorings are invariant under isomorphisms that respect γ. This property then holds for the iterated colorings as well. In particular, whenever (G, γ_G) and (H, γ_H) are isomorphic as colored groups, there is a bijection $f : G^{(k)} \to H^{(k)}$ such that $\chi^{\mathrm{I},k}_{\gamma_G} = \chi^{\mathrm{I},k}_{\gamma_H} \circ f$ (and the same holds for Version II). So we obtain a non-isomorphism test by comparing stable colorings computed by k-WL$_\mathrm{I}$ or k-WL$_\mathrm{II}$ as follows.

► **Definition 3.2.** *Let (G, γ_G) and (H, γ_H) be colored groups. We say G is* distinguished *from H by k-WL$_\mathrm{I}$ if there is no bijection $f : G^{(k)} \to H^{(k)}$ with $\chi^{\mathrm{I},k}_{\gamma_G} = \chi^{\mathrm{I},k}_{\gamma_H} \circ f$. We say k-WL$_\mathrm{I}$* identifies *G if it distinguishes G from all other (non-isomorphic) groups. We write $G \equiv^{\mathrm{I}}_k H$ to indicate that G and H are not distinguished by k-WL$_\mathrm{I}$. Furthermore, for $m \leq k$, tuples of group elements $\bar{g} \in G^{(m)}$ and $\bar{h} \in H^{(m)}$ are* distinguished *by k-WL$_\mathrm{I}$ if they obtain different colors in the respective induced m-colorings $(\chi^{\mathrm{I},k}_{\gamma_G})^{(m)}$ and $(\chi^{\mathrm{I},k}_{\gamma_H})^{(m)}$. All definitions also apply to Version* II *in the obvious way.*

The different versions of WL on groups are closely related: in particular, $(k+1)$-WL$_\mathrm{I}$ subsumes k-WL$_\mathrm{II}$. For the colored versions this is briefly discussed in Lemma 3.4 below.

Finally, we note that in [4], a run time bound of $\mathcal{O}(|G|^{k+1}\log(|G|))$ is given for both versions of k-WL to compute the stable coloring on $G^{(k)}$. The same bound applies to colored groups. In particular, the initial coloring of k-WL$_\mathrm{II}$ is efficiently computable, since we only have to compute isomorphism types of k-generated subgroups *relative* to a fixed and ordered generating set of size k.

3.3 Bijective k-Pebble Games

As with graphs and uncolored groups, WL-algorithms on colored groups can be characterized via pebble games. For details we refer to the full version (contained in the appendix).

► **Lemma 3.3** (see [4, Theorem 3.2]). *Let $J \in \{\mathrm{I}, \mathrm{II}\}$ and $k \geq 2$. Consider colored groups (G, γ_G) and (H, γ_H) with $\bar{g} \in G^{(k)}$ and $\bar{h} \in H^{(k)}$. Then $\chi^{J,k}_{\gamma_G}(\bar{g}) = \chi^{J,k}_{\gamma_H}(\bar{h})$ if and only if Spoiler has a winning strategy in the configuration $[(g_1, \ldots, g_k, \bot), (h_1, \ldots, h_k, \bot)]$ in the $(k+1)$-pebble game (Version J).*

In [4] (see [5, Section 3]), relationships for the different versions of WL for uncolored groups are discussed, for the convenience of the reader we sketch the corresponding statement for colored groups here.

▶ **Lemma 3.4.** *Let (G, γ_G) and (H, γ_H) be colored groups.*

1. *Consider $\bar{g} \in G^{(m)}, \bar{h} \in H^{(m)}$ and $k \geq m$. If $\bar{g}$ is distinguished from $\bar{h}$ by k-WL$_{\mathrm{I}}$ then $\bar{g}$ is distinguished from $\bar{h}$ by k-WL$_{\mathrm{II}}$. If $\bar{g}$ is distinguished from $\bar{h}$ by k-WL$_{\mathrm{II}}$ then $\bar{g}$ is distinguished from $\bar{h}$ by $(k+1)$-WL$_{\mathrm{I}}$.*
2. *It holds that $(G, \gamma_G) \equiv^{\mathrm{I}}_{k+1} (H, \gamma_H) \Longrightarrow (G, \gamma_G) \equiv^{\mathrm{II}}_{k} (H, \gamma_H) \Longrightarrow (G, \gamma_G) \equiv^{\mathrm{I}}_{k} (H, \gamma_H)$.*

Proof Sketch. Part 2) follows from Part 1). For Part 1), the first claim is true by definition. For the second claim, using Lemma 3.4, we compare the $(k+1)$-pebble game (ver. II) and the $(k+2)$-pebble game (ver. I) with initial configurations given by placing pebble pairs on (g_i, h_i) for all i. In the version I game, Spoiler copies the winning strategy from the version II game. By assumption, Spoiler eventually reaches a winning configuration in the version II game, meaning that the pebble pairs in this eventual configuration induce a map that does not extend to an isomorphism between the subgroups generated by the respectively pebbled group elements. Then, for each bijection Duplicator may further choose, there must be a witness of this fact, i.e., a word over the currently pebbled group elements in G that is not mapped multiplicatively by Duplicator's bijection. Spoiler can use the extra pebble to win immediately or reduce the length of the witness. A very similar argument is spelled out in the proof of [5, Lemma 3.10, Part 3)] in full detail. ◀

3.4 Induced Colorings & Refinements

We say that a coloring $\gamma_2 : G^{(k)} \to \mathcal{C}_2$ *refines* a coloring $\gamma_1 : G^{(k)} \to \mathcal{C}_1$, denoted $\gamma_2 \preceq \gamma_1$, if each γ_1-color class is a union of γ_2-color classes.

▶ **Lemma 3.5.** *Let γ_1, γ_2 be colorings on G such that $(\chi^{\mathrm{I},k}_{\gamma_1})^{(G)} \preceq \gamma_2 \preceq \gamma_1$. Then $\chi^{\mathrm{I},k}_{\gamma_1}$ and $\chi^{\mathrm{I},k}_{\gamma_2}$ induce the same color classes on $G^{(k)}$.*

4 WL-Refinement on Quotient Groups

We investigate the interplay between WL and basic group structure, e.g., subgroups, normal closures or quotients. We use *subset selectors* to compare substructures of different groups.

▶ **Definition 4.1.** *A* subset selector *$\mathcal{S}$ associates with each colored group (G, γ) a subset $\mathcal{S}(G, \gamma) \subseteq G$. For each version $J \in \{\mathrm{I}, \mathrm{II}\}$, a subset selector $\mathcal{S}$ is called* k-WL$_J$-detectable*, if $\chi^{J,k}_{\gamma_G}(\mathcal{S}(G, \gamma_G)) \cap \chi^{J,k}_{\gamma_H}(H \setminus \mathcal{S}(H, \gamma_H)) = \emptyset$ holds for all pairs of colored groups $(G, \gamma_G), (H, \gamma_H)$.*

When the dependency of $\mathcal{S}(G, \gamma_G)$ on (G, γ_G) is clear from the context, we also say that $\mathcal{S}(G, \gamma_G)$ *is k-WL$_J$-detectable* (instead of $(G, \gamma) \mapsto \mathcal{S}(G, \gamma)$ being detectable). Examples of 2-WL$_J$-detectable subset selectors include the association of every group with its center ($J = \mathrm{II}$) or the subset selector associating with each group the subset of elements of order 2.

We should remark that in our sense detectable means that the subset of interest is a union of $\chi^{J,k}_{\gamma_G}$-color classes, but we make no statement on how to algorithmically determine which color classes form the set. It might a priori not be clear that the subset is even computable.

If $\mathcal{S}$ is k-WL$_J$-detectable then $\mathcal{S}(G, \gamma_G)$ is $\chi^{J,k}_{\gamma_G}$-induced, hence $\mathrm{Aut}_{\gamma_G}(G)$-invariant. If $\mathcal{S}$ and $\mathcal{T}$ are k-WL$_J$-detectable, so are their union (intersection) in G and $G \setminus \mathcal{S}(G, \gamma_G)$.

▶ **Definition 4.2.** *A* group expression *$\mathcal{E} := (\mathcal{S}_1, \ldots, \mathcal{S}_t; \mathcal{R})$ of length t is a sequence of subset selectors $\mathcal{S}_i$ together with a set $\mathcal{R}$ of words $w(x_1, \ldots, x_t)$ over t variables $x_1, \ldots, x_t$, allowing inverses. Let (G, γ) be a colored group, then a t-tuple $(g_1, \ldots, g_t) \in G^{(t)}$ is a* solution *to $\mathcal{E}$ if for each i it holds that $g_i \in \mathcal{S}_i(G, \gamma)$ and for each $w \in \mathcal{R}$ it holds that $w(g_1, \ldots, g_t) = 1$. Let $Sol_{\mathcal{E}}(G, \gamma) \subseteq G^{(t)}$ denote the set of all solutions to $\mathcal{E}$ over (G, γ).*

▶ **Lemma 4.3.** *Consider a group expression $\mathcal{E} := (\mathcal{S}_1, \dots, \mathcal{S}_t; \mathcal{R})$. Let $k \geq t$ and assume that each $\mathcal{S}_i$ is k-WL$_{\mathrm{II}}$-detectable.*

1. *Let (G, γ_G) and (H, γ_H) be colored groups. Then all t-tuples in $Sol_{\mathcal{E}}(G, \gamma_G)$ can be distinguished from all t-tuples in $H^{(t)} \setminus Sol_{\mathcal{E}}(H, \gamma_H)$ via k-WL$_{\mathrm{II}}$.*
2. *For $1 \leq j \leq t$ and colored groups (G, γ) define*

$$Sol_j^{\exists}(G,\gamma) := \{x \in G \mid \exists (x_1, \dots, x_t) \in Sol_{\mathcal{E}}(G,\gamma) : x_j = x\}$$
$$Sol_j^{\forall}(G,\gamma) := \{x \in G \mid (\forall x_i \in \mathcal{S}_i(G,\gamma))_{1 \leq i \leq t} : (x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_t) \in Sol_{\mathcal{E}}(G,\gamma)\}.$$

Then $Sol_j^{\exists}$ and $Sol_j^{\forall}$ are k-WL$_{\mathrm{II}}$-detectable subset selectors for all j.
The same holds for k-WL$_{\mathrm{I}}$, provided $k > t$.

We can now argue that WL is powerful enough to incorporate various basic group theoretic concepts. This in particular includes generated subgroups, normal closures, powers, conjugacy classes, centralizers, and normalizers. All these statements are relative to inductively detected structures, so the processes can be iterated. Let us record this in the following lemma.

▶ **Lemma 4.4.** *Consider k-WL$_{\mathrm{II}}$-detectable subset selectors $\mathcal{S}, \mathcal{T}$. Then the following subset selectors are k-WL$_{\mathrm{II}}$-detectable:*

1. *$\mathcal{S}^e$ for each $e \in \mathbb{Z}$, where $\mathcal{S}^e(G,\gamma) := \{s^e \mid s \in \mathcal{S}(G,\gamma)\}$,*
2. *$C_{\mathcal{S}}(\mathcal{T})$, where $C_{\mathcal{S}}(\mathcal{T})(G,\gamma) := \{s \in \mathcal{S}(G,\gamma) \mid [s, \mathcal{T}(G,\gamma)] = \{1\}\}$.*

Provided k is at least 3, k-WL$_{\mathrm{II}}$ further detects the following subset selectors:

3. *$\{s_1 \dots s_e \mid s_i \in \mathcal{S}(G,\gamma)\}$ for each $e \in \mathbb{N}$, in particular also $\langle \mathcal{S}(G,\gamma) \rangle$,*
4. *$\{s^t := tst^{-1} \mid s \in \mathcal{S}(G,\gamma), t \in \mathcal{T}(G,\gamma)\}$, hence the normal closure $\langle \mathcal{S}(G,\gamma)^G \rangle$,*
5. *$\mathcal{N}_{\mathcal{S}}(\mathcal{T})$, where $\mathcal{N}_{\mathcal{S}}(\mathcal{T})(G,\gamma) := \{s \in \mathcal{S}(G,\gamma) \mid$
$\mathcal{T}(G,\gamma)^s = \mathcal{T}(G,\gamma)\}$,*
6. *$[\mathcal{S}, \mathcal{T}]$, where $[\mathcal{S}, \mathcal{T}](G,\gamma) := \langle [s,t] \mid s \in \mathcal{S}(G,\gamma), t \in \mathcal{T}(G,\gamma) \rangle$.*

All statements remain true if we replace Version II *by Version* I *everywhere (including the assumptions), provided $k > 2$ in Parts 1 and 2 and $k > 3$ in Parts 3–6.*

We point out how to identify groups as direct products of detectable subgroups.

▶ **Example 4.5.** Let $G \equiv_3^{\mathrm{II}} H$ and assume that $G = G_1 \times G_2$ with $\chi_G^{\mathrm{II},3}$-induced subgroups G_i. We use element-colors in G_i to define a detectable subset selector $K \mapsto K_i := \{x \in K \mid \chi_K^{\mathrm{II},k}(x) \in \chi_G^{\mathrm{II},k}(G_i)\}$. Since $G \equiv_3^{\mathrm{II}} H$, also $H_i \equiv_3^{\mathrm{II}} G_i$. By the previous lemma, 3-WL$_{\mathrm{II}}$ detects $[G_1, G_2]$ and $G_1 \cap G_2$, which are both trivial, as well as $\langle G_1, G_2 \rangle$, which is equal to G. By definition of detectability, the same must hold for H_1 and H_2, thus $H = H_1 \times H_2$.

In Section 6 we discuss the (much harder) case of arbitrary direct decompositions, without the assumption that each direct factor is detectable as a subgroup.

Next, we prove that WL is capable of exploiting properties of quotients over detectable subgroups. Later, this can be inductively leveraged along chains of subgroups.

▶ **Definition 4.6.** *Given a coloring $\gamma : G \to \mathcal{C}$ and a normal subgroup $N \trianglelefteq G$ define the induced* quotient coloring *$\bar{\gamma}$ on G/N via $\bar{\gamma}(gN) := \{\{\gamma(gn) \mid n \in N\}\}$.*

▶ **Lemma 4.7.** *Let $k \geq 4$ and consider colored groups (G, γ_G) and (H, γ_H). Assume that there are normal subgroups $N_G \trianglelefteq G$ and $N_H \trianglelefteq H$ which are induced by γ_G and γ_H, respectively, such that $\gamma_G(N_G) = \gamma_H(N_H)$. Then*

$$\chi_{\bar{\gamma}_G}^{\mathrm{I},k}(g_1 N_G, \dots, g_k N_G) \neq \chi_{\bar{\gamma}_H}^{\mathrm{I},k}(h_1 N_H, \dots, h_k N_H) \Longrightarrow \chi_{\gamma_G}^{\mathrm{I},k}(g_1, \dots, g_k) \neq \chi_{\gamma_H}^{\mathrm{I},k}(h_1, \dots, h_k)$$

for all choices of $g_i \in G$ and $h_i \in H$.

Proof sketch. The idea is to simulate the pebble game on quotient groups in the pebble game on G and H. Spoiler can first win modulo N_G and N_H, respectively, and then use a constant number of pebbles to manipulate the configuration into one that fulfills the winning condition over the original groups. For details, we refer to the full version. ◀

We synthesize the previous results into our first main theorem, stating that whenever $G \equiv_k^{\mathrm{I}} H$ holds, there is a color preserving correspondence between detectable substructures.

▶ **Theorem 4.8.** *Let k be at least 4.*

1. *Consider subset selectors N, U and U/N such that for all (G,γ) it holds that $N(G,\gamma) \trianglelefteq G$, $N(G,\gamma) \leq U(G,\gamma)$ and $U/N(G/N(G),\bar{\gamma}) = U(G)/N(G)$. If N and U/N are k-WL_{I}-detectable then so is U.*
2. *Consider colored groups $(G,\gamma_G) \equiv_k^{\mathrm{I}} (H,\gamma_H)$. Let $\Psi : G \to H$ be a bijection with $(\chi_{\gamma_G}^{\mathrm{I},k})^{(G)} \circ \Psi = (\chi_{\gamma_H}^{\mathrm{I},k})^{(H)}$. Then $M \subseteq G$ is $\chi_{\gamma_G}^{\mathrm{I},k}$-induced if and only if $\Psi(M) \subseteq H$ is $\chi_{\gamma_H}^{\mathrm{I},k}$-induced. In this case it holds that $\Psi(\langle M\rangle) = \langle \Psi(M)\rangle$. In particular, if M is a subgroup then so is $\Psi(M)$ and it holds $(M,\gamma_G|_M) \equiv_k^{\mathrm{I}} (\Psi(M),\gamma_H|_{\Psi(M)})$. Additionally, M is normal if and only if $\Psi(M)$ is and then it also holds that $(G/M,\bar{\gamma_G}) \equiv_k^{\mathrm{I}} (H/\Psi(M),\bar{\gamma_H})$.*

Finally let us point out that detectable substructures can be used to limit Duplicator-strategies. This technique will be needed towards the main result of Section 6. More precisely, we show that Spoiler can "trade off" one pebble pair to enforce that Duplicator's bijections are simultaneously compatible with detectable substructures in the following sense.

▶ **Lemma 4.9.** *Let $k \geq 3$ and $J \in \{\mathrm{I},\mathrm{II}\}$. Consider groups G and H with $G \equiv_k^J H$, so Duplicator has a winning strategy in the $(k+1)$-pebble game (Version J). Assume $\chi_G^{J,k}$ and $\chi_H^{J,k}$ induce chains of subgroups $G_s \leq \cdots \leq G_1 \leq G$ and $H_s \leq \cdots \leq H_1 \leq H$, respectively, such that $\chi_G^{J,k}(G_i) = \chi_H^{J,k}(H_i)$ for all i. Then Duplicator has a winning strategy in the k-pebble game (Version J) on (G,H) such that each bijection $f : G \to H$ chosen by Duplicator's strategy fulfills the following condition: $\forall x \in G\ \forall i : f(xG_i) = f(x)H_i$.*

The proof actually works in a context more general than groups, replacing subgroup chains by nested equipartitions. This generalization might find applications in different contexts.

5 WL-dimension of certain isomorphism invariants

We just briefly summarize our results in what follows. A detailed treatment can be found in the full version, for group theoretic foundations see for example [16].

▶ **Lemma 5.1.** *For $k \geq 2$, k-WL_{II} identifies all finite k-generated groups and all finite abelian groups.*

5.1 Derived & Central Series

▶ **Lemma 5.2.**

1. *For $k \geq 3$, $G' := [G,G]$ is k-WL_{II}-detectable.*
2. *Assume that $k \geq 4$ and $G \equiv_k^{\mathrm{I}} H$ hold. Let $G_0 \geq G_1 \geq \cdots \geq G_t$ denote the derived, upper central or lower central series of G (without redundancies, starting at $G_0 := G$). Define the corresponding series of H via $H_0 \geq \cdots \geq H_s$. Then $s = t$ holds and for all i we have that $G_i \equiv_k^{\mathrm{I}} H_i$, as well as $G_i/G_{i+1} \cong H_i/H_{i+1}$.*

The requirement $k \geq 3$ is necessary in the first statement as computations on SmallGroup(128,171) and SmallGroup(128,1122) from the Small Groups Library in GAP [14] show.

▶ **Corollary 5.3.** *For $k \geq 4$, k-WL_{I} distinguishes solvable from non-solvable groups and k-WL_{I} distinguishes between groups of different nilpotency classes.*

5.2 Radicals

Let $\mathcal{F}$ be a class of finite groups that is closed under isomorphism and normal products. Then the *$\mathcal{F}$-radical* $\mathcal{O}_{\mathcal{F}}(G)$ of G is defined as the largest normal $\mathcal{F}$-subgroup in G.

▶ **Lemma 5.4.** *Let $k \geq 3$. If $\mathcal{F}$ is closed under normal subgroups and $(k-1)$-WL_{II} distinguishes $\mathcal{F}$-groups from all non-$\mathcal{F}$-groups, then k-WL_{II} detects $\mathcal{O}_{\mathcal{F}}(G)$ in G.*

▶ **Lemma 5.5.** *The solvable radical is 2-WL_{II}-detectable. The nilpotent radical $Fit(G)$ and all π-radicals $\mathcal{O}_{\pi}(G)$ (π a collection of primes) are 3-WL_{II}-detectable.*

▶ **Lemma 5.6.** *If G contains a unique maximal abelian normal subgroup, then it is 3-WL_{II}-detectable.*

5.3 Simple Groups & Composition Factors

Recall that finite (almost) simple groups can be generated with 2 (respectively 3) elements [9].

▶ **Lemma 5.7.** *2-WL_{II} identifies finite simple groups. 3-WL_{II} identifies finite almost simple groups and finite direct products of simple groups.*

In the case of simple groups there is a stronger result, stating that simple groups are uniquely identified among all groups up to isomorphism by their order and the orders of their elements [27].

▶ **Theorem 5.8.** *The socle of a finite group G is 4-WL_{II}-detectable. Let $k \geq 5$ and $G \equiv^{\mathrm{I}}_{k} H$, then G and H have the same composition factors (with multiplicities).*

6 WL-Refinement and Direct Products

In this final section we study the detectability of direct product structures in finite groups. The section is organized similar to [19], in the sense that we first consider direct products where one factor is an abelian group (the semi-abelian case) and reduce to these in the general case later on. A crucial difference between our setting and the one in [19] is that in the latter, computations can be executed as long as they are efficient, where in our case, we are analyzing a fixed algorithm that cannot make non-canonical choices.

▶ **Definition 6.1.** *A group G is the* (internal) central product *of subgroups $G_1, G_2 \leq G$, if it holds that $G = \langle G_1, G_2 \rangle$ and $[G_1, G_2] = \{1\}$.*

Our main difficulty is that a group can admit several inherently different central decompositions. In contrast to that recall that indecomposable *direct* decompositions are unique in the following sense.

▶ **Lemma 6.2.** *Let $G = G_1 \times \cdots \times G_m = H_1 \times \cdots \times H_n$ be two decompositions of G into directly indecomposable factors. Then $n = m$ and there is a permutation $\sigma \in S_m$ such that for all i we have $G_i \cong H_{\sigma(i)}$ and $G_i Z(G) = H_{\sigma(i)} Z(G)$.*

Proof. The first part is the well-known Krull-Remak-Schmidt Theorem and the addition that $G_iZ(G) = H_{\sigma(i)}Z(G)$ can be easily derived (see for example [19, Corollary 6]) ◀

In particular, the collection of subgroups $\{G_iZ(G)\}_{1\leq i\leq m}$ is invariant under automorphisms as a whole. Later we show that $\bigcup_{i=1}^{m} G_iZ(G)$ is 5-WL_{I}-detectable.

▶ **Lemma 6.3.** *If $J \in \{\mathrm{I}, \mathrm{II}\}$, $k \geq 3$, $G_1 \equiv_k^J H_1$ and $G_2 \equiv_k^J H_2$, then $G_1 \times G_2 \equiv_k^J H_1 \times H_2$.*

The opposite direction is investigated below and turns out to be highly non-trivial.

6.1 Abelian and Semi-Abelian Case

Direct products with abelian groups serve as a basis for reduction later on.

▶ **Definition 6.4.** *An element $x \in G$* splits *from G if there is a* complement *$H \leq G$ of x in G, i.e., $G = \langle x\rangle \times H$.*

A detailed treatment of splitting elements can be found in the full version.

▶ **Corollary 6.5.** *The set of elements splitting from a finite group is 4-WL_{I}-detectable.*

The splitting of elements can reveal information about direct decompositions of a group.

▶ **Lemma 6.6.** *Consider a direct product $G = G_1 \times G_2$ and a p-element $z := (z_1, z_2) \in Z(G)$. Then z splits from G if and only if z_i splits from G_i for some $i \in \{1, 2\}$ which fulfills $|z_i| = |z|$.*

This can be inductively leveraged to handle the semi-abelian case, by which we mean groups of the form $H \times A$ where A is abelian and H does not have abelian direct factors.

▶ **Lemma 6.7.** *Let $G = H \times A$ with A a maximal abelian direct factor. The isomorphism type of A is identified by 4-WL_{I}, i.e., if $\tilde{G} \equiv_4^{\mathrm{I}} G$ then $\tilde{G}$ has a maximal abelian direct factor isomorphic to A.*

Controlling the non-abelian part is more complicated and led us to introduce a new technical framework.

▶ **Definition 6.8.** *Let $G = L \times R$. A* component-wise filtration *of $U \leq G$ w.r.t. L and R is a chain of subgroups $\{1\} = U_0 \leq \cdots \leq U_r = U$ such that for all $1 \leq i < r$, we have $U_{i+1} \leq U_i(L \times \{1\})$ or $U_{i+1} \leq U_i(\{1\} \times R)$. The filtration is k-WL_{I}-*detectable *if all subgroups in the chain are.*

▶ **Lemma 6.9.** *Let $G = H \times A$ with maximal abelian direct factor A. There exists a component-wise filtration of $Z(G)$ with respect to H and A that is 4-WL_{I}-detectable.*

▶ **Lemma 6.10.** *Consider $G := H \times A$ and $\hat{G} = \hat{H} \times \hat{A}$ where A and $\hat{A}$ are maximal abelian direct factors. Then, for $k \geq 5$, $G \equiv_k^{\mathrm{I}} \hat{G}$ implies $H \equiv_{k-1}^{\mathrm{I}} \hat{H}$.*

6.2 General Case

The general case is reduced to the semi-abelian case. Consider an indecomposable direct decomposition $G = G_1 \times \cdots \times G_d$. We first show that $\bigcup_i G_iZ(G)$ can be detected by WL and then we exploit the fact that the non-commuting graph induces components on $\bigcup_i G_iZ(G)$ which correspond to the groups $G_iZ(G)$.

▶ **Definition 6.11.** *Given a group G, we define the* non-commuting graph *Γ_G with vertex set G, in which two elements $g, h \in G$ are joined by an edge if and only if $[g, h] \neq 1$.*

▶ **Lemma 6.12** ([1], Prop. 2.1). *If G is non-abelian then $\Gamma_G[G \setminus Z(G)]$ is connected.*

We now approximate $\bigcup_i G_i Z(G)$ from below by constructing a canonical central decomposition of G which is WL-detectable.

▶ **Definition 6.13.** *Consider a finite, non-abelian group G. Define $M_1 \subseteq G$ to be the set of non-central elements g whose centralizers $C_G(g)$ have maximal order among all non-central elements. Iteratively define M_{i+1} by adding those elements g to M_i that have maximal centralizer order $|C_G(g)|$ among the remaining elements $G \setminus \langle M_i \rangle$. Set $M := M_\infty$ to be the stable set resulting from this process. Consider the subgraph of Γ_G induced on M and let $K_1 \dots, K_m$ be its connected components. Set $N_i := \langle K_i \rangle$. We call $N_1, \dots, N_m$ the* non-abelian components *of G.*

▶ **Lemma 6.14.** *In the notation of the previous definition, the following hold:*

1. *M is detectable in G by 3-WL_{II}.*
2. *$G = N_1 \cdots N_m$ is a central decomposition of G. For all i, $Z(G) \leq N_i$ and N_i is non-abelian. In particular M generates G.*
3. *If $G = G_1 \times \cdots \times G_d$ is an arbitrary direct decomposition, then for each $1 \leq i \leq m$ there is exactly one $1 \leq j \leq d$ with $N_i \subseteq G_j Z(G)$. Collect all such i for one fixed j in an index set I_j. Then the product over all N_i for $i \in I_j$ is equal to $G_j Z(G)$.*

▶ **Definition 6.15.** *Let $G = N_1 \cdots N_m$ be the decomposition into non-abelian components and let $G = G_1 \times \cdots \times G_d$ be an arbitrary direct decomposition. We say $x \in G$ is* full *for $(G_{j_1}, \dots, G_{j_r})$, if $\{1 \leq i \leq m \mid [x, N_i] \neq 1\} = I_{j_1} \cup \cdots \cup I_{j_r}$. For all $x \in G$ define $C_x := \Pi_{[x,N_i]=\{1\}} N_i$ and $N_x := \Pi_{[x,N_i]\neq\{1\}} N_i$.*

Overall, when grouped adequately, the full elements with maximal centralizers generate the direct factors modulo central elements (see full version).

▶ **Lemma 6.16.** *Let $G = N_1 \cdots N_m$ be the decomposition into non-abelian components and $G = G_1 \times \cdots \times G_d$ a decomposition into indecomposable direct factors. For $k \geq 5$, k-WL_{II} detects the set of elements that are full for only one G_i as well as the pairs of elements that are full for the same collection of direct factors.*

▶ **Corollary 6.17.** *If $G = G_1 \times \cdots \times G_d$ is a decomposition into indecomposable direct factors then $\bigcup_i G_i Z(G)$ is detected in G by 5-WL_{II}.*

▶ **Theorem 6.18.** *Let $G = G_1 \times \cdots \times G_d$ be a decomposition into indecomposable direct factors and $k \geq 5$. If $G \equiv_k^{\mathrm{II}} H$ then there are indecomposable direct factors $H_i \leq H$ such that $H = H_1 \times \cdots \times H_d$ and $G_i \equiv_{k-1}^{\mathrm{II}} H_i$ for all i. Moreover G and H have isomorphic maximal abelian direct factors and $G_i Z(G) \equiv_k^{\mathrm{II}} H_i Z(H)$.*

Proof. Since $\mathcal{F}_G := \bigcup_i G_i Z(G)$ is 5-WL$_{\mathrm{II}}$-detectable, the group H must be decomposable into indecomposable direct factors $H = \times_j H_j$ such that $\mathcal{F}_H = \bigcup_j H_j Z(H) \subseteq H$ is indistinguishable from $\mathcal{F}_G$. Consider the non-commuting graphs of G and H induced on these sets and recall that non-commuting graphs of non-abelian groups are connected (Lemma 6.12). Since different direct factors in a fixed decomposition centralize each other, we obtain that for each non-singleton connected component K of $\Gamma_G[\mathcal{F}_G]$ there exists a unique indecomposable direct factor G_i such that $K = G_i Z(G) \setminus Z(G)$ and thus $\langle K \rangle = G_i Z(G)$. Again by Lemma 6.12, all non-abelian direct factors appear in this way.

The same holds for H and so if G is not distinguishable from H, there must be a bijection between the components of $\Gamma_G[\mathcal{F}_G]$ and $\Gamma_H[\mathcal{F}_H]$, such that the subgroups generated by corresponding components are indistinguishable via 5-WL$_{\mathrm{II}}$. This defines a correspondence $G_i Z(G) \equiv_k^{\mathrm{II}} H_i Z(H)$ after reordering the factors of H in an appropriate way. From Lemma 6.10 it follows that $G_i \equiv_{k-1}^{\mathrm{II}} H_i$. By Lemma 6.9, G and H must have isomorphic maximal abelian direct factors, so for abelian factors we even have $G_i \cong H_i$. ◀

7 Conclusion

We studied the Weisfeiler-Leman dimension of numerous isomorphism invariants of groups, showing that a low dimensional WL-algorithm in fact captures a plethora of isomorphism invariants, characteristic subgroups, and group properties classic to algorithmic group theory. Particularly tricky was the treatment of direct indecomposable factors, for which we had to circumvent the fact that the they do not correspond to canonical substructures of the groups. Our techniques lead us to a canonical maximal central decomposition.

The observation that many efficiently computable isomorphism invariants are captured by a low dimensional WL-algorithm raises the question whether there are actually invariants that are not captured at all. Here we should emphasize that it is an open problem whether some fixed dimension of WL represents a complete invariant. The question is equivalent to the well-known open question whether the Weisfeiler-Leman dimension of groups is bounded in general (stated explicitly in [4]).

For this open question, our results show that it suffices to consider directly indecomposable groups. We wonder whether there are other, similar reductions to confine the search for groups of high WL-dimension.

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