Dynamic Traffic Assignment for Electric Vehicles

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Abstract

We initiate the study of dynamic traffic assignment for electrical vehicles addressing the specific challenges such as range limitations and the possibility of battery recharge at predefined charging locations. We pose the dynamic equilibrium problem within the deterministic queueing model of Vickrey and as our main result, we establish the existence of an energy-feasible dynamic equilibrium. There are three key modeling-ingredients for obtaining this existence result:

1. We introduce a *walk-based* definition of dynamic traffic flows which allows for cyclic routing behavior as a result of recharging events en route.
2. We use abstract convex feasibility sets in an appropriate function space to model the energy-feasibility of used walks.
3. We introduce the concept of *capacitated dynamic equilibrium walk-flows* which generalize the former unrestricted dynamic equilibrium path-flows.

Viewed in this framework, we show the existence of an energy-feasible dynamic equilibrium by applying an infinite dimensional variational inequality, which in turn requires a careful analysis of continuity properties of the network loading as a result of injecting flow into walks.

We complement our theoretical results by a computational study in which we design a fixed-point algorithm computing energy-feasible dynamic equilibria. We apply the algorithm to standard real-world instances from the traffic assignment community illustrating the complex interplay of resulting travel times, energy consumption and prices paid at equilibrium.

2012 ACM Subject Classification Mathematics of computing → Network flows; Networks → Traffic engineering algorithms

Keywords and phrases Electromobility, Dynamic Traffic Assignment, Dynamic Flows, Fixed Point Algorithm

Digital Object Identifier 10.4230/OASIcs.ATMOS.2022.6

Related Version *Full Version*: `https://arxiv.org/abs/2207.04454`

Supplementary Material *Dataset, Software*: `https://github.com/ArbeitsgruppeTobiasHarks/electric-vehicles/tree/f6f67e45c70c3cd2b0690280be3591d044127747`

Funding Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - HA 8041/1-2.

Acknowledgements We are grateful to the anonymous reviewers for their valuable feedback on this paper. Additionally, we thank the organizers and participants of the 2022 Dagstuhl seminar on "Dynamic Traffic Models in Transportation Science" where we had many helpful and inspiring discussions on the topic of this paper.

1 Introduction

Electric vehicles (EVs) are a great promise for the coming decades in order to allow for mobility but at the same time take measures against the climate change by reducing the emissions of classical combustion engines. The wide-spread operation of EVs, however, is by

22nd Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2022).
Editors: Mattia D'Emidio and Niels Lindner; Article No. 6; pp. 6:1–6:15
OpenAccess Series in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

far not fully resolved as the battery technology comes with several complications, some of which are listed below:

- The limited battery capacity implies a limited driving range of EVs resulting in complex resource-constrained routing behavior taking the feasibility of routes w.r.t. the energy consumption into account (cf. [4, 17]).
- Feasible routes may contain cycles if the possibility of recharging at predefined charging stations is included (see [1, 18, 17]). The necessity of multiple recharging operations is especially relevant for longer trips such as long-haul trucking or for the use of EVs in urban logistics [4].
- The recharging strategy itself can be quite complex involving *mode choices* ranging from low-power supply modes (22 kW) to high-power supply modes (350 kW) (cf. [19]). Different modes may come with substantially different recharging times and prices (cf. [16]).
- For a selected recharge mode, the *duration* of the recharge determines both, the resulting battery state (and hence the subsequent reach of the vehicle), and the corresponding total recharge price and, thus, adds a further strategic dimension.

While some of the above challenges have been partly addressed within the "battery-constrained routing" community (cf. [1, 4, 6, 18, 15, 17, 13] and references therein), the majority of these works rely on a *static* and mostly *decoupled* view on traffic assignment: Each vehicle is routed independently (subject to battery related side constraints) and the interaction of vehicles in terms of congestion effects with increased travel times is not considered. Only a few works (such as [22, 23]) take congestion effects of routing EVs into account, yet, still relying on a static routing model.

In a realistic traffic system, vehicles travel *dynamically* through the network and the route choices of vehicles are mutually dependent as the propagation of traffic flow leads to congestion at bottlenecks and in turn determines the route choices to avoid congestion. This complex and self-referential dependency has been under scrutiny in the traffic assignment community for a long time and it is usually resolved by *dynamic traffic assignments (DTA)* under which – roughly speaking – at any point in time, no driver can opt to a better route. As a result, the actual equilibrium travel times do depend on the collective route choices of all vehicles and even more strikingly, the equilibrium routes determine the actual energy consumption profile of an EV leading to a complex coupled dynamic system. Note that emergent congestion effects are even relevant for the pure recharging process of an EV, since with the rapid growth rates of EVs compared to the relatively scarce recharging infrastructure, significant waiting times at recharging stations are anticipated (cf. [19]).

DTA models have been studied in the transportation science community for more than 50 years with remarkable success in deriving a concise mathematical theory of dynamic equilibrium distributions, yet there is no such theory for DTA models addressing the specific characteristics of EVs. Let us quote a recent survey article by Wang, Szeto, Han and Friesz [20] that mentions the lack of DTA models for the operation of EVs: "To our best knowledge, a DTA model with path distance constraints for electric vehicles remains undeveloped; so do the corresponding solution algorithms." This research gap might have good mathematical reasons: virtually all known existence results in the DTA literature rely on the assumption that paths must be *acyclic* in order to obtain a well defined *path-delay operator* mapping the path-inflows to the experienced travel time (cf. [2, 3, 5, 9, 25, 12, 14]). As explained above, the range-limitation of EVs requires recharging stops and, thus, leads to cyclic routing behavior with path length restrictions requiring a new approach to establish equilibrium existence.

Our Contribution

In this paper, we study a dynamic traffic assignment problem that addresses the operation of electrical vehicles including their range-limitations caused by limited battery energy and necessary recharging stops. Our contributions can be summarized as follows:

1. We propose a DTA model tailored to the operation of EVs that combines the Vickrey deterministic queueing model with graph-based gadgets modeling complex recharging procedures such as mode choices and recharge durations. A combined routing and recharging strategy of an EV can be reduced to choosing an energy-feasible walk within this extended network.
2. A feasible walk may contain cycles and the set of feasible walks that respect the battery-constraints may be quite complex. After establishing some fundamental properties of the resulting network loading when flow is injected into walks, we introduce abstract convex, closed and bounded feasibility sets in an appropriate function space to describe the resulting feasible dynamic walk-flows. These feasibility sets are used to set up the formal definition of a capacitated dynamic equilibrium in which also the monetary effect of prices charged at recharging stations is integrated in the utility function of agents.
3. With the formalism of the network loading and the notion of a capacitated dynamic equilibrium, we then proceed to the key question of equilibrium existence. We show that the walk-delay operator that maps the walk-inflows to resulting travel times is sequentially weak-strong continuous on the convex feasibility space (which corresponds to *weakly-continuous* as previously used by Zhu and Marcotte [25] for paths under the strict FIFO-condition). This allows us to apply a variational inequality formulation by Lions [11] to establish the existence of dynamic equilibria. While the general variational inequality approach dates back to Friesz et al. [5], our result generalizes previous works on side-constraint dynamic equilibria (e.g. Zhong et al. [24]), because we do not assume a priori compactness of the underlying convex restriction set, nor strict FIFO as in [24, 25].
4. We finally develop a fixed-point algorithm for the computation of energy-feasible dynamic equilibria and apply the algorithm to several real-world instances from the literature. To the best of our knowledge, this work is among the first to compute dynamic traffic equilibria for electric vehicles and it can serve as the basis for evaluating the interplay between congestion, travel times and used energy in a dynamic traffic equilibrium.

2 The Model

We now introduce our model for electric vehicles in which we combine the Vickrey deterministic queuing model with graph-based extensions in order to model the key characteristics of the battery recharging technology for electric vehicles. The complex strategic decision of an EV involves

1. the route choice – possibly involving necessary recharging stops and cycles,
2. the mode choice of the battery-recharge (e.g., Level 1, 2, 3),
3. the actual duration of each battery-recharge en route, which determines the resulting battery state and the recharge cost while also adding to the EV's total travel time.

We model this complex decision space by using several graph-based gadgets inside the Vickrey network model leading to the *battery-extended network*. This way, we can reduce the complex strategy choice of an EV to selecting a *feasible walk* inside the battery-extended network. We will now start with the physical Vickrey flow model and then discuss the battery-extended network.

The Physical Vickrey Network Model. The physical Vickrey network model is based on a finite directed graph $G' = (V', E')$ with positive rate capacities $\nu_e \in \mathbb{R}_+$ and positive transit times $\tau_e \in \mathbb{R}_+$ for every edge $e \in E'$. There is a finite set of commodities $I = [n] \coloneqq \{1, \dots, n\}$, each with a commodity-specific source node $s_i \in V'$ and a commodity-specific sink node $t_i \in V'$. The (infinitesimally small) agents of every commodity $i \in I$ each represent a vehicle (electric or combustion engine) and they enter the network according to a bounded and integrable network inflow rate function $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with bounded support. We denote by $T \coloneqq \sup \{ \theta \in \mathbb{R}_{\geq 0} \mid \exists i \in I : u_i(\theta) > 0 \}$ the last time a vehicle enters the network. If the total inflow into an edge $e = vw \in E'$ exceeds the rate capacity ν_e, a queue builds up and agents need to wait in the queue before they are forwarded along the edge. The total travel time along e is thus composed of the waiting time spent in the queue plus the physical transit time τ_e.

The Battery-Extended Network. For vehicles corresponding to a commodity $i \in I$, we assume that they all have an equal initial battery state of level $b_i > 0, i \in I$. If an agent of commodity i travels along an edge $e \in E$, it comes with a (flow-independent) battery cost of $b_{i,e} \in \mathbb{R}$ which may be positive (energy consumption) or negative (recuperation). The maximum battery capacity is denoted by $b_i^{\max}$. Note that the assumption that battery cost is independent of congestion is well justified, since the engine of an EV completely turns off when a vehicle stands still leading to negligible energy consumption while queueing up. Yet, the chosen route does depend on the perceived travel time, thus, also the realized energy consumption does (indirectly) depend on congestion.

Recharging may occur using different *modes* ranging from relatively low power supply (up to 3.7 kilowatts (kW), Level 1) to medium supply (up to 22 kW, Level 2) up to high supply (25 kW to more than 350 kW, Level 3) or even complete battery swaps. Each mode may result in different recharging times for a fixed targeted state of charging (SOC), and also the resulting prices may significantly vary not only among modes but also among recharge locations.[1] Besides the recharge location and mode choice, the planned duration for the recharge is an important decision as it directly affects the journey time, the resulting SOC and the price paid. Given a tariff for recharging,[2] we can model the set of possible combinations of recharge times, battery states and recharge prices via tuples of the form $(\tau, b_i, p_i), i \in I$, where $\tau \in \mathbb{N}$ is the time (in minutes) spent for recharging, $b_i \equiv b_i(\tau)$ is the resulting increase of the battery level and $p_i \equiv p_i(\tau) \in \mathbb{R}_+$ is the charged price for a vehicle of commodity $i \in I$. Note that the functions $b_i(\tau)$, $p_i(\tau)$ can be directly derived from the SOC function for recharging and the resulting tariffs, respectively (cf. Xiao et al [21]). Recharging stations are identified with subsets of nodes of V' denoted by $C_i \subseteq V', i \in I$, where C_i depends on $i \in I$ to allow for different recharging technologies, that is, some vehicles may only recharge at stations that have the required technology. By introducing copies of commodities it is again without loss of generality to assume that every agent of commodity i uses the same technology. For a recharging location $v \in C_i, i \in I$, we introduce a subgraph as depicted in Figure 1. For $v \in C_i$, the parallel edges leaving v correspond to the different recharging modes available and the subsequent edges model the different recharging times with corresponding recharge

[1] The statistics for 2021 for the recharging prices in Germany show for instance a significant price span for the "cents per kWh tariff" ranging from 35 Euro cents at public stations to 79 cents at private stations (cf. [16]).

[2] Pricing happens frequently on the basis of a per-minute tariff, other tariffs charge on a per kWh basis or on a per-session basis, see [16] for an overview on pricing schemes in Germany.

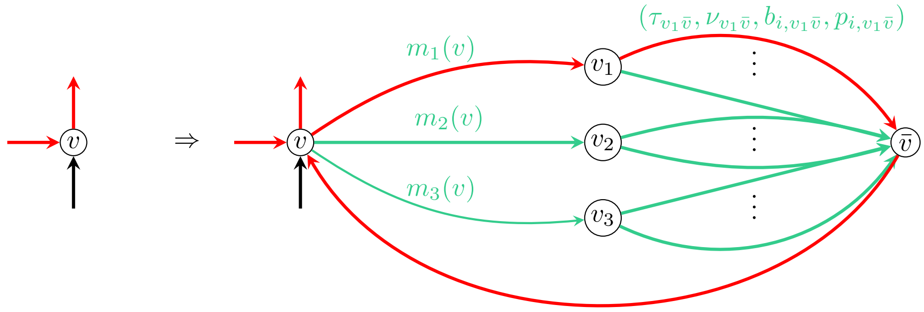

Figure 1 Left: Initial vertex v with an EV using a walk (red edges) without recharging. Right: Expansion of node v using a graph-based gadget modeling the recharging options. There are three recharging modes, say a low, medium or high power supply (Level 1, Level 2, Level 3) leading to the first three edges $m_1(v), m_2(v), m_3(v)$. The subsequent parallel edges model the different charging times and resulting increase of the battery levels. The red edges describe one cycle inside the gadget and represent a recharge using mode 1 for time $\tau_{v_1\bar{v}}$ with resulting battery level increase of $|b_{v_1\bar{v}}|$ at price $p_{v_1\bar{v}}$.

states and prices.[3] At the end of this series-parallel graph-gadget, a backwards arc towards v is introduced. We associate with every edge a tuple of the form $(\tau_e, \nu_e, b_{i,e}, p_{i,e})$, where τ_e is the travel time (or recharge duration for a gadget edge), ν_e the inflow capacity, $b_{i,e}$ the battery recharge and $p_{i,e}$ the price paid for the used recharge on edge e. Note that we have $p_{i,e} \equiv p_{i,e}(\tau_e)$ and $b_{i,e} \equiv b_{i,e}(\tau_e)$ for corresponding pricing and recharging functions, respectively. Any cycle in such a gadget is in one-to-one correspondence to a mode (e), recharge duration (τ_e), battery recharge $(b_{i,e})$ and price decision $(p_{i,e})$. If a mode is not compatible with the recharging technology used by EVs of type $i \in I$, we can set $b_{i,e} = +\infty$ to close the corresponding recharge edge for $i \in I$. For every $i \in N$, we denote the newly constructed vertices and edges, respectively, by $V(C_i)$, $E(C_i)$, $i \in I$.

▶ **Definition 1.** *The* battery-extended network *is a tuple $\mathcal{N} = (G, \nu, \tau, b, p)$, where*
- *$G = (V, E)$ is the battery-extended graph with $V := V' \cup_{i \in I} V(C_i)$ and $E := E' \cup_{i \in I} E(C_i))$,*
- *$\nu_e \in \mathbb{R}_+, e \in E$ denotes the inflow-capacities,*
- *$\tau_e \in \mathbb{R}_+, e \in E$ denotes the travel times or recharge durations,*
- *$b_{i,e} \in \mathbb{R}, i \in I, e \in E$ denotes the battery-consumption values,*
- *$p_{i,e} \in \mathbb{R}_+, i \in I, e \in E$ denotes the recharge prices.*

An s_i-t_i walk in the battery-extended graph G corresponds to a route choice in the original graph G' together with recharging decisions corresponding to cycles inside the gadgets, see Figure 1 for an example.

Feasible Walks in the Battery-Extended Network. Assume that we are given the battery-extended network $\mathcal{N}$. Let $W = (e_1, \ldots, e_k)$ be a sequence of edges in the graph G. We call W a *walk* if its edges can be traversed in this order i.e. if we have $e_j = v_{j-1}v_j$ for all $j \in [k]$ for $k \in \mathbb{N}$. We assume that all walks considered in this paper are finite and just use the term *walk* to denote a finite walk. Note, that a walk is allowed to contain self-loops and/or

[3] For the sake of a simple illustration we allow parallel arcs but by introducing further dummy nodes subdividing an edge, one obtains a simple graph so that an edge can uniquely be represented by a tuple vw for $v, w \in V$.

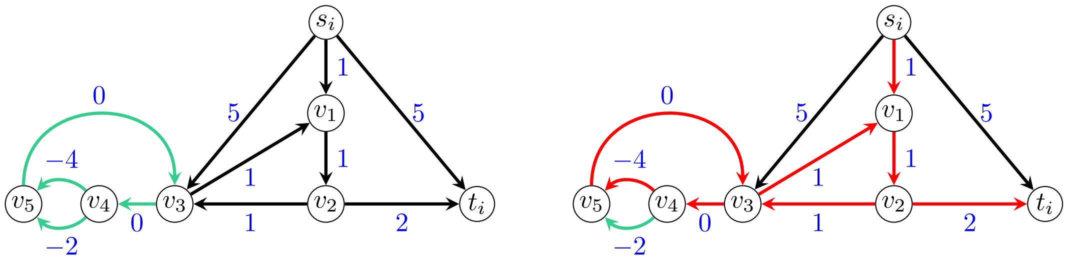

Figure 2 Example of an instance with start node s_i and sink node t_i, $b_i = 3$, $b_i^{\max} = 4$. The green edges represent the recharging gadget. Blue numbers at edges indicate the energy consumption values $b_{i,e}$. The shortest energy-feasible walk (assuming positive travel times on edges) is illustrated with red edges on the right which contains two simple cycles $C_1 := \{v_3, v_4, v_5, v_3\}$ and $C_2 := \{v_1, v_2, v_3, v_1\}$, where the first cycle is contained in the recharging gadget and represents a mode and duration choice.

nontrivial cycles as required for a recharge operation. We denote by $k_W := k$ the length of W and by e_j^W the j-th edge of walk W. W is an s_i-t_i *walk*, if $v_0 = s_i$ and $v_k = t_i$. We denote by $\mathcal{W}_i$ the set of all s_i,t_i-walks and assume that this set is always non-empty, i.e. that every commodity has at least one walk from its source to its sink. Finally, we denote by $\mathcal{W} := \{\, (i, W) \mid i \in I, W \in \mathcal{W}_i \,\}$ the set of all commodity-walk pairs. The set $\mathcal{W}_i$ represents the set of strategies for a particle of commodity $i \in I$ and, thus, a complete strategy profile is a family of walk inflow rates for all commodities and all walks such that for every commodity the sum of its walk inflow rates matches its network inflow rate. We denote the set of all such strategy profiles by

$$K := \left\{ h \in \left(L^2_{\geq 0}([0,T])\right)^{\mathcal{W}} \;\middle|\; \forall i \in I : \sum_{W \in \mathcal{W}_i} h_i^W(\theta) = u_i(\theta) \text{ for almost all } \theta \in \mathbb{R}_{\geq 0} \right\},$$

where $L^2_{\geq 0}([0,T])$ denotes the set of L^2-integrable non-negative functions and any $h \in K$ is called a *walk-flow*. The crucial point when modeling electric vehicles is the energy-feasibility of a walk, that is, the battery must not fully deplete when traversing a walk. We capture this property in the following definition.

▶ **Definition 2.** *A walk $W = (e_1, \dots, e_k) \in \mathcal{W}_i$ is* energy-feasible *for commodity $i \in I$, if $b_W(v_j) \in [0, b_i^{\max}]$ holds for all $j = 1, \dots, k$, where $b_W(v_j)$ is defined inductively as $b_W(v_1) = b_i$ and $b_W(v_{j+1}) = \min\{b_W(v_j) - b_{i,e_{j+1}}, b_i^{\max}\}$.*

We assume that for every $i \in I$ there is at least one energy-feasible walk and denote their collection by $\mathcal{W}_{i,b} := \{W \in \mathcal{W}_i | W \text{ is energy feasible for } i\}$. This set represents the set of energy-feasible strategies for a particle of commodity $i \in I$. Thus, a complete energy-feasible strategy profile is a family of walk inflow rates for all commodities and all walks such that for every commodity the sum of its walk inflow rates matches its network inflow rate. We further define $\mathcal{W}_b = \{(i, W) | i \in I, W \in \mathcal{W}_{i,b}\}$ to be the set of commodity and energy-feasible walk pairs. Note that the set $\mathcal{W}_b$ need not be finite. In Figure 2, we give an example illustrating that walking along cycles might indeed be necessary to reach the sink.

3 Dynamic Equilibria with Convex Constraints

So far, we have reduced the strategy space of every player involving the routing and recharging decisions to the set of feasible walks inside the battery-extended graph G. What is still missing to formally introduce the traffic assignment problem, or equivalently, the dynamic equilibrium problem, is the precise form of the utility function for an agent. We assume that agents want to travel from s_i to t_i but have preferences over travel time and recharge prices. While the recharge prices can be directly derived from the chosen walk W, the resulting travel time can only be described, if the walk-choices of all agents have been unfolded over time giving the resulting queueing times of a walk. This dynamic unfolding of the traffic inflow is usually termed as the *network loading* which is discussed in the following paragraphs.

Edge-Walk-Based Flows over Time. Given a feasible walk-flow $h \in K$, we develop the theoretical basis for the resulting *network loading*. This network loading provides then the basis for *time dependent label functions* $\mu_i^W : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which for every time θ provide us with the travel time for a particle entering walk W at time θ. These label functions will then be used for our dynamic equilibrium concept which takes energy-feasibility of walks and their resulting travel time into account. Let $\mathcal{R} := \{\, (i, W, j) \mid i \in I, W \in \mathcal{W}_i, j \in [k_W] \,\}$ denote the set of triplets consisting of the commodity identifier, walk and edge position in the walk, respectively. A flow over time is then a tuple $f = (f^+, f^-)$, where $f^+, f^- \in \left(L^2_{\geq 0}(\mathbb{R}_{\geq 0})\right)^{\mathcal{R}}$ are vectors of L^2-integrable non-negative functions modeling the inflow rate $f_{i,j}^{W,+}(\theta)$ and outflow rate $f_{i,j}^{W,-}(\theta)$ of commodity i on the j-th edge of some walk $W \in \mathcal{W}_i$ at time θ. For any such flow over time we define the *aggregated* edge in- and outflow rates of an edge $e \in E$ as

$$f_e^+(\theta) := \sum_{(i,W,j) \in \mathcal{R} : e_j^W = e} f_{i,j}^{W,+}(\theta) \quad \text{and} \quad f_e^-(\theta) := \sum_{(i,W,j) \in \mathcal{R} : e_j^W = e} f_{i,j}^{W,-}(\theta) \tag{1}$$

and the cumulative edge in- and outflows by $F_e^+(\theta) := \int_0^\theta f_e^+(z)dz$, $F_e^-(\theta) := \int_0^\theta f_e^-(z)dz$, $F_{i,j}^{W,+}(\theta) := \int_0^\theta f_{i,j}^{W,+}(z)dz$ and $F_{i,j}^{W,-}(\theta) := \int_0^\theta f_{i,j}^{W,-}(z)dz$. Note, that $F_e^+, F_e^-, F_{i,j}^{W,+}$ and $F_{i,j}^{W,-}$ are non-decreasing, absolute continuous functions which satisfy

$$F_e^+(\theta) = \sum_{(i,W,j) \in \mathcal{R} : e_j^W = e} F_{i,j}^{W,+}(\theta) \quad \text{and} \quad F_e^-(\theta) = \sum_{(i,W,j) \in \mathcal{R} : e_j^W = e} F_{i,j}^{W,-}(\theta).$$

Furthermore, we define the *queue length* of an edge e at time θ by $q_e(\theta) := F_e^+(\theta) - F_e^-(\theta + \tau_e)$. Then, for any flow particle entering an edge $e = vw$ at time time θ, its travel time on this edge is $c_e(\theta) := \tau_e + \frac{q_e(\theta)}{\nu_e}$ and its exit time from edge e is given by $T_e(\theta) := \theta + c_e(\theta)$. Now, given some feasible walk-flow $h \in K$ we call a flow over time f a *feasible flow over time associated with* h if it satisfies the following constraints (2)–(6): The walk inflow rates of h and f match, i.e., for every $i \in I$, $W \in \mathcal{W}_i$ we have

$$f_{i,1}^{W,+}(\theta) = h_i^W(\theta) \text{ for almost all } \theta \in \mathbb{R}_{\geq 0}. \tag{2}$$

The flow satisfies a balancing constraint at every intermediate node, i.e. for every $i \in I$, $W \in \mathcal{W}_i$ and any $1 \leq j < k_W$ we have

$$f_{i,j}^{W,-}(\theta) = f_{i,j+1}^{W,+}(\theta) \text{ for almost all } \theta \in \mathbb{R}_{\geq 0}. \tag{3}$$

The aggregated outflow respects the edges capacity, i.e. for every edge e we have

$$f_e^-(\theta + \tau_e) \leq \nu_e \text{ for almost all } \theta \in \mathbb{R}_{\geq 0}, \tag{4}$$

as well as weak flow conservation over edges, i.e. for every edge e we have

$$F_e^-(\theta + \tau_e) \leq F_e^+(\theta) \text{ for all } \theta \in \mathbb{R}_{\geq 0}. \tag{5}$$

And, finally, the flow has to satisfy the following link transfer equation for every $i \in I$, $W \in \mathcal{W}_i$ and any $1 \le j \le k_W$:

$$F_{i,j}^{W,-}\left(T_{e_j^W}(\theta)\right) = F_{i,j}^{W,+}(\theta) \text{ for all } \theta \in \mathbb{R}_{\geq 0}. \tag{6}$$

It turns out that every feasible walk-flow $h \in K$ has a *unique* associated feasible flow over time which we can obtain by a natural network loading procedure. This has been shown by Cominetti et al. in [2, Proposition 3] for the case of flows using only simple paths, but the same proof can also be applied to the case of general walks.

▶ **Lemma 3.** *For any $h \in K$ there is a unique (up to changes on a subset of measure zero) associated flow over time f.*

For any fixed network we denote by $\mathcal{F}$ the set of all feasible flows over time associated with some $h \in K$. Lemma 3 then provides us with a one-to-one mapping between K and $\mathcal{F}$.

Capacitated Dynamic Equilibria. For a given walk-flow h with associated feasible flow over time f, we are in the position to compute for every commodity type (i, W) with $W = (e_1^W, \dots, e_{k_W}^W)$ a label function giving at time θ for any node on that walk the arrival time at t_i. Let $\hat{W} = (v_0, \dots, v_{k_W})$ denote the representation of W as a sequence of nodes satisfying $e_j^W = v_{j-1}v_j, j \in [k_W]$ with $v_0 = s_i, v_{k_W} = t_i$. As a node can appear multiple times in W, we use the subindex $j \in [k_W]$ as a unique identifier of the position of that node in the walk. With this notation we can unambiguously and recursively define the following label function:

$$\begin{aligned} \ell_{i,k_W}^W(\theta) &:= \theta, \text{ for all } \theta \geq 0, \\ \ell_{i,j}^W(\theta) &:= \ell_{i,j+1}^W(T_{e_{j+1}^W}(\theta)), \text{ for } j = [k_W] - 1, \dots, 0 \text{ and all } \theta \geq 0 \end{aligned} \tag{7}$$

where $\ell_{i,j}^W$ is the label function of the $(j+1)$-th node when traversing the walk $\hat{W}$ beginning with the starting node at position 0. Since W is a walk with end node t_i, the value $\ell_{i,0}^W(\theta)$ measures the arrival time at t_i for a particle entering W at time θ (assuming that the particle follows W). Note that $\ell_{i,j}^W$ is only defined for nodes contained in W and a node v in $\hat{W}$ may be associated with several label functions whose number is equal to the number of occurrences of v in $\hat{W}$. We can easily compute the total travel time for a vehicle of commodity $i \in I$ leaving at time θ as $\mu_i^W(\theta) := \ell_{i,0}^W(\theta) - \theta$. Finally, to determine the total cost of any particle each commodity $i \in I$ has an associated *aggregation function* c_i which can be any continuous, non-decreasing function $c_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.[4] The total cost of a particle of commodity i starting at time θ on walk W is then $c_i(\mu_i^W(\theta), \sum_{e \in W} p_{i,e})$.

Now, instead of letting particles choose *any* walk between their respective source and sink node, we impose further restrictions to only use walk-flows from some closed, convex restriction set $S \subseteq L^2([0,T])^{\mathcal{W}}$. Using such S we can, for example, not only model battery constraints but also temporary road closures or restrictions on the set of feasible flows itself (as every h corresponds to a unique flow) – though, in the latter case it is in general not obvious whether the resulting set S satisfies convexity. We now want to express that some $h \in S$ is an equilibrium, if no particle can improve its total cost (i.e. aggregate of travel time and total price) by deviating from its current path while staying within S. However, since individual particles are infinitesimally small, the deviation of a single particle does not

[4] A simple example of such a function would be a weighted sum of the two arguments.

influence the feasibility w.r.t. S. Instead, we will consider deviations of arbitrarily small but positive volumes of flow leading to the notion of *saturated* and *unsaturated* walks as used in the static Wardropian model by Larsson and Patriksson [10]. To do that we first define for any given walk-inflow h, commodity i, walks $W, Q \in \mathcal{W}_i$, time $\bar{\theta} \geq 0$ and constants $\varepsilon, \delta > 0$ the walk-inflow obtained by shifting flow of commodity i from walk W to walk Q at a rate of ε during the interval $[\bar{\theta}, \bar{\theta}+\delta]$ by $H_i^{W \to Q}(h, \bar{\theta}, \varepsilon, \delta) := (h'_R)_{R \in \mathcal{W}}$ with

$$\begin{aligned} &h'^W_i = [h^W_i - \varepsilon \mathbb{1}_{[\bar{\theta}, \bar{\theta}+\delta]}]_+, \quad h'^Q_i = h^Q_i + h^W_i - h'^W_i \text{ and} \\ &h'^R_{i'} = h^R_{i'} \text{ for all } (i', R) \in \mathcal{W} \setminus \{(i,Q),(i,W)\}, \end{aligned}$$

where $\mathbb{1}_{[\bar{\theta}, \bar{\theta}+\delta]} : [0,T] \to \mathbb{R}$ is the indicator function of the interval $[\bar{\theta}, \bar{\theta}+\delta]$ and for any function $g : [0,T] \to \mathbb{R}$ the function $[g]_+$ is the non-negative part of g, i.e. the function $[g]_+ : [0,T] \to \mathbb{R}, \theta \mapsto \max\{g(\theta), 0\}$. Using this notation, we can define the set of unsaturated alternatives to some fixed walk W of some commodity i with respect to some $h \in S$ at time $\bar{\theta} \geq 0$ as

$$D_i^W(h, \bar{\theta}) := \left\{ Q \in \mathcal{W}_i \,\middle|\, \forall \delta' > 0 : \exists \delta \in (0, \delta'], \varepsilon > 0 : H_i^{W \to Q}(h, \bar{\theta}, \varepsilon, \delta) \in S \right\}. \tag{8}$$

With this definition we are now able to formally introduce the concept of a dynamic equilibrium in our model.

▶ **Definition 4.** *Given a network $\mathcal{N} = (G, \nu, \tau, p)$, a set of commodities I, a restriction set S and for every commodity an associated source-sink pair $(s_i, t_i) \in V \times V$ as well as an aggregation function c_i, a feasible walk-flow $h \in S \cap K$ is a* capacitated dynamic equilibrium*, if for all $(i, W) \in \mathcal{W}$ and almost all $\bar{\theta} \geq 0$ it holds that*

$$h_i^W(\bar{\theta}) > 0 \implies c_i\left(\mu_i^W(\bar{\theta}), \sum_{e \in W} p_{i,e}\right) \leq c_i\left(\mu_i^Q(\bar{\theta}), \sum_{e \in Q} p_{i,e}\right) \text{ for all } Q \in D_i^W(h, \bar{\theta}). \tag{9}$$

Note that, in the case where all inflows are allowed (i.e. $S = L^2([0,T])^{\mathcal{W}}$) the above definition is equivalent to the classic definition of dynamic equilibria. For a battery-extended network we can use $S := \{\, h \in L^2([0,T])^{\mathcal{W}} \mid h_i^W \equiv 0 \text{ for all } W \in \mathcal{W} \setminus \mathcal{W}_b \,\}$ and will call a capacitated dynamic equilibrium an *energy-feasible dynamic equilibrium*.

4 Existence of Capacitated Dynamic Equilibria

In this section, we will show the existence of capacitated dynamic equilibria using an infinite dimensional variational inequality as pioneered by Friesz et al. [5] and also used by Cominetti et al. [2]. Since we use a more general equilibrium concept and allow for flow to use arbitrary walks (from an a priori *infinite* set of possible walks) instead of just simple paths, we have to adjust several technical steps of the proof. See Figure 2 for a simple instance where travelling along cycles is already necessary and [18] for an extensive discussion of this topic.

The general structure of the proof will be as follows: First, we introduce the concept of dominating sets of walks which will allow us to only consider some finite subset $\mathcal{W}'$ of the set of all walks. We then define a function $\mathcal{A} : h \mapsto c_i\left(\mu_i^W(\_), \sum_{e \in W} p_{i,e}\right)$ mapping walk-flows to costs of particles of commodity i using walk W. Using this mapping we can then formulate a variational inequality for which we can show that any solution to it is a capacitated dynamic equilibrium. Finally, a result by Lions [11] guarantees the existence of such solutions given that the mapping $\mathcal{A}$ satisfies an appropriate continuity property which we will show to hold for our model. We start by giving the definition of dominating walks and sets and then formally state our main theorem:

▶ **Definition 5.** *A walk $(i,Q') \in \mathcal{W}$ is a* dominating walk *for another walk (i,Q) with respect to S if for any walk-flow $h \in K \cap S$ and all times $\theta \in [0,T]$ we have $c_i\left(\mu_i^{Q'}(\bar{\theta}), \sum_{e\in Q'} p_{i,e}\right) \leq c_i\left(\mu_i^{Q}(\bar{\theta}), \sum_{e\in Q} p_{i,e}\right)$ and, additionally, $Q \in D_i^W(h,\bar{\theta})$ always implies $Q' \in D_i^W(h,\bar{\theta})$ for any walk $(i,W) \in \mathcal{W}$.*

A subset $\mathcal{W}' \subseteq \mathcal{W}$ is a dominating set with respect to S *if for any walk $(i,Q) \in \mathcal{W}$, there exists a* dominating walk $(i,Q') \in \mathcal{W}'$.

▶ **Theorem 6.** *Let $\mathcal{N} = (G,\nu,\tau,p)$ be any network and I a finite set of commodities each associated with an aggregation function c_i and a source-sink pair (s_i,t_i). Let $S \subseteq L^2([0,T])^{\mathcal{W}}$ be a restriction set which is closed, convex and has non-empty intersection with K, and there exists some finite dominating set $\mathcal{W}' \subseteq \mathcal{W}$ with respect to S. Then there exists a capacitated dynamic equilibrium in $\mathcal{N}$.*

In order to prove this theorem we first need some additional definitions and notation: We will make use of two function spaces, namely the space $L^2([a,b])$ of L^2-integrable functions from an interval $[a,b]$ to $\mathbb{R}$ and the space $C([a,b])$ of continuous functions from $[a,b]$ to $\mathbb{R}$. The former is a Hilbert space with the natural pairing $\langle .,. \rangle : L^2([a,b]) \times L^2([a,b]) \to \mathbb{R}, (g,h) \mapsto \langle g,h \rangle := \int_a^b g(x)h(x)\,\mathrm{d}x$. The latter is a normed space with the uniform norm $\|f\|_\infty := \sup_{\theta\in[a,b]} |f(\theta)|$. Both, the natural pairing and the norm, can be extended in a natural way to $L^2([a,b])^d$ and $C([a,b])^d$, respectively, for any $d \in \mathbb{N}$. In particular, all these spaces are topological vector spaces. We say that a sequence h^k of functions in $L^2([a,b])^d$ *converges weakly* to some function $h \in L^2([a,b])^d$ if for any function $g \in L^2([a,b])$ we have $\lim_{k\to\infty}\langle h^k, g\rangle = \langle h, g\rangle$. For any topological space X (in the following this will be either $L^2([a,b])^d$ or $C([a,b])^d$) and any subset $C \subseteq L^2([a,b])^d$ a mapping $\mathcal{A} : C \to X$ is called *sequentially weak-strong continuous* if it maps any weakly converging sequence of functions in C to a (strongly) convergent sequence in X.

With this, we can now describe the kind of variational inequality we will use to show the existence of capacitated dynamic equilibria. Namely, given an interval $[a,b] \subseteq \mathbb{R}_{\geq 0}$, a number $d \in \mathbb{N}$, a subset $C \subseteq L^2([a,b])^d$ and a mapping $\mathcal{A} : C \to L^2([a,b])^d$, the variational inequality $\mathrm{VI}(C,\mathcal{A})$ is the following:

$$\text{Find } h^* \in C \text{ such that } \langle \mathcal{A}(h^*), \bar{h} - h^* \rangle \geq 0 \text{ for all } \bar{h} \in C. \tag{VI($C,\mathcal{A}$)}$$

Conditions to guarantee the existence of such an element h^* are given by Lions in [11, Chapitre 2, Théorème 8.1] which, following Cominetti et al. [2], can be restated as follows:

▶ **Theorem 7.** *Let C be a non-empty, closed, convex and bounded subset of $L^2([a,b])^d$. Let $\mathcal{A} : C \to L^2([a,b])^d$ be sequentially weak-strong continuous. Then, the variational inequality* (VI($C,\mathcal{A}$)) *has a solution $h^* \in C$.*

For our proof we choose $C := \pi(S \cap K \cap \iota(\left(L^2([0,T])\right)^{\mathcal{W}'}))$, where $\iota : \left(L^2([0,T])\right)^{\mathcal{W}'} \to \left(L^2([0,T])\right)^{\mathcal{W}}$ is the canonical embedding (i.e. augmenting $\mathcal{W}'$-dimensional vectors with zero functions to $\mathcal{W}$-dimensional vectors) and $\pi : \left(L^2([0,T])\right)^{\mathcal{W}} \to \left(L^2([0,T])\right)^{\mathcal{W}'}$ the canonical projection. For ease of notation we will usually omit these embeddings/projections from our notation and assume that they are implicitly applied, whenever we switch between elements of $\left(L^2([0,T])\right)^{\mathcal{W}'}$ and $\left(L^2([0,T])\right)^{\mathcal{W}}$. Next, we define a mapping $\mathcal{A} : C \to L^2([0,T])^{\mathcal{W}'}$ by defining for every walk-flow $h \in C$, commodity $i \in I$ and walk $W \in \mathcal{W}'_i := \{\, W \in \mathcal{W}_i \mid (i,W) \in \mathcal{W}' \,\}$ the continuous function $\mathcal{A}_i^W(h)$ given by

$$\mathcal{A}_i^W(h) : \theta \mapsto c_i\left(\mu_i^W(\bar{\theta}), \sum_{e\in W} p_{i,e}\right) - \min_{Q\in\mathcal{W}'_i} c_i\left(\mu_i^Q(\bar{\theta}), \sum_{e\in Q} p_{i,e}\right).$$

Clearly, the assumptions on S and the fact that K is bounded, closed and convex imply that C is a non-empty, closed, convex and bounded subset of $L^2([0,T])^{\mathcal{W}'}$. Thus, in order to be able to apply Theorem 7 it only remains to show that $\mathcal{A}$ is sequentially weak-strong continuous. Since taking differences and minima of sequentially weak-strong continuous mappings results again in such a mapping, it suffices to show that the maps $h \mapsto c_i\left(\mu_i^W(\_), \sum_{e \in W} p_{i,e}\right)$ are sequentially weak-strong continuous from C to $L^2([0,T])$.

▶ **Lemma 8.** *The map $C \mapsto L^2([0,T]), h \mapsto \big([0,T] \to \mathbb{R}, \theta \mapsto c_i\big(\mu_i^W(\theta), \sum_{e \in W} p_{i,e}\big)\big)$ is sequentially weak-strong continuous for every $W \in \mathcal{W}_i', i \in I$.*

The proof of this lemma follows along similar lines as [2, Lemmas 3-7] by Cominetti et al. but requires some adjustments due to the differences between the models (in particular, the fact that we allow for walks involving cycles). The main steps of the proof are first to determine a (flow-independent) bound on the residence time of particles in the network, then decompose the lemma's map into several simpler maps and, finally, show appropriate continuity properties for those. The details of this proof can be found in the full version of the paper. With this lemma at hand we can now prove our main theorem.

Proof of Theorem 6. By Lemma 8 the map $h \mapsto c_i\left(\mu_i^W(\_), \sum_{e \in W} p_{i,e}\right)$ is weak-strong continuous from C to $L^2([0,T])$ for each $W \in \mathcal{W}_i', i \in I$. Taking the minimum of finitely many weak-strong continuous mappings results in a weak-strong continuous mapping again and, finally, the difference of two weak-strong continuous mappings is also weak-strong continuous. Thus, $\mathcal{A}$ is sequentially weak-strong-continuous from C to $L^2([0,T])^{\mathcal{W}'}$. Applying Theorem 7 provides a solution h^* for $\mathrm{VI}(C, \mathcal{A})$. It remains to show that this is, in fact, a capacitated dynamic equilibrium. We do this by contradiction, i.e. assume that h^* is not a capacitated dynamic equilibrium. Then, by using some technical measure theoretic arguments, we can get an alternative walk inflow $\bar{h} := H_i^{W \to Q}(h^*, \bar{\theta}, \varepsilon, \delta) \in S$ with $\int_{\bar{\theta}}^{\bar{\theta}+\delta} \min\{\, {h^*}_i^W(\theta), \varepsilon \,\}\, \mathrm{d}\theta > 0$ and $c_i\left(\mu_i^W(\theta), \sum_{e \in W} p_{i,e}\right) - c_i\left(\mu_i^Q(\theta), \sum_{e \in Q} p_{i,e}\right) \geq \gamma$ for all $\theta \in [\bar{\theta}, \bar{\theta}+\delta]$ and some $\gamma > 0$. Since $\bar{h}$ only uses walks that are already used in h^* and additionally walk Q, all walks used by $\bar{h}$ are in $\mathcal{W}'$. Thus, we can conclude that $\bar{h} \in C$. But at the same time a direct calculation shows that $\langle \mathcal{A}(h^*), \bar{h} - h^* \rangle = \int_{\bar{\theta}}^{\bar{\theta}+\delta} \left(\mathcal{A}(h^*)_Q(\theta) - \mathcal{A}(h^*)_W(\theta)\right) \cdot \min\{\, {h^*}_i^W(\theta), \varepsilon \,\}\, \mathrm{d}\theta < 0$, which is a contradiction to h^* being a solution to $(\mathrm{VI}(C, \mathcal{A}))$. Therefore, h^* already is a capacitated dynamic equilibrium. ◀

We conclude by discussing two special cases for which our existence theorem can be applied by suitable choices of the abstract restriction set S: Dynamic equilibria and energy-feasible dynamic equilibria.

Dynamic Equilibria. If we choose $S = L^2([0,T])^{\mathcal{W}}$ then capacitated dynamic equilibria are exactly the dynamic equilibria as defined in [2, 5, 9, 25, 12]. To see this, note, that in this case we always have $D_i^W(h, \bar{\theta}) = \mathcal{W}_i$. Thus, (9) translates to the constraint that whenever there is positive inflow into some walk W, this walk has to be a shortest walk at that time. Since dynamic flows in the Vickrey-model satisfy FIFO, the set of simple paths is a dominating set for the set of all walks with respect to $S = L^2([0,T])^{\mathcal{W}}$ (i.e. removing a cycle from a walk can never increase its aggregated cost). As the set of simple paths is clearly finite, one can use Theorem 6 to show existence of dynamic equilibria. Note that the classical existence proofs for dynamic equilibria (e.g. by Han et al. [8] or Cominetti et al. [2]) usually have the restriction to simple paths as part of the model itself, i.e. they only allow walk-inflows from $L^2([0,T])^{\mathcal{W}'}$ where $\mathcal{W}'$ is the set of simple source-sink paths.

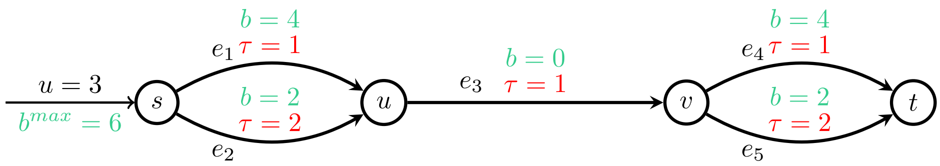

Figure 3 An instance where overtaking can occur in an energy-feasible dynamic equilibrium. If the capacities are chosen such that no queues form, the paths $P_1 = (e_1, e_3, e_5)$ and $P_2 = (e_2, e_3, e_4)$ are the shortest energy-feasible paths and any flow split between these two paths is an equilibrium. Note, that in such a flow particles travelling along path P_1 will temporally overtake particles travelling on path P_2 even though both paths have the same total travel time. If we add suitable edge capacities, this may lead to particles travelling on path P_2 being delayed on edge e_3 by *later starting* particles travelling on path P_1.

Energy-Feasible Dynamic Equilibria. Now let us turn to the case of energy-feasible dynamic equilibria, i.e. equilibria of flows in battery-extended networks. We show that Theorem 6 implies the existence of energy-feasible dynamic equilibria.

▶ **Theorem 9.** *Let $\mathcal{N}$ be an battery-extended network and $S := \iota(L^2([0,T])^{\mathcal{W}_b}) \subseteq L^2([0,T])^{\mathcal{W}}$. Then, there exists an energy-feasible dynamic equilibrium in $\mathcal{N}$, i.e. a capacitated dynamic equilibrium with respect to S.*

Proof. First, it is quite obvious that S is closed, convex and has non-empty intersection with K (using our assumption that every commodity has at least one energy-feasible source-sink walk). For the existence of a finite dominating set, we will show that due to the FIFO condition in the Vickrey model, there exists a constant κ_i such that for every agent playing against any walk choices of all other agents there exists an optimal strategy which enters any (recharging) node at most κ_i times. We begin by defining the minimum positive energy increment for $i \in I$ along any simple cycle by $\alpha_i := \min_{E' \subseteq E} \left\{\sum_{e \in E'} b_{i,e} \middle| \sum_{e \in E'} b_{i,e} > 0\right\}$ and then choosing $\kappa_i := \max\left\{\frac{b_i^{\max}}{\alpha_i}\right\}$. Now, suppose there is some node v, which is visited $k \in \mathbb{N}$ times by a walk W of commodity i. By renaming indices, we can assume that v appears in W in the order $v_1, \ldots, v_k$ with $v_j = v, j \in [k]$. Clearly, whenever we have $b_W(v_\ell) \geq b_W(v_j)$ for some $\ell < j$, we can delete the cycles between v_ℓ and v_j to obtain another energy-feasible walk W' of the same commodity. Due to FIFO and the fact that the aggregation function c_i is non-decreasing, the new walk W' then has at most the same aggregated cost as W. Thus, commodity i always has an optimal walk where the sequence $b_W(v_1) < \cdots < b_W(v_k)$ is monotonically increasing with increments of at least $\alpha_i > 0$. With $b_W(v_k) \leq b_i^{\max}$, we get $k \leq \kappa_i$ as wanted. Consequently, choosing $\mathcal{W}'$ as the (finite!) subset of $\mathcal{W}$ containing only walks which visit any particular node at most κ_i times provides us with the required dominating walk set. Thus, all conditions of Theorem 6 are satisfied and we obtain the existence of an energy-feasible dynamic equilibrium. ◀

5 Computational Study and Conclusion

While Theorem 9 guarantees the existence of energy-feasible dynamic equilibria, the non-constructive nature of our proof (or more precisely the non-constructive existence result for the variational inequality) means that it is not clear how to actually compute such equilibria. Moreover, in contrast to dynamic equilibria, even in the single-commodity case it seems unlikely that energy-feasible dynamic equilibria exhibit a simple phase structure which would allow for a stepwise construction by repeatedly extending a given partial equilibrium as it is

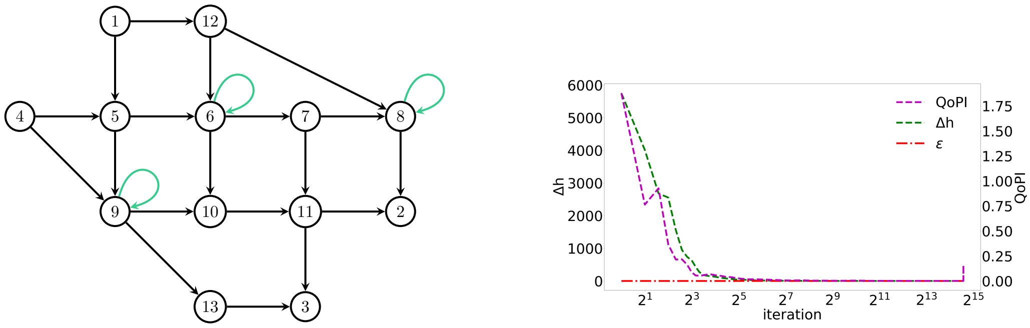

Figure 4 Left: The Nguyen-network with three recharging stations (green loops). Right: Convergence of quality measures during the algorithm (change of flow between consecutive iterations Δh and regret QoPI; ε denotes the desired quality at which the algorithm terminates).

possible for dynamic equilibria (cf. [9]). Namely, even in simple toy instances (e.g. Figure 3) simultaneous starting particles may overtake each other at intermediate nodes while still arriving at the sink at the same time. Consequently, if one were to extend a given equilibrium flow, particles starting within the new extension period might overtake particles of a previous phase and then form a queue, hereby increasing the travel time of those earlier particles and possibly leading to violations of the equilibrium condition in the previously calculated part of the flow. Consequently, to compute an energy-feasible dynamic equilibrium the whole time-horizon $[0, T]$ has to be taken into account at once. This makes it unlikely, that an exact computation of energy-feasible dynamic equilibria is possible.

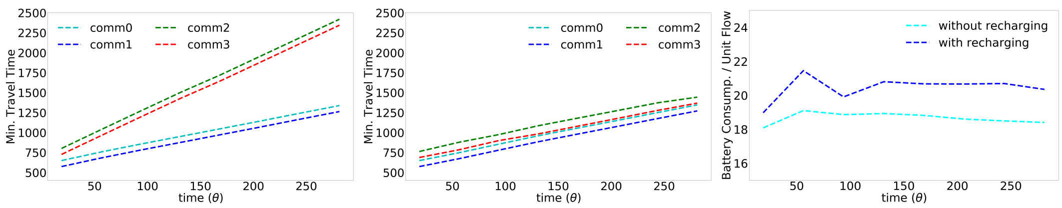

Figure 5 Left: Travel times of the four commodities in the Nguyen network without recharging. Middle: Travel times with recharging. Right: Energy consumption per unit flow with and without recharging. Note, that allowing recharging can reduce the travel times of some of the commodities (as more routes become feasible) – for the price of increased total energy consumption.

Thus, we instead compute approximate equilibria by discretizing time and employing a walk-flow based fixed point algorithm similar to the one used by Han et al. in [7] for dynamic equilibria. We apply this algorithm to a set of real-world instances and are able to compute flows which are very close to energy-feasible dynamic equilibria (in the sense that particles only use walks which are close to shortest energy-feasible walks in hindsight). We demonstrate this convergence of the flows to approximate dynamic equilibria in terms of certain quality measures and show the applicability of our algorithm to moderate sized instances like the Nguyen network with up to 20 commodities (see Figures 4 and 5 for some of the results for four commodities). On the negative side we observe a sharp increase in computation time with larger networks and/or more recharging stations as the number of walks we have to consider increases exponentially. More detailed results of our computational study can be found in the full version of our paper.

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