Cover and Hitting Times of Hyperbolic Random Graphs

Marcos Kiwi ✉ 🏠 🆔
Department of Industrial Engineering and Center for Mathematical Modeling,
Universidad de Chile, Santiago, Chile

Markus Schepers ✉ 🏠 🆔
Institut für Medizinische Biometrie, Epidemiologie und Informatik,
Johannes-Gutenberg-University Mainz, Germany

John Sylvester ✉ 🏠 🆔
School of Computing Science, University of Glasgow, UK

Abstract

We study random walks on the giant component of Hyperbolic Random Graphs (HRGs), in the regime when the degree distribution obeys a power law with exponent in the range $(2, 3)$. In particular, we focus on the expected times for a random walk to hit a given vertex or visit, i.e. cover, all vertices. We show that up to multiplicative constants: the cover time is $n(\log n)^2$, the maximum hitting time is $n \log n$, and the average hitting time is n. The first two results hold in expectation and a.a.s. and the last in expectation (with respect to the HRG).

We prove these results by determining the effective resistance either between an average vertex and the well-connected "center" of HRGs or between an appropriately chosen collection of extremal vertices. We bound the effective resistance by the energy dissipated by carefully designed network flows associated to a tiling of the hyperbolic plane on which we overlay a forest-like structure.

1 Introduction

In 2010, Krioukov et al. [52] proposed the Hyperbolic Random Graph (HRG) as a model of "real-world" networks such as the Internet (also referred to as complex networks). Early results via non-rigorous methods indicated that HRGs exhibited several key properties empirically observed in frequently studied networks (such as networks of acquaintances, citation networks, networks of autonomous systems [13], etc.). Many of these properties were later established formally, among these are power-law degree distribution [37], short graph distances [1, 46] (a.k.a. small world phenomena), and strong clustering [17, 31, 37]. Many

other fundamental parameters of the HRG model have been studied since its introduction (see the related work section), however notable exceptions are key quantities concerning the behaviour of random walks. This paper is a first step in redressing this situation. The random walk is the quintessential random process, and studies of random walks have proven relevant for algorithm design and analysis; this coupled with the aforementioned appealing aspects of the HRG model motivates this research.

The (simple) random walk is a stochastic process on the vertices of a graph, which at each time step uniformly samples a neighbour of the current vertex as its next state [3, 56]. A key property of the random walk is that, for any connected graph, the expected time it takes for the walk to visit a given vertex (or to visit all vertices) is polynomial in the number of vertices in the graph. These times are known as the hitting and cover times, respectively. This ability of a random walk to explore an unknown connected graph efficiently using a small amount of memory was, for example, used to solve the undirected $s - t$ connectivity problem in logarithmic space [4]. Other properties such as the ability to sample a vertex independently of the start vertex after a polynomial (often logarithmic) number of steps (mixing time) helped random walks become a fundamental primitive in the design of randomized and approximation algorithms [59]. In particular, random walks have been applied in tasks such as load balancing [68], searching [35], resource location [44], property testing [26, 53, 54], graph parameter estimation [8] and biological applications [38].

One issue to keep in mind when working with HRGs is that for the most relevant range of parameters of the model (that one for which it exhibits the properties observed in "real-world" networks) the graphs obtained are disconnected with probability that tends to 1 as the order of the graph goes to infinity. Quantities such as average hitting time and commute time are not meaningful for disconnected graphs (i.e., they are trivially equal to infinity). However, again for the range of parameters we are interested in, Bode, Fountoulakis and Müller [11] showed that it is very likely the graph has a component of linear size. This result was then complemented by the first author and Mitsche [46] who showed that all other connected components were of size at most polylogarithmic in the order of the graph. This justifies referring to the linear size component as the *giant component*. With this work being among the first study of characteristics of simple random walks in HRGs, it is thus natural and relevant to understand their behavior in the giant component of such graphs. This is the main challenge we undertake in this paper.

Among our main contributions are the determination of the order of the hitting and cover times of random walks on the giant component of HRGs. To achieve this, we appeal to a connection of the former to effective resistances in the graph [56, Section 9]. The effective resistance is a metric, and the resistances between all pairs of vertices uniquely determines the graph [40]. The effective resistance has also found applications to graph clustering [5], spectral sparsification [69], graph convolutional networks [2], and flow-based problems in combinatorial optimization [6, 18, 61].

1.1   Main Results

Our main contributions are to determine several quantities related to random walks on the largest connected component $\mathcal{C}_{\alpha,\nu}(n)$ of the (Poissonized) hyperbolic random graph $\mathcal{G}_{\alpha,\nu}(n)$. We refer to this component as the *giant* and note that it is known to have $\Theta(n)$ vertices a.a.s. [11]. The primary probability space we will be working in is the one induced by the HRG and we use $\mathbb{P}$ for the associated measure. We also deal with the expected stopping times of random walks, and we use bold type (e.g. $\mathbf{E}$) for the expectation with respect to the

random walk on a fixed graph. We say that a sequence of events (w.r.t. the HRG) holds *asymptotically almost surely (a.a.s.)* if it occurs with probability going to 1 as $n \to \infty$. We give brief descriptions of the objects we study here, for full definitions see Section 2.

The *effective resistance* $\mathcal{R}(x \leftrightarrow y)$ between two vertices x, y of a graph G is the energy dissipated by a unit current flow from x to y. Due to a connection with simple random walks, we consider effective resistance in the case when all edges have unit resistances, see Section 2.5 for a formal definition. The sum of all resistances in G is the *Kirchhoff index* $\mathcal{K}(G)$, this has found uses in centrality [57], noisy consensus problems [67], and social recommender systems [73]. Our first result shows the expected effective resistance between two vertices of the giant chosen uniformly at random is bounded, and gives the expected order of the Kirchhoff index.

▶ **Theorem 1.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then*

$$\mathbb{E}(\mathcal{K}(\mathcal{C})) = \Theta(n^2), \qquad and \qquad \mathbb{E}\Big(\frac{1}{|V(\mathcal{C})|^2} \sum_{u,v \in V(\mathcal{C})} \mathcal{R}(u \leftrightarrow v)\Big) = \Theta(1).$$

The upper bounds in Theorem 1 are established by exploiting the well known relation between effective resistance and energy dissipated by network flows. The two results in this theorem are very closely related but do not directly imply each other as $|V(\mathcal{C})|$, the size of the center component, is a random variable that is not independent from the resistances.

Our construction of low energy flows relies on a tiling of the hyperbolic plane. In this regard, it bears some similarity to how various authors have obtained estimates of the size of the giant and upper bounds on the diameter of the HRG [64]. However, when constructing a desirable flow one often needs multiple paths (as opposed to just one when bounding the diameter) or else the energy dissipated by the flow could be too large to get a tight bound on the effective resistance. Abdullah et al. [1] showed that hyperbolic random graphs of expected size $\Theta(n)$ have typical distances of length $\Theta(\log \log n)$ (within the same component), in contrast we show that typical resistances are $\Theta(1)$. The diameter of the HRG when $\frac{1}{2} < \alpha < 1$ was only recently determined precisely [64], though the lower bound, non-tight upper bounds, and the diameter for other values of α, were established earlier [46, 33]. The tight $\mathcal{O}(\log n)$ upper bound for the diameter of the giant of the HRG when $\frac{1}{2} < \alpha < 1$ [64] was proved using a coupling with the Fountoulakis-Müller upper half-plane HRG model [30] and is also based in a tiling-construction. The tiling on which we rely to construct flows is closely related to the Fountoulakis-Müller tiling of the half-plane model. In fact, our tiling is approximately equal to the latter (see the discussion in the last paragraph of the detailed description of our tiling in Secton 3.1).

The *target time* $t_\odot(G)$ of a graph G (also known as *Kemeny's constant*) is the expected time for a random walk to travel between two vertices chosen independently from the stationary distribution π, see Section 2.4. When considering a random walk on a graph, the stationary distribution is arguably the most natural measure on the vertices. Thus the target time should be considered as the "average" hitting time. We show that on the giant of the HRG this notion of average hitting time is of order n in expectation.

▶ **Theorem 2.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then $\mathbb{E}(t_\odot(\mathcal{C})) = \Theta(n)$.*

The *hitting time* of a vertex v from a vertex u in a graph G is the expected time it take a random walk started from u to first visit v. Let $t_{\mathsf{hit}}(G)$ denote the *maximum hitting time*, this is the maximum over all pairs of vertices u, v in $V(G)$ of the hitting time of u from v. We let the *cover time* $t_{\mathsf{cov}}(G)$ be the expected time for the walk to visit all vertices of G (taken from a worst case start vertex), see Section 2.4. We show that both of these quantities concentrate on the giant of the HRG.

▶ **Theorem 3.** *For any $\frac{1}{2} < \alpha < 1$ and $\nu > 0$, if $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$, then a.a.s. and in expectation*

$$t_{\mathsf{hit}}(\mathcal{C}) = \Theta(n \log n) \qquad and \qquad t_{\mathsf{cov}}(\mathcal{C}) = \Theta(n \log^2 n).$$

The result above also establishes that the maximum resistance between two vertices of the giant is $\Theta(\log n)$ a.a.s., compared to $\Theta(1)$ for a typical pair by Theorem 1. This discrepancy between the maximum and the average resistances is also seen in graph distances in the giant, as the maximum and average distances are $\Theta(\log n)$ [64] and $\Theta(\log \log n)$ [1] a.a.s., respectively. Interestingly, there are enough (polynomially many) pairs of vertices with resistance matching the maximum to ensure that the cover time is a factor $\Theta(\log^2 n)$ larger than the average hitting time, many random graphs (e.g., connected Erdős-Rényi, preferential attachment) are expanders and do not have this feature.

Stating additional contributions of this paper, as well as providing more detail about those already stated, requires a bit more terminology and notation, which we introduce below after discussing the related literature.

1.2   Further Related Work and Our Techniques

Over the last two decades, the cover time of many random graph models has been determined. These networks include the binomial random graph [20, 22], random geometric graph [23], preferential attachment model [21], configuration model [25], random digraphs [24] and the binomial random intersection graph [10]. These results were all proven using Cooper and Frieze's *first visit lemma*, see the aforementioned papers or [62]. This result is based on expressing the probability that a vertex has been visited up-to a given time by a function of the return probabilities. One (simplified) condition required to easily apply the first visit lemma is that $t_{\mathsf{rel}} \cdot \max_{v \in V} \pi(v) = o(1)$, where π is the stationary distribution and $t_{\mathsf{rel}}(G) := \frac{1}{1-\lambda_2}$ is the *relaxation time* of G, and λ_2 is the second-largest eigenvalue of the transition matrix of the (lazy) random walk on G. However, inserting the best known bounds on t_{rel} and $\max_{v \in V} \pi(v)$ for the HRG, by [47] and [37] respectively, gives $t_{\mathsf{rel}} \cdot \max_{v \in V} \pi(v) \leqslant (n^{2\alpha-1} \log n) \cdot n^{\frac{1}{2\alpha}-1+o(1)}$ which is not $o(1)$ for any $\frac{1}{2} \leqslant \alpha \leqslant 1$.

Another key ingredient of the first visit lemma is good bounds on the expected number of returns to a vertex before the walk mixes, i.e. $\sum_{t=0}^{t_{\mathsf{mix}}} P_{v,v}^t$ where $P_{x,y}^t$ is the probability a (lazy) random walk from x is at vertex y after exactly t steps. Obtaining such bounds in the HRG appears challenging due to the large mixing time and irregular local structure of the HRG. This also effects arguably the most natural approach to obtaining bounds on the average hitting time, that is applying the formula $\pi(v)E_\pi[\tau_v] = \sum_{t=0}^{\infty}[P_{v,v}^t - \pi(v)]$, see [3, Lemma 2.11], as this involves the same sum (which only needs to be considered up to relaxation/mixing time).

Given the perceived difficulty in determining the cover time using the return probabilities as described above, the approach taken in this paper is to determine the hitting and cover times via the effective resistances $\{\mathcal{R}\,(u \leftrightarrow v)\}_{u,v \in V}$. There is an intimate connection between reversible Markov chains and electrical networks as certain quantities in each setting are determined by the same harmonic equations. Classically this connection has been exploited to determine whether random walks on infinite graphs are transient or recurrent [60, Chapter 2], and more recently the effective resistance metric has been understood to relate the blanket times of random walks on finite graphs to the Gaussian free field [28]. The main connection we shall use is that the commute time (sum of hitting times in either direction) between two vertices is equal to the number of edges times the effective resistance between the two points [16, 71]. This result has been used to bound hitting and cover times in several random graph models, notably in the binomial random graph [41, 70] and the geometric random graph [7]. Luxburg et al. [72] recently refined a previous bound of Lovász [59] to give

$$\left| \mathcal{R}\left(u \leftrightarrow v\right) - \frac{1}{d(u)} - \frac{1}{d(v)} \right| \leqslant \frac{t_{\mathsf{rel}} + 2}{d_{\mathsf{min}}} \left(\frac{1}{d(u)} + \frac{1}{d(v)} \right), \tag{1}$$

for any non-bipartite graph G and $u, v \in V(G)$, where $d(v)$ is the degree of the vertex v and $d_{\mathsf{min}} = \min_{v \in V} d(v)$ is the minimum degree. For the HRG with parameter $\frac{1}{2} < \alpha < 1$, with high probability, $t_{\mathsf{rel}} \geqslant n^{2\alpha-1}/(\log n)^{1+o(1)}$ [47] and the average degree is constant - thus (1) does not give a good bound.

Since their introduction in 2010 [52], hyperbolic random graphs have been studied by various authors. Apart from the results already mentioned (power-law degree distribution, short graph distances, strong clustering, giant component, spectral gap and diameter), connectivity was investigated by Bode et al. [12]. Further results exist on the number of k-cliques and the clique number [32], the existence of perfect matchings and Hamilton cycles [29], the tree-width [9] and sub-tree counts [65]. Two models, commonly considered closely related to the hyperbolic random graphs, are scale-free percolation [27] and geometric inhomogeneous random graphs [14].

Few random processes on HRGs have been rigorously studied. Among the notable exceptions is the work by Linker et al. [58] which studies the contact processes in the HRG model, bootstrap percolation by Candellero and Fountoulakis [15] and Marshall et al. [63], and, for geometric inhomogeneous random graphs, by Koch and Lengler [50]. Komjáthy and Lodewijks [51] studied first passage percolation on scale free spatial network models.

To the best of our knowledge, the only work that explicitly studies random walks that deals with (a more general model of) HRGs is the work by Cipriani and Salvi [19] on mixing time of scale-free percolation. However, some aspects of simple random walks have been analyzed on infinite versions of HRGs. Specifically, Heydenreich et al. study transience and recurrence of random walks in the scale-free percolation model [39] (also known as heterogeneous long-range percolation) which is a "lattice" version of the HRG model. For similar investigations, but for more general graphs on Poisson point processes, see [36]. Additionally, the first author, Linker, and Mitsche [45] have studied a dynamic variant of the HRG generated by stationary Brownian motions.

2 Preliminaries

In this section we introduce notation, define some objects and terms we will be working with, and collect, for future reference, some known results concerning them. We adopt some conventions in Section 2.1, we recall a large deviations bound in Section 2.2, then we formally define the HRG model in Section 2.3, we discuss random walks in Section 2.4 and electrical networks in Section 2.5.

2.1 Conventions

Throughout, we use standard notions and notation concerning the asymptotic behavior of sequences. If $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are two sequences of real numbers, we write $a_n = \mathcal{O}(b_n)$ to denote that for some constant $C > 0$ and $n_0 \in \mathbb{N}$ it holds that $|a_n| \leqslant C|b_n|$ for all $n \geqslant n_0$. Also, we write $a_n = \Omega(b_n)$ if $b_n = \mathcal{O}(a_n)$, and $a_n = \Theta(b_n)$ if $a_n = \mathcal{O}(b_n)$ and $a_n = \Omega(b_n)$.

Unless stated otherwise, all asymptotics are as $n \to \infty$ and all other parameters are assumed fixed. Expressions given in terms of other variables that depend on n, for example $\mathcal{O}(R)$, are still asymptotics with respect to n. As we are interested in asymptotics, we only claim and prove inequalities for n sufficiently large. So, for simplicity, we always assume n sufficiently large. For example, we may write $n^2 > 5n$ without requiring $n > 5$.

An event, more precisely a family of events parameterized by $n \in \mathbb{N}$, is said to hold *with high probability (w.h.p.)*, if for every $c > 0$ the event holds with probability at least $1 - \mathcal{O}(n^{-c})$.

We shall follow standard notation, such as denoting the vertex and edge sets of G by $V(G)$ and $E(G)$, respectively. We use $d_G(u, v)$ to denote the graph distance between two vertices $u, v \in V(G)$, let $N(v) := \{u \in V \mid d_G(u, v) = 1\}$ denote the neighbourhood of a vertex, and let $d(v) := |N(v)|$.

2.2 Poisson Random Variables

We will be working with a Poissonized model, where the number of points within a given region is Poisson-distributed. Thus, we will need some elementary results for Poisson random variable. The first is a (Chernoff) large deviation bound.

▶ **Lemma 4.** *Let P have a Poisson distribution with mean μ. The following holds*
   (i) $\mathbb{P}(P \leqslant \frac{1}{2}\mu) \leqslant e^{-\frac{1}{8}\mu}$.
   (ii) *If $\delta \geqslant e^{\frac{3}{2}}$, then $\mathbb{P}(P \geqslant \delta\mu) \leqslant e^{-\frac{1}{2}\delta\mu}$.*

Several times, when bounding various expectations, we use the following crude but useful bound on the raw moments of Poisson random variables.

▶ **Lemma 5.** *Let X be a Poisson random variable with mean μ. Then, for any real $\kappa \geqslant 1$, we have $\mathbb{E}(X^{\kappa}) \leqslant \mu^{\kappa} \cdot \left(40 \cdot \min\left\{\frac{\kappa}{5\mu}, 1\right\}\right)^{\kappa}$.*

2.3 The HRG model

We represent the hyperbolic plane (of constant Gaussian curvature -1), denoted $\mathbb{H}^2$, by points in $\mathbb{R}^2$. Elements of $\mathbb{H}^2$ are referred to by the polar coordinates (r, θ) of their representation as points in $\mathbb{R}^2$. The point with coordinates $(0, 0)$ will be called the *origin* of $\mathbb{H}^2$ and denoted O. When alluding to a point $u \in \mathbb{H}^2$ we denote its polar coordinates by (r_u, θ_u). The hyperbolic distance $d_{\mathbb{H}^2}(u, v)$ between two points $u, v \in \mathbb{H}^2$ is determined via the Hyperbolic Law of Cosines as the unique solution of

$$\cosh d_{\mathbb{H}^2}(u, v) = \cosh r_u \cosh r_v - \sinh r_u \sinh r_v \cos(\theta_u - \theta_v).$$

In particular, the hyperbolic distance between the origin and a point $u \in \mathbb{H}^2$ equals r_u. For a point $p \in \mathbb{H}^2$ the ball of radius $\rho > 0$ centered at p will be denoted $B_p(\rho)$, i.e.,

$$B_p(\rho) := \{q \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(p, q) < \rho\}.$$

We will work in the Poissonized version of the HRG model which we describe next. For a positive integer n and positive constant ν we consider a Poisson point process on the hyperbolic disk centered at the origin O and of radius $R := 2\ln(n/\nu)$. The intensity function at polar coordinates (r, θ) for $0 \leqslant r < R$ and $\theta \in \mathbb{R}$ equals

$$\lambda(r, \theta) := \nu e^{\frac{R}{2}} f(r, \theta) = n f(r, \theta) \tag{2}$$

where $f(r, \theta)$ is the joint density function of independent random variables θ and r, with θ chosen uniformly at random in $[0, 2\pi)$ and r chosen according to the following density function:

$$f(r) := \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} \cdot \mathbf{1}_{[0,R)}(r) \qquad \text{where } \mathbf{1}_{[0,R)}(\cdot) \text{ is the indicator of } [0, R).$$
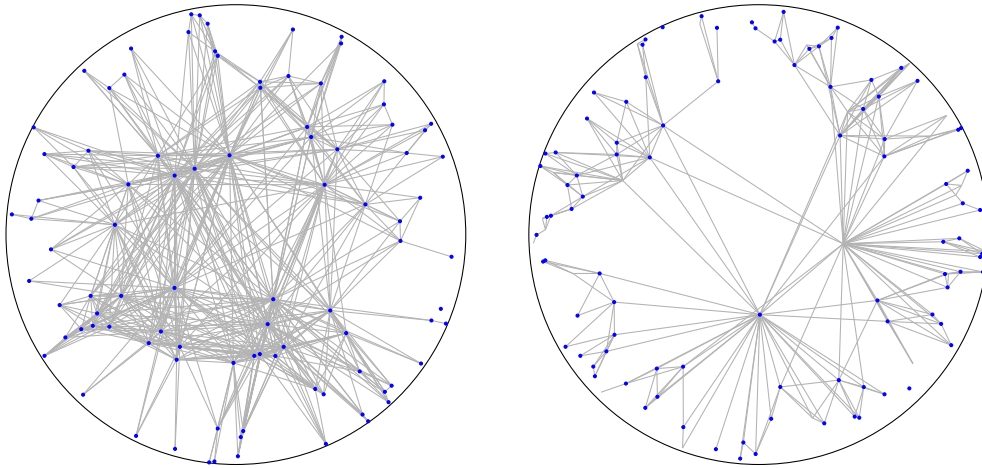
Figure 1 Instances of $\mathcal{G}_{\alpha,\nu}(n)$ for $n = 100$, $\nu \approx 1.832$, $\alpha = 0.6$ (left) and $\alpha = 0.9$ (right).

The parameter $\alpha > 0$ controls the distribution: For $\alpha = 1$, the distribution is uniform in $\mathbb{H}^2$, for smaller values the vertices are distributed more towards the center of $B_O(R)$ and for bigger values more towards the border. (See Figure 1 for an illustration of instances of $\mathcal{G}_{\alpha,\nu}(n)$ for two distinct values of α.)

We shall need the following useful approximation to the density $f(\cdot)$.

▶ **Lemma 6** ([33, Equation (3)]). $f(r) = \alpha e^{-\alpha(R-r)} \cdot (1 + \Theta(e^{-\alpha R} + e^{-2\alpha r})) \cdot \mathbf{1}_{[0,R)}(r)$.

We denote the point set of the Poisson process by V and we identify elements of V with the vertices of a graph whose edge set E is the collection of vertex pairs uv such that $d_{\mathbb{H}^2}(u,v) < R$. The probability space over graphs (V,E) thus generated is denoted by $\mathcal{G}_{\alpha,\nu}(n)$ and referred to as the HRG. Note in particular that $\mathbb{E}|V| = n$ since

$$\int_{B_O(R)} \lambda(r,\theta) \, \mathrm{d}\theta \mathrm{d}r = \nu e^{\frac{R}{2}} \int_0^\infty f(r) \, \mathrm{d}r = n.$$

The parameter ν controls the average degree of $\mathcal{G}_{\alpha,\nu}(n)$ which, for $\alpha > \frac{1}{2}$, is $(1+o(1))\frac{2\alpha^2\nu}{(\alpha-1/2)^2}$ (see [37, Theorem 2.3]).

The Hyperbolic Law of Cosines turns out to be complicated to work with when computing distances in hyperbolic space. Instead, it is more convenient to consider the maximum angle $\theta_R(r_u, r_v)$ that two points $u, v \in B_O(R)$ can form with the origin O and still be within (hyperbolic) distance at most R provided u and v are at distance r_u and r_v from the origin, respectively.

▶ **Remark 7.** Replacing in (7) the terms $d_{\mathbb{H}^2}(u,v)$ by R and $\theta_u - \theta_v$ by $\theta_R(r_u, r_v)$, taking partial derivatives on both sides with respect to r_u and some basic arithmetic gives that the mapping $r_u \mapsto \theta_R(r_u, r_v)$ is continuous and strictly decreasing in the interval $[0, R)$. Since $\theta_R(r_u, r_v) = \theta_R(r_v, r_u)$, the same is true of the mapping $r_v \mapsto \theta_R(r_u, r_v)$. (See [48, Remark 2.1] for additional details.)

The following estimate of $\theta_R(r, r')$, due to Gugelmann, Panagiotou and Peter is very useful and accurate (especially when $R - (r + r')$ goes to infinity with n).

▶ **Lemma 8** ([37, Lemma 6]). *If $0 \leqslant r \leqslant R$ and $r + r' \geqslant R$, then $\theta_R(r, r') = 2e^{\frac{1}{2}(R-r-r')}(1 + \Theta(e^{R-r-r'}))$.*

We will need estimates for measures of regions of the hyperbolic plane, and specifically for the measure of balls. We denote by $\mu(S)$ the measure of a set $S \subseteq \mathbb{H}^2$, i.e., $\mu(S) := \int_S f(r, \theta)\,\mathrm{d}r\mathrm{d}\theta$. The following approximation of the measures of the ball of radius ρ centered at the origin and the ball of radius R centered at $p \in B_O(R)$, both also due to Gugelmann et al., will be used frequently in our analysis.

▶ **Lemma 9** ([37, Lemma 7]). *For $\alpha > \frac{1}{2}$, $p \in B_O(R)$ and $0 \leqslant r \leqslant R$ we have*

$$\mu(B_O(r)) = e^{-\alpha(R-r)}(1 + o(1)),$$

$$\mu(B_p(R) \cap B_O(R)) = \frac{2\alpha e^{-\frac{1}{2}r_p}}{\pi(\alpha - \frac{1}{2})}\big(1 + \mathcal{O}(e^{-(\alpha - \frac{1}{2})r_p} + e^{-r_p})\big).$$

Next, we state a result that is implicit in [11] (later refined in [30]) concerning the size and the "geometric location" of the giant component of $\mathcal{G}_{\alpha,\nu}(n)$. First, observe that the set of vertices of $\mathcal{G}_{\alpha,\nu}(n)$ that are inside $B_O(\frac{R}{2})$ are within distance at most R of each other. Hence, they form a clique and in particular belong to the same connected component. The graph induced by the connected component of $\mathcal{G}_{\alpha,\nu}(n)$ to which the vertices in $B_O(\frac{R}{2})$ belong will be referred to as the *center component* of $\mathcal{G}_{\alpha,\nu}(n)$.

▶ **Theorem 10** ([11, Theorem 1.4],[48, Theorem 1.1]). *If $\frac{1}{2} < \alpha < 1$, then a.a.s. the center component of $\mathcal{G}_{\alpha,\nu}(n)$ has size $\Theta(n)$ and all other connected components of $\mathcal{G}_{\alpha,\nu}(n)$ are of size polylogarithmic in n.*

The previous result partly explains why we focus our attention on simple random walks in the center component of $\mathcal{G}_{\alpha,\nu}(n)$. In the following remark we justify formally why we, henceforth, consider both the giant and the center component as being the same component, and consequently denote both of them by $\mathcal{C}_{\alpha,\nu}(n)$.

▶ Remark 11. Let $\mathcal{S}_n$ be the event that the giant (the largest component) is equal to the center component, then $\mathbb{P}(\mathcal{S}_n) = 1 - e^{-\Omega(n)}$ by [11]. It follows immediately that all of our results holding a.a.s. for the center component also hold for the giant component. For statements of the form $\mathbb{E}(X(\mathcal{C}))$, where $X(G)$ is a function of a graph satisfying $1 \leqslant X(G) \leqslant |V(G)|^\kappa$ for some fixed $\kappa > 0$, for example (non-trivial) cover/hitting times, the results also carry over. That is, if $\mathcal{C}'$ is the giant of the HRG then $\mathbb{E}(X(\mathcal{C}')) = (1 + o(1))\mathbb{E}(X(\mathcal{C}))$. To see this, since $\mathbb{E}(|V(\mathcal{G})|^\kappa) = \mathcal{O}(n^\kappa)$ by Lemma 5, we have the following by Cauchy–Schwartz,

$$\mathbb{E}(X(\mathcal{C}')) = \mathbb{E}(X(\mathcal{C}')\mathbf{1}_{\mathcal{S}_n}) + \sqrt{\mathbb{E}|V(\mathcal{C}')|^{2\kappa} \cdot \mathbb{P}(\mathcal{S}_n^c)} \leqslant \mathbb{E}(X(\mathcal{C})) + n^\kappa e^{-\Omega(n)} = (1 + o(1))\mathbb{E}(X(\mathcal{C})).$$

This also holds with the roles of $\mathcal{C}'$ and $\mathcal{C}$ reversed, giving the result.

The condition $\alpha > \frac{1}{2}$ guarantees that $\mathcal{G}_{\alpha,\nu}(n)$ has constant average degree depending on α and ν only [37, Theorem 2.3]. If on the contrary $\alpha \leqslant \frac{1}{2}$, then the average degree grows with n. If $\alpha > 1$, the largest component of $\mathcal{G}_{\alpha,\nu}(n)$ is sublinear in n [11, Theorem 1.4]. For $\alpha = 1$ whether the largest component is of size linear in n depends on ν [11, Theorem 1.5]. Hence, the parameter range where $\frac{1}{2} < \alpha < 1$ is where HRGs are always sparse, exhibit power law degree distribution with exponent between 2 and 3 and the giant component is, a.a.s., unique and of linear size. All these are characteristics ascribed to so called "social" or "complex networks" which HRGs purport to model. Our main motivation is to contribute to understand processes that take place in complex networks, many of which, as already discussed in the introduction, either involve or are related to simple random walks on such networks. Thus, we restrict our study exclusively to the case where $\frac{1}{2} < \alpha < 1$, but in order to avoid excessive repetition, we omit this condition from the statements we establish.

The last known property of HRGs that we recall is that, w.h.p. the center component has a linear in n number of edges.

▶ **Lemma 12** ([47, Lemma 15]). *For $\frac{1}{2} < \alpha < 1$, w.h.p. $|E(\mathcal{C}_{\alpha,\nu}(n))|$ is $O(n)$.*

To conclude this section we point out that everything we do throughout this paper can be easily adapted to the case where $\mathbb{H}^2$ has Gaussian curvature $-\eta^2$ instead of -1 and all stated results hold provided α is replaced by α/η everywhere.

2.4 Random Walks

The simple random walk $(X_t)_{t \geqslant 0}$ on a graph $G = (V, E)$ is a discrete time random process on V which at each time moves to a neighbour of the current vertex $u \in V$ uniformly with probability $1/d(u)$. We use $\mathbf{P}(\cdot) := \mathbf{P}^G(\cdot)$ to denote the law of the random walk on a graph G (as opposed to $\mathbb{P}$ for the random graph). For a probability distribution μ on V we let $\mathbf{P}^G_\mu(\cdot) := \mathbf{P}^G(\cdot \mid X_0 \sim \mu)$ be the conditional probability of the walk on G started from a vertex sampled from μ. In the case where the walk starts from a single vertex $u \in V$ we write u instead of μ, for example $\mathbf{E}^G_u(\cdot) := \mathbf{E}^G(\cdot \mid X_0 = u)$. We shall drop the superscript G when the graph is clear from the context. We now define the *cover time* $t_{\mathsf{cov}}(G)$ of G by

$$t_{\mathsf{cov}}(G) := \max_{u \in V} \mathbf{E}^G_u(\tau_{\mathsf{cov}}), \qquad \text{where} \qquad \tau_{\mathsf{cov}} := \inf\left\{ t \geqslant 0 : \bigcup_{i=0}^t \{X_i\} = V(G) \right\}.$$

Similarly, for $u, v \in V$ we let $\mathbf{E}_u(\tau_v)$, where $\tau_v := \inf\{t \geqslant 0 \mid X_t = v\}$, be the *hitting time* of v from u. We shall use π to denote the *stationary distribution* of a single random walk on a connected graph G, given by $\pi(v) := \frac{d(v)}{2|E|}$ for $v \in V$. Let

$$t_{\mathsf{hit}}(G) := \max_{u,v \in V} \mathbf{E}^G_u(\tau_v), \quad \text{and} \quad t_\odot(G) := \sum_{u,v \in V(G)} \mathbf{E}^G_u(\tau_v)\,\pi(u)\pi(v),$$

denote the *maximum hitting time* and the *target time*, respectively. We define each of these times to be 0 if G is either the empty graph or consists of just a single vertex. The target time $t_\odot(G)$, also given by $\mathbf{E}^G_\pi(\tau_\pi)$, is the expected time for a random walk to travel between two vertices sampled independently from the stationary distribution [56, Section 10.2]. We will consider the random walk on the center component $\mathcal{C} := \mathcal{C}_{\alpha,\nu}(n)$ of the HRG and so each of the expected stopping times $t_{\mathsf{cov}}(\mathcal{C}), t_{\mathsf{hit}}(\mathcal{C})$ and $t_\odot(\mathcal{C})$ will in fact be random variables. We shall also refer to $\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u)$ as the *commute time* between the vertices u, v.

2.5 Electrical Networks & Effective Resistance

An electrical network, $N := (G, C)$, is a graph G and an assignment of conductances $C : E(G) \to \mathbb{R}^+$ to the edges of G. For an undirected graph G we define $\vec{E}(G) := \{\vec{xy} \mid xy \in E(G)\}$ as the set of all possible oriented edges for which there is an edge in G. For some $S, T \subseteq V(G)$, a *flow* from S to T (or (S, T)-*flow*, for short) on N is a function $f : \vec{E}(G) \to \mathbb{R}$ satisfying $f(\vec{xy}) = -f(\vec{yx})$ for every $xy \in E(G)$ as well as Kirchoff's node law for every vertex apart from those that belong to S and T, i.e.

$$\sum_{u \in N(v)} f(\vec{uv}) = 0 \qquad \text{for each } v \in V \setminus (S \cup T).$$

A flow from S to T is called a *unit* flow if, in addition, its strength is 1, that is,

$$\sum_{s \in S} \sum_{u \in N(s)} f(\vec{su}) = 1.$$

We say that the (S, T)-flow is *balanced* if

$$\sum_{u \in N(s)} f(\vec{su}) = \sum_{u \in N(s')} f(\vec{s'u}) \quad \text{and} \quad \sum_{u \in N(t)} f(\vec{ut}) = \sum_{u \in N(t')} f(\vec{ut'}) \quad \text{for all } s, s' \in S \text{ and } t, t' \in T.$$

When $S = \{s\}$ and $T = \{t\}$ we write (s, t)-flow instead of $(\{s\}, \{t\})$-flow. Note that (s, t)-flows are always balanced. Given two flows f and g on $N := (G, C)$, we say that *g can be concatenated to f* if $f + g$ is a flow on N and for every oriented edge $\vec{e} \in \vec{E}$ either $f(\vec{e})$ and $g(\vec{e})$ are both 0, or they have opposite signs, so $(f(\vec{e}) + g(\vec{e}))^2 \leqslant (f(\vec{e}))^2 + (g(\vec{e}))^2$.

For the network $N := (G, C)$ and a flow f on N define the *energy dissipated* by f, denoted $\mathcal{E}(f)$, by

$$\mathcal{E}(f) := \sum_{e \in \vec{E}(G)} \frac{f(e)^2}{2C(e)}, \tag{3}$$

For future reference, we state the following easily verified fact:

▷ **Claim 13.** Let $N := (G, C)$ be an electrical network and $S, M, T \subseteq V(G)$. If f and g are balanced (S, M) and (M, T) flows on N, respectively, and g can be concatenated to f, then $f + g$ is a balanced (S, T)-flow on N and $\mathcal{E}(f + g) \leqslant \mathcal{E}(f) + \mathcal{E}(g)$. Moreover, if f and g are unit flows, so is $f + g$.

For $S, T \subseteq V(G)$, the *effective resistance* between S and T, denoted $\mathcal{R}_C(S \leftrightarrow T)$, is

$$\mathcal{R}_C(S \leftrightarrow T) := \inf \{\mathcal{E}(f) \mid f \text{ is a unit flow from } S \text{ to } T\}. \tag{4}$$

The set of conductances C defines a reversible Markov chain [60, Chapter 2]. In this paper we shall fix $C(e) = 1$ for all $e \in E(G)$ as this corresponds to a simple random walk. In this case, we write $\mathcal{R}_G(S \leftrightarrow T)$ (or $\mathcal{R}(S \leftrightarrow T)$ if the graph is clear) instead of $\mathcal{R}_C(S \leftrightarrow T)$. The following is well known.

▶ **Proposition 14** ([56, Corollary 10.8]). *The effective resistances form a metric space on any graph, in particular they satisfy the triangle inequality.*

Choosing a single shortest path P between any two vertices s, t (if one exists) in a network (with $C(e) = 1$ for each $e \in E$) and routing a unit flow down the edges of P we obtain, straight from the definition (4) of $\mathcal{R}(s \leftrightarrow t)$, the following basic but useful result.

▶ **Lemma 15** ([16, Lemma 3.2]). *For any graph G and $s, t \in V(G)$, we have $\mathcal{R}(s \leftrightarrow t) \leqslant d_G(s, t)$.*

Another very useful tool is Rayleigh's monotonicity law (RML).

▶ **Theorem 16** (Rayleigh's Monotonicity Law [56, Theorem 9.12]). *Let $\{C(e)\}_{e \in E}$ and $\{C'(e)\}_{e \in E}$ be sets of conductances on the edges of the same graph $G = (V, E)$. If $C(e) \geqslant C'(e)$ for all $e \in E$, then*

$$\mathcal{R}_C(S \leftrightarrow T) \leqslant \mathcal{R}_{C'}(S \leftrightarrow T) \qquad \text{for all } S, T \subseteq V.$$

The *Kirchhoff index* $\mathcal{K}(G)$ of a graph G is the sum of resistances in the graph, that is

$$\mathcal{K}(G) = \sum_{u, v \in V(G)} \mathcal{R}(u \leftrightarrow v).$$

The Kirchhoff index has applications in mathematical chemistry, see [66] and citing papers.

The following result allows us to relate hitting times to effective resistance.

▶ **Lemma 17** ([56, Proposition 10.6]). *For any graph G and any pair of vertices $u, v \in V(G)$ we have*

$$\mathbf{E}_u(\tau_v) + \mathbf{E}_v(\tau_u) = 2|E(G)| \cdot \mathcal{R}(u \leftrightarrow v). \qquad \text{(Commute time identity)}$$

3   Overview of the Proofs of the Main Results

Owing to space we give a brief (heavily abridged, non-rigorous) overview of the ideas used in the proofs of the main theorems from Section 1.1. Proofs of all claims can be found in the full version of this paper [49].

3.1   Theorems 1 & 2: Average Effective Resistance & Target Time

To control the resistance $\mathcal{R}(s \leftrightarrow t)$ between two vertices s and t of the center component $\mathcal{C}_{\alpha,\nu}(n)$ we bound the energy dissipated by a carefully designed (s,t)-flow $f_{s,t}$ associated to a tiling of the hyperbolic plane on which we overlay a forest-like structure. We shall first describe the tiling, then the flow, before explaining how to bound the resistance from this flow.

Tiling

We define a set of tiles $\{T_{i,j}\}_{i,j \geqslant 0}$ of the hyperbolic plane $\mathbb{H}^2$ centered around the origin O. This tiling is spherical in nature (see Figure 2) and, roughly speaking, tile $T_{i,j}$ is i tiles from the origin (we say it is at *level i*) and it is the jth tile, at level i, when going clockwise from a fixed ray emanating from O, at 0 degrees "north". A region of $\mathbb{H}^2$ between two rays emanating from the origin O will be called a *sector*, and refer to the angle of the sector as the (smallest) angle between the two rays.

There will be a distinguished collection of tiles, called *root* tiles $\{T_{0,j}\}_{j \in N_0}$, corresponding to the elements of the equipartition of $B_O(\frac{R}{2})$ into $N_0 = \Theta(1)$ sectors, hence each sector has angle $\theta_0 = 2\pi/N_0$. We then define three sequences N_i, θ_i and h_i, for $i \in \mathbb{N}$ where $N_i := 2^i N_0$, $\theta_i := 2\pi/N_i$, and (very roughly speaking) $h_0 = R/2$ and $h_i \approx R/2 + i \cdot \ln 2$. The rest of the tiling is specified by setting $T_{i,j}$ to be the region at distance between h_{i-1} and h_i from O that lies in the smallest sector between rays at angles $\theta_i \cdot j$ and $\theta_i \cdot (j+1)$. Thus each sector has central angle θ_i and there is a total of $N_i = 2N_{i-1}$ sectors for a given i, see Figure 2.

We say that $T_{i,j}$ is the *parent* tile of both $T_{i+1,2j}$ and $T_{i+1,2j+1}$ and refer to the latter two tiles as *children* of tile $T_{i,j}$ (root tiles are assumed to be their own parent). For a tile $T_{i,j}$ we refer to the tiles of height $i' \leqslant i$ that intersect a ray from O that passes though $T_{i,j}$ as the *ancestors* of $T_{i,j}$. A tile T will be said to be a *descendant* of another tile T' if the latter is an ancestor of the former. Given a tile $T_{i,j}$ let $\{T_{i,j}^0, T_{i,j}^1\}$ be the "natural" equipartition of $T_{i,j}$ into two tiles along a ray from O though the center of $T_{i,j}$. We refer to $T_{i,j}^0, T_{i,j}^1$ as the *half-tiles* of $T_{i,j}$ and say $T_{i,j}^0$ is the *twin* of $T_{i,j}^1$, and vice versa. Given a tile T we call H the *parent half-tile* of T if it is a half-tile of the parent of T and a line segment from the origin to any point in the interior of T intersects H.

Recall that two points at distance at most R are connected by an edge of $\mathcal{G}_{\alpha,\nu}(n)$. The sequences N_i, θ_i and h_i are chosen in such a way so that the tiling satisfies the following two properties.

(i) Two points in a given tile are within distance at most R of each other.
(ii) Any point in a tile is within distance at most R from any point in its parent half-tile.

These two properties allow us too describe a flow based on how it moves between blocks, rather than getting bogged down with specific vertices.

Comparison with a Tiling Due To Fountoulakis and Müller [30]

In this article we work in the so called *Gans model* [34] or *native model* [52] of hyperbolic space, in contrast to [30] where the *Poincaré half-plane model* is used. Although the two tilings are approximately equal, ours is a partition of $\mathbb{R}^2$ instead of the half-plane, i.e., each
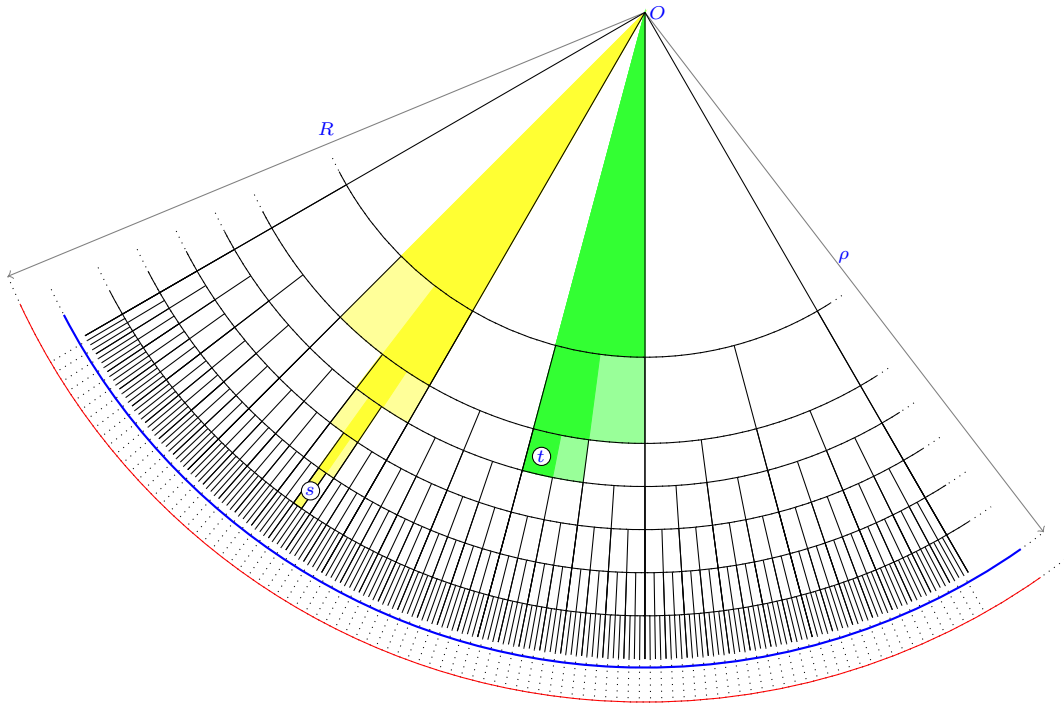
■ **Figure 2** (a) Partial illustration of a tiling of $B_O(R)$ (not at scale). (b) Depiction of flow between vertices s and t with no common ancestor tile. Coloured regions contain vertices through which flow from s to t is routed. Flow is pushed radially towards the origin O from a yellow shaded tile to its parent half-tile. Flow is pushed in an angular direction from dark to light yellow shaded half-tiles. The direction of flow is reversed in green shaded region.

tiling partitions the set used to represent hyperbolic space. Since both representations are alternative models of $\mathbb{H}^2$, both tilings can be cast in one or the other. Doing so, one gets that the similarity of both tilings increases the further towards infinity their tiles are, i.e., further from the origin in the Gans model and closer to the half-plane boundary in the Poincaré half-plane model. For points close to the origin in the former or far from the boundary in the latter representation, both tilings differ significantly, although this is irrelevant for the analyses performed either here or in [30, 64]. However, working with our tiling avoids having to perform, as in [30], a coupling between the HRG and a point process in the half-plane and also avoids some of the approximations incurred by working with the idealized model of Müller and Staps [64]. We believe this explains why we can guarantee that most of our results hold with probability $1 - \mathcal{O}(n^{-c})$ for all $c > 0$ instead of the same probability but just for some $c > 0$.

Definition of the Flow

We now sketch the construction of our unit (s,t)-flow, between two distinct vertices s and t of $\mathcal{G}_{\alpha,\nu}(n)$. The energy dissipated by this flow yields bounds on the effective resistance between s and t.

The flow $f_{s,t}$ is built up as a sum of five separate flows, that is

$$f_{s,t} := f_s + f_s^t + f^{s,t} + f_t^s + f_t. \tag{5}$$

The rough idea is to find the tile T that is a common ancestor of s and t at furthest distance from O and create flows to this tile from s, and from this tile to t. If s and t have no common ancestor then the flows meet up in the center $B_O(\frac{R}{2})$ (which is a clique), with the centre essentially taking the place of T. The first term f_s in (5) spreads flow from s evenly across the half-tile it is contained in, likewise f_t collects flow from vertices in the half-tile containing t and sends it to t. The term f_s^t moves flow towards the centre by first moving flow from the current half-tile to its twin, then from the current (full) tile to its parent half-tile, and repeating like this until reaching the half-tile of the common ancestor (or a root half-tile in the center $B_O(\frac{R}{2})$). The term f_s^t does the same in reverse, taking flow out from the center to t, see the yellow and green parts of Figure 2 for an illustration. Finally the term $f^{s,t}$ moves the flow from the half-tile in the common ancestor in the ray that intersects s to its twin (or across root tiles), this flow is zero if s and t lie in a ray from O.

The main ideas of this construction are that it spreads the flow over as evenly as possible over the vertices in each tile. Its modular construction also makes it easy to analyse, in particular when bounding the energy dissipated. One can show that if each half-tile encountered in the above sketched construction of $f_{s,t}$ is non-empty, then $f_{s,t}$ indeed gives a valid (s,t)-flow. Moreover, since this flow is balanced and the parts can be concatenated, we have

$$\mathcal{E}(f_{s,t}) \leqslant \mathcal{E}(f_s) + \mathcal{E}(f_s^t) + \mathcal{E}(f^{s,t}) + \mathcal{E}(f_t^s) + \mathcal{E}(f_t).$$

Validity and Energy of the Flow

At this juncture we turn our attention to determining conditions under which $f_{s,t}$ exists (i.e., every tile contains a vertex), and more importantly is a good flow in the sense that it dissipates a small amount of energy. Clearly, the larger the number of vertices in each half-tile, the smaller the energy dissipated by the flow.

We say that a half-tile H is *sparse* if the number of vertices it contains is fewer than half the ones expected, i.e., if $|V \cap H| < \frac{1}{2}\mathbb{E}|V \cap H|$. We say that a tile T is *faulty* if either one of its two associated half-tiles is sparse. For $C > 0$ a large constant to be determined, let

$$\rho := R - \frac{\ln(C\frac{R}{\nu})}{1-\alpha}. \tag{6}$$

Using standard arguments concerning Poisson point processes we argue that,

w.h.p., none of the tiles T contained in $B_O(\rho)$ are faulty. $\tag{7}$

We say that a tile T *is robust* if none of its ancestors (including itself) is faulty. Thus, (7) implies that w.h.p. every tile T contained in $B_O(\rho)$ is robust. The condition $T \subseteq B_O(\rho)$ cannot be relaxed significantly, so we will have to settle for a weaker statement. For $C' > 0$, let

$$\rho' := R - \frac{\ln(\frac{2C'}{\nu})}{1-\alpha}. \tag{8}$$

Thus some points in $B_O(\rho')$ are only a constant distance from the boundary. We show that

a tile contained in $B_O(\rho')$ has at least a constant probability of being robust. $\tag{9}$

We finally establish the following:

if s and t belong to robust tiles then $f_{s,t}$ is a unit (s,t)-flow and $\mathcal{E}(f_{s,t}) = \mathcal{O}(1)$. $\tag{10}$

Bounding Average Resistance and Target Time

Together, (9) and (10) show that a constant fraction of pairs of points in $B_O(\rho')$ have bounded resistance between them. We are yet far from done, as a significant fraction of the vertices of the giant component of $\mathcal{G} := \mathcal{G}_{\alpha,\nu}(n)$ fall outside the ball $B_O(\rho')$. For any vertex $w \in V(\mathcal{G})$ let $\Upsilon(w)$ be the smallest sector containing w whose closure contains vertices in $V(\mathcal{G}) \cap (B_O(\rho') \setminus B_O(\rho' - \ln 2))$ both clockwise and anti-clockwise of w that reside in robust tiles. We then prove the following.

$$\text{Let } w \text{ be a vertex in the giant and } w' \in V(\mathcal{G}) \cap B_O(\rho') \text{ be a robust vertex which} \tag{11}$$
$$\text{is closest (in graph distance) to } w. \text{ Then, } d(w, w') \leqslant 1 + |V(G) \cap \Upsilon(w)|.$$

By the triangle inequality (as $\mathcal{R}(\cdot \leftrightarrow \cdot)$ is a metric), and (11) and (10), for any s, t in the giant

$$\mathcal{R}(s \leftrightarrow t) \leqslant |V(\mathcal{G}) \cap \Upsilon(s)| + |V(\mathcal{G}) \cap \Upsilon(t)| + \mathcal{O}(1). \tag{12}$$

Armed with (12) we can now bound the resistance between s and t by bounding the number of vertices in the smallest sectors containing s and t defined by rays through robust vertices. We prove some further (more technical) results in the spirit of (9) that give improved bounds on the probability of a vertex being robust as a function of its location. Using these bounds, together with some elaborate arguments discussed in the full version of this article [49], for a vertex $w \in V(\mathcal{G}) \setminus B_O(\rho)$ we can bounds the tails of $|V(\mathcal{G}) \cap \Upsilon(w)|$ by a stretched exponential function. That is, roughly speaking, we can show that there exists constants $\kappa_1, \kappa_2 > 0$ such that for any $w \in V(\mathcal{G}) \setminus B_O(\rho)$ and $p \in \mathbb{N} \setminus \{0\}$,

$$\mathbb{P}(|V(\mathcal{G}) \cap \Upsilon(w)| \geqslant \kappa_1 \cdot 2^p) \leqslant \exp(-2^{\kappa_2 \cdot p}). \tag{13}$$

Applying the Campbell-Mecke formula [55, Theorem 4.4], a powerful tool from point process theory expressing expectations of functions of point processes as integrals of their mean measure, with (13) we show that $\mathbb{E}(\sum_{w \in V(\mathcal{G}) \setminus B_O(\rho)} |V(G) \cap \Upsilon(w)|^\kappa) = \mathcal{O}(n)$ for any fixed real $\kappa \geqslant 1$. Taking $\kappa = 1$ now gives Theorem 1. With some additional effort and a bound on the raw moments of degrees in the HRG we can also prove Theorem 2 using the commute time identity and Hölders inequality.

3.2   Theorem 3: Max Hitting and Cover Times

Due to the well known bound $\mathbf{E}_x[\tau_y] \leqslant d(x,y) \cdot 2|E|$, we obtain $t_{\mathsf{hit}} = \mathcal{O}(n \log n)$ since a.a.s. the diameter is $\Theta(\log n)$ and there are $\Theta(n)$ many edges. A bound of $\mathcal{O}(n \log^2 n)$ on the cover time then follows from Matthews bound.

The lower bounds are significantly more demanding and the basic Matthews lower bound gives a bound on the cover time that is a polynomial factor off the upper bound. We establish the following result which is implicit in [46]:

$$\text{w.h.p. there are } n^{\Omega(1)} \text{ maximal dangling paths of length at least } \Omega(\ln n) \text{ in } \mathcal{C}_{\alpha,\nu}(n), \tag{14}$$

and since the resistance between any pair of endpoints of the paths in (14) is $\Omega(\log n)$, we deduce that the commute time is $\Omega(n \log n)$. A refinement of Matthews bound due to Kahn, Kim, Lovász and Vu [42, Theorem 1.3] then gives the desired bound on the cover time from which the claimed lower bound on the hitting time immediately follows.

4 Concluding remarks

In this paper we determined the expected order of the maximum hitting time, cover time, target time and effective resistance between two uniform vertices, with the first two results also holding a.a.s. (w.r.t. the HRG). Our main finding to take away is that (in expectation) there are order $\log n$ gaps between each of the quantities. This indicates that most vertices in the giant are well-connected to the center of the graph, but a significant proportion are not.

We restricted our study to the giant component of the graph in the regime $\frac{1}{2} < \alpha < 1$, although this is arguably the most interesting regime it would still be interesting to determine the aforementioned quantities on the other smaller components or when $\alpha \notin (\frac{1}{2}, 1)$. Another problem we leave open is to discover the leading constants hidden behind our asymptotic notation, if the expression for the clustering coefficient of the HRG [31] is anything to go by these constants may have very rich and complex expressions as functions of α and ν. An interesting problem is to determine the order of meeting time, that is, the expected time it takes two (lazy) random walks to occupy the same vertex when started from the worst case start vertices [43]. Finally, the mixing time of a (lazy) random walk on the giant HRG is known up to polylogarithmic factors by [47]. Closing this gap is of great importance, but it may well be quite challenging.

References

1 Mohammed Amin Abdullah, Michel Bode, and Nikolaos Fountoulakis. Typical distances in a geometric model for complex networks. *Internet Math.*, page 38, 2017.

2 Tasweer Ahmad, Lianwen Jin, Luojun Lin, and Guozhi Tang. Skeleton-based action recognition using sparse spatio-temporal GCN with edge effective resistance. *Neurocomputing*, 423:389–398, 2021. `doi:10.1016/j.neucom.2020.10.096`.

3 David Aldous and James Allen Fill. Reversible Markov chains and random walks on graphs, 2002. Unfinished monograph, recompiled 2014. URL: `https://www.stat.berkeley.edu/~aldous/RWG/book.html`.

4 Romas Aleliunas, Richard M. Karp, Richard J. Lipton, László Lovász, and Charles Rackoff. Random walks, universal traversal sequences, and the complexity of maze problems. In *20th Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico, 29-31 October 1979*, pages 218–223. IEEE Computer Society, 1979. `doi:10.1109/SFCS.1979.34`.

5 Vedat Levi Alev, Nima Anari, Lap Chi Lau, and Shayan Oveis Gharan. Graph clustering using effective resistance. In Anna R. Karlin, editor, *9th Innovations in Theoretical Computer Science Conference, ITCS 2018, January 11-14, 2018, Cambridge, MA, USA*, volume 94 of *LIPIcs*, pages 41:1–41:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. `doi:10.4230/LIPIcs.ITCS.2018.41`.

6 Nima Anari and Shayan Oveis Gharan. Effective-resistance-reducing flows, spectrally thin trees, and asymmetric TSP. In Venkatesan Guruswami, editor, *IEEE 56th Annual Symposium on Foundations of Computer Science, FOCS 2015, Berkeley, CA, USA, 17-20 October, 2015*, pages 20–39. IEEE Computer Society, 2015. `doi:10.1109/FOCS.2015.11`.

7 Chen Avin and Gunes Ercal. On the cover time and mixing time of random geometric graphs. *Theor. Comput. Sci.*, 380(1-2):2–22, 2007. `doi:10.1016/j.tcs.2007.02.065`.

8 Anna Ben-Hamou, Roberto I. Oliveira, and Yuval Peres. Estimating graph parameters via random walks with restarts. In Artur Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 1702–1714. SIAM, 2018. `doi:10.1137/1.9781611975031.111`.

9 Thomas Bläsius, Tobias Friedrich, and Anton Krohmer. Hyperbolic random graphs: Separators and treewidth. In *24th Annual European Symposium on Algorithms (ESA 2016)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

10   Mindaugas Bloznelis, Jerzy Jaworski, and Katarzyna Rybarczyk. The cover time of a sparse random intersection graph, 2019. `doi:10.48550/ARXIV.1910.09639`.

11   Michel Bode, Nikolaos Fountoulakis, and Tobias Müller. On the largest component of a hyperbolic model of complex networks. *Electron. J. Comb.*, 22(3):P3.24, 2015. `doi:10.37236/4958`.

12   Michel Bode, Nikolaos Fountoulakis, and Tobias Müller. The probability of connectivity in a hyperbolic model of complex networks. *Random Struct. Algorithms*, 49(1):65–94, 2016.

13   Marián Boguná, Fragkiskos Papadopoulos, and Dmitri Krioukov. Sustaining the internet with hyperbolic mapping. *Nature communications*, 1(1):1–8, 2010.

14   Karl Bringmann, Ralph Keusch, and Johannes Lengler. Geometric inhomogeneous random graphs. *Theoretical Computer Science*, 760:35–54, 2019.

15   Elisabetta Candellero and Nikolaos Fountoulakis. Bootstrap percolation and the geometry of complex networks. *Stochastic Processes and their Applications*, 126(1):234–264, 2016.

16   Ashok K. Chandra, Prabhakar Raghavan, Walter L. Ruzzo, Roman Smolensky, and Prasoon Tiwari. The electrical resistance of a graph captures its commute and cover times. *Comput. Complex.*, 6(4):312–340, 1997. `doi:10.1007/BF01270385`.

17   Jordan Chellig, Nikolaos Fountoulakis, and Fiona Skerman. The modularity of random graphs on the hyperbolic plane, 2021. `arXiv:2104.01041`.

18   Paul F. Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In Lance Fortnow and Salil P. Vadhan, editors, *Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, 6-8 June 2011*, pages 273–282. ACM, 2011. `doi:10.1145/1993636.1993674`.

19   Alessandra Cipriani and Michele Salvi. Scale-free percolation mixing time, 2021. `doi:10.48550/ARXIV.2111.05201`.

20   Colin Cooper and Alan M. Frieze. The cover time of sparse random graphs. *Random Struct. Algorithms*, 30(1-2):1–16, 2007. `doi:10.1002/rsa.20151`.

21   Colin Cooper and Alan M. Frieze. The cover time of the preferential attachment graph. *J. Comb. Theory B*, 97(2):269–290, 2007. `doi:10.1016/j.jctb.2006.05.007`.

22   Colin Cooper and Alan M. Frieze. The cover time of the giant component of a random graph. *Random Struct. Algorithms*, 32(4):401–439, 2008. `doi:10.1002/rsa.20201`.

23   Colin Cooper and Alan M. Frieze. The cover time of random geometric graphs. *Random Struct. Algorithms*, 38(3):324–349, 2011. `doi:10.1002/rsa.20320`.

24   Colin Cooper and Alan M. Frieze. Stationary distribution and cover time of random walks on random digraphs. *J. Comb. Theory B*, 102(2):329–362, 2012. `doi:10.1016/j.jctb.2011.11.001`.

25   Colin Cooper, Alan M. Frieze, and Eyal Lubetzky. Cover time of a random graph with a degree sequence II: Allowing vertices of degree two. *Random Struct. Algorithms*, 45(4):627–674, 2014. `doi:10.1002/rsa.20573`.

26   Artur Czumaj, Morteza Monemizadeh, Krzysztof Onak, and Christian Sohler. Planar graphs: Random walks and bipartiteness testing. *Random Struct. Algorithms*, 55(1):104–124, 2019. `doi:10.1002/rsa.20826`.

27   Maria Deijfen, Remco Van der Hofstad, and Gerard Hooghiemstra. Scale-free percolation. *Annales de l'IHP Probabilités et statistiques*, 49(3):817–838, 2013.

28   Jian Ding, James R. Lee, and Yuval Peres. Cover times, blanket times, and majorizing measures. *Ann. of Math. (2)*, 175(3):1409–1471, 2012. `doi:10.4007/annals.2012.175.3.8`.

29   Nikolaos Fountoulakis, Dieter Mitsche, Tobias Müller, and Markus Schepers. Hamilton cycles and perfect matchings in the KPKVB model. *Stochastic Processes and their Applications*, 131:340–358, 2021.

30   Nikolaos Fountoulakis and Tobias Müller. Law of large numbers for the largest component in a hyperbolic model of complex networks. *Ann. Appl. Probab.*, 28(1):607–650, 2018. `doi:10.1214/17-AAP1314`.

31   Nikolaos Fountoulakis, Pim van der Hoorn, Tobias Müller, and Markus Schepers. Clustering in a hyperbolic model of complex networks. *Electron. J. Probab.*, 26:Paper No. 13, 132, 2021. `doi:10.1214/21-ejp583`.

32   Tobias Friedrich and Anton Krohmer. Cliques in hyperbolic random graphs. In *2015 IEEE Conference on Computer Communications (INFOCOM)*, pages 1544–1552. IEEE, 2015.

33   Tobias Friedrich and Anton Krohmer. On the diameter of hyperbolic random graphs. *SIAM J. Discret. Math.*, 32(2):1314–1334, 2018. `doi:10.1137/17M1123961`.

34   David Gans. A new model of the hyperbolic plane. *The American Mathematical Monthly*, 73(3):291–295, 1966.

35   Christos Gkantsidis, Milena Mihail, and Amin Saberi. Hybrid search schemes for unstructured peer-to-peer networks. In *INFOCOM 2005. 24th Annual Joint Conference of the IEEE Computer and Communications Societies, 13-17 March 2005, Miami, FL, USA*, pages 1526–1537. IEEE, 2005. `doi:10.1109/INFCOM.2005.1498436`.

36   Peter Gracar, Markus Heydenreich, Christian Mönch, and Peter Mörters. Recurrence versus transience for weight-dependent random connection models, 2021. `arXiv:1911.04350`.

37   Luca Gugelmann, Konstantinos Panagiotou, and Ueli Peter. Random hyperbolic graphs: Degree sequence and clustering. In *Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Part II*, volume 7392 of *Lecture Notes in Computer Science*, pages 573–585. Springer, 2012. `doi:10.1007/978-3-642-31585-5_51`.

38   Brieuc Guinard and Amos Korman. Tight bounds for the cover times of random walks with heterogeneous step lengths. In Christophe Paul and Markus Bläser, editors, *37th International Symposium on Theoretical Aspects of Computer Science, STACS 2020, March 10-13, 2020, Montpellier, France*, volume 154 of *LIPIcs*, pages 28:1–28:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. `doi:10.4230/LIPIcs.STACS.2020.28`.

39   Markus Heydenreich, Tim Hulshof, and Joost Jorritsma. Structures in supercritical scale-free percolation. *Ann. Appl. Probab.*, 27(4):2569–2604, 2017. `doi:10.1214/16-AAP1270`.

40   Jeremy G. Hoskins, Cameron Musco, Christopher Musco, and Babis Tsourakakis. Inferring networks from random walk-based node similarities. In Samy Bengio, Hanna M. Wallach, Hugo Larochelle, Kristen Grauman, Nicolò Cesa-Bianchi, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 31: Annual Conference on Neural Information Processing Systems 2018, NeurIPS 2018, December 3-8, 2018, Montréal, Canada*, pages 3708–3719, 2018. URL: `https://proceedings.neurips.cc/paper/2018/hash/2f25f6e326adb93c5787175dda209ab6-Abstract.html`.

41   Johan Jonasson. On the cover time for random walks on random graphs. *Comb. Probab. Comput.*, 7(3):265–279, 1998. URL: `http://journals.cambridge.org/action/displayAbstract?aid=46637`.

42   Jeff Kahn, Jeong Han Kim, László Lovász, and Van H. Vu. The cover time, the blanket time, and the Matthews bound. In *41st Annual Symposium on Foundations of Computer Science, FOCS 2000*, pages 467–475. IEEE Computer Society, 2000. `doi:10.1109/SFCS.2000.892134`.

43   Varun Kanade, Frederik Mallmann-Trenn, and Thomas Sauerwald. On coalescence time in graphs: When is coalescing as fast as meeting? In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 956–965. SIAM, 2019. `doi:10.1137/1.9781611975482.59`.

44   David Kempe, Jon M. Kleinberg, and Alan J. Demers. Spatial gossip and resource location protocols. In Jeffrey Scott Vitter, Paul G. Spirakis, and Mihalis Yannakakis, editors, *Proceedings on 33rd Annual ACM Symposium on Theory of Computing, July 6-8, 2001, Heraklion, Crete, Greece*, pages 163–172. ACM, 2001. `doi:10.1145/380752.380796`.

45   Marcos Kiwi, Amitai Linker, and Dieter Mitsche. Tail bounds for detection times in mobile hyperbolic graphs, 2022. `doi:10.48550/ARXIV.2203.00209`.

46 Marcos Kiwi and Dieter Mitsche. A bound for the diameter of random hyperbolic graphs. In *Proceedings of the Twelfth Workshop on Analytic Algorithmics and Combinatorics, ANALCO 2015*, pages 26–39. SIAM, 2015. `doi:10.1137/1.9781611973761.3`.

47 Marcos Kiwi and Dieter Mitsche. Spectral gap of random hyperbolic graphs and related parameters. *Ann. Appl. Probab.*, 28(2):941–989, 2018.

48 Marcos Kiwi and Dieter Mitsche. On the second largest component of random hyperbolic graphs. *SIAM J. Discrete Math.*, 33(4):2200–2217, 2019. `doi:10.1137/18M121201X`.

49 Marcos Kiwi, Markus Schepers, and John Sylvester. Cover and hitting times of hyperbolic random graphs, 2022. `doi:10.48550/ARXIV.2207.06956`.

50 Christoph Koch and Johannes Lengler. Bootstrap percolation on geometric inhomogeneous random graphs. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, volume 55 of *LIPIcs*, pages 147:1–147:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016. `doi:10.4230/LIPIcs.ICALP.2016.147`.

51 Júlia Komjáthy and Bas Lodewijks. Explosion in weighted hyperbolic random graphs and geometric inhomogeneous random graphs. *Stochastic Processes and their Applications*, 130(3):1309–1367, 2020. `doi:10.1016/j.spa.2019.04.014`.

52 D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguñá. Hyperbolic geometry of complex networks. *Phys. Rev. E*, 82(3):036106, 18, 2010. `doi:10.1103/PhysRevE.82.036106`.

53 Akash Kumar, C. Seshadhri, and Andrew Stolman. Finding forbidden minors in sublinear time: A $n^{1/2+o(1)}$-query one-sided tester for minor closed properties on bounded degree graphs. In Mikkel Thorup, editor, *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*, pages 509–520. IEEE Computer Society, 2018. `doi:10.1109/FOCS.2018.00055`.

54 Akash Kumar, C. Seshadhri, and Andrew Stolman. Random walks and forbidden minors II: A poly(d/ε)-query tester for minor-closed properties of bounded degree graphs. In Moses Charikar and Edith Cohen, editors, *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*, pages 559–567. ACM, 2019. `doi:10.1145/3313276.3316330`.

55 G. Last and M. Penrose. *Lectures on the Poisson Process*. IMS Textbooks. Cambridge University Press, 2018.

56 David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009.

57 Huan Li and Zhongzhi Zhang. Kirchhoff index as a measure of edge centrality in weighted networks: Nearly linear time algorithms. In Artur Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 2377–2396. SIAM, 2018. `doi:10.1137/1.9781611975031.153`.

58 Amitai Linker, Dieter Mitsche, Bruno Schapira, and Daniel Valesin. The contact process on random hyperbolic graphs: metastability and critical exponents. *Ann. Probab.*, 49(3):1480–1514, 2021. `doi:10.1214/20-aop1489`.

59 László Lovász. Random walks on graphs: A survey. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 353–397. János Bolyai Math. Soc., Budapest, 1996.

60 Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016. `doi:10.1017/9781316672815`.

61 Aleksander Madry, Damian Straszak, and Jakub Tarnawski. Fast generation of random spanning trees and the effective resistance metric. In Piotr Indyk, editor, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 2019–2036. SIAM, 2015. `doi:10.1137/1.9781611973730.134`.

62 Francesco Manzo, Matteo Quattropani, and Elisabetta Scoppola. A probabilistic proof of Cooper & Frieze's "First visit time lemma". *ALEA Lat. Am. J. Probab. Math. Stat.*, 18(2):1739–1758, 2021. `doi:10.30757/alea.v18-64`.

63 Christine Marshall, Colm O'Riordan, and James Cruickshank. Targeting influential nodes for recovery in bootstrap percolation on hyperbolic networks. In *European Network Intelligence Conference*, pages 3–16. Springer, 2017.

64 Tobias Müller and Merlijn Staps. The diameter of KPKVB random graphs. *Adv. Appl. Probab.*, 51(2):358–377, 2019. `doi:10.1017/apr.2019.23`.

65 Takashi Owada and D Yogeshwaran. Sub-tree counts on hyperbolic random geometric graphs. *To appear in Advances in Applied Probability*, 2022+.

66 José Luis Palacios. Closed-form formulas for Kirchhoff index. *International J. of Quantum Chemistry*, 81(2):135–140, 2001.

67 Stacy Patterson and Bassam Bamieh. Consensus and coherence in fractal networks. *IEEE Trans. Control. Netw. Syst.*, 1(4):338–348, 2014. `doi:10.1109/TCNS.2014.2357552`.

68 Thomas Sauerwald and He Sun. Tight bounds for randomized load balancing on arbitrary network topologies. In *53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012*, pages 341–350. IEEE Computer Society, 2012. `doi:10.1109/FOCS.2012.86`.

69 Daniel A. Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. *SIAM J. Comput.*, 40(6):1913–1926, 2011. `doi:10.1137/080734029`.

70 John Sylvester. Random walk hitting times and effective resistance in sparsely connected Erdős-Rényi random graphs. *J. Graph Theory*, 96(1):44–84, 2021. `doi:10.1002/jgt.22551`.

71 Prasad Tetali. Random walks and the effective resistance of networks. *J. Theor. Probab.*, 4(1):101–109, 1991. `doi:10.1007/BF01046996`.

72 Ulrike von Luxburg, Agnes Radl, and Matthias Hein. Hitting and commute times in large random neighborhood graphs. *J. Mach. Learn. Res.*, 15(52):1751–1798, 2014.

73 Felix Ming Fai Wong, Zhenming Liu, and Mung Chiang. On the efficiency of social recommender networks. *IEEE/ACM Trans. Netw.*, 24(4):2512–2524, 2016. `doi:10.1109/TNET.2015.2475616`.