On the Complexity of Rainbow Vertex Colouring Diametral Path Graphs

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Abstract

Given a graph and a colouring of its vertices, a rainbow vertex path is a path between two vertices such that all the internal nodes of the path are coloured distinctly. A graph is rainbow vertex-connected if between every pair of vertices in the graph there exists a rainbow vertex path. We study the problem of deciding whether a given graph can be coloured using k or less colours such that it is rainbow vertex-connected. Note that every graph G needs at least $\text{diam}(G) - 1$ colours to be rainbow vertex connected.

Heggernes et al. [MFCS, 2018] conjectured that if G is a graph in which every induced subgraph has a dominating diametral path, then G can always be rainbow vertex coloured with $\text{diam}(G) - 1$ many colours. In this work, we confirm their conjecture for chordal, bipartite and claw-free diametral path graphs. We complement these results by showing the conjecture does not hold if the condition on *every* induced subgraph is dropped. In fact we show that, in this case, even though $\text{diam}(G)$ many colours are always enough, it is NP-complete to determine whether a graph with a dominating diametral path of length three can be rainbow vertex coloured with two colours.

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1 Introduction

Rainbow colouring is a concept that was first introduced by Chartrand et al. [2] in 2008, to study connectivity in edge coloured graphs. A rainbow colouring of a graph is a colouring of its edges such that, between any two vertices, there exists a path no two edges of which have the same colour. The study of this topic garnered a lot of attention and led Krivelevich and Yuster [7] to define a variation which applied to vertex colourings, instead of edge colourings.

In this variant, we are concerned with *rainbow vertex paths*, which are paths between two vertices in which all the *internal* nodes have different colours. We say a graph is *rainbow vertex connected* if between every pair of vertices there exists a rainbow vertex path. Resulting from this we have the decision problem Rainbow Vertex Colouring (RVC) which takes as input a graph G and some integer k with the task to decide if G can be coloured using k or less colours such that it is rainbow vertex-connected. The minimum number of colours needed to make a graph G rainbow vertex-connected is denoted by $\mathbf{rvc}(G)$, and it is called the rainbow vertex-connection number of G.

It is easy to see that any rainbow vertex colouring needs at least $\text{diam}(G) - 1$ many colours, where $\text{diam}(G)$ denotes the diameter of the graph G. On the other hand, the size of a minimum connected dominating set provides an upper bound on $\mathbf{rvc}$. Indeed, if we give each vertex of a connected dominating set D a distinct colour, and an arbitrary colour to the remaining vertices, we obtain a rainbow vertex colouring of G, as between any two pairs of vertices there is a path whose all internal vertices lie in D. This relation led Heggernes et

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al. [6] to propose the following conjecture on diametral path graphs. A *diametral path* is a shortest path whose length is equal to the diameter of the graph. A graph is a *diametral path graph* if every induced subgraph contains a dominating diametral path.

▶ **Conjecture 1** (Heggernes et al. [6]). *If G is a diametral path graph, then* $\mathbf{rvc}(G) = \operatorname{diam}(G) - 1$.

To support their conjecture, they showed it to be true for two subclasses of diametral path graphs, namely bipartite permutation graphs and interval graphs. Later on, the above conjecture was also shown to be true for permutation graphs [11].

Our results

We show the conjecture to be true for chordal, bipartite and claw-free diametral path graphs. In fact, we show a slightly more general result, as our algorithms require only the input graph to have a dominating diametral path, instead of every induced subgraph.

▶ **Theorem 2.** *If G is a chordal, bipartite or claw-free graph that contains a dominating diametral path, then* $\mathbf{rvc}(G) = \operatorname{diam}(G) - 1$. *Moreover, a rainbow vertex colouring can be computed in time that is polynomial on the size of G.*

Note that our results for chordal graphs, bipartite graphs and claw-free graphs with a dominating diametral path generalize the results for interval graphs, bipartite permutation graphs and unit interval graphs, respectively, obtained by Heggernes et al. [6]. It is also interesting to notice that these results contrast with the fact that Rainbow Vertex Colouring is NP-complete on bipartite graphs for any fixed $k \geq 3$ [6, 9] and on chordal graphs for any fixed $k \geq 2$ [6].

We complement these results by showing that Conjecture 1 does not hold if we drop the condition on every induced subgraph. We give an example of a graph with a dominating diametral path that requires $\operatorname{diam}(G)$ many colours to be rainbow vertex connected. Moreover, we show that, in this graph class, even though $\operatorname{diam}(G)$ many colours always suffice, already when $\operatorname{diam}(G) = 3$, it is NP-complete to decide whether $\mathbf{rvc}(G) = 2$.

▶ **Theorem 3.** *If G is a graph that has a dominating diametral path, then* $\mathbf{rvc}(G) \leq \operatorname{diam}(G)$.

▶ **Theorem 4.** *Let G be a graph with a dominating diametral path of length three. It is* NP*-complete to decide whether* $\mathbf{rvc}(G) \leq 2$.

A summary of these results is presented in Figure 1. In the figure we shorten dominating diametral path with the abbreviation DDP.

A variant of the RVC problem concerned with shortest paths was introduced by Li et al. [10]. We say a graph is *strongly rainbow vertex-connected* if between every pair of vertices there exists a shortest path which is also rainbow vertex path. The resulting decision problem is called Strong Rainbow Vertex Colouring (SRVC) and the analogous parameter is denoted by $\mathbf{srvc}(G)$. Every strong rainbow vertex colouring is also a rainbow vertex colouring, thus $\mathbf{rvc}(G) \leq \mathbf{srvc}(G)$. Heggernes et al. [6] showed that if G is a unit interval graph, then $\mathbf{srvc}(G) = \operatorname{diam}(G) - 1$. Their result for interval graphs, however, applied only to the standard variant of the problem. In this work we strengthen their result by showing the following.

▶ **Theorem 5.** *If G is an interval graph, then* $\mathbf{srvc}(G) = \operatorname{diam}(G) - 1$ *and an optimal strong rainbow vertex colouring can be found in time that is linear in the size of G.*

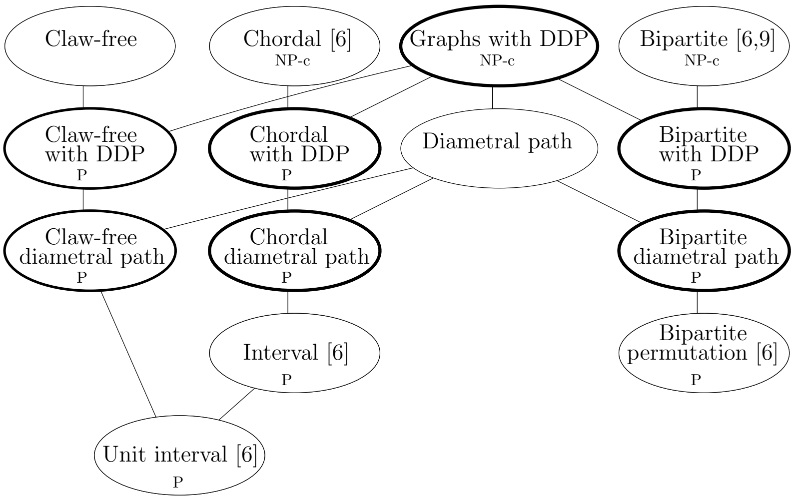

Figure 1 An overview of the complexity of RVC restricted to the graph classes considered here. A thick line represents results obtained in this work. A line connecting two graph classes means the lower one is a subclass of the upper one.

This paper is organized is follows. In Section 2 we define the notation and define basic concepts that will be used throughout the text. In Section 3 we prove Theorems 3 and 4, and provide the example of a graph with a dominating diametral path that requires $\text{diam}(G)$ many colours in any rainbow vertex colouring. The proof of Theorem 2 is split in Sections 4, 5 and 6. Finally, in Section 7, we consider interval graphs and prove Theorem 5. Due to space constraints, statements marked with ♣ have their proofs deferred to the full version of this work.

Other related work

As mentioned above, RVC is NP-complete for any fixed $k \geq 2$ on chordal graphs (in fact, even on split graphs) and any fixed $k \geq 3$ on bipartite graphs [6]. These results also apply to SRVC. In terms of positive results, both problems are polynomial-time solvable on bipartite permutation graphs, unit interval graphs, and block graphs [6], on planar graphs for every fixed k [8, 5], and on split strongly chordal graphs [11]. RVC was furthermore showed to be polynomial-time solvable on interval graphs [6] and on permutation graphs and powers of trees [11].

Although rainbow vertex colouring may seem abstract and almost purely theoretical it does have some quite interesting applications to real world problems especially in terms of encryption and data security. One example of this mentioned in [4] can be seen in onion routing, a technique for anonymously browsing online. In onion routing the goal is to prevent an adversary from knowing what sites you are connecting to. The way this is achieved is by sending the message through a path of intermediaries before accessing the server you requested. This message will be sent using multiple layers of encryption [12], where each node of the path can only decrypt a single layer of this encryption. Assigning decryption keys to the network draws parallels to rainbow vertex colouring.

2 Preliminaries

For basic graph terminology we refer the reader to [1]. We denote a set of consecutive integers from 1 to k as $[k]$. Given some path $P = ux_1x_2...x_mv$ the vertices $x_1x_2...x_m$ are called the *internal vertices* of P. A path is a rainbow vertex path if for every pair of internal vertices x_i, x_j they are coloured such that $c(x_i) \neq c(x_j)$. For some graph G if $\text{diam}(G) = 2$, then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = 1$. The reason each node of the G can have the same colour is that between every pair of vertices there will always exist some path such that there is only one internal node.

Breadth-first search (BFS) is an algorithm for traversing graph structures. The algorithm works by exploring the graph in layers, starting from some arbitrary root node v_0. The layers are indicated L_i, where L_0 is the layer of the root vertex. Thus all vertices in layer $v_i \in L_i$ of the BFS-structure have a distance of i to v_0. Throughout this text we will often, for the sake of convenience, indicate a vertex v belonging to some layer L_i of a BFS-structure with v_i.

A *diametral path* is a shortest path whose length is equal to $\text{diam}(G)$. A dominating diametral path is a path $P = x...y$ such that $\text{dist}(x, y) = \text{diam}(G)$ and the set $V(P) = \{x, ..., y\}$ is a dominating set of G.

We say a graph is a *diametral path graph* if every connected induced subgraph has a dominating diametral path. It has been shown that the existence of a dominating diametral path in a graph can be checked and the path itself can be computed in $\mathcal{O}(n^3m)$ time [3]. Throughout this text we will often run a BFS-search on one end of the dominating diametral path. We denote the vertices of the diametral path p_i, where p_i belongs to layer L_i of the BFS-tree. We will also often refer to vertices as being *dominated to the right*, *dominated to the left* or *dominated in layer*. For some vertex $v \in L_i$ we say it is dominated to the right if $(v, p_{i+1}) \in E(G)$, we say it is dominated to the left if $(v, p_{i-1}) \in E(G)$ and we say it is dominated in layer if $(v, p_i) \in E(G)$.

A graph is *chordal* if it contains no induced cycle of a length which is greater than three. A well known subclass of the chordal graph is that of *interval graphs.* A graph G is an interval graph if its vertices can be associated with a set of intervals in the real line in such a way that two vertices are adjacent if and only if the corresponding intervals intersect. The set of intervals associated with a graph G is called an *interval model for G*. Interval graphs is a subclass of the chordal graphs with a dominating diametral path.

If the vertices of graph can be divided into two disjoint subsets such that each of the subsets are independent the graph is said to be a *bipartite graph.* A *claw* is a bipartite graph on four vertices a, b, c and d, with edges (a, b), (a, c) and (a, d). A graph is *claw-free* if it contains no induced *claw.* When referring to claws throughout this text we will write them $\{abcd\}$ such that $\{b, c, d\} \subset N(a)$, or said in another way the center of the claw will be written first.

Basic properties of graphs with a dominating diametral path

We now present some basic properties of the layers of a BFS performed on a graph with a dominating diametral path. These properties will come in useful in all the algorithms presented in this work. Let G be a graph containing a dominating diametral path to which a BFS is performed on one end of the diametral path. We let $k = \text{diam}(G)$ thus the BFS will have $k + 1$ layers. For each layer L_i, for $0 \leq i \leq k$, we let p_i indicate the vertex of the dominating diametral path in layer i, where p_0 is the root vertex. The following is an easy property of a BFS.

▶ **Observation 6.** *A vertex $x \in L_i$ can only have neighbours in L_{i-1}, L_i and L_{i+1}.*

We will now introduce two new definitions which are used repeatedly throughout the text. Recall that for convenience of notation we indicate vertices by the layer they belong to, i.e. a vertex $v \in L_i$ is denoted v_i.

▶ **Definition 7.** *We say a path between $x \in L_i$ and $y \in L_j$ is a direct path if the length of the path is $j - i$, that is, it intersects every layer from L_i to L_j only once.*

▶ **Definition 8.** *We say a path between two vertices $x \in L_i$ and $y \in L_j$ zig-zags in L_l if it is of the following structure: $x ... v_{l-1} v_l v'_{l-1} v'_l ... y$, where $v_{l-1} \neq v'_{l-1}$, $v_l \neq v'_l$ and the paths x to v_l and v'_l to y are direct paths.*

This next observation comes naturally as a result of p_0 being the root vertex of the BFS.

▶ **Observation 9.** *The root vertex p_0 of the BFS will have a direct path to every vertex in the graph.*

The following claims concerns colourings in which the vertices of the diametral path are coloured in a specific way.

▷ Claim 10 (♣). Let $c : V \to [k-1]$ be a colouring for G in which each vertex p_i of the diametral path is coloured $c(p_i) = i$, for $1 \leq i < k$. Furthermore, $c(p_0) = k - 1$ and $c(p_k) = 1$. The vertices not in the diametral path are coloured with arbitrary colours in $[k-1]$. For pairs of vertices $x \in L_i$ and $y \in L_j$, where $2 < i \leq j \leq k$, a rainbow path can always be found between x and y whose all internal vertices are contained in the diametral path.

The consequence of this claim is that when looking at pairs of vertices $u \in L_i$ and $v \in L_j$, where $0 \leq i \leq j \leq k$ if the colouring of the diametral path is equal to c then we only have to examine cases when $i < 3$. This next claim ensures that when $i = j$ for two vertices we only have to examine the case when the diameter of the graph is three.

▷ Claim 11 (♣). Let $c : V \to [k-1]$ be a colouring for G in which each vertex p_i of the diametral path is coloured $c(p_i) = i$, for $1 \leq i < k$. Furthermore, $c(p_0) = k - 1$ and $c(p_k) = 1$. The vertices not in the diametral path are coloured with arbitrary colours in $[k-1]$. For a pair of vertices $x, y \in L_i$ in G there will always be a rainbow path connecting them, unless $\text{diam}(G) = 3$.

3 Graphs with a dominating diametral path

In this section, we first prove Theorem 3, stating that $\text{diam}(G)$ many colours are always enough to rainbow vertex colour a graph G that has a dominating diametral path. We then proceed to give an example of one such graph that in fact *needs* $\text{diam}(G)$ many colours to be rainbow vertex connected. To round off this section, we present the proof of Theorem 4, stating that even when $\text{diam}(G) = 3$, testing whether two colours are enough to make G rainbow vertex connected is an NP-complete problem.

▶ **Theorem 3.** *If G is a graph that has a dominating diametral path, then $\mathbf{rvc}(G) \leq \text{diam}(G)$.*

Sketch of the proof (♣). To construct the colouring $c : V \to [k]$, we run a BFS on some end-point of the diametral path. We let k equal the number of layers of the tree excluding p_0 and let p_i denote the vertex of the diametral in layer L_i. We assign the colouring for $v \in L_i$: $c(v) = k$ if $i = 0$ or $i = k$; and $c(v) = i$ otherwise. For a proof that G is rainbow vertex coloured under c we refer the reader to the full version of this work. ◀

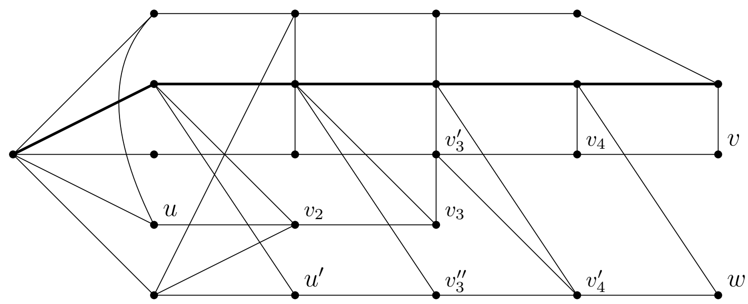

Figure 2 The graph has a diameter of 5, while also needing 5 colours to be rainbow vertex-coloured. The dominating diametral path is highlighted.

We now give an example of a graph with a dominating diametral path that needs $\text{diam}(G)$ many colours to be rainbow vertex connected. The graph is depicted in Figure 2. It should be noted that this does not disprove the conjecture as the example graph in Figure 2 is not a diametral path graph. It is rather a graph *containing a dominating diametral path.* The distinction can be seen by observing that the graph in the figure contains two induced cycles of a length greater than six, which is a forbidden structure in diametral path graphs [3].

▶ **Proposition 12.** *If G is the graph shown in Figure 2, then* $\mathbf{rvc}(G) \geq \text{diam}(G)$

Proof. Note that the diameter of the graph in Figure 2 is five. Assume for a contradiction that G can be rainbow vertex coloured with four colours. First we highlight the three paths of interest in the graph: $uv_2v_3v_3'v_4v$, $uv_2v_3v_3'v_4'w$ and $u'v_3''v_4'v_3'v_4$. What makes these paths interesting for our purposes is that they are the only paths of length less than or equal to the diameter between their two endpoints. Thus, if there exists a rainbow vertex colouring for the graph using 4 colours all of these paths must be rainbow coloured simultaneously. We begin by giving the uv-path an arbitrary rainbow colouring, the internal vertices $v_2v_3v_3'v_4$ receive the colouring $1, 2, 3, 4$ such that $c(v_2) = 1$, $c(v_3) = 2$, $c(v_3') = 3$ and $c(v_4) = 4$. For the uw-path we see that v_4' must be coloured 4 for it to be rainbow, but this means that the $u'v$-path will not be rainbow as there are two nodes of its internal vertices which are coloured 4. Thus we see that there is no way for all of these paths to be rainbow at the same time. ◀

To conclude this section, we prove the following.

▶ **Theorem 4.** *Let G be a graph with a dominating diametral path of length three. It is* NP*-complete to decide whether* $\mathbf{rvc}(G) \leq 2$.

Proof. We give a reduction from Hypergraph 2-Colouring. In this problem, we are given a hypergraph $\mathcal{H}$ and we want to colour its vertices in such a way that no hyperedge is monochromatic. The Hypergraph Colouring problem is known to be NP-hard for any fixed number of colours $k \geq 2$.

Given an instance $\mathcal{H}$ of Hypergraph 2-Colouring with $V(\mathcal{H}) = [n]$ and $E(\mathcal{H}) = \{e_1, \ldots, e_m\}$, we construct an instance (G, k) of Rainbow Vertex Colouring, with $k = 2$, as follows. The graph G has four vertices $p_0, \ldots, p_3$ forming an induced path; two pendant vertices a_1 and a_2 attached to p_1 and p_2, respectively; two independent sets S_1 and S_2, where each vertex of S_i corresponds to a hyperedge of $\mathcal{H}$; and a clique C, each vertex of which corresponds to an element of $V(\mathcal{H})$.

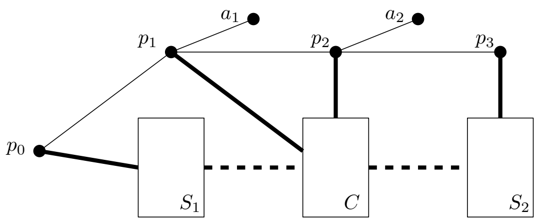

Figure 3 The graph constructed in the proof of Theorem 4. Thick lines indicate a vertex is complete to the set. Dashed lines indicate the two sets are attached.

We now give some definitions to describe the edge set of G. We denote by s_{iq} the vertex of S_i corresponding to the hyperedge e_q of $\mathcal{H}$, and by v_q the vertex of C corresponding to the element $q \in V(\mathcal{H})$. We say a set S_i is *attached* to C if for every $1 \leq q \leq m$, s_{iq} is adjacent to all the vertices of C corresponding to elements of e_q. That is, $(s_{iq}, v_\ell) \in E(G)$ if and only if $\ell \in e_q$ in $\mathcal{H}$. In words, the adjacencies between S_i and C encode the structure of $\mathcal{H}$.

The edges of G can then be described as follows: $p_0p_1 \dots p_3$ is an induced path; $(p_i, a_i) \in E(G)$ for $i \in \{1, 2\}$; C is a clique; S_1 and S_2 are attached to C; p_0 dominates S_1; p_1 and p_2 dominate C; and p_3 dominates S_2.

See Figure 3 for an illustration of the structure of G.

▷ Claim 13 (♣). If $\mathcal{H}$ is 2-colourable, then G has rainbow vertex colouring with two colours.

▷ Claim 14. If G has a rainbow vertex colouring with two colours, then $\mathcal{H}$ is 2-colourable.

Proof. Let c be a rainbow colouring for G with two colours. Since p_1 and p_2 are cut vertices in G, they must have distinct colours under c. So we may assume $c(p_1) = 1$ and $c(p_2) = 2$. To retrieve a 2-colouring for $\mathcal{H}$ from c we consider the restriction of c to C. That is, we define a 2-colouring for $\mathcal{H}$ as follows: for every $q \in V(\mathcal{H})$, $\phi(q) = c(v_q)$. To see that $\mathcal{H}$ has no monochromatic edge of colour 1 under ϕ, consider rainbow paths between a_1 and the vertices of S_2. Since $N(S_2) = C \cup \{p_3\}$ and $\text{dist}(a_1, p_3) = 3$, every such rainbow path has to use a vertex from C. Note also that this path must contain p_1, and $c(p_1) = 1$. If there exists a monochromatic hyperedge e_j with colour 1 under ϕ, then there is no rainbow path between s_{2j} and a_1, a contradiction. Hence ϕ has no monochromatic hyperedge of colour 1. By a similar argument, we see that $\mathcal{H}$ has no monochromatic edge of colour 2. Indeed, since $N(S_1) = C \cup \{p_0\}$ and $\text{dist}(p_0, a_2) = 3$, any path between a vertex of S_1 and a_2 must contain p_2 and a vertex from C. Since $c(p_2) = 2$ and S_1 is attached to C, the claim follows. ◁

To conclude the proof of the theorem, note that $\text{diam}(G) = 3$ and that $p_0p_1p_2p_3$ is a dominating diametral path in G. ◀

4 Bipartite graphs with a dominating diametral path

In this section, we investigate the complexity of RVC on bipartite graphs with a dominating diametral path, and prove Conjecture 1 for this graph class. Recall that for bipartite graphs RVC has already been proven to be NP-complete for any fixed $k \geq 3$. The result we achieve for this graph class is a direct improvement on the earlier result of Heggernes et al. [6] on bipartite permutation graphs, a known subclass of the bipartite graphs with a dominating diametral path.

▶ **Theorem 15** (♣). *If G is a bipartite graph with a dominating diametral path, then* $\mathbf{rvc}(G) = \text{diam}(G) - 1$ *and the corresponding rainbow vertex colouring can be found in time that is polynomial in the size of G.*

5 Chordal graphs with a dominating diametral path

In a similar vein to the previous section, we are now going to consider the complexity of RVC on chordal graphs with a dominating diametral path. In particular, we show that Conjecture 1 is true for chordal diametral path graphs. Recall that RVC is NP-complete on chordal graphs for $k \geq 2$. Our algorithm for chordal graphs with a dominating diametral path generalizes the existing result for interval graphs of Heggernes et al. [6]. We also point out that an example of a graph that is a chordal diametral path graph and *not* an interval graph is the 3-sun, a graph on vertex set $\{a, b, c, d, e, f\}$ in which $\{a, b, c\}$, $\{b, d, e\}$ and $\{c, d, f\}$ induce triangles.

▶ **Theorem 16.** *If G is a chordal graph with a dominating diametral path, then* $\mathbf{rvc}(G) = \text{diam}(G) - 1$ *and the corresponding rainbow vertex colouring can be found in time that is polynomial in the size of G.*

Proof. Let $G = (V, E)$ be a chordal graph with a dominating diametral path. We run a BFS search on one of the endpoints of the diametral path which we will call p_0. We let k denote the number of layers of the BFS tree and let p_i be the vertex of the diametral path in L_i where $0 \leq i \leq k$. To construct a rainbow colouring $c : V \to [k-1]$ for G we assign the colouring for $v \in L_i$:

$$c(v) = \begin{cases} k-1 & \text{if } i = 0 \\ 1 & \text{if } i = k \text{ or if } i = 2 \text{ and for some } v_2 \in (N(v) \cap L_2) : \text{dist}(v_2, p_k) = k \\ i & \text{otherwise.} \end{cases}$$

Once a BFS is performed on a chordal graph with a dominating diametral path, we will see that some special edges must exist within a layer, otherwise we would have long induced cycles. We will formulate some of these properties before delving into the details of the proof as they will be of great use.

▷ Claim 17 (♣). If we have a path $P_3 = abc$ such that $a, c \in L_i$ and $b \in L_{i+1}$ then the edge (a, c) must be in $E(G)$.

▶ **Observation 18** (♣). *A vertex $x \in L_i$ in G cannot only be dominated to the right.*

▷ Claim 19 (♣). If we have some $P_4 = abcd$ where $a, d \in L_i$ and $b, c \in L_{i+1}$, then the edge (a, d) must be in $E(G)$.

▷ Claim 20 (♣). If a vertex $x \in L_i$, where $x \neq p_i$, has a neighbour $y \in L_{i+1}$ then x must be dominated in layer.

▶ **Observation 21** (♣). *A vertex $x \in L_2$ with a direct path to p_k will always be coloured 2.*

▷ Claim 22 (♣). There will never exist a direct path between some vertex $x \in L_2$ coloured 1 and some vertex $y \in L_i$, $2 < i \leq k$, where y is only dominated in layer.

▷ Claim 23 (♣). If for a pair of vertices $x \in L_i$ and $y \in L_j$ where $j > i$ and $\text{dist}(x, y) = j - i + 1$, then there will always exist some path between x and y such that it repeats layers in L_i.

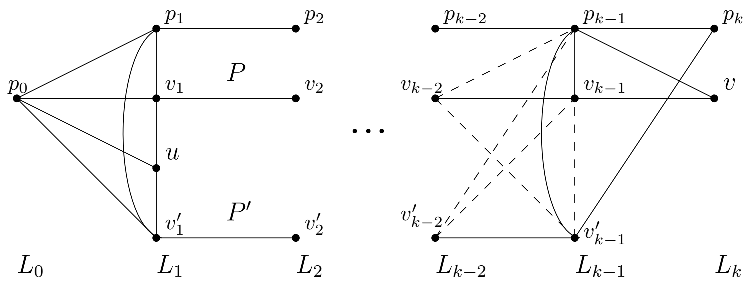

Figure 4 Illustration of the Case 2 in the proof of Theorem 16.

To argue the correctness of the colouring let us look at a pair of arbitrary vertices $u \in L_i$ and $v \in L_j$. Firstly we consider the case when $i = j$. By Claim 11 we know there will be a rainbow path for all cases of G except for when $\text{diam}(G) = 3$. For this instance there are three possibilities for i and j.

1. $i = j = 1$. The path up_0v will always exist and is rainbow.
2. $i = j = 2$. Because of Observation 18 we know u and v must be dominated by either p_1, p_2 or both. If they are dominated by the same vertex there is a rainbow path, while if they are dominated by different vertices either path up_1p_2v and up_2p_1v is rainbow.
3. $i = j = 3$. The vertices are either dominated by p_2, p_3 or both. If they are dominated by the same vertex there is a rainbow path, and if they are dominated by different vertices either path up_2p_3v and up_3p_2v is rainbow.

We will now examine all cases of u and v such that $0 \leq i < j \leq k$. First we can observe by Claim 10 that when $i \geq 3$ we can always use the dominating path. We will look at the cases where the dominating path cannot be arbitrarily used, i.e. when $i < 3$ more closely.

Case 1. $i = 0$**.** The only case we cannot use the dominating path is if $j = k$ and v is only dominated in layer. We know by Observation 9 that p_0 must have a direct path to every vertex including v. By Claim 22, this path will never intersect some vertex coloured 1 in L_2 and thus is rainbow.

Case 2. $i = 1$**.** If u is dominated to the right the diametral path will be rainbow for every case of v. This leaves us two cases of u.

1. u is dominated in layer. The only case of v we have to look at is when $j = k$ while v is only dominated in layer. There are two subcases we have to examine for this case.
   a. $\text{dist}(u, v) = k - 1$. Since the path between u and v is direct and v is only dominated in layer we know by Claim 22 that no internal vertex is coloured 1 and thus the path is rainbow.
   b. $\text{dist}(u, v) = k$. Using Claim 23 we know there will be some path repeating in L_1 and from there go directly to v. Since the path from L_1 to v is direct we can again use Claim 22 and thus we know that all internal vertices are distinctly coloured.
2. u is only dominated by p_0. For this case in both instances when $j = k - 1$ and $j = k$ we cannot arbitrarily use the diametral path.
   a. $j = k - 1$. If v is dominated to the left we can use the diametral path, thus let us assume v is only dominated in layer. From p_0 there must exist some direct path to v. We can thus go from u to p_0 and then traverse the direct path to v. By Claim 22, since v is dominated in layer, the vertex in L_2 of the direct path will never be coloured 1.

b. $j = k$. First we observe that the combination of u not being dominated in layer and Claim 20 means that $\text{dist}(u, v) = k$. By Claim 23 we therefore know there must exist some path $P = uv_1v_2...v_{k-1}v$, where $v_1 \neq p_1$. If v is only dominated in layer, Claim 22 states that v_2 must be coloured 2 and thus P is rainbow. Therefore let us assume v is dominated to the left. As argued before, the fact that u is not dominated in layer and Claim 20, imply that by Claim 23 we can identify that the following path $P' = uv'_1v'_2...v'_{k-1}p_k$ also must be in G. By Observation 21 v'_2 is not coloured 1 as this vertex has a direct path to p_k. If for some $i < k$, $v'_i = p_i$ then we have the following rainbow path: $uv'_1...v'_{i-1}p_i...p_{k-1}v$, thus let us assume for all $i < k$, $v'_i \neq p_i$. Using Claim 20 we know both v'_{k-1} and v_{k-1} must be dominated in layer. With P and P' we can as a result see the formation of a large cycle starting at u and meeting at p_{k-1}. In particular in L_{k-1} we have a P_3 on the vetices $v'_{k-1}p_{k-1}v_{k-1}$. We let P'' denote the shortest path between v_{k-1} and v'_{k-1} in $G' = G[\{u\} \cup \{v_1, v'_1\} \cup ... \cup \{v_{k-2}, v'_{k-2}\} \cup \{v_{k-1}, v'_{k-1}\}]$. If $|E(P'')| = 2$ then either v_{k-2} or v'_{k-2} is an internal node of P''. If v_{k-2} is the internal node this means the edge (v_{k-2}, v'_{k-1}) is in $E(G)$, which means there is a direct path from v_2 to p_k and therefore v_2 would be coloured 2, making P a rainbow path. We therefore check for when v'_{k-2} is the internal node. This means that (v'_{k-2}, v_{k-1}) is an edge in $E(G)$, but this results in the rainbow path: $uv'_2...v'_{k-2}v_{k-1}v$. Since $|E(P'')| = 2$ always results in a rainbow path between u and v we check for when $|E(P'')| = 3$. This means that (v_{k-2}, v'_{k-2}) is an edge in $E(G)$. We already established earlier that neither edge (v'_{k-2}, v_{k-1}) or (v_{k-2}, v'_{k-1}) can be in $E(G)$. If (v_{k-2}, p_{k-1}) is in the graph then, again, v_2 will have a direct path to p_k – resulting in v_2 being coloured 2. If (v'_{k-2}, p_{k-1}) is an edge in the graph then there is a rainbow path $uv'_1...v'_{k-2}p_{k-1}v$ between u and v, thus we can omit these two edges. As we currently have a cycle $v_{k-2}v_{k-1}p_{k-1}v'_{k-1}v'_{k-2}$ and no other edges can be added to G' in these layers we see that for $|E(P'')| > 3$ we will find an even longer induced cycle. Therefore $|E(P'')| = 1$ and by applying Claim 19 on the $P_4 = v_{k-2}v_{k-1}v'_{k-1}v'_{k-2}$ we conclude (v_{k-2}, v'_{k-2}) is an edge in G. However as argued above, (v_{k-2}, v'_{k-1}) and (v'_{k-2}, v_{k-1}) are not in G. We then have an induced cycle of length four in G, a contradiction since G is chordal. All the potential edges in the cycle is shown in Figure 4.

The proof of **Case 3**, that is, when $i = 2$, is deferred to the full version of this work.

This proves c is indeed a rainbow colouring for G with $\text{diam}(G) - 1$ colours. Finding the root vertex of the dominating diametral path is a polynomial time operation while the BFS takes linear time. Checking if nodes in L_2 must be coloured 1 is done by checking for each vertex in L_2 their distance to p_k. This is a $\mathcal{O}(n^2)$ time operation, thus when summarizing all these times together we end up with a polynomial time algorithm. ◀

6 Claw-free graphs with a dominating diametral path

In this section, we give a polynomial-time algorithm to optimally rainbow vertex colour claw-free graphs with a dominating diametral path. In particular, this proves Conjecture 1 for claw-free diametral path graphs. This result also improves on the previous algorithm for unit interval graphs [6].

▶ **Theorem 24.** *If G is a claw-free graph with a dominating diametral path, then $\mathbf{rvc}(G) = \text{diam}(G) - 1$ and the corresponding rainbow vertex colouring can be found in time that is polyonmial in the size of G.*

Proof. Let G be a claw-free diametral path graph. We run a BFS on one of the ends of the diametral path. We denote this vertex p_0 and we let k denote the number of layers of the BFS-tree excluding p_0. The vertex of the diametral path in each layer we denote p_i, $1 \leq i \leq k$. To construct a colouring $c : V \to [k-1]$ on G for $v \in L_i$ we assign the colouring:

$$c(v) = \begin{cases} k-1 & \text{if } i = 0 \\ 1 & \text{if } i = k \text{ or if } N(v) \cap L_{i+1} = \emptyset \\ i & \text{otherwise.} \end{cases}$$

Before arguing the correctness of our colouring we make the following observations.

▶ **Observation 25** (♣). *A direct path between vertices $x \in L_i$ and $y \in L_j$, where $j > i$, cannot intersect some vertex $z \in L_l$ coloured 1, $0 < i \leq l < j$.*

▷ Claim 26 (♣). If a vertex $x \in L_i$, with $i > 1$, is dominated to the left, then x is also dominated in layer.

To argue the correctness of the colouring we will look at a pair of arbitrary vertices $u \in L_i$ and $v \in L_j$. We start by looking at the case where $i = j$. By Claim 11 we know that for all cases except for when $\text{diam}(G) = 3$ there will be a rainbow path between u and v. There are three possible cases for i and j:

1. $i = j = 1$. Both u and v are connected to p_0, leading to the rainbow path up_0v.
2. $i = j = 2$. Claim 26 states that neither u nor v can only be dominated to the left, which means they are either dominated by p_2, p_3 or both. If the vertices are dominated by the same vertex there is a rainbow path, and for the case they are dominated by different vertices either path up_2p_3v and up_3p_2v is rainbow.
3. $i = j = 3$. Since u and v are in layer L_3 they cannot be dominated to the right. Claim 26 tells us u and v cannot only be dominated left so we therefore know that both u and v must be dominated in layer. We, as a result, have the rainbow path up_3v.

For the rest of the proof we will assume $0 \leq i < j \leq k$. For cases when $i \geq 3$ we know by Claim 10 that there exists a rainbow path between u and v. When $i = 0$ we know by Observation 9 that there must always be a direct path to v, and this path will never intersect some vertex coloured 1, other than in L_1, as this would by Observation 25 imply the path not being direct. When $i = 2$ we know from Claim 26 that u must be dominated in layer and thus for all cases of j the diametral path will have a rainbow path to v. We are thus left with only one case to examine, which is when $i = 1$ while u is not dominated to the right.

1. $j = k - 1$. Consider the path using the edge (u, p_0) and then a direct path between p_0 and v. The direct path between p_0 and v will by Observation 25 never intersect a vertex coloured 1 in layers other than L_1 and thus the path is rainbow.
2. $j = k$. If $\text{dist}(u, v) = k - 1$ then the path goes directly to v and by Observation 25 this path will not intersect some vertex coloured 1, thus it is rainbow. If $\text{dist}(u, v) = k$ the path must repeat layers at some point. If the repetition of layers happens in L_i or L_j the path will be rainbow. Therefore let us assume this repetition happens in some layer L_l where $i < l < j$. We have the following path: $P = u...v_{l-1}v_lv'_lv'_{l+1}...v$. If v_l has no neighbour in L_{l+1} it will be coloured 1, making P rainbow, thus let us assume P has some neighbour $v_{l+1} \in L_{l+1}$. If $(v_{l-1}, v'_l) \notin E(G)$ and $(v'_l, v_{l+1}) \notin E(G)$ then $\{v_lv_{l-1}v'_lv_{l+1}\}$ induces a claw in G. As a result, either (v_{l-1}, v'_l) or (v'_l, v_{l+1}) must be an edge in $E(G)$. The former case implies a direct path $u...v_{l-1}v'_lv'_{l+1}...v$ between u and v, thus let us assume only $(v'_l, v_{l+1}) \in E(G)$. Since v'_l must have some neighbour v'_{l-1} in L_{i-1}

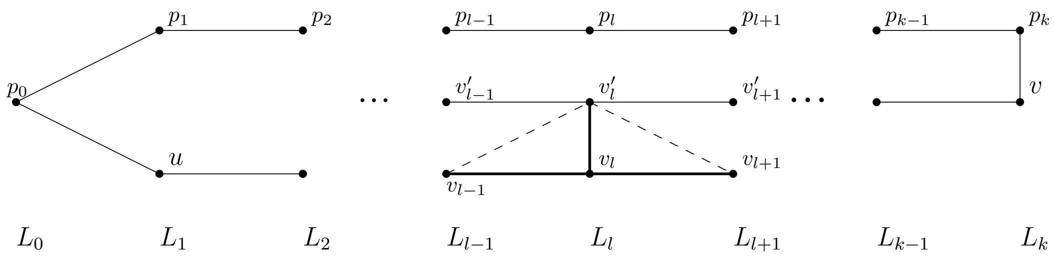

Figure 5 Scenario where the path repeats in layer L_l. If v_l has no neighbours in L_{l+1} it is coloured 1, otherwise one of two possible edges (dashed lines) must be in the graph.

another claw becomes apparent: $\{v'_l v'_{l-1} v'_{l+1} v_{l+1}\}$. In a similar argument to the proof of Claim 26 we have three possible edges to prevent an induced claw, but only one, as seen by Observation 6 is allowed within the BFS-structure – thus (v_{l+1}, v'_{l+1}) must be an edge within $E(G)$. The inclusion of this edge means the repetition of layers can be shifted one step further to the right, to L_{i+1}, with the following path: $u ... v_{l-1} v_l v_{l+1} v'_{l+1} ... v$. For this new path either v_{l+1} has a neighbour in L_{l+2}, which means the repetition can be shifted a further step to the right, the argument being identical to the one presented just now, or v_{l+1} has no neighbour in L_{i+2} meaning its coloured 1, making this new path rainbow. If the path repeats in L_k, in the scenario where the repetition has shifted all the way to the right, the path will also be rainbow. This scenario where the path between u and v repeats in some layer L_l is shown in Figure 5.

This proves that c is a rainbow colouring for G using $\text{diam}(G) - 1$ colours. It is simple to see that our colouring algorithm only takes polynomial time. The BFS-search is linear, while finding vertices which are not dominated by the following layer is also linear. This added with the time it takes to find the dominating diametral path results in a polynomial time algorithm. ◀

7 Interval graphs

In Section 5, we presented a proof which generalized an earlier result concerning interval graphs. For this section we will expand on the theme of interval graphs as we prove that also SRVC can be efficiently solved on this graph class. Only an algorithm for computing an optimal strong rainbow verex colouring on unit interval graphs was previously known [6].

▶ **Theorem 5.** *If G is an interval graph, then $\mathbf{srvc}(G) = \text{diam}(G) - 1$ and an optimal strong rainbow vertex colouring can be found in time that is linear in the size of G.*

8 Conclusion

In this work we have given efficient algorithms for optimally colouring multiple subclasses of graphs with dominating diametral paths. We have shown that for a graph G with a dominating diametral path, and one of the following properties: chordal, claw-free or bipartite, we can find a colouring in polynomial time using $\text{diam}(G) - 1$ colours. But the status of the of the conjecture of Heggernes et al. [6] still remains an open problem.

We believe the conjecture is true because diametral path graphs seem to be a graph class that is more dense than graphs with a dominating diametral path. As we have seen for the claw-free and chordal graphs, the dominating diametral path in conjunction with some property making for a more dense graph class seems to help in terms of finding a rainbow vertex colouring using $\mathrm{diam}(G) - 1$ colours. A step towards solving Conjecture 1 would be to consider other widely studied subclasses of diametral path graphs, such as co-comparability graphs and AT-free graphs.

On the other hand, we showed Conjecture 1 is not true for a closely related graph class, namely if we assume that only the input graph has a dominating diametral path, as opposed to every induced subgraph of it. We showed an example of a graph with a dominating diametral path that requires $\mathrm{diam}(G)$ many colours in any rainbow vertex colouring. We also showed that RVC becomes NP-complete on this graph class. In the light of our algorithms for chordal, bipartite and claw-free graphs with a dominating diametral path, it would be interesting to understand which other properties ensure efficient algorithms for RVC on graphs with a dominating diametral path.

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