Parameterized Complexity of Perfectly Matched Sets

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Abstract

For an undirected graph $G$, a pair of vertex disjoint subsets $(A, B)$ is a pair of perfectly matched sets if each vertex in $A$ (resp. $B$) has exactly one neighbor in $B$ (resp. $A$). In the above, the size of the pair is $|A|$ ($= |B|$). Given a graph $G$ and a positive integer $k$, the Perfectly Matched Sets problem asks whether there exists a pair of perfectly matched sets of size at least $k$ in $G$. This problem is known to be NP-hard on planar graphs and $W[1]$-hard on general graphs, when parameterized by $k$. However, little is known about the parameterized complexity of the problem in restricted graph classes. In this work, we study the problem parameterized by $k$, and design FPT algorithms for: i) apex-minor-free graphs running in time $2^{O(k)} \cdot n^{O(1)}$, and ii) $K_{b,b}$-free graphs. We obtain a linear kernel for planar graphs and $O(kd)$-sized kernel for $d$-degenerate graphs. It is known that the problem is $W[1]$-hard on chordal graphs, in fact on split graphs, parameterized by $k$. We complement this hardness result by designing a polynomial-time algorithm for interval graphs.

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1 Introduction

MATCHING is one of the very classical polynomial-time solvable problems in Computer Science with varied applications. Finding a matching with additional structure, such as an induced matching has been well studied both in classical complexity as well as parameterized complexity, see, for instance, [4, 9, 18, 20, 24, 24, 27, 28] (list is only illustrative, and not comprehensive). In this article, we are interested in a matching that is slightly weaker than the structure of an induced matching but still more structured than a matching.

For a graph $G$, a pair of vertex disjoint subsets, $(A, B)$ is a pair of perfectly matched sets in $G$ if each vertex in $A$ has exactly one neighbor in $B$ and each vertex in $B$ has exactly one neighbor in $A$; the size of the pair is $|A|$ ($= |B|$). Note that there can be edges between vertices of $A$ (resp. $B$), which is forbidden in the case of induced matching. We study the problem called Perfectly Matched Sets, which is defined below.

| Perfectly Matched Sets | Parameter: $k$
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<td><strong>Input:</strong> An undirected graph $G$ and an integer $k$.</td>
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<td><strong>Question:</strong> Does there exist a pair of perfectly matched sets of size at least $k$ in $G$?</td>
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Parameterized Complexity of Perfectly Matched Sets

This problem was first introduced in [27] where it was named as Maximum TR-matching problem (Transmitter-Receiver problem). The paper showed that this problem is $\text{NP}$-complete when restricted to graphs having degree 3. Evan, Goldreich, and Tong in [13] showed that TR-matching is $\text{NP}$-complete on bipartite graphs. This problem was revisited by Aravind and Saxena in 2021, [1] where they called the problem as Perfectly Matched Sets. They designed FPT algorithms for this problem parameterized by the structural parameters such as distance to cluster, distance to co-cluster, and treewidth. They also prove that the problem is $\text{NP}$-hard on planar graphs and $\text{W}[1]$-hard parameterized by the solution size $k$, when restricted to bipartite graphs and split graphs.

The Perfectly Matched Sets problem is also closely related to the problem Perfect Matching Cut where we want edge cuts of size $k$, such that the vertices participating in these edges induce a matching and a perfect matching, respectively. We remark that in Perfectly Matched Sets, we do not insist that the edges between the pair of perfectly matched sets $\{A, B\}$ is a cut in the graph. The Matching Cut and Perfect Matching Cut problems have been investigated in the literature even when restricted to well-studied graph classes, see, for instance, [2, 6, 7, 21, 22, 25].

**Our Results.** In this paper, we investigate the parameterized complexity of the Perfectly Matched Sets problem when the input graph is from a structured graph family, for several choices of well-studied graph families. The starting point of our work is the result by Aravind and Saxena [1]. The paper showed that the problem is $\text{W}[1]$-hard even on split graphs, which is an important subclass of chordal graphs. Inspired by this negative result, we turn to interval graphs, which is arguably the most well-studied subclass of chordal graphs. We obtain the following result by using a dynamic programming based algorithm.

**Theorem 1.** Perfectly Matched Sets on interval graphs admits an algorithm running in time $O(n^5)$.

Aravind and Saxena [1] showed that Perfectly Matched Sets is $\text{NP}$-complete even when the input graph is planar. Inspired by this we design an FPT algorithm for a strictly more general class of apex-minor-free graphs. A graph $H$ is an apex graph if there is $v \in V(H)$, such that $H - \{v\}$ is planar. Consider any finite set $\mathcal{H}$ of graphs that contains at least one apex graph, and let $\mathcal{F}_H$ be the family of graphs that do not contain any graph from $\mathcal{H}$ as a minor. The $\mathcal{H}$-Minor Free PMS problem is the Perfectly Matched Sets problem problem with an additional guarantee that the input graph belongs to $\mathcal{F}_H$. Note that for $\mathcal{H} = \{K_5, K_{3,3}\}$, $\mathcal{F}_H$ is the family of planar graphs. We obtain the following result:

**Theorem 2.** For any (fixed) finite set $\mathcal{H}$ of graphs that contains at least one apex graph, $\mathcal{H}$-Minor Free PMS has an FPT algorithm running in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

We remark that the same approach used in obtaining the above result can be used to obtain an FPT algorithm on bounded genus graphs, due to bidimensionality [10]. We remark that having a pair of perfectly matched sets of size at least $k$ is expressible in MSO (actually, even in FO). So, there is an FPT algorithm on the much more general nowhere dense classes (admittedly with a worse running time) [16].

For $b \in \mathbb{N}$, a graph is $K_{b,b}$-free if it does not contain a bi-clique with $b$ vertices on each side as a subgraph. We obtain the following result by using an approach similar to random separation [3], in combination with a result of Dabrowski et al. [9].

**Theorem 3.** For any fixed $b \in \mathbb{N}$, Perfectly Matched Sets on $K_{b,b}$-free graphs admits an FPT algorithm, when parameterized by $k$. 
Kanj et al. [18] and Erman et al. [20] independently designed $O(k^c)$ kernels for the Induced Matching problem for graphs of arboricity bounded by $c$. The authors [18] also showed that any twinless graph of average degree $d$ and bounded chromatic number contains an induced matching of size $\Omega(n^{1/d})$. The core of their proof is the system of strong representatives of a set family. This combinatorial tool also forms the backbone of our following result.

\textbf{Theorem 4.} Perfectly Matched Sets admits a $k^{O(d)}$-sized kernel on $d$-degenerate graphs.

As planar graphs are 5-degenerate, the theorem above directly gives us a polynomial kernel for Perfectly Matched Sets on these graphs. Following an approach by Kanj et al. [18] for obtaining a linear kernel for Induced Matching on planar graphs, we obtain a linear kernel (improving upon the already obtained polynomial kernel) for Perfectly Matched Sets on this graph class.

\section{Preliminaries}

\textbf{Sets and graph notations.} We use $\mathbb{N} = \{1,2,\ldots\}$ to denote the set of natural numbers. We use $[k]$ as a shorthand for $\{1,2,\ldots,k\}$ and use $[k]_0$ for $\{k\} \cup \{0\}$, where $k \in \mathbb{N}$. In this article, we only consider simple undirected graphs. Given a graph $G$, we denote the vertex set and edge set of $G$ by $V(G)$ and $E(G)$ respectively. Unless specified, $n$ and $m$ denote the number of vertices and edges of the graph $G$. Two vertices $u,v$ are said to be adjacent if there is an edge (denoted by $[u,v]$) between $u$ and $v$ in $G$. For $X \subseteq V(G)$, $G[X]$ denotes the induced subgraph of $G$ with vertex set $X$ and edge set $\{[u,v] \mid u,v \in X \text{ and } [u,v] \in E(G)\}$. $G - X$ denotes the subgraph $G[V(G) \setminus X]$. For an edge set $E' \subseteq E$, $V(E')$ denotes the set of all the vertices of $G$ having at least one edge in $E'$ incident on it. $E(A,B)$ denotes the set of edges with one endpoint in $A$ and the other in $B$. The open neighborhood of a vertex $v$, denoted by $N_G(v)$, is the set of vertices adjacent to $v$. The closed neighborhood of $v$ is defined as $N_G[v] = N_G(v) \cup \{v\}$. The subscript in the notation for neighborhood is omitted if the graph under consideration is clear. For $X \subseteq V(G)$, $N[X]$ denotes the set of vertices $\bigcup_{v \in X} N[v]$. Two distinct vertices $u,v$ is said to be a pair of false twins if $N_G(u) = N_G(v)$ and true twins if $N_G[u] = N_G[v]$. A clique in graph $G$ is a set of vertices such that there is an edge between every pair of vertices in the set. An independent set in the graph $G$ is a set of vertices such that there is no edge between any pair of vertices in the set. $K_{n,m}$ is the complete bipartite graph, also known as a biclique, with partitions of size $n$ and $m$. A $k$-biclique is a $2k$-vertex complete bipartite graph. A subset $D \subseteq V(G)$ is said to be a dominating set of $G$ if $N[D] = V(G)$. A vertex cover of a graph is a set of vertices that includes at least one endpoint of every edge of the graph. The cardinality of the smallest size dominating set is called as domination number of $G$. $D$ is said to be a 2-dominating set if $N[N[D]] = V(G)$. $G\setminus e$ denotes the graph obtained by contracting the edge $e$ in $G$. The contraction of an edge $[u,v]$ in the graph involves the deletion of vertices $u$ and $v$ from $G$ and the addition of a new vertex $w$, which is adjacent to all the vertices of $N(u) \cup N(v)$. For two graphs $G_1$ and $G_2$, we denote $G_1 \subseteq G_2$ if $G_1$ is an induced subgraph of $G_2$.

\textbf{Graph classes.} A graph is planar if it can be drawn in the plane without edge intersections except at the endpoint). A graph $G$ is a $d$-degenerate graph if every induced subgraph of $G$ contains a vertex of degree at most $d$. A $K_{b,b}$-free graph is a graph that does not contain biclique $K_{b,b}$ as a subgraph (not necessarily induced). An apex graph is a graph that can
be made planar by removing one of its vertices. Apex-minor-free graphs are basically those graphs that exclude a fixed apex graph as a minor. More precisely, \( G \) is apex-minor-free graph class if there exists some apex graph \( H \) such that no graph from \( G \) admits \( H \) as a minor. An interval graph is an undirected graph formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. It is the intersection graph of the intervals. [5]. For standard graph definition and notations, we refer to the graph theory book by R. Diestel [11]. For parameterized complexity terminology, we refer to the parameterized algorithms book by Cygan et al. [8].

**Treewidth.** A tree decomposition of a graph \( G = (V, E) \) is a pair \((T, X)\) where \( T \) is a tree on vertex set \( V(T) \). The vertices of \( V(T) \) are called nodes. Also, \( X = \{\{X_i | i \in V(T)\}\} \) is a collection of subsets of \( V \) such that -
1. Every vertex of \( G \) is contained in at least one bag. \( \bigcup_{i \in V(T)} X_i = V \),
2. For every edge \( \{u, v\} \in E \), there exists a node \( i \in V(T) \) such that bag \( X_i \) contains both \( u \) and \( v \).
3. For each \( u \in V \), the set of nodes whose bags contain \( u \), \( T_u = \{i \in V(T) : i \in X_i\} \) forms a connected subtree of \( T \).

The width of a tree decomposition \((T, \{(X_i | i \in V(T))\})\) is equal to the maximum size of its bag minus 1, \( \max_{i \in V(T)} |X_i| - 1 \). The treewidth of a graph \( G \), \( tw(G) \) is the minimum width of a tree decomposition over all tree decompositions of \( G \).

**Perfectly matched sets.** A matching in a graph \( G \) is a set of edges \( M \) such that no two edges in \( M \) share the same endpoint. A matching \( M \) is maximal if \( G - V(M) \) is edge less. A matching \( M \) is said to be an induced matching if the subgraph induced by the vertices in \( M \) contains only the edges of \( M \). If \( M \) is maximal then \( V(M) \) is a vertex cover of \( G \), and it is easy to verify that \( tw(G) \leq |V(M)| \). For a pair \((A, B)\) of disjoint subsets of vertices of \( V(G) \), we say \((A, B)\) is a pair of perfectly matched sets if every vertex in \( A \) (resp. \( B \)) has exactly one neighbor in \( B \) (resp. \( A \)). The size of the pair is \( |A| = |B| \).

**Parameterized problems and kernels.** A parameterized problem \( \Pi \) is a subset of \( \Gamma^* \times \mathbb{N} \) for some finite alphabet \( \Gamma \). An instance of a parameterized problem consists of \((X, k)\), where \( k \) is called the parameter. The notion of kernelization is formally defined as follows. A kernelization algorithm, or in short, a kernelization, for a parameterized problem \( \Pi \subseteq \Gamma^* \times \mathbb{N} \) is an algorithm that, given \((X, k) \in \Gamma^* \times \mathbb{N} \), outputs in time polynomial in \( |X| + k \) a pair \((X', k') \in \Gamma^* \times \mathbb{N} \) such that (a) \((X, k) \in \Pi \) if and only if \((X', k') \in \Pi \) and (b) \(|x'|, |k| \leq g(k)\), where \( g \) is some computable function depending only on \( k \). The output of kernelization \((X', k') \) is referred to as the kernel and the function \( g \) is referred to as the size of the kernel. If \( g(k) \in k^O(1) \), then we say that \( \Pi \) admits a polynomial kernel. We refer to the monographs [12, 14, 26] for a detailed study of the area of kernelization.

### 3 Polynomial-time Algorithm for Interval Graphs

Recall that Perfectly Matched Sets is \( W[1] \)-hard when parameterized by the solution size \( k \) even when restricted to split graphs (and thus, chordal graphs). Interval graphs belong to the class of chordal graphs. In this section, we present a polynomial-time dynamic programming algorithm that computes a maximum-sized pair of perfectly matched sets for any given interval graph.
Let $G$ be an interval graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since $G$ is an interval graph, there exists a corresponding geometric intersection representation of $G$, where each vertex $v_i \in V(G)$ is associated with an interval $I_i = [\ell(I_i), r(I_i)]$ in the real line, where $\ell(I_i)$ and $r(I_i)$ denote left and right endpoints, respectively in $I_i$. Two vertices $v_i$ and $v_j$ are adjacent in $G$ if and only if their corresponding intervals $I_i$ and $I_j$ intersect with each other. We can also assume that along with the graph, we are also given the corresponding underlying intervals on the real line, as there are well-known linear-time algorithms that compute such a representation [19]. We use $I$ to denote the set $\{I_i: v_i \in V\}$ of intervals and $P$ to denote the set of all endpoints of these intervals, i.e., $P = \cup_{I \in \mathcal{I}} \{\ell(I), r(I)\}$. In the remaining section, we will use $v_i$ and $I_i$ interchangeably. Note that we can assume that the endpoints of all the intervals in the interval representation are distinct — otherwise, we can slightly perturb the endpoints of the intervals to obtain a new interval representation of the graph in which this is true.

**Proposition 5.** Let $G$ be a connected interval graph. There exists an ordering, $\prec$, of $V(G)$ such that for $u, v, w \in V(G)$ if $u \prec v \prec w$ and $\{u,w\} \in E(G)$ then $\{v, w\} \in E(G)$.

We remark that such an ordering in Proposition 5 can be obtained based on the right endpoints of intervals, more specifically the set $\{r(I_i)\}$ and the ordering is as follows: for any two vertices $v_i$ and $v_j$, we have $v_i \prec v_j$ if and only if $r(I_i) < r(I_j)$. We call such an ordering, the right-end ordering of $V(G)$.

**Lemma 6.** Let $G$ be an interval graph with a right-end ordering, $\prec$, of $V(G)$. Consider any distinct pair of edges $\{u, v\}$ and $\{u', v'\}$ in a pair of perfectly matched sets $(A, B)$ where $u \prec v$ and $u' \prec v'$. If $u \prec u'$, then $v \prec u'$.

**Proof.** Towards a contradiction suppose there are edges $\{u, v\}, \{u', v'\}$ in the pair of perfectly matched sets $(A, B)$, where $u \prec v$, $u' \prec v'$, $u \prec u'$ and $u' \prec v$. Then, either $u \prec u' \prec v' \prec v$ or $u \prec u' \prec v < v'$. In either of these cases, by Proposition 5, $v$ is adjacent to both $u'$ and $v'$ which is a contradiction to the fact that $(A, B)$ is perfectly matched sets in $G$.

Lemma 6 directly implies the following remark.

**Remark 7.** Let $\{u_i, v_i\}: 1 \leq i \leq k$ be a set of $k$ edges in a pair $(A, B)$ of perfectly matched sets in $G$ with $u_1 \prec u_2 \prec \ldots \prec u_k$ and $u_i < v_i$, for each $i \in [k]$. Then, $u_1 \prec v_1 < u_2 \prec v_2 < \ldots < u_k < v_k$.

**Algorithm and its Correctness.** We define a table for our dynamic-programming algorithm. Let $v_1 \prec v_2 \prec \ldots \prec v_n$ be the right-end ordering of the vertex set of $G$. For every tuple $(v_i, v_j, t)$, where $\{v_i, v_j\} \in E(G), i, j \in [n], i < j$ and $t \in [[n/2]]$, we define two Boolean values: (i) $\text{PM}[(v_i, A), (v_j, B); t]$ and (ii) $\text{PM}[(v_i, B), (v_j, A); t]$.

The entry $\text{PM}[(v_i, A), (v_j, B); t]$ is true if there exists a pair $(A, B)$ of perfectly matched sets of size $t$ such that $v_i \in A$, $v_j \in B$ and for all the vertices $v \in (A \cup B) \setminus \{v_i, v_j\}$, we have $v \prec v_i$. Similarly the entry $\text{PM}[(v_i, B), (v_j, A); t]$ is true if there exists a pair $(A, B)$ of perfectly matched sets of size $t$ such that $v_i \in B$, $v_j \in A$ and for all the vertices $u \in (A \cup B) \setminus \{v_i, v_j\}$, we have $u < v_i$.

In the base case, both $\text{PM}[(v_i, A), (v_j, B); 1]$ and $\text{PM}[(v_i, B), (v_j, A); 1]$ are true for every possible pair $v_i$ and $v_j$ (note because of the way the entry is defined, $\{v_i, v_j\}$ must be an edge in $G$). We will use the convention that empty OR is 0. In the lemma below, we give a recursive formula for computing the values $\text{PM}[(v_i, A), (v_j, B); t]$ for $t > 1$.

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1. $A$ and $B$ in these entries are just symbols, added for added for clarity.
Lemma 8. For every integer \( t \in [[n/2]] \setminus \{1\} \), and every pair of adjacent vertices \( v_i, v_j \) in \( G \) where \( i < j \), the following recurrence holds:

\[
\text{PM}[(v_i, A), (v_j, B); t] = \bigvee_{\{x, y\} \in E(G)} \left( \left( \text{PM}[(x, A), (y, B); t-1] \land \left[ \begin{array}{l}
\{x, v_j\} \notin E(G) \\
\{y, v_i\} \notin E(G)
\end{array} \right] \right) \lor \left( \text{PM}[(x, B), (y, A); t-1] \land \left[ \begin{array}{l}
\{x, v_j\} \notin E(G) \\
\{y, v_i\} \notin E(G)
\end{array} \right] \right) \right)
\]

Proof. In the forward direction let us assume that \( \text{PM}[(v_i, A), (v_j, B); t] = \text{true} \). So according to the definition of our dynamic-programming table, \( \{v_i, v_j\} \in E(G) \) and there exists a pair \( (A, B) \) of perfectly matched sets of size \( t \) such that \( v_i \in A \), \( v_j \in B \) and for all the vertices \( v \in (A \cup B) \setminus \{v_i, v_j\} \), we have \( v < v_i \). Now consider the pair \( (A' = A \setminus \{v_j\}, B' = B \setminus \{v_j\}) \). It is easy to see that this pair is a perfectly matched sets of size \( t - 1 \) and all the vertices \( v \) in the pair having the property that \( v < v_i \). Consider the last vertex in the right-end ordering of \( V(G) \) which occurs in the vertex set \( A' \cup B' \). Let this vertex be \( y \) and \( x \) be its (only) neighbour in \( B' \). Note that \( x < y \) and for any vertex \( v \in (A' \cup B') \setminus \{x, y\} \), it must hold that \( v < x \) (see Remark 7). If \( y \in B' \), then clearly, \( \{x, v_j\} \notin E(G) \), \( \{y, v_i\} \notin E(G) \), and \( \text{PM}[(x, A), (y, B); (t-1)] = \text{true} \). Otherwise, \( y \in A' \), and then \( \{x, v_i\} \notin E(G) \), \( \{y, v_j\} \notin E(G) \), and \( \text{PM}[(x, B), (y, A); (t-1)] = \text{true} \).

In the reverse direction, assume that there exists a pair of vertices \( x < y \), \( \{x, y\} \in E(G) \) such that \( \text{PM}[(x, A), (y, B); (t-1)] = \text{true} \) and \( \{x, v_j\} \notin E(G) \), \( \{y, v_i\} \notin E(G) \). (The case when \( \text{PM}[(x, B), (y, A); (t-1)] = \text{true} \) and \( \{x, v_j\} \notin E(G) \), \( \{y, v_i\} \notin E(G) \) can be argued symmetrically.) The above means that there is a pair of perfectly matched sets \( (A', B') \) with \( t - 1 \) edges such that: \( \{x, y\} \in E(G) \), \( x \in A' \), \( y \in B' \), and for each \( v \in (A' \cup B') \setminus \{x, y\} \), we have \( v < x \). Let \( A = A' \cup \{v_i\} \) and \( B = B' \cup \{v_j\} \). Note that we have \( x < y < v_i < v_j \), and thus, for each \( v \in A' \cup B' \), we have \( v < v_i < v_j \). For a contradiction suppose that we have \( v \in B' \), such that \( \{v, v_j\} \in E(G) \). Note that \( v < x < y < v_i \), as \( \{y, v_i\} \notin E(G) \) (see Remark 7). But then from Lemma 6, we can obtain that \( \{v, x\} \notin E(G) \), which contradicts that \( (A', B') \) is a pair of perfectly matched sets. Similarly, towards a contradiction suppose that we have \( v \in A' \), such that \( \{v, v_j\} \in E(G) \). Then, \( v < x < y < v_j \), and thus, \( \{y, v_i\} \in E(G) \), which is a contradiction. From the above discussions, we can conclude that \( \text{PM}[(v_i, A), (v_j, B); t] = \text{true} \).

Similarly, we have a recursive formula for computing the values \( \text{PM}[(v_i, B), (v_j, A); t] \) for \( t > 1 \). The correctness proof is similar to that of Lemma 8.

Lemma 9. For every integer \( t \in [[n/2]] \setminus \{1\} \), and every pair of adjacent vertices \( v_i, v_j \) in \( G \) where \( i < j \), the following recurrence holds:

\[
\text{PM}[(v_i, B), (v_j, A); t] = \bigvee_{\{x, y\} \in E(G)} \left( \left( \text{PM}[(x, A), (y, B); t-1] \land \left[ \begin{array}{l}
\{x, v_j\} \notin E(G) \\
\{y, v_i\} \notin E(G)
\end{array} \right] \right) \lor \left( \text{PM}[(x, B), (y, A); t-1] \land \left[ \begin{array}{l}
\{x, v_j\} \notin E(G) \\
\{y, v_i\} \notin E(G)
\end{array} \right] \right) \right)
\]

We can compute all the entries of our dynamic programming table using the recurrence relations given by Lemma 8 and Lemma 9.

Time Complexity. For a pair of adjacent vertices \( v_i, v_j \), where \( i < j \), the time required to compute \( \text{PM}[(v_i, A), (v_j, B); t] \) and \( \text{PM}[(v_i, B), (v_j, A); t] \), once we have computed the entries till the values at most \( t - 1 \), is bounded by \( O(n^2) \). As \( t < n \), the number of entries we have to compute is bounded by \( O(n^3) \), thus bounding the total running time of our algorithm by \( O(n^5) \). This proves Theorem 1.
4 FPT Algorithm for Apex-Minor-Free Graphs

Consider any (fixed) finite set $H$ of graphs that contains at least one apex graph; we will work with this fixed family throughout this section. Recall that $\mathcal{F}_H$ is the family of graphs that do not contain any graph from $H$ as a minor, and the $H$-MINOR FREE PMS problem is the same as the PERFECTLY MATCHED SETS problem with an additional guarantee that the input graph belongs to $\mathcal{F}_H$. In this section, we prove Theorem 2 by designing a simple FPT algorithm with the desired running time. Let $(G, k)$ be an instance of $H$-MINOR FREE PMS. Our algorithm will begin by greedily trying to construct a solution, if we succeed then the algorithm halts. Otherwise, we will be able to bound the size of a 2-dominating in $G$ by $O(k)$. This together with a result of Fomin [15]) will imply that the treewidth of $G$ is bounded by $O(\sqrt{k})$. Now we can use the algorithm of Aravind and Saxena [1] for PERFECTLY MATCHED SETS parameterized by treewidth to obtain the proof of the theorem. We begin by stating the two useful results.

- **Proposition 10 (Lemma 2, [15]).** For an $H$-minor free graph $G$, if $t$ is the size of a minimum 2-dominating set of $G$, then the treewidth of $G$ is bounded by $c_H \cdot \sqrt{t}$, where $c_H$ is a constant depending on $H$.

- **Proposition 11 (Theorem 7, [1]).** There exists an algorithm that calculates maximum perfectly matched sets for an $n$ vertex graph with treewidth at most $w$ in time $O(12^w \cdot \text{poly}(n))$.

The next lemma gives the procedure that either resolves the instance or obtains a small 2-dominating set in $G$.

- **Lemma 12.** There is a polynomial time algorithm that either correctly concludes that $(G, k)$ is a yes-instance of $H$-MINOR FREE PMS, or outputs a 2-dominating set $Q$ of $G$ where $|Q| \leq 2 \cdot (k - 1)$.

**Proof.** Let $(G, k)$ be an instance of the problem. If $G$ has an isolated vertex, then such a vertex is not part of any perfectly matched set, and thus we remove it. We will next create a sequence of perfectly matched sets $S_0 \subset S_1 \subset \cdots \subset S_q$ and graphs $G_0 \supset G_1 \supset \cdots \supset G_q$, which, intuitively speaking, will be constructed by greedily adding an edge (one at a time) to form a perfectly matched set.

Initialize $S_0 = \emptyset$ and $G_0 = G$. Iteratively do the following: if there is an edge $e_i = \{u_i, v_i\} \in E(G_i)$, then set $S_{i+1} = S_i \cup \{e\}$ and $G_{i+1} = G_i - (N_G[u_i] \cup N_G[v_i])$. The $q$ be an integer where the above procedure stops, which is the case when $G_q$ has no edges. Notice that for any $i \in [q_0]$, each $S \in \{S_j | S_j \subset \{i + 1, i + 2, \ldots, q\}\}$ is a pair of perfectly matched sets in $G_i$. The above in particular implies that $S_q$ is a pair of perfectly matched sets in $G = G_0$. Also, for each $i \in [q_0]$, $|S_i| = i$. If $q \geq k$, then we have obtained a pair of perfectly matched sets in $G$ of size at least $k$, and thus we can conclude that the instance is a yes-instance. Otherwise $q \leq k - 1$, and we let $Q = \{u_i, v_i \mid i \in [q]\}$. Consider any vertex $u \in V(G) \setminus N_G[Q]$. Since $G$ has no isolated vertices, $u$ must have a neighbor $v$ in $G$. Note that $v \notin Q$, as $u \notin V(G) \setminus N_G[Q]$. Also, if $v \notin N_G(Q)$, then $(u, v)$ is an edge in $G_q$, which contradicts that $G_q$ has no edges. The above discussions imply that $Q$ is a 2-dominating set in $G$ of size at most $|Q| \leq 2 \cdot (k - 1)$.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Consider an instance $(G, k)$ of $H$-MINOR FREE PMS. If Lemma 12 returns that the instance is a yes-instance, then we are done. Otherwise, it returns a 2-dominating set in $G$ of size at most $2 \cdot (k - 1)$. From Proposition 10, the treewidth of $G$ is...
bounded by $c_H \cdot \sqrt{2 \cdot (k - 1)}$, where $c_H$ is a constant depending on the family $H$. Now using Lemma 7.4 of [8], we compute a nice tree decomposition of width at most $c_H \cdot \sqrt{2 \cdot (k - 1)}$ in time bounded by $O(nk)$. Now we can use Proposition 11 to resolve the instance. ▶

5 FPT Algorithm for $K_{b,b}$-free Graphs

The goal of this section is to prove Theorem 3. Consider any fixed number $b \in \mathbb{N}$. Recall that a graph is $K_{b,b}$-free if it does not contain a subgraph isomorphic to $K_{b,b}$. We obtain an FPT algorithm for Perfectly Matched Sets on $K_{b,b}$-free graphs by using an approach similar to random separation [3], in combination with the below-stated result of Dabrowski et al. [9].

Proposition 13 (Lemma 2, [9]). For any natural numbers $s, t$ and $p$, there is a number $N'(s, t, p)$ such that every graph with a matching of size at least $N'(s, t, p)$ contains either a clique $K_s$, an induced bi-clique $K_{s,t}$ or an induced matching of size $p$. Here, $N'(s, t, p) = R(s, R(s, N(t, p)))$ where $R(s, t)$ is the non-symmetric Ramsey number.

Let $(G, k)$ be an instance of Perfectly Matched Sets, where $G$ is a $K_{b,b}$-free graph with $n$ vertices. We color the vertices of $V(G)$ independently and randomly using two colors, red and blue (with equal probability). This forms a random partition $V_R \uplus V_B$ of the vertices of $G$, where $V_R$ and $V_B$ are the set of vertices colored with red and blue color, respectively. We call these two partitions as color classes. Next, we obtain the graph $G'$ from $G$ by removing all the edges between the vertices of the same color class. Thus, the edges in $G'$ have endpoints of differing colors, and thus it is bipartite. We compute (in polynomial time) a maximum sized matching $M$ in $G'$ [23]. We will next argue that either $M$ has at most $N'(3, b, k)$ edges, or we can conclude that $(G, k)$ is a yes-instance.

Case 1. Firstly suppose that $M$ has at least $N'(3, b, k)$ edges. Recall that $G$ is bipartite, so it does not have any $K_3$. Moreover, as $G$ is $K_{b,b}$-free, we can obtain that $G'$ has no induced $K_{b,b}$. As the size of a maximum matching in $G'$ is at least $N'(3, b, k)$, using Proposition 13 we can obtain that $G'$ has an induced matching $M_I$ of size at least $k$. Now using the next observation we can conclude that $(G, k)$ is a yes-instance of the problem.

Observation 14. $(V_R \cap V(M_I), V_B \cap V(M_I))$ is a pair of perfectly matched sets in $G$ of size at least $k$.

Proof. Consider $x \in V_R \cap V(M_I)$, where $x$ has a neighbor $y \in V_B \cap V(M_I)$ and $\{x, y\}$ is an edge in $M_I$. Let $z \neq y$ be another neighbor of $x$ in $V_B \cap V(M_I)$. Then since $x$ is colored with red and $z$ is colored with blue, the edge $(x, z) \in E(G')$. But this is a contradiction to the fact that $M_I$ is an induced matching. From the above discussions, we can obtain that each vertex in $V_R \cap V(M_I)$ has exactly one neighbor in $V_B \cap V(M_I)$ and vice-versa. ▶

Case 2. Now suppose that in $G'$ the matching $M$ has less than $N'(3, b, k)$ edges, and thus, $\text{tw}(G') \leq 2 \cdot N'(3, b, k)$. Now in $G'$, we look for a pair of perfectly matched sets $(X, Y)$ where $X \subseteq V_R$ and $Y \subseteq V_B$. Let us denote this version of Perfectly Matched Sets as the colored-Perfectly Matched Sets problem. Aravind et al. [1] designed an FPT algorithm for Perfectly Matched Sets parameterized by the treewidth of the given graph. They use a nice tree decomposition of the graph, where in each bag $\beta(t)$, $X \cap \beta(t)$ and $Y \cap \beta(t)$ play a crucial role in the construction of their algorithm. To adapt their algorithm for colored-Perfectly Matched Sets, we only need to enforce that $X \cap \beta(t)$ and $Y \cap \beta(t)$ are selected from $V_R$ and $V_B$, respectively. Precisely in Section 5.3 of their draft [1], $A \cap \beta(t) = A_t$.
and $B \cap \beta(t) = B_t$ can be replaced by $A \cap (\beta(t) \cap V_R) = A_t$ and $B \cap (\beta(t) \cap V_B) = B_t$, respectively, to obtain an algorithm for the colored version. Notice that they denote the desired perfectly matched sets by $(A, B)$ while we do it by $(X, Y)$. Hence we have an FPT algorithm running in time $2^\mathcal{O}(\text{tw}(G')) \cdot n^\mathcal{O}(1)$ to obtain a pair of perfectly matched sets $(X, Y)$ of $G'$ of size $k$ where $X \subseteq V_R$, $Y \subseteq V_B$. We remark that the algorithm given by [1] can actually compute such a set by the standard backtracking technique, and thus even for our colored case, we can compute a pair of perfectly matched sets in $G'$. Now we claim the following.

\textbf{Observation 15.} $(X, Y)$ is also a pair of perfectly matched sets of $G$.

\textbf{Proof.} Suppose $(X, Y)$ is not a pair of perfectly matched sets of $G$. Notice that $E(G') \subseteq E(G)$ and hence there is a vertex $v$ in $X$ with more than one neighbor in $Y$ or there is a vertex $u$ in $Y$ with more than one neighbor in $X$. Without loss of generality let such a vertex $v$ be in $X$. Let two of its neighbors in $Y$ be $y_1$ and $y_2$. But the edges $\{v, y_1\}$ and $\{v, y_2\}$ are also in $G'$ as they have endpoints with differing colors. But this contradicts the fact that $(X, Y)$ is a pair of perfectly matched sets of $G'$.

In the construction of $G'$ from $G$, we delete edges with endpoints in the same color classes. Hence a pair of perfectly matched sets of $G$ may not remain a pair of perfectly matched sets of $G'$. But in the claim below, we show that for a fixed size of perfectly matched sets, the chances of such an event happening stays low.

\textbf{Observation 16.} Any $k$-sized perfectly matched sets $(X, Y)$ of $G$ is also a perfectly matched sets of $G'$ with probability at least $2^{-2k}$.

\textbf{Proof.} The probability that all vertices of $X$ are colored red and all vertices of $Y$ are colored blue is at least $2^{-2k}$. Thus we can obtain that with probability at least $2^{-2k}$ $(X, Y)$ is also a perfectly matched sets of $G'$.

The proof of the following lemma follows from Observations 14, 15 and 16 with the standard trick of making independent runs of the discussed algorithm.

\textbf{Lemma 17.} There exists a randomized FPT algorithm running in time $2^{\mathcal{O}(N'(3,b,k)+k)} \cdot n^{\mathcal{O}(1)}$ that, given a Perfectly Matched Sets instance $(G, k)$ on $K_{b,b'}$-free graphs, either reports a failure or finds a pair of perfectly matched sets in $G$ of size at least $k$. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.

We now explain the derandomization procedure for the above algorithm. It involves deterministically constructing a family $\mathcal{F}$ of coloring functions $f : [n] \rightarrow [2]$ rather than selecting a random coloring $\chi : [n] \rightarrow [2]$ such that it is assured that one of the functions from $\mathcal{F}$ colors one set from a pair of perfectly matched sets of size $k$ (when $(G, k)$ is a yes-instance) with color 1 and the other set with color 2. To this end, we will use the following.

\textbf{Definition 18 (Definition 5.19, [8]).} An $(n, k)$-universal set is a family $\mathcal{U}$ of subsets of $[n]$ such that for each $S \subseteq [n]$ of size $k$, the family $\{A \cap S : A \in \mathcal{U}\}$ contains all $2^k$ subsets of $S$.

\textbf{Proposition 19 (Theorem 5.20, [8]).} For any $n, k \geq 1$, we can construct an $(n, k)$-universal set of size $2^k k^{\mathcal{O}(\log k)} \log n$ in time $2^k k^{\mathcal{O}(\log k)} n \log n$.

We assume that $V(G) = [n]$ (otherwise we can relabel the vertices). We first construct an $(n, 2k)$-universal set, $\mathcal{U}$, using the above proposition. Now we construct a family of function $\mathcal{F}$ from $[n]$ to $\{1, 2\}$ as follows, where $\mathcal{F}$ is initialized to $\emptyset$. For each $U \in \mathcal{U}$, add the function...
Parameterized Complexity of Perfectly Matched Sets

Let \( f_U : [n] \rightarrow [2] \), where \( f_U^{-1}(1) = U \). Note that if \( G \) has a pair of perfectly matched sets \((A, B)\) of size \( k \), then there is \( U \in \mathcal{U} \), such that \((A \cup B) \cap U = A\). Thus at least one function in \( \mathcal{F} \) is the correct coloring for us. We can iterate over each of the colorings given by \( \mathcal{F} \), and this leads us to the following result.

\[ \text{Proposition 22.} \quad \text{Consider any family } \mathcal{F} \text{ with more than } \binom{r+k}{k} \text{ distinct sets of sizes at most } r. \text{ Then, at least } k+2 \text{ sets in this family have a strong system of distinct representatives.} \]

The following property of a \( d \)-degenerate graph follows directly from the definition.

\[ \text{Proposition 23.} \quad \text{A } d \text{-degenerate graph on } n \text{ vertices has at most } dn \text{ edges.} \]

Next, we give a lower bound on the number of low-degree vertices in a \( d \)-degenerate graph.

\[ \text{Lemma 24.} \quad \text{Let } G \text{ be } d \text{-degenerate graph with } n \geq 6 \text{ vertices. Then } G \text{ has strictly more than } 5n/6 \text{ vertices of degree at most } 12d. \]

\[ \text{Proof.} \quad \text{Let } G \text{ be } d \text{-degenerate graph with } n \text{ vertices. By Proposition 23, the number of edges in } G \text{ is at most } dn. \text{ So the sum of the degrees of the vertices in } G \text{ is bounded by } 2dn. \text{ Assume that, there are at most } 5n/6 \text{ vertices of degree at most } 12d \text{ in } G. \text{ Then we have a set } U \subseteq V(G) \text{ of at least } n/6 \geq 1 \text{ vertices of degree strictly more than } 12d. \text{ Now the sum of the degrees of the vertices in } U \text{ is strictly more than } (n/6) \cdot 12d = 2dn, \text{ a contradiction. Hence there are strictly more than } 5n/6 \text{ vertices of degree at most } 12d \text{ in } G. \]

\[ \text{Observation 25.} \text{ In a pair of perfectly matched sets } (A, B) \text{ of a graph } G, \text{ there are at most two non-adjacent vertices } x, y \in A \cup B \text{ such that } N(x) = N(y). \]

\[ \text{Proof.} \quad \text{Let } x, y, z \in A \cup B \text{ be three pairwise non-adjacent vertices such that } N(x) = N(y) = N(z). \text{ At least two of these vertices are either in } A \text{ or } B. \text{ Without loss of generality let } x, y \in A. \text{ But then } x \text{ and } y, \text{ both have the exactly same neighbors in } B, \text{ which contradicts that } A \cup B \text{ is a pair of perfectly matched sets of } G. \]

With Observation 25, we obtain the following reduction rule.

\[ \text{Reduction Rule 1.} \quad \text{Let } u, v, w \text{ be three distinct vertices in } V(G) \text{ such that } N(u) = N(v) = N(w), \text{ then reduce } (G, k) \text{ to } (G - w, k). \]
Lemma 26. Reduction Rule 1 is safe.

Proof. Consider an application of Reduction Rule 1 in which a vertex, say $w \in V(G)$ was deleted because there are two distinct vertices $u$ and $v$ other than $w$ such that $N(u) = N(v) = N(w)$. We will prove that $(G, k)$ is a yes-instance of Perfectly Matched Sets if and only if $(G - w, k)$ is a yes-instance of Perfectly Matched Sets.

If $(G - w, k)$ is a yes-instance, any pair of perfectly matched sets in $G - w$ is also a pair of perfectly matched sets in $G$, thus $(G, k)$ must also be a yes-instance. For the other direction suppose that $(G, k)$ is a yes-instance of the problem, and we have two disjoint sets $A, B \subseteq V(G)$ such that every vertex in $A$ has exactly one neighbor in $B$ and vice-versa. If $w \notin A \cup B$, then $(A, B)$ is a pair of perfectly matched sets in $G - w$ of size $k$, and we are done. Else, exactly one of $A$ and $B$ must contain $w$. Without loss of generality we assume that $w \in A$. From Observation 25, we know that $|(A \cup B) \cap \{u, v, w\}| \leq 2$. Now neither $v$ nor $u$ belongs to $A$. If $B \cap \{u, v\} = \emptyset$, then $(A \setminus \{w\} \cup \{u\}, B)$ is a pair of perfectly matched sets in $G - w$ of size $k$. Else, exactly one of $v$ or $u$ belongs to $B$, say $u \in B$ (the other case is symmetric). Then, $(A \setminus \{w\} \cup \{v\}, B)$ is a pair of perfectly matched sets in $G - w$ of size $k$. ▶

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let $(G, k)$ be an instance of Perfectly Matched Sets where $G$ is a $d$-degenerate graph. If Reduction Rule 1 on $(G, k)$ is applicable, then we apply it in polynomial time and reduced the number of vertices. When the reduction rule is no longer applicable, we do the following. Let $X$ be the set of vertices of with degree at most $12d$, and let $t = |X|$. Consider the family $\mathcal{F} = \{N(u) \mid u \in X\}$ (with repetitions removed). By the non-applicability of Reduction Rule 1 and Lemma 24, we can obtain that $|\mathcal{F}| \geq t/2 \geq (5n/6)/2 = 5n/12$. Also note that each set in $\mathcal{F}$ has size at most $12d$.

If $|\mathcal{F}| \leq \binom{12d+k}{k}$, then $5n/12 < \mathcal{F} \leq \binom{12d+k}{k}$. Therefore, i.e., the number of vertices in $G$ is bounded by $k^{O(d)}$. Otherwise, $|\mathcal{F}| > \binom{12d+k}{k}$, and we argue that $(G, k)$ is a yes-instance. From Proposition 22, at least $k + 2$ of these sets form $\mathcal{F}$ have a strong system of distinct representatives, say these sets are $N(v_1), N(v_2), \ldots, N(v_{k+2})$ and $(u_1, u_2, \ldots, u_{k+2})$ is its strong system of distinct representatives. Let $A = \{v_1, v_2, \ldots, v_{k+2}\}$ and $B = \{u_1, u_2, \ldots, u_{k+2}\}$. Note that for each $i \in [k + 2]$, we have $\{v_i, u_j\} \in E(G)$. For any $i \in [k + 2]$ and $j \in [k]\{i\}$, $\{v_i, u_j\} \notin E(G)$, as $u_j \notin N(v_i)$ by the definition of a strong system of distinct representatives. Thus, $(A, B)$ is a pair of perfectly matched sets of size at least $(k + 2)$ in $G$. ▶

As planar graphs are 5-degenerate, the above result directly gives us a polynomial kernel (which is not linear!) for planar graphs. We next obtain a linear kernel for planar graphs.

Linear Kernel on Planar Graphs. We describe a procedure to obtain a linear-sized vertex kernel for planar graphs. To this end, we state the following useful result.

Proposition 27 (Theorem 4.11, [18]). A twinless planar graph with $n \geq 2$ vertices contains an induced matching of size at least $n/40$.

From Proposition 27, we have the following observation.

Observation 28. Let $G$ be a planar graph on $n \geq 4$ vertices such that there are no three vertices that are pairwise false twins. Then $G$ contains a pair of perfectly matched sets of size at least $n/80$. ▶
Proof. From $G$, we can construct a twinless planar graph $G'$ by keeping exactly one of the false twins i.e. for any two false twins $u$ and $v$, we delete exactly one of them. Hence $G'$ is a twinless planar graph with size at least $n/2 \geq 2$ vertices. From Proposition 27, $G'$ has an induced matching of size at least $n/80$, which is also an induced matching in $G$. But such an induced matching gives us a pair of perfectly matched sets of size $n/80$. □

Theorem 29. Perfectly Matched Sets on planar graphs admits an $O(k)$-sized kernel.

Proof. Consider an instance $(G,k)$ of the problem, where $G$ is a planar graph with $n$ vertices. Apply Reduction Rule 1 as long as it is applicable. If $|V(G)| < 2$, then we are done. Otherwise, from Observation 28, $G$ has a pair of perfectly matched sets with size at least $n/80$. If $k \leq n/80$, then the given instance is a yes-instance, and otherwise $|V(G)| < 80k$. □

References


