On the Hardness of Generalized Domination Problems Parameterized by Mim-Width

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Abstract

For nonempty $\sigma, \rho \subseteq \mathbb{N}$, a vertex set S in a graph G is a (σ, ρ)-*dominating set* if for all $v \in S$, $|N(v) \cap S| \in \sigma$, and for all $v \in V(G) \setminus S$, $|N(v) \cap S| \in \rho$. The Min/Max (σ, ρ)-Dominating Set problems ask, given a graph G and an integer k, whether G contains a (σ, ρ)-dominating set of size at most k and at least k, respectively. This framework captures many well-studied graph problems related to independence and domination. Bui-Xuan, Telle, and Vatshelle [TCS 2013] showed that for finite or co-finite σ and ρ, the Min/Max (σ, ρ)-Dominating Set problems are solvable in XP time parameterized by the *mim-width* of a given branch decomposition of the input graph. In this work we consider the parameterized complexity of these problems and obtain the following: For minimization problems, we complete several scattered W[1]-hardness results in the literature to a full dichotomoy into polynomial-time solvable and W[1]-hard cases, and for maximization problems we obtain the same result under the additional restriction that σ and ρ are finite sets. All W[1]-hard cases hold assuming that a *linear* branch decomposition of bounded mim-width is given, and with the solution size being an additional part of the parameter. Furthermore, for all W[1]-hard cases we also rule out $f(w)n^{o(w/\log w)}$-time algorithms assuming the Exponential Time Hypothesis, where f is any computable function, n is the number of vertices and w the mim-width of the given linear branch decomposition of the input graph.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms; Mathematics of computing → Graph algorithms

Keywords and phrases generalized domination, linear mim-width, W[1]-hardness, Exponential Time Hypothesis

Digital Object Identifier 10.4230/LIPIcs.IPEC.2022.3

Related Version *First author's Master's thesis*: `https://bora.uib.no/bora-xmlui/handle/11250/3001140` [2]

Funding *Lars Jaffke*: Supported by the Norwegian Research Council (project number 274526).

1 Introduction

Maximum induced matching width [35], or mim-width for short, is a width measure of graphs based on branch decompositions over the vertex set. On the one hand, mim-width has high expressive power, while on the other hand, it allows for efficient algorithms for many fundamental NP-hard problems when the input graph is given together with a decomposition of small width. Mim-width strictly generalizes tree-width and clique-width, in the sense that a bound on each of the latter measures implies a bound on the mim-width, while there are n-vertex graphs that have clique-width $\Omega(\sqrt{n})$ and mim-width 1 [3, 24]. Mim-width and twin-width [9] are incomparable. Moreover, the mim-width remains bounded by a constant on several deeply studied graph classes such as interval graphs, permutation graphs, and some of their generalizations, see e.g. [3, 10, 27, 35], as well as several graph classes excluding

Table 1 Some examples of MIN/MAX (σ, ρ)-DOMINATING SET problems and their complexity when parameterized by the mim-width of a given (linear) branch decomposition of the input graph plus solution size. In all occurrences, the value of d is a fixed constant.

Standard name	σ	ρ	MIN	MAX
INDEPENDENT SET	$\{0\}$	$\mathbb{N}$	P	W[1]-h [19]
DOMINATING SET	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	W[1]-h [19]	P
INDEPENDENT DOMINATING SET	$\{0\}$	$\mathbb{N} \setminus \{0\}$	W[1]-h [19]	W[1]-h [19]
TOTAL DOMINATING SET	$\mathbb{N} \setminus \{0\}$	$\mathbb{N} \setminus \{0\}$	W[1]-h [28]	P
STRONG STABLE SET/2-PACKING	$\{0\}$	$\{0,1\}$	P	W[1]-h [This]
PERFECT CODE	$\{0\}$	$\{1\}$	W[1]-h [This]	W[1]-h [This]
TOTAL NEARLY PERFECT SET	$\{0,1\}$	$\{0,1\}$	P	W[1]-h [This]
WEAKLY PERFECT DOMINATING SET	$\{0,1\}$	$\{1\}$	W[1]-h [This]	W[1]-h [This]
TOTAL PERFECT DOMINATING SET	$\{1\}$	$\{1\}$	W[1]-h [This]	W[1]-h [This]
INDUCED MATCHING	$\{1\}$	$\mathbb{N}$	P	W[1]-h [28]
DOMINATING INDUCED MATCHING	$\{1\}$	$\mathbb{N} \setminus \{0\}$	W[1]-h [28]	W[1]-h [28]
PERFECT DOMINATING SET	$\mathbb{N}$	$\{1\}$	W[1]-h [This]	?
d-DOMINATING SET	$\mathbb{N}$	$\{d, d+1, \ldots\}$	W[1]-h [28]	?
INDUCED d-REGULAR SUBGRAPH	$\{d\}$	$\mathbb{N}$	P	W[1]-h [28]
SUBGRAPH OF MIN DEGREE $\geq d$	$\{d, d+1, \ldots\}$	$\mathbb{N}$	P	?
INDUCED SUBG. OF MAX DEGREE $\leq d$	$\{0, 1, \ldots, d\}$	$\mathbb{N}$	P	W[1]-h [28]

small graphs as induced subgraphs [11]. This implies that algorithms for graphs of bounded mim-width often unify and extend several algorithmic results on graph classes from the literature.

In recent years, an increasing number of problems has been shown to admit such algorithms [4, 5, 6, 13, 20, 25, 28, 29, 30]. However, all of these algorithms run in XP time when parameterized by the mim-width of the given branch decomposition of the input graph, and the parameterized complexity of these problem is much less understood. In this work, we contribute to the systematic study of the parameterized complexity of problems parameterized by the mim-width of a given (linear) branch decomposition of the input graph, by showing dichotomies into polynomial-time solvable and W[1]-hard cases for locally checkable minimization and maximization problems.

The *locally checkable vertex subset problems*, or (σ, ρ)*-domination problems* [34], capture many problems related to independence and domination in graphs in a unified framework. Here, a problem is formulated by prescribing for its solutions, which are vertex sets, for each vertex v in the graph, how many neighbors it has to have in the set, depending on whether v is in the set or not. Concretely, for two nonempty sets $\sigma, \rho \subseteq \mathbb{N}$, a (σ, ρ)*-dominating set* in a graph G is a set of vertices S such that for each vertex in S, the number of neighbors it has in S is an element of σ, and for each vertex outside of S, the number of neighbors it has in S is an element of ρ. The MIN/MAX (σ, ρ)-DOMINATING SET problems ask, given a graph G and an integer k, whether G contains a (σ, ρ)-dominating set of size at most k and at least k, respectively. Observe for instance that the MIN $(\mathbb{N}, \mathbb{N} \setminus \{0\})$-DOMINATING SET problem is the MINIMUM DOMINATING SET problem, and that the MAX $(\{0\}, \mathbb{N})$-DOMINATING SET problem is the MAXIMUM INDEPENDENT SET problem. Many more problems can be expressed in this way, see Table 1 for examples. While such problems are often NP-complete, several (σ, ρ)-domination problems are trivial to solve – for instance all MIN (σ, ρ_0)-DOMINATING SET problems, where $0 \in \rho_0$. This is simply because the empty set is a solution to any such MIN (σ, ρ_0)-DOMINATING SET problem.

The (σ, ρ)-domination problems play a central role in the algorithmic study of mim-width. They are among the first problems that have been shown to be solvable in XP time parameterized by the mim-width of a given branch decomposition of the input graph by Bui-Xuan et al. [13] (whenever σ and ρ are finite or co-finite), and contain the first problems for which W[1]-hardness in this parameterization was shown. Fomin et al. [19] proved that Maximum Independent Set and Minimum Dominating Set are W[1]-hard parameterized by the mim-width of a given linear branch decomposition of the input graph. The W[1]-hardness of several other (σ, ρ)-domination problems was shown by Jaffke et al. [28]. However, these results are far from complete dichotomies. For minimization problems, we achieve such a dichotomy in this work, and for maximization problems, whenever σ and ρ are finite. In both cases, hardness already holds for the more restrictive parameterization by *linear* mim-width.

▶ **Theorem 1.** *Let $\sigma, \rho \subseteq \mathbb{N}$ be nonempty. If $0 \in \rho$, then Min (σ, ρ)-Dominating Set is polynomial-time solvable, otherwise it* W[1]*-hard parameterized by the mim-width of a given linear branch decomposition of the input graph plus solution size.*

▶ **Theorem 2.** *Let $\sigma, \rho \subseteq \mathbb{N}$ be nonempty and finite. If $\rho = \{0\}$, then Max (σ, ρ)-Dominating Set is polynomial-time solvable, otherwise it is* W[1]*-hard parameterized by the mim-width of a given linear branch decomposition of the input graph plus solution size.*

Note that since the solution size can be a part of the parameter in the previous theorems, they extend several hardness results for Min/Max (σ, ρ)-Dominating Set problems parameterized by solution size due to Golovach et al. [23]. They obtained a dichotomy into polynomial-time solvable and W[1]-complete for Min (σ, ρ)-Dominating Set when σ and ρ are finite.

Mim-width and the Exponential Time Hypothesis. All known XP-algorithms for problems parameterized by mim-width, except for the ones in [6], run in $n^{O(w)}$ time, where n is the number of vertices of the input graph, and w the mim-width of the given branch decomposition. A natural follow-up question to Theorems 1 and 2 is whether the dependence on w can be improved, in particular if one of these problems admits an $n^{o(w)}$ time algorithm. Several of the reductions given in [19, 28] start from the Multicolored Clique problem parameterized by the number of color classes k, and the mim-width of the instance constructed in the reduction is quadratic in k. This is due to the fact that the gadgeteering depends on the number of edges in the (complete) quotient graph associated with the color partition of the input graph. Therefore these reductions only rule out $f(w)n^{o(\sqrt{w})}$ time algorithms under the Exponential Time Hypothesis (ETH). We can observe that the same reduction works if we start from the Partitioned Subgraph Isomorphism problem parameterized by the number of edges h in the pattern graph, and the mim-width of the reduced instance remains $O(h)$;[1] this gives a strengthened lower bound of $f(w)n^{o(w/\log w)}$ time by a theorem due to Marx [31]. All reductions presented in this work start from the Partitioned Subgraph Isomorphism problem and give the improved lower bounds under the ETH. It remains an open problem to close the gap between the $f(w)n^{o(w/\log w)}$ time lower bounds and the $n^{O(w)}$ time algorithms.

[1] For a worked out example, see [2].

▶ **Corollary 3.** *Let $\sigma, \rho \subseteq \mathbb{N}$ be nonempty. If $0 \notin \rho$, then* Min (σ,ρ)-Dominating Set *does not admit $f(w)n^{o(w/\log w)}$ time algorithms, for any computable function f, on n-vertex graphs given with a linear branch decomposition of mim-width w, unless the* ETH *is false. If σ and ρ are finite and $\rho \neq \{0\}$, then the same holds for* Max (σ,ρ)-Dominating Set.

Related work. Bui-Xuan et al. [13] showed that the Min/Max (σ,ρ)-Dominating Set problems are XP-time solvable parameterized by the mim-width of a given branch decomposition of the input graph, whenever σ and ρ are either finite or co-finite. The first W[1]-hardness proofs for several Min/Max (σ,ρ)-Dominating Set problems were given in [19, 28]. However, several other problems have been shown to be even harder on graphs of bounded mim-width, which often follows from the NP-completeness of problems on graph classes that have constant mim-width [3]. For instance, the following problems are para-NP-hard parameterized by the mim-width of a given linear branch decomposition of the input graph: Clique and Co-Dominating Set [22, 35], Graph Coloring [21], Maximum Cut [1], and Hamiltonian Path [29]. The NP-completeness of Min/Max (σ,ρ)-Dominating Set problems has been systematically studied by Telle [33], and Golovach et al. [23] considered their complexity parameterized by solution size.

Methods. As mentioned above, all reductions we give start from the Partitioned Subgraph Isomorphism (PSI) problem. Here, we are given two graphs G and K, and a partition of $V(G)$ where each part is associated with a vertex from K, and the question is whether G contains K as a subgraph witnessed by an isomorphism that respects the partition of $V(G)$. This problems is known to be W[1]-hard parameterized by $h = |E(K)|$ and not to have $f(h)n^{o(h/\log h)}$-time algorithms, where $n = |V(G)|$, unless the Exponential Time Hypothesis fails [18, 31, 32].

We give a high level outline of how we reduce the PSI problem to any (σ,ρ)-Dominating Set problem. As the overall strategy is the same for all choices of σ and ρ, and whether we are concerned with minimization or maximization, we do not specify which case we are in for now. Let (G,K) be an instance of PSI and for ease of reference, suppose that $V(K) = \{1,\ldots,k\}$, and let V_i be the part of the partition of $V(G)$ corresponding to vertex i. The graph H of the (σ,ρ)-Dominating Set instance contains, for each V_i, a set $\hat{S}_i$ of *selection vertices* that encodes which vertex of V_i is chosen in a potential solution to (G,K). For each edge $ij \in E(K)$, we add a subgraph to H that preserves information about adjacencies between the vertices in V_i and V_j, but induces cuts that have no induced matchings of size larger than two. This construction is adapted from the work of Fomin et al. [19]. It ensures that once a (σ,ρ)-dominating set D contains precisely one vertex from each $\hat{S}_i$, then the remainder of the vertices in D witness the existence of a K-subgraph in G.

To ensure that each solution to the (σ,ρ)-Dominating Set instance picks precisely one vertex from each $\hat{S}_i$, we add gadgets to H that depend on the choice of σ and ρ and whether we are concerned with a minimization or a maximization problem. These gadgets are constructed carefully enough so that the linear mim-width of H does not increase prohibitively. In the end, we have a partition of H such that each subgraph induced by a part has linear mim-width that only depends on the some fixed constants contained in σ and ρ, and such that the cuts between the parts do not contain large induced matchings either. By adapting a lemma of Brettell et al. [12] to linear mim-width, and with a slightly more careful analysis, we conclude that we can construct in polynomial time a linear branch decomposition of H that has mim-width $O(h)$.

The high degree of generality in our reductions is achieved by the following: the construction combined with the budget are tight enough so that, roughly speaking, in minimization problems, each vertex can only be *minimially dominated* and in maximization problems, each vertex has to be *maximally dominated*. This means that in either case, σ and ρ only contain one relevant value for feasible solutions: for minimization that is $\varsigma = \min \sigma$ and $\varrho = \min \rho$, and for maximization ς becomes $\max \sigma$ and ϱ becomes $\max \rho$. The linear mim-width of H depends on ς and ϱ, and in the case of minimization problems, these are always constants. In the case of maximization, however, ς and ϱ are only constant when σ and ρ are finite.

Throughout the paper, proofs of statements marked with "♣" and full proofs of sketches are deferred to the full version.

2 Preliminaries

For basic background in graph theory, we refer to [15], and for basics in parameterized complexity, we refer the reader to [14, 16]. We use the following notation: $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $[n] = \{1, 2, ..., n\}$, $[n]_0 = \{0, 1, 2, ..., n\}$. For a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, we let $G[A, B]$ be the biparite graph with vertex set $A \cup B$ and edge set $\{ab \mid ab \in E(G), a \in A, b \in B\}$. A *matching* in a graph G is a set $M \subseteq E(G)$ of pairwise disjoint edges. We say that M is *induced* if there are no additional edges between the endpoints of the edges in M; that is, if u is an endpoint of some edge in M and v is some endpoint of some edge in M, then either $uv \in M$ or $uv \notin E(G)$. For a graph G, we denote by $\mathsf{cc}(G)$ the set of its connected components. For additional clarification of basic graph theoretic concepts and notation we refer to the full version.

Mim-Width. For a graph G and $A, B \subseteq V(G)$ with $A \cap B = \emptyset$ we define $\mathsf{cutmim}_G(A, B)$ to be the largest size of any induced matching in $G[A, B]$. For a set $A \subseteq V(G)$, we let $\mathsf{mim}_G(A) = \mathsf{cutmim}_G(A, V(G) \setminus A)$.

A *branch decomposition* of a graph G is a pair $(T, \mathcal{L})$, where T is a tree where all of whose vertices have degree at most 3, and $\mathcal{L}$ a bijection mapping the vertices of the graph $V(G)$ to the leaves of the tree T. For a subtree T' of T, we denote by $V_{T'}$ the vertices of G that are mapped to leaves of T'. The *mim-width of* $(T, \mathcal{L})$ is $\mathsf{mimw}_G(T, \mathcal{L}) = \max_{e \in E(G), T' \in \mathsf{cc}(T-e)} \mathsf{mim}_G(V_{T'})$. The *mim-width of* G, denoted by $\mathsf{mimw}(G)$, is the minimum mim-width over all its branch decompositions.

A branch decomposition $(T, \mathcal{L})$ is called *linear* if T is a caterpillar graph, i.e., a tree containing an induced path P such that each vertex in $V(T) \setminus V(P)$ has precisely one neighbor on P. The *linear mim-width* of a graph G, denoted by $\mathsf{linmimw}(G)$, is the minimum mim-width over all its linear branch decompositions. Linear branch decompositions can be equated with linear orderings of the vertex set of a graph. For a linear order Λ of the vertices of G, we will therefore write $\mathsf{mimw}_G(\Lambda)$ for the mim-width of the linear branch decomposition corresponding to Λ. In all definitions given in these last paragraphs, we may drop G as a subscript if it is clear from the context.

Exponential-Time Hypothesis. The Exponential-Time Hypothesis (ETH) is a conjecture about the complexity of the 3-SAT problem, which given a boolean formula in conjunctive normal form and clauses of size at most three, asks whether it has a satisfying assignment.

▶ **Conjecture 4** (ETH [26], informal). *The 3-SAT problem cannot be solved in $2^{o(n)}$ time, where n is the number of variables of the input formula.*

2.1 Generalized dominating set problems

Let $\sigma, \rho \subseteq \mathbb{N}$, and let G be a graph. A vertex set $S \subseteq V(G)$ is a (σ, ρ)-*dominating set*, if for all $v \in V(G)$: If $v \in S$, then $|N(v) \cap S| \in \sigma$, and if $v \notin S$,then $|N(v) \cap S| \in \rho$. The computational problems associated with (σ, ρ)-dominating sets we consider in this work are:

Min/Max (σ, ρ)-Dominating Set

Input:	Graph G, integer k
Question:	Does G contain a (σ, ρ)-dominating set of size at most/at least k?

Many maximization and minimization problems formulated in this manner are computationally hard, in the sense that they are NP-hard and W[1]-hard with solution size as a parameter. We now discuss the exceptions that are relevant for this work, i.e. some cases when the Min/Max (σ, ρ)-Dominating Set problems are polynomial-time solvable.

Trivial minimization problems. Whenever $0 \in \rho$, the empty set is a solution of the Min (σ, ρ)-Dominating Set problem. This case is then trivial as any algorithm can always return the empty set as a valid optimal solution. These are the only trivial cases for minimization.

Trivial maximization problems. We focus here on trivial cases where σ and ρ are finite, since these are the cases for which we show hardness in this work. Note however that there are more trivial cases when σ and ρ need not be finite, for instance when $\sigma = \mathbb{N}$: in this case, the entire vertex set of the input graph is a valid optimal solution.

If $\rho = \{0\}$, then any solution has to consist of connected components of the input graph. Suppose S is a $(\sigma, \{0\})$-dominating set of a graph G and let $C_1, C_2 \in \mathsf{cc}(G)$. Then, whether or not $C_1 \subseteq S$ is independent of whether or not $C_2 \subseteq S$. Furthermore, for any connected component $C \in \mathsf{cc}(G)$, we can verify in polynomial time whether or not C can be contained in a $(\sigma, \{0\})$-dominating set: we only have to check for all $v \in C$ that $\deg(v) \in \sigma$. Therefore we can use a greedy algorithm to solve the problem, by first identifying all connected components of the input graph followed by greedily including each connected component C in the solution if it passes the aforementioned check.

▶ **Observation 5.** *Let $\sigma, \rho \subseteq \mathbb{N}$. If $0 \in \rho$, then Min (σ, ρ)-Dominating Set is polynomial-time solvable, and if $\rho = \{0\}$, then Max (σ, ρ)-Dominating Set is polynomial-time solvable.*

2.2 Problem Definitions

We collect here the definitions of the problems that are relevant to this work. The following parameterized variant of the Min/Max (σ, ρ)-Dominating Set problems is the main object of study.

Min/Max (σ, ρ)-Dominating Set[LMim + Sol]

Input:	Graph G, integer k, linear order Λ of $V(G)$.
Parameter:	$\mathsf{mimw}(\Lambda) + k$.
Question:	Does G contain a (σ, ρ)-dominating set of size at most/at least k?

The starting point of our reductions will be the Partitioned Subgraph Isomorphism problem, which is known to be W[1]-hard and not to have $f(h)n^{o(h/\log h)}$-time algorithms, unless the ETH is false [31].

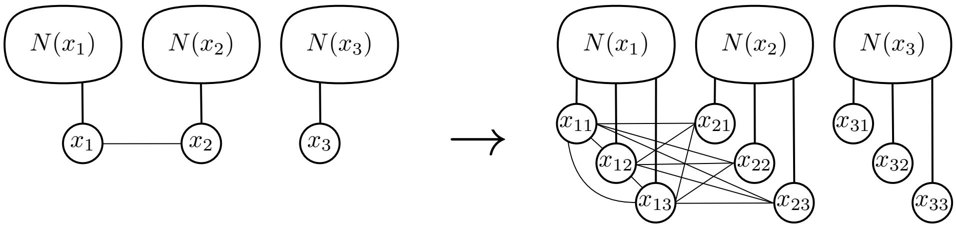

Figure 1 2-Blowups of the vertices x_1, x_2, and x_3. In vertex x_1, a clique blowup was performed and in x_2 and x_3 and independent blowup.

PARTITIONED SUBGRAPH ISOMORPHISM

Input:	(G, K, ϕ), where G and K are graphs, and $\phi\colon V(G) \to V(K)$.
Parameter:	$h = \|E(K)\|$.
Question:	Is there an injective function $f : V(K) \to V(G)$ such that $ab \in E(K) \Rightarrow f(a)f(b) \in E(G)$ for all $a, b \in V(K)$, and $\phi(f(a)) = a$ for all $a \in V(K)$?

We introduce some notation that will be useful when talking about instances of PARTITIONED SUBGRAPH ISOMORPHISM. We say a function $f : V(K) \to V(G)$ *preserves neighbors* if $ab \in E(K) \Rightarrow f(a)f(b) \in E(G)$ for all $a, b \in V(K)$, and f *preserves colors* (relative to $\phi : V(G) \to V(K)$) if $\phi(f(a)) = a$ for all $a \in V(K)$. As these above mentioned hardness result from [31] also holds when the pattern graph K is connected, we we will commonly make this assumption throughout the paper.

3 Graph operations and bounds on the linear mim-width

The following lemma can be seen as an analogue of a lemma due to Brettel et al. [12] for linear mim-width.

▶ **Lemma 6** (♣, Cf. Lemma 7 in [12]). *Let G be a graph, let $X = (X_1, \ldots, X_p)$ be a partition of $V(G)$ such that* $\mathsf{cutmim}_G(X_i, X_j) \le c$ *for all distinct $i, j \in [p]$, and let G/X be the quotient graph of X. Then,*

$$\mathsf{linmimw}(G) \le |E(G/X)| \cdot c + \max_{i\in[p]} \mathsf{linmimw}(G[X_i]).$$

Moreover, if for all $i \in [p]$, Λ_i is a linear order of X_i, then one can in polynomial time construct a linear order Λ of G with

$$\mathsf{mimw}(\Lambda) \le |E(G/X)| \cdot c + \max_{i\in[p]} \mathsf{mimw}(\Lambda_i).$$

The following operation is illustrated in Figure 1.

▶ **Definition 7** (Blowup). *Let G be a graph, $v \in V(G)$, and $k \in \mathbb{N}$. A* clique/independent k-blowup of v *is the operation of adding k twins of v which form a clique/independent set. We call an operation simply a* blowup *if it is either a clique k-blowup or an independent k-blowup for some $k \in \mathbb{N}$.*

We show that performing blowups cannot increase the mim-width by more than 1. Note in the following lemma that we consider a series of blowups performed at once instead of a single blowup.

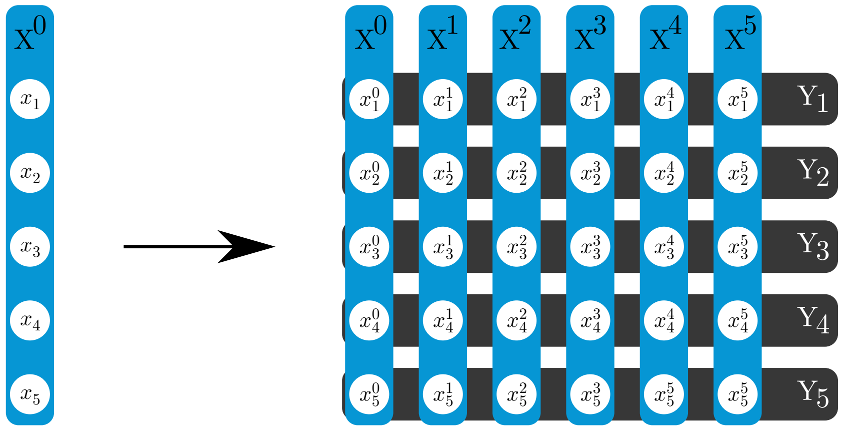

Figure 2 A depth-5 grid of cliques implant at $\{x_1, \ldots, x_5\}$. Shaded regions indicate cliques.

▶ **Lemma 8** (♣). *Let G be a graph, and let Λ be a linear ordering of G. Let G' be obtained from G by a series of blowups. Then, there is a linear order Λ' of $V(G')$ computable in polynomial time from Λ such that* $\mathsf{mimw}(\Lambda') \leq \mathsf{mimw}(\Lambda) + 1$.

We define another operation that will find a similar use in the later sections.

▶ **Definition 9** (Depth-ℓ grid of cliques implant). *Let G be a graph, let $X = \{x_1, \ldots, x_k\} \subseteq V(G)$ be a clique in G, and let $\ell \in \mathbb{N}$. For all $i \in [k]$, let $x_i = x_i^0$. The operation of*

- *adding, for all $i \in [\ell]$, vertices $x_1^i, \ldots, x_k^i$,*
- *for each $i \in [\ell]$, making $\{x_1^i, \ldots, x_k^i\} = X^i$ a clique (called the i-th column), and*
- *for each $j \in [k]$, making $\{x_j^0, \ldots, x_j^\ell\} = Y_j$ a clique (called the j-th row),*

is called a depth-ℓ grid of cliques implant (at X in G).

For an illustration of the previous operation see Figure 2.

▶ **Lemma 10** (♣). *Let G be a graph, let Λ be a linear ordering of G. Let G' be obtained from G by a depth-ℓ grid of cliques implant. There exists a linear ordering Λ' of G' computable in polynomial time from Λ, such that* $\mathsf{mimw}(\Lambda') \leq \mathsf{mimw}(\Lambda) + \ell$.

4 Hardness of (σ, ρ)-Dominating Set problems

In this section we discuss the main results of this work, which are the hardness results for non-trivial MIN/MAX (σ, ρ)-DOMINATING SET problems. Note that the cases that are not covered by the following theorem ($0 \in \rho$ for minimization and $\rho = \{0\}$ for maximization) have been observed to be trivial in Observation 5.

▶ **Theorem 11.** *Let $\sigma, \rho \subseteq \mathbb{N}$ be nonempty where $0 \notin \rho$. Then, the MIN (σ, ρ)-DOMINATING SET[LMIM + SOL] problem is* W[1]*-hard. Moreover, unless the* ETH *is false, it cannot be solved in $f(w)n^{o(w/\log w)}$ time, where f is any computable function, on n-vertex graphs given with a linear ordering of mim-width w.*

Furthermore, if σ and ρ are nonempty, finite, and $\rho \neq \{0\}$, then the MAX (σ, ρ)-DOMINATING SET[LMIM + SOL] problem is W[1]*-hard, and cannot be solved in $f(w)n^{o(w/\log w)}$ time, where f, n, and w are as above, unless the* ETH *is false.*

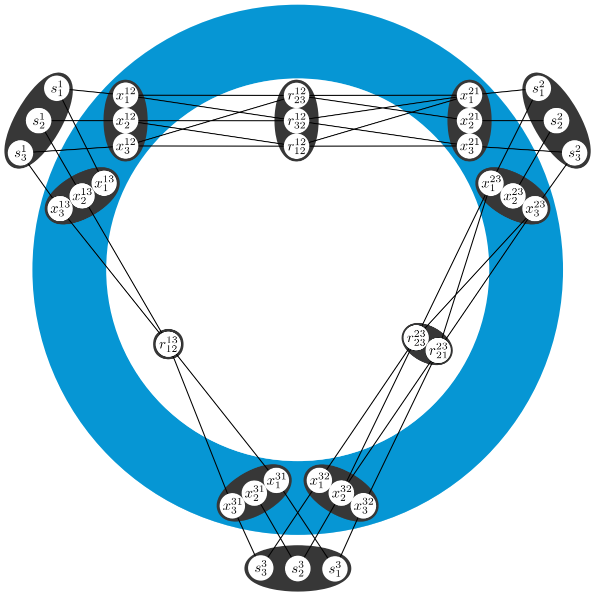

Figure 3 Example of $\mathcal{H}$ for $p = 3$, and $k = 3$, and K is the complete graph with three vertices. Colored regions indicate cliques.

The proof is by a reduction from the W[1]-hard problem Partitioned Subgraph Isomorphism, where first a core graph $\mathcal{H}$ is constructed. Afterwards the graph is modified to obtain either H_0, H_1, H_2, or H_3 depending on σ and ρ in such a manner that all of the above mentioned cases are captured. These modifications use, among other things, the two operations described in Section 3.

4.1 The core graph $\mathcal{H}$

Let (K, G, ϕ) be an instance of the Partitioned Subgraph Isomorphism problem. Recall that for the sake of our reduction, we can assume that K is connected. Throughout, we assume that $V(K) = \{1, \ldots, k\}$, and that $(V_1, \ldots V_k)$ is the partition of $V(G)$ according to ϕ, that is, for all $i \in [k]$, $V_i = \{v \in V(G) \mid \phi(v) = i\}$.

We describe how to construct from it the above mentioned core graph $\mathcal{H}$. We can assume that $|V_i| = p$, for all $i \in [k]$, where $p = \max\{|V_i| \mid i \in [k]\}$. If this is not the case then we can simply add isolated vertices to the sets whose cardinality is less than p. Isolated vertices clearly do not affect the Partitioned Subgraph Isomorphism instance, as K has no isolated vertices as we assumed it was connected. For all $i \in [k]$, we let $V_i = \{v_1^i, ..., v_p^i\}$. The core graph $\mathcal{H}$ is constructed as follows:

1. For all $i, j \in [k]$ such that $ij \in E(K)$, and for all $a \in [p]$, we add the vertex x_a^{ij} to $V(\mathcal{H})$. We let $X^{ij} = \{x_a^{ij} \mid a \in [p]\}$.
2. For all $i, j \in [k]$ such that $ij \in E(K)$ and for all $a, b \in [p]$ such that $v_a^i v_b^j \in E(G)$, we add the vertex $r_{ab}^{ij} = r_{ba}^{ji}$ to $V(\mathcal{H})$. We connect r_{ab}^{ij} to all the vertices in $\{x_{a'}^{ij} \mid a' \neq a, a' \in [p]\}$, and all the vertices in $\{x_{b'}^{ji} \mid b' \neq b, b' \in [p]\}$. We let $R^{ij} = \{r_{ab}^{ij} \mid v_a^i v_b^j \in E(G)\} = R^{ji}$.
3. For all $i \in [k]$ and for all $a \in [p]$ we add the vertex s_a^i to $V(\mathcal{H})$. Furthermore for all $j \in [k]$ such that $ij \in E(K)$, and all $a \in [p]$ we connect s_a^i to x_a^{ij}. We let $S^i = \{s_a^i \mid a \in [p]\}$.
4. For all $i \in [k]$, we make S^i a clique. For all $ij \in E(K)$, we make R^{ij} a clique. We let $X = \bigcup_{ij \in E(K)} X^{ij}$ and make X a clique.

See Figure 3 for an illustration. Notice that $R^{ij} = R^{ji}$ but $X^{ij} \neq X^{ji}$ for all $ij \in E(K)$.

The notation $\mathcal{I}$, $\mathcal{J}$, Z^α, and z^α_β. As S^i and R^{ij} have many similar properties in our reduction, we use the following slight abuse of notation. We let $\mathcal{I} = [k] \cup \{ij \mid ij \in E(K)\}$, and for $\alpha \in \mathcal{I}$, we let Z^α be S^i if $\alpha = i$ for some $i \in [k]$, and we let Z^α be R^{ij} if $\alpha = ij$ for some $ij \in E(K)$. We let $\mathcal{J} = [p] \cup [p] \times [p]$. For a pair $\alpha \in \mathcal{I}$, $\beta \in \mathcal{J}$, the vertex z^α_β is s^i_a if $\alpha \in [k]$ and $\beta \in [p]$ and r^{ij}_{ab} if $\alpha \in E(K)$ and $\beta \in [p] \times [p]$. Note that for the case $\alpha \in [k]$ and $\beta \in [p] \times [p]$ (or $\alpha \in E(K)$ and $\beta \in [p]$), the vertex z^α_β is not defined.

As outlined above, it is essential that the core graph has bounded linear mim-width. We sketch how to obtain a linear order whose mim-width is linear in the number of edges in K.

▷ Claim 12. There is a linear order Λ of $V(\mathcal{H})$ computable in polynomial time such that $\mathsf{mimw}(\Lambda) \leq 4|E(K)| + 4$.

Proof (sketch). Consider the following partition of $V(\mathcal{H})$: Let $\Gamma = \{\Gamma_\alpha \mid \alpha \in \mathcal{I}\}$, where

- for all $i \in [k]$, $\Gamma_i = S^i \cup \bigcup_{ij \in E(K)} X^{ij}$ and
- for all $ij \in E(K)$, $\Gamma_{ij} = R^{ij}$.

We give the vertices in Γ_i the ordering

$$\Lambda_i : s^i_1 < x^{i1}_1 < x^{i2}_1 < \cdots < x^{ik}_1 < s^i_2 < x^{i1}_2 < \cdots < x^{ik}_2 < \cdots < s^i_p < x^{i1}_p < \cdots < x^{ik}_p,$$

and we give the vertices in Γ_{ij} any linear ordering Λ_{ij}. As $\Gamma_{ij} = R^{ij}$ is a clique, any linear ordering of Γ_{ij} has mim-width 1. Furthermore, one can show that the mim-width of Λ_i, for each $i \in [k]$, is at most 3. Considering the subgraph $\mathcal{H}'$ of $\mathcal{H}$ obtained by removing all edges between X^{ij} and $X^{i'j'}$ for distinct $ij, i'j' \in E(K)$, we can prove that $\mathsf{cutmim}_{\mathcal{H}'}[\Gamma_\alpha, \Gamma_{\alpha'}] \leq 2$ for any distinct $\alpha, \alpha' \in \mathcal{I}$. The number of edges in $\mathcal{H}'/\Gamma$ is equal to $2|E(K)|$, so by Lemma 6 we can obtain a linear order of the vertices of $\mathcal{H}'$ whose mim-width is at most $4|E(K)| + 3$. (Take any linear order of $V(\mathcal{H}')$ that respects each Λ_α, $\alpha \in \mathcal{I}$.) To get the bound for $\mathcal{H}$, note that turning a set of vertices into a clique can increase the mim-width of any cut by at most 1. ◁

4.2 Minimization problems

We now turn to the case when we want to show hardness for a Min (σ, ρ)-Dominating Set problem, and describe how the core graph $\mathcal{H}$ will be enhanced/transformed to give the graph of the resulting instance. This construction crucially depends on the minimum values of σ and ρ. Therefore, we let $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$. Note that $\varsigma + \varrho = O(1)$. In each of the cases, we show how the graph of the resulting instance is constructed, and give the budget. We state three claims, one regarding the linear mim-width of the constructed graph, and two claims that assert the correctness of the reduction. We exemplify these proofs in Section 4.2.2, where all arguments for the case treated there are given. The remaining proofs are deferred to the full version.

4.2.1 When $\varrho = \varsigma + 1$ and $\varsigma \geq 1$

We transform $\mathcal{H}$ into the graph solution size pair (H_0, k_0), where

$$k_0 = (2\varsigma + 2)(k + |E(K)|) + (\varsigma + 1)$$

and H_0 is constructed as follows. Recall that Z^α is either S^i when $\alpha = i \in [k]$, or R^{ij} when $\alpha = ij \in E(K)$.

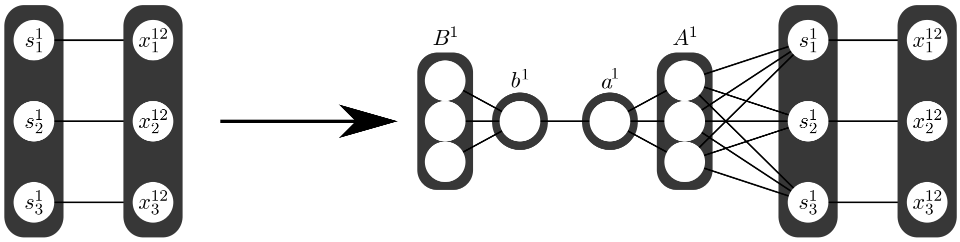

Figure 4 Example of how $S^1 \cup X^{12}$ is transformed for minimization problems when $p = 3$, $\varsigma = 3$, $\varrho = 4$. Circles indicate vertices, and grey colored regions indicate cliques.

1. For all $\alpha \in \mathcal{I}$, we create two cliques A^α and B^α which both have size ς, where A^α is adjacent to all of Z^α.
2. We add two adjacent vertices a^α and b^α. The vertex a^α is adjacent to all vertices in A^α, and b^α is adjacent to all vertices in B^α.
3. We add a clique $\mathcal{X}$ of size $\varsigma + 1$ which is partitioned in two parts: $\mathcal{X}_1$ and $\mathcal{X}_2$, where $|\mathcal{X}_2| = 1$. Every vertex in $\mathcal{X}_1$ is adjacent to all the vertices in X, and every vertex in $\mathcal{X}_2$ is only adjacent to the vertices in $\mathcal{X}$.

For an illustration see Figure 4.

▷ Claim 13 (♣). There is a linear order Λ_0 of $V(H_0)$ computable in polynomial time such that $\mathsf{mimw}(\Lambda_0) = O(|E(K)|)$.

▷ Claim 14 (♣). If (K, G, ϕ) is a YES-instance of the PARTITIONED SUBGRAPH ISOMORPHISM problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_0 in H_0.

▷ Claim 15 (♣). If there exists a (σ, ρ)-dominating set of size at most k_0 in H_0, then (K, G, ϕ) is a YES-instance of the PARTITIONED SUBGRAPH ISOMORPHISM problem.

4.2.2 When $\varrho > \varsigma + 1$ and $\varsigma \geq 1$

Let $\varrho' = \varrho - \varsigma$. In this case, we create the graph solution size pair (H_1, k_1), where

$$k_1 = (\varrho'\varsigma + \varrho')(k + |E(K)|) + (\varsigma + 1),$$

and H_1 is obtained from $\mathcal{H}$ as follows. Recall the operation of a blowup, Definition 7, and see Figure 5 for an illustration of the following.

1. For each $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$ such that $z_\beta^\alpha \in V(\mathcal{H})$, we perform an independent $(\varrho' - 1)$-blowup of z_β^α. We call the twins of z_β^α: $z_{\beta 2}^\alpha, \ldots, z_{\beta \varrho'}^\alpha$, and we let $z_\beta^\alpha = z_{\beta 1}^\alpha$.
2. For each $\alpha \in \mathcal{I}$ and $\ell \in [\varrho']$, we add a clique A_ℓ^α of size ς, where every vertex in A_ℓ^α is adjacent to every vertex in $Z_{*\ell}^\alpha = \{z_{\beta\ell}^\alpha \mid \beta \in \mathcal{J} \text{ s.t. } z_\beta^\alpha \in V(\mathcal{H})\}$. We let $A^\alpha = \bigcup_{\ell \in [\varrho']} A_\ell^\alpha$.
3. We add a clique $\mathcal{X}$ of size $\varsigma + 1$ to H_1; this clique is partitioned into two parts $\mathcal{X}_1$ and $\mathcal{X}_2$, where $|\mathcal{X}_2| = 1$. Every vertex in $\mathcal{X}_1$ is adjacent to all vertices in X, and the vertex in $\mathcal{X}_2$ is only adjacent to $\mathcal{X}$.

We use the following notation. We call the set containing z_β^α with its $\varrho' - 1$ twins $Z_{\beta *}^\alpha = \{z_{\beta\ell}^\alpha \mid \ell \in [\varrho']\}$, and we let $Z_{**}^\alpha = \bigcup_{\ell \in [\varrho']} Z_{*\ell}^\alpha$. Note that the vertices in $Z_{\beta *}^\alpha$ are not adjacent to any other vertex in $Z_{\beta *}^\alpha$, however they are all adjacent to $Z_{\beta' *}^\alpha$ for all $\beta' \neq \beta$.

▷ Claim 16. There is a linear order Λ_1 of $V(H_1)$ computable in polynomial time such that $\mathsf{mimw}(\Lambda_1) = O(|E(K)|)$.

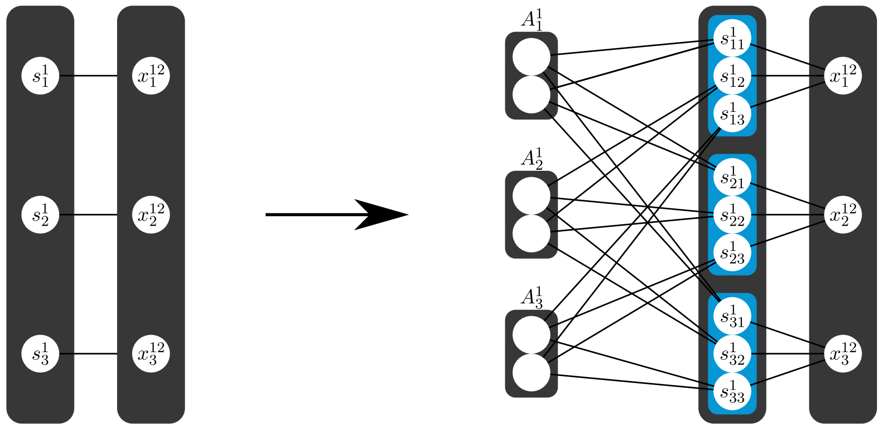

Figure 5 Example of the modification of $S^1 \cup X^{12}$ for $p = 3$, $\varsigma = 2$, $\varrho = 5$. Circles indicate vertices, and grey colored regions indicate cliques. The blue regions indicate independent sets.

Proof. Let Λ be a linear order of $\mathcal{H}$ of mim-width $O(|E(K)|)$ obtained from Claim 12 in polynomial time. H_1 is constructed from $\mathcal{H}$ by blowing up all vertices $z_\beta^\alpha \in V(\mathcal{H})$, where $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$, and adding $|E(K)| + k + 1 = O(|E(K)|)$ vertex sets of constant size. We can place the latter vertices anywhere in the ordering Λ without increasing the mim-width by more than $O(|E(K)|)$, call the resulting ordering Λ'. From Λ' we can obtain a linear order of H_1 in polynomial time whose mim-width is at most one larger using Lemma 8. $\lhd$

We now show the correctness of the reduction in the following two claims.

$\rhd$ Claim 17. If (K, G, ϕ) is a Yes-instance of the Partitioned Subgraph Isomorphism problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_1 in H_1.

Proof. Let $f : V(K) \to V(G)$ be the injective function preserving neighbors and colors. Let $f(i) = v_{c_i}^i$ for all $i \in [k]$ and for some $c_1, ..., c_k \in [p]$. Note that $ij \in E(K)$ implies that $v_{c_i}^i v_{c_j}^j \in E(G)$ further implying that $r_{c_i c_j}^{ij} \in V(\mathcal{H}) \subseteq V(H_1)$. We argue that

$$D = \mathcal{X} \cup \bigcup\nolimits_{i \in [k]} S_{c_i *}^i \cup \bigcup\nolimits_{ij \in E(K)} R_{c_i c_j *}^{ij} \cup \bigcup\nolimits_{\alpha \in \mathcal{I}} A^\alpha$$

is a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_1 in H_1. First, we observe that

$$|D| = \varsigma + 1 + k \cdot \varrho' + |E(K)| \cdot \varrho' + (k + |E(K)|)\varsigma\varrho' = k_1.$$

The sets $\mathcal{X}$, $\bigcup_{\alpha \in \mathcal{I}} Z_{**}^\alpha$, $\bigcup_{\alpha \in \mathcal{I}} A^\alpha$, $\bigcup_{ij \in E(K)} X^{ij}$ form a partition of $V(H_1)$. First consider any vertex x in $\mathcal{X} \subseteq D$, and recall that $\mathcal{X}$ is a clique of size $\varsigma + 1$. If x is the unique vertex in $\mathcal{X}_2$, then $N(x) = \mathcal{X}_1 \subseteq D$, so x has ς neighbors in D. If $x \in \mathcal{X}_1$, then $N(x) = X \cup \mathcal{X} \setminus \{x\}$, and since $X \cap D = \emptyset$, we have that x has ς neighbors in D as well.

Next, consider a vertex z in $\bigcup_{\alpha \in \mathcal{I}} Z_{**}^\alpha$. There are two cases. In the first case, $\alpha = i \in [k]$, and $z = s_{j\ell}^i$ for some $j \in [p]$ and $\ell \in [\varrho']$. If $j = c_i$, then $z = s_{c_i \ell}^i \in D$, and $N(s_{c_i \ell}^i) \cap D = A_\ell^i$, and therefore $|N(z) \cap D| = \varsigma$. (Note that $s_{c_i \ell}^i$ is not adjacent to any vertex in $S_{c_i *}^i \subseteq D$.) If $j \neq c_i$, then $N(z) \cap D = A_\ell^i \cup S_{c_i}^i$, so $|N(z) \cap D| = \varsigma + \varrho' = \varrho$. The second case, when $\alpha = ij \in E(K)$, can be argued in the same way.

Now let a be a vertex in $A^\alpha \subseteq D$, for some $\alpha \in \mathcal{I}$, and assume that $\alpha = i \in [k]$. Then we have that $a \in A_\ell^i$ for some $\ell \in [\varrho']$, and the intersection of $N(a)$ with D consists of $A_\ell^i \setminus \{a\}$, and the vertex $s_{c_i \ell}^i$. We conclude that $|N(a) \cap D| = \varsigma$; the case when $\alpha = ij \in E(K)$ is the same.

Finally, consider some vertex in X^{ij}, where $ij \in E(K)$, in particular such a vertex is x_a^{ij} for some $a \in [p]$. Since $D \cap X = \emptyset$, we have that $x_a^{ij} \notin D$. Furthermore, $N(x_a^{ij})$ contains $\mathcal{X}_1 \subseteq D$. Now, suppose that $a = c_i$. Then, $S^i_{c_i *}$ is also in $N(x_a^{ij}) \cap D$. The only other neighbors of vertices in X^{ij} that are not in $X \cup S^i_{**} \cup \mathcal{X}$ are in R^{ij}_{**}. However, the only vertices in $D \cap R^{ij}_{**}$ are in $R^{ij}_{c_i c_j *}$, and by construction, $x^{ij}_{c_i}$ is not adjacent to any of them. Therefore, $x^{ij}_{c_i}$ has $\varsigma + \varrho' = \varrho$ neighbors in D. If $a \neq c_i$, the argument is similar, but with the roles of S^i_a and $R^{ij}_{c_i c_j *}$ exchanged. ◁

▷ Claim 18. If there exists a (σ, ρ)-dominating set of size at most k_1 in H_1, then (K, G, ϕ) is a Yes-instance of the Partitioned Subgraph Isomorphism problem.

Proof. Let $D \subseteq V(H_1)$ be the (σ, ρ)-dominating set of size at most k_1 in H_1. We show that for all $\alpha \in \mathcal{I}$ there is some $\beta \in \mathcal{J}$ such that $Z^\alpha_{\beta *} \subseteq D$. From these pairs (α, β), we will then derive a solution to (K, G, ϕ). Recall that $|Z^\alpha_{\beta *}| = \varrho'$ for all such α, β, so as a first step we show that

$$\text{for all } \alpha \in \mathcal{I}, |Z^\alpha_{**} \cap D| = \varrho'. \tag{1}$$

Let $\alpha \in \mathcal{I}$. We first show that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| = \varrho'(\varsigma + 1)$, and narrow down to prove (1) afterwards. Towards this, we argue that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'(\varsigma + 1)$. Observe that

$$\text{for all } \ell \in [\varrho'], A^\alpha_\ell \subseteq D \text{ or } |N[A^\alpha_\ell] \cap D| \geq \varrho. \tag{2}$$

Indeed, for all $v \in A^\alpha_\ell$, either $v \in D$, or $|N(v) \cap D| \geq \varrho$. This in turn means that either $A^\alpha \subseteq D$ or that there is some $\ell \in [\varrho']$, such that $|N[A^\alpha_\ell] \cap D| \geq \varrho$. Let a be the number of $\ell \in [\varrho']$ such that $A^\alpha_\ell \not\subseteq D$. Since $|A^\alpha_\ell| = \varsigma$ and for distinct $\ell, \ell' \in [\varrho']$, $N[A^\alpha_\ell] \cap N[A^\alpha_{\ell'}] = \emptyset$, this implies together with (2) that

$$|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq (\varrho' - a)\varsigma + a\varrho = \varrho'(\varsigma + a). \tag{3}$$

Now, if $a \geq 1$, then we can conclude immediately that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'(\varsigma + 1)$, so suppose $a = 0$. Then, by (2), $A^\alpha_\ell \subseteq D$ for all $\ell \in [\varrho']$. However, $|A^\alpha_\ell| = \varsigma$, so there has to be at least one more vertex in $N(A^\alpha_\ell) \cap D$. Since $N(A^\alpha_\ell) \subseteq Z^\alpha_{**}$, and for distinct $\ell, \ell' \in [\varrho']$ $N(A^\alpha_\ell) \cap N(A^\alpha_{\ell'}) = \emptyset$, it follows that D has to contain at least another ϱ' vertices from Z^α_{**}; therefore, also when $a = 0$, we have that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'(\varsigma + 1)$.

We show that by the choice of k_1, the inequality we just argued is an equality. To do so, consider $\mathcal{X}$. Since there is a vertex in $\mathcal{X}$ whose degree is ς, and since $\varsigma < \varrho$, we conclude that $\mathcal{X} \subseteq D$. We have argued that

$$|D| \geq \varsigma + 1 + |\mathcal{I}| \cdot \varrho' \cdot (\varsigma + 1) = \varsigma + 1 + (k + |E(K)|)(\varrho'\varsigma + \varrho') = k_1,$$

and since $|D| \leq k_1$ by assumption, we have that $|D| = k_1$ and

$$\text{for all } \alpha \in \mathcal{I}, |(Z^\alpha_{**} \cup A^\alpha) \cap D| = \varrho'(\varsigma + 1). \tag{4}$$

Note that since the vertices considered so far already use up all the budget, we also have $X \cap D = \emptyset$.

As a last step, we argue that $A^\alpha \subseteq D$, which together with (4) implies (1). (Recall that $|A^\alpha| = \varrho'\varsigma$.) In other words, we want to show that $a = 0$. If $a > 1$, then by (3) we get a contradiction with (4). So suppose that $a = 1$, and let $\ell \in [\varrho']$ be such that $A^\alpha_\ell \not\subseteq D$. For each $\ell' \in [\varrho'] \setminus \{\ell\}$, $A^\alpha_{\ell'} \subseteq D$; and since $|A^\alpha_{\ell'}| = \varsigma$, there is at least one more vertex in

$N(A^\alpha_{\ell'}) \cap D$. Similar to above, this allows us to conclude that $|D'| \geq (\varrho' - 1)(\varsigma + 1)$, where $D' = D \cap \bigcup_{\ell' \in [\varrho'] \setminus \{\ell\}} N[A^\alpha_{\ell'}]$. By (2), we have that $|N[A^\alpha_\ell] \cap D| \geq \varrho$, and by construction $N[A^\alpha_\ell] \cap D' = \emptyset$. Together with (4) this means that

$$\varrho'(\varsigma + 1) = |(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq (\varrho' - 1)(\varsigma + 1) + \varrho = \varrho'(\varsigma + 1) + \varrho' - 1,$$

which only holds if $\varrho' \leq 1$. However, $\varrho > \varsigma + 1$, so $\varrho' = \varrho - \varsigma > 1$, a contradiction. We have argued that $a = 0$, and therefore $A^\alpha \subseteq D$, proving (1) due to (4).

Now that we know that $|Z^\alpha_{**} \cap D| = \varrho'$, it remains to show that there is some $\beta \in \mathcal{J}$ such that $Z^\alpha_{\beta *} \subseteq D$. Suppose not, then there exists some $\gamma \in \mathcal{J}$ such that $1 \leq |Z^\alpha_{\gamma *} \cap D| < \varrho'$. Let $z^\alpha_{\gamma \ell} \notin D$ where $\ell \in [\varrho']$. The neighborhood of $z^\alpha_{\gamma\ell}$ is contained in A^α_ℓ, Z^α_{**}, and X. So, $z^\alpha_{\gamma\ell}$ has ς neighbors in $D \cap A^\alpha_\ell$, no neighbors in $D \cap X$ (recall that $X \cap D = \emptyset$), and at most $\varrho' - 1$ neighbors in $D \cap Z^\alpha_{**}$. The latter is due to the fact that $Z^\alpha_{\gamma *}$ contains at least one vertex from D, and the fact that $Z^\alpha_{\gamma *}$ is an independent set. So $|N(z^\alpha_{\gamma\ell}) \cap D| \leq \varsigma + \varrho' - 1 = \varrho - 1$, a contradiction with D being a (σ, ρ)-dominating set.

Then for all $i \in [k]$ there exists a $c_i \in [p]$ such that $S^i_{c_i *} \subseteq D$, and for all $ij \in E(K)$ there exists $d_i, d_j \in [p]$ such that $R^{ij}_{d_i d_j *} \subseteq D$. Suppose that $c_i \neq d_i$ then notice the vertex $x^{ij}_{d_i}$ is only being dominated by the $\varsigma < \varrho$ vertices in $\mathcal{X} \cap D$, but $\varsigma \notin \rho$. Therefore $c_i = d_i$, and by a similar argument $c_j = d_j$. We can conclude that the edges $\{v^i_{c_i} v^j_{c_j} \mid i, j \in [k]\}$ exist in G. Then the function $f : V(K) \to V(G)$ where $f(i) = v^i_{c_i}$, is a function preserving neighbors and colors. ◁

4.2.3 When $\varrho < \varsigma + 1$

Let $\varsigma' = \varsigma - \varrho + 1$. In this case, we construct the graph solution size pair: (H_2, k_2), where

$$k_2 = (\varsigma + 1) \cdot (|E(K)| + k) + \varsigma + 1.$$

The graph H_2 is obtained from $\mathcal{H}$ by the modifications given below. Recall the operation of a depth-ℓ grid of cliques implant, see Definition 9; and for convenience, for all $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$ such that $z^\alpha_\beta \in V(\mathcal{H})$, let $z^\alpha_\beta = z^\alpha_{\beta 0}$.

1. For each $\alpha \in \mathcal{I}$, we perform a depth-ς' grid of cliques implant at Z^α. We call the ℓ-th column $Z^\alpha_{*\ell}$, for all $\ell \in [\varsigma']_0$, and the β-th row $Z^\alpha_{\beta *}$, for all β such that $z^\alpha_\beta \in Z^\alpha$. Let $Z^\alpha_{**} = \bigcup_{\ell \in [\varsigma']_0} Z^\alpha_{*\ell}$.
2. For each $\alpha \in \mathcal{I}$: If $\varrho > 1$, then we add a clique A^α of size $\varrho - 1$, where the vertices in A^α are adjacent to all vertices in Z^α_{**}. If $\varrho = 1$ then $A^\alpha = \emptyset$.
3. We add a clique $\mathcal{X}$ of size $\varsigma + 1$ to H_2. This clique is partitioned into two parts $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1$ has size $\varrho - 1$ and all its vertices are adjacent to all vertices in X. The vertices in $\mathcal{X}_2$ are only adjacent to all all vertices in $\mathcal{X}$ and $\mathcal{X}_2$ has size $\varsigma' + 1$.[2]

▷ Claim 19 (♣). There is a linear order Λ_2 of $V(H_2)$ computable in polynomial time such that $\mathsf{mimw}(\Lambda_2) = O(|E(K)|)$.

▷ Claim 20 (♣). If (K, G, ϕ) is a YES-instance of the PARTITIONED SUBGRAPH ISOMORPHISM problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_2 in H_2.

▷ Claim 21 (♣). If there exists a (σ, ρ)-dominating set of size at most k_2 in H_2, then (K, G, ϕ) is a YES-instance of the PARTITIONED SUBGRAPH ISOMORPHISM problem.

[2] The set $\mathcal{X}$ is not needed for correctness when $\varsigma = 0$, but for simplicity we include it anyway.

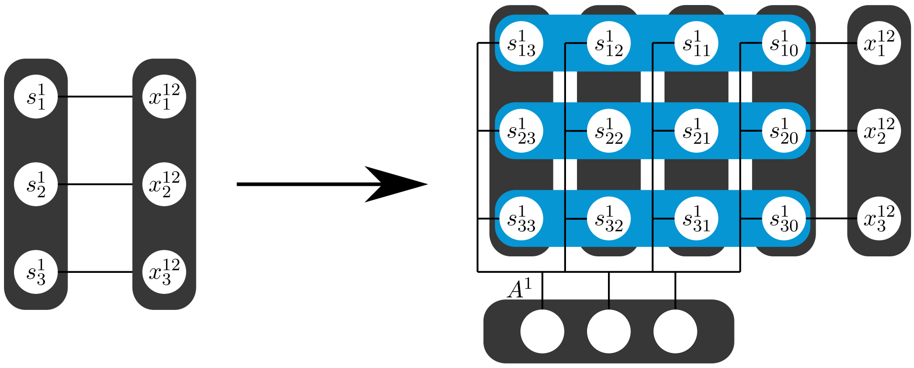

Figure 6 Example of the modifications to $S^1 \cup X^{12}$ for $p = 3$, $\varsigma = 6$, $\varrho = 4$.

4.2.4 When $\varrho \geq 1$ and $\varsigma = 0$

In this case, we construct the graph solution size pair (H_3, k_3), where $k_3 = \varrho(k + |E(K)|)$, and H_3 is constructed from $\mathcal{H}$ follows.

1. For each $\alpha \in \mathcal{I}$, $\beta \in \mathcal{J}$ such that $z^\alpha_\beta \in V(\mathcal{H})$, we perform an independent $(\varrho - 1)$-blowup of z^α_β.[3] We call the twins of z^α_β: $z^\alpha_{\beta 2}, ..., z^\alpha_{\beta\varrho}$, and we let $z^\alpha_\beta = z^\alpha_{\beta 1}$. We let $Z^\alpha_{\beta *} = \{z^\alpha_{\beta\ell} \mid \ell \in [\varrho]\}$, $Z^\alpha_{*\ell} = \{z^\alpha_{\beta\ell} \mid \beta \in \mathcal{J} \text{ s.t. } z^\alpha_\beta \in V(\mathcal{H})\}$, and $Z^\alpha_{**} = Z^\alpha \cup \bigcup_{\ell\in[\varrho]} Z^\alpha_{*\ell}$.
2. For all $\alpha \in \mathcal{I}$, we add a clique A^α of size ϱ, and we connect all of its vertices to to all the vertices in Z^α_{**}.

▷ Claim 22 (♣). There is a linear order Λ_3 of $V(H_3)$ computable in polynomial time such that $\mathsf{mimw}(\Lambda_3) = O(|E(K)|)$.

▷ Claim 23 (♣). If (K, G, ϕ) is a Yes-instance of the Partitioned Subgraph Isomorphism problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_3 in H_3.

▷ Claim 24 (♣). If there exists a (σ, ρ)-dominating set of size at most k_3 in H_3, then (K, G, ϕ) is a Yes-instance of the Partitioned Subgraph Isomorphism problem.

4.3 Maximization problems

For maximization problems, we can reuse the constructions as in the previous section; however we let $\varsigma = \max(\sigma)$ and $\varrho = \max(\rho)$ (instead of taking the minima of σ and ρ). This is why we require σ and ρ to be finite.

When $\varsigma < \varrho$, we construct (H_1, k_1) as in Section 4.2.2. Therefore the mim-width bound follows by the same arguments, and one direction of the correctness proof is already shown in Claim 17. A bit of attention is necessary in case $\varrho' = 0$, but the arguments still work after some minor tweaks. In case $\varsigma \geq \varrho$, we construct (H_2, k_2) as in Section 4.2.3. Again, the mim-width bound and one direction of the correctness proof (Claim 20) are already taken care of. The remaining proofs and other details are given in the full version.

[3] If $\varrho = 1$ then this step is skipped.

5 Conclusion

In this work, we proved that each Min (σ, ρ)-Dominating Set problem is either polynomial-time solvable or W[1]-hard parameterized by the mim-width of a given linear branch decomposition of the input graph plus solution size, and that the same holds for Max (σ, ρ)-Dominating Set problems whenever σ and ρ are finite. An immediate open question is whether we can complete the dichotomy for maximization problems to the cases when σ and/or ρ are infinite.

▶ **Open Problem 1.** *Is it true that for all $\sigma, \rho \subseteq \mathbb{N}$, including infinite sets, Max (σ, ρ)-Dominating Set is either polynomial-time solvable or* W[1]*-hard when parameterized by the mim-width of a given linear branch decomposition of the input graph?*

For all the W[1]-hard cases, our reductions also ruled out $f(w)n^{o(w/\log w)}$-time algorithms under the ETH, for any computable f, where n is the number of vertices of the input graph and w the mim-width of the given linear branch decomposition. Since the algorithms for finite and co-finite Min/Max (σ, ρ)-Dominating Set problems run in $n^{O(w)}$ time [13], it is a natural question to close this gap.

▶ **Open Problem 2.** *Are there finite or co-finite sets $\sigma, \rho \subseteq \mathbb{N}$ such that an algorithm for the Min/Max (σ, ρ)-Dominating Set problem that is* W[1]*-hard parameterized by the mim-width w of a given (linear) branch decomposition of the input n-vertex graph, running in $n^{o(w)}$ time, would refute the* ETH*?*

In this work, we only considered minimization and maximization variants of (σ, ρ)-Dominating Set problems. A third variant, say the Exact (σ, ρ)-Dominating Set problem, asks for a (σ, ρ)-dominating set of size *exactly* k. While all hardness proofs given in this work also work for Exact (σ, ρ)-Dominating Set problems, these problems are *not* trivial to solve when $0 \in \rho$, as the empty set is not a solution in this case (unless, of course, $k = 0$). We therefore ask the following question, and remark that the analogous question parameterized by solution size was asked by Golovach et al. [23].

▶ **Open Problem 3.** *Are there some (finite or co-finite) $\sigma, \rho \subseteq \mathbb{N}$ with $0 \in \rho$ such that Exact (σ, ρ)-Dominating Set parameterized by the mim-width of a given (linear) branch decomposition is* W[1]*-hard?*

In a recent work [17], Eiben et al. introduced a framework of width measures based on branch decompositions over the vertex set. There, given a family $\mathcal{F}$ of biparite graphs, the value of a cut is determined as the largest graph in $\mathcal{F}$ that appears as a semi-induced subgraph across the cut. Mim-width is an instantiation of this framework where $\mathcal{F}$ is the family of matchings. Our hardness proofs greatly rely on the fact that mim-width is not closed under taking the complement of the graph. It would be interesting to see what happens to the complexity of the problems in this work when one considers the width measure obtained by letting $\mathcal{F}$ be the union of the family of matchings *and anti-matchings* as the parameter, which results in a parameter related to mim-width that is closed under the complement.

Lastly, we want to point out that we cannot expect to prove W[1]-completeness for the W[1]-hard cases of Min/Max (σ, ρ)-Dominating Set parameterized by linear mim-width considered in this work. In a recent work, Bodlaender et al. [7] showed that the Minimum Dominating Set and Maximum Independent Set problems parameterized by the mim-width of a given linear branch decomposition of the input graph are XNLP-complete [8]. This in turn implies that these problems are W[t]-hard for all t, which makes containment in

W[1] unlikely. Furthermore, we believe that the ideas used in our work and those from [7] can be combined to show that all W[1]-hard cases from our work are indeed XNLP-hard. Membership in XNLP can be derived for all finite or co-finite σ and ρ, in a similar way as it is done for Maximum Independent Set and Minimum Dominating Set in [7].

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