# Quantitative Hennessy-Milner Theorems via Notions of Density

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## Abstract

The classical *Hennessy-Milner theorem* is an important tool in the analysis of concurrent processes; it guarantees that any two non-bisimilar states in finitely branching labelled transition systems can be distinguished by a modal formula. Numerous variants of this theorem have since been established for a wide range of logics and system types, including quantitative versions where lower bounds on behavioural distance (e.g. in weighted, metric, or probabilistic transition systems) are witnessed by quantitative modal formulas. Both the qualitative and the quantitative versions have been accommodated within the framework of *coalgebraic logic*, with distances taking values in quantales, subject to certain restrictions, such as being so-called *value quantales*. While previous quantitative coalgebraic Hennessy-Milner theorems apply only to liftings of set functors to (pseudo)metric spaces, in the present work we provide a quantitative coalgebraic Hennessy-Milner theorem that applies more widely to functors native to metric spaces; notably, we thus cover, for the first time, the well-known Hennessy-Milner theorem for continuous probabilistic transition systems, where transitions are given by Borel measures on metric spaces, as an instance of such a general result. In the process, we also relax the restrictions imposed on the quantale, and additionally parametrize the technical account over notions of *closure* and, hence, *density*, providing associated variants of the Stone-Weierstraß theorem; this allows us to cover, for instance, behavioural ultrametrics.

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## 1 Introduction

Modal logic in general is an established tool in the analysis of concurrent systems. One of its uses is as a means to distinguish non-equivalent states; for instance, the classical Hennessy-Milner theorem [18] guarantees that any two non-bisimilar states in finitely branching labelled transition systems can be distinguished by a formula in a modal logic naturally associated to labelled transition systems. Similar theorems have subsequently proliferated, having been established, for instance, for probabilistic transition systems [24], neighbourhood structures [17], and open bisimilarity in the $\pi$-calculus [3]. As a recent example application, the counterproof for unlinkability in the ICAO 9303 standard for e-passports [13] is based on providing a distinguishing modal formula in an intuitionistic modal logic.

For systems featuring quantitative data, such as probabilistic or weighted systems or metric transition systems, *behavioural distance* provides a more fine-grained measure of agreement between systems than two-valued bisimilarity (e.g. [15, 33, 32, 10]). In analogy to the classical Hennessy-Milner theorem, behavioural distances can often be characterized by quantitative modal logics, in the sense that the behavioural distance of any two states can be approximated by the difference in value of quantitative modal formulae on these states (that is, for states with distance $> r$ one can find a quantitative modal formula on which the states disagree by at least $r$). Such theorems, to which we refer as *quantitative Hennessy-Milner theorems*, have been proved, e.g., for probabilistic transition systems [33, 32] and for metric transition systems [10].

*Universal coalgebra* [29] serves as a generic framework for concurrent systems, based on the key abstraction of encapsulating the system type in a functor, whose coalgebras then correspond to the systems of interest. Both two-valued and quantitative Hennessy-Milner theorems have been established at the level of generality offered by *coalgebraic modal logic*. The two-valued coalgebraic Hennessy-Milner theorem [27, 30] covers all coalgebraic system types, under the assumption of having a *separating* set of modalities; instances include the mentioned Hennessy-Milner theorems for probabilistic systems [24] and neighbourhood structures [17]. Various quantitative coalgebraic Hennessy-Milner theorems have been established fairly recently [23, 34, 35, 22]. These existing theorems are tied to considering liftings of set functors to metric spaces (or in fact more general topological categories [22]); our contribution in the present work is to complement these theorems by a result that instead applies to unrestricted functors on metric spaces (we give a more detailed comparison in the related work section). In particular, our result covers, for the first time, the original expressivity result for probabilistic modal logic on continuous probabilistic transition systems (where "continuous" refers to the structure of the state space) [33, 32] as an instance of a general coalgebraic result. We work not only in coalgebraic generality but also parametrize the development over the choice of a *quantale* $\mathcal{V}$, in which distances and truth values are taken; this covers the case of standard bounded real-valued distances by taking $\mathcal{V}$ to be the unit interval, and the classical two-valued case by taking $\mathcal{V}$ to be the set of Boolean truth values. Previous work on quantalic distances [35] needed to restrict to so-called value quantales [14]; we relax this assumption, covering, for instance, all finite quantales (such as the four-valued quantale used in some paraconsistent logics; see, e.g., [28]), and the square of the unit interval.

Technically, our results are additionally parametrized over the choice of *closure* operators on sets of $\mathcal{V}$-valued predicates, which induce a notion of *density*. The notion of density is the key ingredient that lets our results apply beyond discrete state spaces (e.g. to the mentioned continuous probabilistic transition systems); by varying the notion of closure, we cover, for instance, both standard metric spaces and ultrametric spaces (which in turn are induced by different quantale structures on the unit interval).

Proofs are mostly omitted and can be found in arXiv version of the paper.

**Related work.** As indicated above, quantitative coalgebraic Hennessy-Milner theorems exist in previous work [23, 34, 35, 22], from which our present work is distinguished in that it applies to functors that live natively on metric spaces (such as tight Borel distributions) rather than only to liftings of set functors (such as finitely supported distributions). We detach the technical development from both lax extensions [34, 35] and fixpoint induction [23, 34, 35], which work only for monotone modalities; we thereby cover also systems requiring non-monotone modalities, such as weighted transition systems with negative weights. A recent general framework for Hennessy-Milner theorems based on Galois connections between real-valued predicates and (pseudo)metrics is aimed primarily at generality over a linear-time/branching-time spectrum [7].

The framework of codensity liftings developed by Komorida et al. [21, 22] works at a very high level of generality, and in fact applies to topological categories (or $\mathsf{CLat}_{\sqcap}$-fibrations, in the terminology of *op. cit.*) beyond metric spaces, such as uniform spaces. Our present framework is on the one hand more general in that we do not restrict to functors lifted from the category of sets, but on the other hand less general in that we cover only (quantalic) behavioural distances. In terms of the main technical result, we provide a coalgebraic quantitative Hennessy-Milner theorem that is stated in fairly simple terms, and can be instantiated to concrete logics and systems by just verifying a few fairly straightforward conditions that concern only the functor and the modalities. In particular, we have no conditions requiring that certain sets of formula evaluations on a given coalgebra are *approximating* (cf. [22, Theorems IV.5, IV.7]); instead, we prove similar properties as *lemmas* along the way, using the key notions of closure and density. In fact, one of the conditions of our Hennessy-Milner theorem can be seen as a form of Stone-Weierstraß property, and in particular concerns density of sets of functions closed under suitable propositional combinations; this property depends only on the quantale, not on the functor or the modalities, and we give general Stone-Weierstraß-type theorems for several classes of quantales.

## 2 Preliminaries

Basic familiarity with category theory will be assumed [1, 5]. More specifically, we make extensive use of topological categories (e.g. [1]). We recall some central notions for convenience.

**Universal coalgebra.** For an endofunctor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$, an **$\mathsf{F}$-coalgebra** $(X, \alpha)$ consists of an object $X$ of $\mathcal{C}$, thought of as an object of *states*, and a morphism $\alpha\colon X \to \mathsf{F}X$, thought of as assigning structured collections (sets, distributions, etc.) of *successors* to states. A **coalgebra morphism** from $(X, \alpha)$ to $(Y, \beta)$ is a morphism $f\colon X \to Y$ such that $\beta \circ f = \mathsf{F}f \circ \alpha$. A **concrete category** over $\mathsf{Set}$ comes equipped with a faithful functor $|{-}|\colon \mathcal{C} \to \mathsf{Set}$, which allows us to speak about individual *states*, as elements of $|X|$. Given a coalgebra $(X, \alpha)$ and states $x, y \in |X|$, we say that $x$ and $y$ are **behaviourally equivalent** if there are a coalgebra $(Z, \gamma)$ and a coalgebra morphism $f\colon X \to Z$ such that $|f|(x) = |f|(y)$. (For brevity, we restrict the treatment of both behavioural equivalence and behavioural distances to states in the same coalgebra; in all our examples the extension to states in different coalgebras can be accommodated by taking coproducts.) The notion of behavioural equivalence is strictly two-valued, meaning that different states are either behaviourally equivalent or not. The downside of this notion is thus that in systems dealing with quantitative information, any slight change can render two states behaviourally distinct, even though they may be virtually indistinguishable in any practical context. The rest of this paper is concerned with quantifying the degree to which states differ from each other, as well as with logics to witness these degrees.

► **Example 1.**

1. Labelled transition systems w.r.t. a set $A$ of actions are coalgebras for the $\mathsf{Set}$-functor $\mathcal{P}(A \times -)$. Behavioural equivalence coincides with the classical notion of (strong) bisimilarity.
2. We write $\diamond$ for the four-element diamond-shaped lattice, i.e. $\diamond = \{\bot, \mathsf{N}, \mathsf{B}, \top\}$, ordered by $\bot < \mathsf{N} < \top$, $\bot < \mathsf{B} < \top$. Let $\mathcal{B}$ be the *$\diamond$-valued powerset functor*, which sends a set $X$ to the set of all maps $q \colon X \to \diamond$, and a map $f \colon X \to Y$ to the map $\mathcal{B}f$ given by $\mathcal{B}f(q)(y) = \bigvee_{f(x)=y} q(x)$. A map $q \colon X \to \diamond$ can be seen as a $\diamond$-valued fuzzy subset of $X$, which for every element $x$ tells us either that $x$ is in the set ($q(x) = \top$), or that $x$ is not in the set ($q(x) = \bot$), or that there is evidence both that $x$ is in the set and that $x$ is not in the set ($q(x) = \mathsf{B}$), or that nothing is known ($q(x) = \mathsf{N}$). $\mathcal{B}$-coalgebras have been used in a Kripke-style semantics of paraconsistent modal logics [28] (our $\bot$, $\top$, $\mathsf{N}$, $\mathsf{B}$ respectively correspond to $\mathsf{f}$, $\mathsf{t}$, $\bot$, $\top$, and $\leq$ to $\leq_t$ in op. cit.).
3. We denote by $\mathcal{D}$ the functor that maps a set X to the set of finitely supported probability distributions on $X$. Coalgebras for the functor $FX = (1 + \mathcal{D}X)^A$, for a finite set $A$ of actions, are probabilistic transition systems [24, 11]. In this context, behavioural equivalence instantiates to probabilistic bisimilarity [20].
4. We consider weighted transition systems with possibly negative weights (e.g. [8]): Let $\mathcal{W}$ be the functor on 1-bounded metric spaces that maps every set $X$ to the set of finite $[-1,1]$-weighted sets over $X$. That is, the elements of $\mathcal{W}X$ are functions $t \colon X \to [-1,1]$ such that $t(x) = 0$ for all but finitely many $x$, and $\sum_{x \in A} t(x) \in [-1,1]$ for all $A \subseteq X$. On morphisms, $\mathcal{W}$ acts by summing over preimages, that is, $\mathcal{W}g(t)(y) = \sum_{x \in g^{-1}(y)} t(x)$ for $g \colon X \to Y$, $t \in \mathcal{W}X$.
   The distance of $s, t \in \mathcal{W}X$ is given by $d(s,t) = \frac{1}{2} \bigvee_f \sum_{x \in X} s(x)f(x) - t(x)f(x)$ where the join ranges over all nonexpansive functions $f \colon X \to [0,1]$. Then $(\mathcal{W}-)^A$ coalgebras are $([-1,1], +, 0)$-weighted $A$-labelled transition systems. Behavioural equivalence instantiates to weighted bisimilarity [20].
5. Consider the following variation of the Kantorovich functor $\mathcal{K}$ [32, 2]. We say that a probability measure $\mu$ on the Borel $\sigma$-algebra of a (pseudo)metric space $(X,d)$ is *tight* if for every $\epsilon > 0$, there is a totally bounded subset $Y \subseteq X$ such that $\mu(X \setminus Y) < \epsilon$. The Kantorovich functor $\mathcal{K}$ maps a (pseudo)metric space $(X,d)$ to the set of tight probability measures on $(X,d)$, equipped with the Kantorovich metric, defined as $d_{\mathcal{K}X}(\mu,\nu) = \sup_f \left\{\int f \,\mathrm{d}\mu - \int f \,\mathrm{d}\nu\right\}$ for $\mu, \nu \in \mathcal{K}X$, where again $f$ ranges over all nonexpansive maps $X \to [0,1]$. On morphisms, $\mathcal{K}$ acts by taking image measures, i.e. for $f \colon X \to Y$ we have $\mathcal{K}f(\mu)(Y') = \mu(f^{-1}(Y'))$ for Borel sets $Y' \subseteq Y$. Given a finite set $A$ of actions, $A$-labelled continuous probabilistic transition systems are $\mathcal{K}(1 + -)^A$-coalgebras [32, 33] (so the term *continuous* applies to the state space, not the system evolution), and behavioural equivalence instantiates to probabilistic bisimilarity of continuous systems.

Consider the probabilistic transition systems depicted in Figure 1. If $\epsilon > 0$, then the root states are not probabilistically bisimilar, as they have different probabilities of reaching a deadlock state. Still one would like to say that their difference in behaviour is small if $\epsilon$ is small. We will review formal definitions of such concepts in Section 4.


**Figure 1** Probabilistic transition systems with behaviourally inequivalent root states.

**Topological Categories.** Let $\mathsf{F}\colon \mathcal{C} \to \mathcal{X}$ be a faithful functor. Given $\mathcal{C}$-objects $C, D$, we say that an $\mathcal{X}$-morphism $f\colon \mathsf{F}C \to \mathsf{F}D$ **is a morphism** $C \to D$ if $f = \mathsf{F}\bar{f}$ for some (necessarily unique) $\bar{f}\colon C \to D$. A cone $(f_i\colon C \to C_i)_{i\in I}$ in $\mathcal{C}$ (with $I$ a class) is **initial** if the following hods: whenever, given a $\mathcal{C}$-object $B$ and $g\colon \mathsf{F}B \to \mathsf{F}C$, $f_i \circ g$ is a morphism $B \to C_i$ for all $i \in I$, then $g$ is a morphism $B \to C$. A morphism is **initial** if the corresponding singleton cone is initial. A functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}$ **preserves initial morphisms** if $\mathsf{F}f$ is initial whenever $f$ is initial. Now, an ($\mathsf{F}$-)**structured morphism** is a pair $(f, C)$ consisting of a $\mathcal{C}$-object $C$ and a morphism $f\colon X \to \mathsf{F}C$, typically written just as $f\colon X \to \mathsf{F}C$, and an ($\mathsf{F}$-)**structured cone** is a family $\mathcal{S} = (f_i\colon X \to \mathsf{F}C_i)_{i\in I}$ of structured morphisms. An **initial lift** of $\mathcal{S}$ is an initial cone $(\bar{f}_i\colon C \to C_i)$ such that $\mathsf{F}\bar{f}_i = f_i$ for all $i$. The functor $\mathsf{F}$ is **topological** if every $\mathsf{F}$-structured cone has an initial lift; we then also say that $\mathcal{C}$ is **topological over** $\mathcal{X}$, leaving $\mathsf{F}$ implicit. (We note that we have assumed faithfulness of $\mathsf{F}$ only for ease of presentation, and in fact one can show under a prima facie more general definition that all topological functors are faithful [1, Theorem 21.3].) Typical examples of topological categories are topological spaces, pseudometric spaces, and preordered sets (while subcategories of such categories that are determined by separation conditions, e.g. Hausdorff spaces, metric spaces, or ordered sets, are typically only **monotopological**, in the sense that only monic cones are guaranteed to have initial lifts [1]). For instance, the initial lift of a structured cone $(f_i\colon X \to (Y_i, \leq_i))_{i\in I}$ of preordered sets is the preorder $\leq$ on $X$ given by $x \leq z$ iff $f_i(x) \leq_i f_i(z)$ for all $i$.

Recent work on codensity liftings [21, 22] employs $\mathsf{CLat}_\sqcap$-*fibrations*, which are easily seen to be essentially equivalent to topological functors (more precisely, to *amnestic* topological functors [1]). Topological functors come with a rich and well-developed theory, on which we will draw to some degree in our technical treatment. Basic facts to note are that topological functors lift limits, so topological categories are complete if their underlying category is complete, and that topological functors are also cotopological, so the same holds for colimits.

## 3 Quantales and Quantale-Enriched Categories

A central notion of our development are quantales, which will serve as objects of both truth values and distance values, subsuming in particular the two-valued and the real-valued case. A **quantale** $(\mathcal{V}, \bigvee, \otimes, k)$, more precisely a commutative and unital quantale, is a complete lattice $\mathcal{V}$ that carries the structure of a commutative monoid $(\mathcal{V}, \otimes, k)$, with $\otimes$ called **tensor** and $k$ called **unit**, such that for every $u \in \mathcal{V}$, the map $u \otimes -\colon \mathcal{V} \to \mathcal{V}$ preserves suprema, which entails that every $u \otimes -\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $\hom(u, -)\colon \mathcal{V} \to \mathcal{V}$, characterized by the property $u \otimes v \leq w \iff v \leq \hom(u, w)$. We denote by $\top$ and $\bot$ the greatest and the least element of a quantale respectively. A quantale is **non-trivial** if $\bot \neq \top$, and **integral** if $\top = k$.

▶ **Example 2.**

1. Every frame (i.e. a complete lattice in which binary meets distribute over infinite joins) is a quantale with $\otimes = \wedge$ and $k = \top$. In particular, every finite distributive lattice is a quantale, prominently 2, the two-element lattice $\{\bot, \top\}$. In this case, the hom operation is implication in the usual sense.

2. Every left continuous $t$-norm [4] defines a quantale on the unit interval equipped with its natural order.
3. The previous clause further specializes as follows (up to isomorphism):
   a. The quantale $[0,\infty]_+ = ([0,\infty], \inf, +, 0)$ of non-negative real numbers with infinity, ordered by the greater or equal relation, and with tensor given by addition.
   b. The quantale $[0,\infty]_{\max} = ([0,\infty], \inf, \max, 0)$ of non-negative real numbers with infinity, ordered by the greater or equal relation, and with tensor given by maximum.
   c. The quantale $[0,1]_\oplus = ([0,1], \inf, \oplus, 0)$ of the unit interval, ordered by the greater or equal order, and with tensor given by truncated addition. In this case as well as in $[0,\infty]_+$, the hom operation is truncated addition: $\hom(u,v) = \max(v-u, 0)$.

   (Note that in all these examples, the quantalic order is dual to the standard numeric order.)
4. Every commutative monoid $(M, \cdot, e)$ generates a quantale structure on $(\mathcal{P}M, \bigcup)$, the free quantale on $M$. The tensor $\otimes$ on $\mathcal{P}M$ is defined by $A \otimes B = \{a \cdot b \mid a \in A \text{ and } b \in B\}$, for all $A, B \subseteq M$. The unit of this multiplication is the set $\{e\}$.
5. For every quantale $\mathcal{V}$ and every partially ordered set $X$, the set of monotone maps $\mathsf{Pos}(X, \mathcal{V})$ ordered pointwise becomes a quantale with tensor defined pointwise. For instance, $\mathsf{Pos}(2, [0,1]_\oplus)$ with discrete 2 yields the quantale $[0,1]_\oplus^2$, and by replacing 2 with the two-element chain $0 \geq 1$ we obtain the quantale $\mathcal{I}([0,1]_\oplus)$ of non-empty closed subintervals of $[0,1]$ [35].

Category theory highlights preordered sets as 2-enriched categories. By replacing 2 with a quantale $\mathcal{V}$, we enrich the relevant preorders with a quantitative extent: A **$\mathcal{V}$-category** is pair $(X, a)$ consisting of a set $X$ and a map $a\colon X \times X \to \mathcal{V}$ that satisfies the inequalities $k \leq a(x,x)$ and $a(x,y) \otimes a(y,z) \leq a(x,z)$ for all $x, y, z \in X$. We think of $a$ as providing a generalized notion of *similarity* (which under the reverse ordering becomes a notion of *distance*). The quantale $\mathcal{V}$ itself is canonically a $\mathcal{V}$-category $(\mathcal{V}, \hom)$, which we also denote by just $\mathcal{V}$. A **$\mathcal{V}$-functor** $f\colon (X,a) \to (Y,b)$ is a map $f\colon X \to Y$ such that $a(x,y) \leq b(f(x), f(y))$ for all $x, y \in X$. $\mathcal{V}$-categories and $\mathcal{V}$-functors form the category $\mathcal{V}$-$\mathsf{Cat}$.

Every $\mathcal{V}$-category $(X,a)$ carries a **natural order** defined by $x \leq y$ whenever $k \leq a(x,y)$, which induces a faithful functor $\mathcal{V}$-$\mathsf{Cat} \to \mathsf{Ord}$ into the category $\mathsf{Ord}$ of partially ordered sets.

A $\mathcal{V}$-category $(X,a)$ is **symmetric** if $a(x,y) = a(y,x)$ for all $x, y \in X$, and **separated** if its natural order is antisymmetric. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ the full subcategories of $\mathcal{V}$-$\mathsf{Cat}$ determined by the symmetric and the symmetric separated $\mathcal{V}$-categories, respectively. For real-valued $\mathcal{V}$, $\mathcal{V}$-$\mathsf{Cat}$, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$, and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ correspond to categories of hemimetric, pseudometric, and metric spaces, respectively. We will use $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ as the main device to formalize examples and state our main results, although most of these results can be meaningfully reinterpreted for $\mathcal{V}$-$\mathsf{Cat}$.

▶ **Example 3.**
1. The category 2-$\mathsf{Cat}$ is equivalent to the category $\mathsf{Ord}$ of **preordered sets** and monotone maps.
2. As noted by Lawvere [25], metric, ultrametric, and bounded metric spaces can be seen as quantale-enriched categories:
   a. The category $[0,\infty]_+$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is equivalent to the category $\mathsf{Met}$ of generalized **metric spaces** and non-expansive maps.
   b. The category $[0,\infty]_{\max}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is equivalent to the category $\mathsf{UMet}$ of generalized **ultrametric spaces** and non-expansive maps.
   c. The category $[0,1]_\oplus$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is equivalent to the category $\mathsf{BMet}$ of **bounded-by-1 metric spaces** and non-expansive maps.

**Table 1** $\mathcal{V}$-categorical notions in the qualitative and the quantitative setting. The prefix "pseudo" refers to absence of separatedness, and the prefix "hemi" additionally indicates absence of symmetry.

| General $\mathcal{V}$ | Qualitative ($\mathcal{V} = 2$) | Quantitative ($\mathcal{V} = [0,1]_\oplus$) |
|---|---|---|
| $\mathcal{V}$-category | preorder | bounded-by-1 hemimetric space |
| symmetric $\mathcal{V}$-category | equivalence | bounded-by-1 pseudometric space |
| $\mathcal{V}$-functor | monotone map | non-expansive map |
| initial $\mathcal{V}$-functor | order-reflecting monotone map | isometry |
| L-dense $\mathcal{V}$-functor | monotone map that is surjective up to the induced equivalence | non-expansive map with dense image |
| L-closure | closure under the induced equivalence | topological closure |

3. Categories enriched in the powerset of the monoids underlying the quantales of the previous example can be thought of as spaces where a non-deterministic distance is assigned to each pair of points.
4. Categories enriched in the quantale $\mathcal{I}([0,1]_\oplus)$ can be thought of as spaces where a distance range is assigned to each pair of points.

The examples $\mathcal{V} = 2$ and $\mathcal{V} = [0,1]_\oplus$ are particularly instructive, as they represent the most established qualitative and quantitative aspects quantales aim to generalize. Table 1 provides some instances of generic quantale-based concepts (either introduced above or to be introduced presently) in these two cases, for further reference.

The forgetful functor $|-|\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$ is topological (Section 2): The **initial lift** $(X,a)$ of a structured cone $\mathcal{S} = (f_i\colon X \to |(X_i,a_i)|)_{i\in I}$, with $a$ referred to as the **initial structure** w.r.t. $\mathcal{S}$, is given by $a(x,y) = \bigwedge_{i\in I} a_i(f_i(x), f_i(y))$ for $x,y \in X$. If all $a_i$ are symmetric, then $a$ is symmetric, and if all $a_i$ are separated *and* the cone $\mathcal{S}$ is monic, then $a$ is separated. Thus, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ is topological over $\mathsf{Set}$, while $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$ and $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$ are monotopological over $\mathsf{Set}$ (but not topological). It follows by general results [1] that all these categories are reflective in $\mathcal{V}\text{-}\mathsf{Cat}$. In particular, the reflector

$$(-)_q\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$$

quotients $(X,a)$ by its natural preorder, which for symmetric $(X,a)$ is an equivalence. The category $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ is also coreflective in $\mathcal{V}\text{-}\mathsf{Cat}$; the coreflector

$$(-)_s\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$$

sends a $\mathcal{V}$-category $(X,a)$ to its **symmetrization** $(X,a)_s = (X,a_s)$, where $a_s(x,y) = a(x,y) \wedge a(y,x)$ (keep in mind that in Example 2.3, the order is the dual of the numeric order). In particular, $\mathcal{V}_s$ is the set $\mathcal{V}$ equipped with the $\mathcal{V}$-category structure $a$ given by $a(u,v) = \hom(u,v) \wedge \hom(v,u)$. For instance, for $\mathcal{V} = [0,1]_\oplus$ (Example 2.3), the $\mathcal{V}$-category structure of $\mathcal{V}_s$ is just the usual Euclidean distance $|u-v|$ on $[0,1]$.

For a set $X$ and a $\mathcal{V}$-category $(Y,b)$, we write $(Y,b)^X$ for the set of all maps $X \to Y$, equipped with the $\mathcal{V}$-category structure $[-,-]$ given by $[h,l] = \bigwedge_{x\in X} b(h(x), l(x))$ for $h,l\colon X \to Y$ (in the *numeric* ordering on real-valued quantales, this corresponds to the usual supremum metric on functions). For a $\mathcal{V}$-category $(X,a)$, we moreover write $(Y,b)^{(X,a)}$ for $\mathcal{V}\text{-}\mathsf{Cat}((X,a),(Y,b))$, equipped with the $\mathcal{V}$-category structure inherited from $(Y,b)^X$. We note that we will often designate $\mathcal{V}$-categories just by single letters such as $X$; disambiguation between spaces of maps and spaces of $\mathcal{V}$-functors should nevertheless always be clear from the context. For instance, if $X$ is a $\mathcal{V}$-category, then $\mathcal{V}_s^X$ is the space of all $\mathcal{V}$-functors $X \to \mathcal{V}_s$, while $\mathcal{V}_s^{|X|}$ is the space of all maps $X \to \mathcal{V}_s$.

Quantale-enriched categories come equipped with a canonical closure operator. A $\mathcal{V}$-functor $m\colon M \to X$ is **L-dense** [19] if for all $\mathcal{V}$-functors $f, g\colon X \to \mathcal{V}$, $f \cdot m = g \cdot m$ implies $f = g$. The composite of L-dense $\mathcal{V}$-functors is again L-dense. For a subset $A$ of $X$, the **L-closure** $\overline{A}$ of $A$ in $(X, a)$ is the largest $\mathcal{V}$-subcategory of $(X, a)$ in which $A$ is L-dense; this can be explicitly computed as

$$\overline{A} = \{x \in X \mid k \leq \textstyle\bigvee_{y \in A} a(x, y) \otimes a(y, x)\}.$$

A subset $A \subseteq X$ of a $\mathcal{V}$-category $(X, a)$ is **L-closed** if $A = \overline{A}$, and **L-dense in** $(X, a)$ if $\overline{A} = X$. A function $f\colon X \to Y$ between $\mathcal{V}$-categories $(X, a)$ and $(Y, b)$ is said to be **L-continuous** if $f[\overline{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$. It is easy to see that the notion of L-closure is so designed that for metric-like examples, it coincides with the topological closure w.r.t. the open-ball topology, and continuity in the sense defined above coincides with continuity w.r.t. this topology. For a preorder $(X, \leq)$, $A \subseteq X$ is L-closed iff it is closed under the induced equivalence. It is easy to check that every $\mathcal{V}$-functor is L-continuous, in generalization of the standard fact that every non-expansive map of metric spaces is continuous.

▶ **Proposition 4.** *For every $\mathcal{V}$-category $X$, $\mathcal{V}\text{-}\mathsf{Cat}(X, \mathcal{V})$ is L-closed in $\mathcal{V}^{|X|}$. For every symmetric $\mathcal{V}$-category $X$, $\mathcal{V}\text{-}\mathsf{Cat}(X, \mathcal{V}_s)$ is L-closed in $\mathcal{V}_s^{|X|}$.*

Maps that are continuous w.r.t. L-closure or other closures will be essential in Section 6.

## 4 Quantitative Coalgebraic Modal Logics

We proceed to introduce a variant of (quantitative) coalgebraic logic [27, 30, 9, 23, 34], which in particular follows the paradigm of interpreting modalities via predicate liftings, in this case of $\mathcal{V}$-valued predicates.

Given a cardinal $\kappa$, a classical $\kappa$-ary predicate lifting for a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ is a natural transformation $\lambda\colon \mathsf{Set}(-, 2^\kappa) \to \mathsf{Set}(\mathsf{F}-, 2)$, where we see a map $X \to 2^\kappa$ as $\kappa$ many predicates on $X$; we switch freely between predicates and subsets. For example, the Kripke semantics of the modal logic $K$ can be couched in terms of the diamond modality $\Diamond$, which we identify with the unary predicate lifting $\Diamond_X(A) = \{B \subseteq X \mid A \cap B \neq \varnothing\}$ for the powerset functor. This notion of predicate lifting naturally extends to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$-functors:

▶ **Definition 5.** Given a cardinal $\kappa$, a $\kappa$-ary **predicate lifting** for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ is a natural transformation of type $\lambda\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(-, \mathcal{V}_s^\kappa) \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(\mathsf{F}-, \mathcal{V}_s)$.

A $\kappa$-ary predicate lifting thus lifts $\kappa$-many $\mathcal{V}$-functorial $\mathcal{V}$-valued predicates on $X$ to a $\mathcal{V}$-functorial predicate on $\mathsf{F}X$.

▶ Remark 6 (Yoneda Lemma). By the Yoneda lemma, a $\kappa$-ary predicate lifting $\lambda$ for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ is completely determined by its action on the identity map on $\mathcal{V}_s^\kappa$.

For the sake of brevity, we restrict the technical treatment to unary predicate liftings ($\kappa = 1$) henceforth; we do occasionally use non-unary liftings in examples, in particular nullary ones.

The syntax of quantitative coalgebraic modal logic can now be defined by the grammar

$$\phi ::= \top \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid u \otimes \phi \mid \hom_s(u, \phi) \mid \lambda(\phi) \qquad (u \in \mathcal{V}, \lambda \in \Lambda)$$

where $\Lambda$ is a set of *modalities*, which we identify, by abuse of notation, with predicate liftings for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$. We view all other connectives as propositional operators. Let $\mathcal{L}(\Lambda)$ be the set of modal formulas thus defined. The semantics is given by assigning to each formula $\phi \in \mathcal{L}(\Lambda)$ and each coalgebra $\alpha\colon X \to \mathsf{F}X$ the *interpretation* of $\phi$ over $\alpha$, the $\mathcal{V}$-functor $[\![\phi]\!]_\alpha\colon X \to \mathcal{V}_s$ recursively defined as follows:

- for $\phi = \top$, we take $[\![\top]\!]_\alpha$ to be the $\mathcal{V}$-functor given by the constant map into $\top$;
- for an $n$-ary propositional operator $p$, we put $[\![p(\phi_1, \dots, \phi_n)]\!]_\alpha = p([\![\phi_1]\!]_\alpha, \dots, [\![\phi_n]\!]_\alpha)$, with $p$ interpreted using the lattice structure of $\mathcal{V}$ and the $\mathcal{V}$-categorical structure $\hom_s$ of $\mathcal{V}_s$, respectively, on the right-hand side;
- for $\lambda \in \Lambda$, we put $[\![\lambda(\phi)]\!]_\alpha = \lambda([\![\phi]\!]_\alpha) \cdot \alpha$.

Given a coalgebra $(X, \alpha)$, we denote the set of all maps of the form $[\![\phi]\!]_\alpha$ by $[\![\mathcal{L}]\!]_\alpha$. Interpreting $\top$ as $\top$ and not as $k$ is essential for non-integral quantales, for which the constant map with value $k$ fails to be a $\mathcal{V}$-functor.

▶ **Example 7.** For a first example instance of the generic logic introduced above (more examples are seen in Section 8), recall the functor $\mathcal{K}$ from Example 1(5), which assigns to a pseudometric space $X$ the space of tight probability measures on $X$, and put $\mathsf{F} = \mathcal{K}(1 + -)^A$ for a set $A$ of actions; then, $\mathsf{F}$-coalgebras are pseudometric probabilistic labelled transition systems [33]. We have the expectation predicate lifting $\mathbb{E}$ for $\mathcal{K}$, given by

$$\mathbb{E}_X(f)(\mu) = \textstyle\int_X f(x)\, d\mu(x)$$

for $\mu \in \mathcal{K}X$ and non-expansive $f\colon X \to [0,1]$. From $\mathbb{E}$, we define a set $\Lambda = \{\mathbb{E}^{a,+1} \mid a \in A\}$ of predicate liftings for $\mathsf{F}$, given by $\mathbb{E}^{a,+1}_X(f)(l) = \mathbb{E}_{1+X}(f^{+1})(l_a)$ for an $A$-indexed family $l$ of tight probability measures $l_a$ on $1 + X$ and non-expansive $f\colon X \to [0,1]$, where $f^{+1}\colon X + 1 \to [0,1]$ acts like $f$ on $X$ and sends the other element to 0. The arising instance of quantitative coalgebraic modal logic is van Breugel and Worrell's quantitative probabilistic modal logic [33].

Quantitative coalgebraic modal logic is invariant under coalgebra morphisms:

▶ **Proposition 8.** *Let $\Lambda$ be a set of predicate liftings for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$, and let $f\colon (X, \alpha) \to (Y, \beta)$ be a morphism of $\mathsf{F}$-coalgebras. Then, for every formula $\phi \in \mathcal{L}(\Lambda)$, $[\![\phi]\!]_\alpha = [\![\phi]\!]_\beta \cdot f$.*

The established approach to coalgebraic behavioural distances is to start with a set functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ and obtain a $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$-functor as a lifting of $\mathsf{F}$. In the quantalic setting, this approach may take the following shape. Topological properties of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ entail that every set $\Lambda$ of predicate liftings for a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ induces a functor $\mathsf{F}^\Lambda\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$, known as the **Kantorovich lifting** of $\mathsf{F}$ w.r.t. $\Lambda$ [6]. Concretely, $\mathsf{F}^\Lambda$ sends a $\mathcal{V}$-category $(X, a)$ to the $\mathcal{V}$-category determined by the initial structure on $\mathsf{F}X$ w.r.t. the structured cone of all maps $\lambda(f)\colon \mathsf{F}X \to |\mathcal{V}_s|$ with $\lambda \in \Lambda$ and $f\colon (X, a) \to \mathcal{V}_s \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$. As the name indicates, $\mathsf{F}^\Lambda$ is indeed a **lifting** of $\mathsf{F}$ to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$, that is, $|-| \cdot \mathsf{F}^\Lambda = \mathsf{F} \cdot |-|$ where $|-|\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathsf{Set}$ is the forgetful functor. Every predicate lifting $\lambda \in \Lambda$ for $\mathsf{F}$ becomes a predicate lifting for the Kantorovich lifting $\mathsf{F}^\Lambda$.

Kantorovich liftings are crucial prerequisites for existing expressivity results of quantitative coalgebraic logics for $\mathsf{Set}$-functors (e.g. [34, 35, 22]). It turns out that the Kantorovich property can be usefully detached from the notion of functor lifting:

▶ **Definition 9** (Kantorovich Functor)**.** Let $\Lambda$ be a set of predicate liftings for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$. The functor $\mathsf{F}$ is **$\Lambda$-Kantorovich** if for every $\mathcal{V}$-category $X$, the cone of all $\mathcal{V}$-functors $\lambda(f)\colon \mathsf{F}X \to \mathcal{V}_s$ with $\lambda \in \Lambda$ and $f \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X, \mathcal{V}_s)$ is initial.

Clearly, every Kantorovich lifting $F^\Lambda$ is $\Lambda$-Kantorovich.

▶ **Example 10.** Recall the finite distribution functor $\mathcal{D}\colon \mathsf{Set} \to \mathsf{Set}$ from Example 1(3); in analogy to Example 7, we have the expectation predicate lifting $\mathbb{E}$ for $\mathcal{D}$, given by $\mathbb{E}_X(f)(\mu) = \sum_{x\in X} f(x)\mu(x)$ for $\mu \in \mathbb{D}X$ and $f\colon X \to [0,1]$. The corresponding lifting $\mathcal{D}^{\mathbb{E}}$ is the usual Kantorovich distance on finite distributions. The closely related functor $\mathcal{K}$ from Example 1(5) already lives on bounded pseudometric spaces (that is, on $[0,1]_{\oplus}$-$\mathsf{Cat}_{\mathsf{sym}}$), and is not a lifting of any set functor. However, $\mathcal{K}$ is $\Lambda$-Kantorovich for $\Lambda = \{\mathbb{E}\}$ where $\mathbb{E}$ is the expectation predicate lifting for $\mathcal{K}$ as in Example 7. The functor $\mathcal{D}^{\mathbb{E}}$ is a subfunctor of $\mathcal{K}$, and the components of the associated inclusion natural transformation are initial.

Using the above, one easily checks that, similarly, the functor $\mathsf{F} = \mathcal{K}(1 + -)^A$ as in Example 7 is $\Lambda$-Kantorovich for $\Lambda = \{\mathbb{E}^{a,+1} \mid a \in A\}$, and the functor $\mathcal{D}^{\mathbb{E}}(1 + -)^A$ is the Kantorovich lifting of the functor $\mathcal{D}(1 + -)^A\colon \mathsf{Set} \to \mathsf{Set}$ w.r.t $\Lambda = \{\mathbb{E}^{a,+1} \mid a \in A\}$, where the predicate liftings $\mathbb{E}^{a,+1}$ for the $\mathcal{D}(1 + -)^A$ are given analogously to the predicate liftings $\mathbb{E}^{a,+1}$ for $\mathcal{K}(1 + -)^A$.

It has been shown recently that the Kantorovich functors (w.r.t. a class of predicate liftings of possibly ininite arities) are precisely the ones that preserve initial morphisms [16, Theorem 5.3].

## 5 Behavioural and Logical Distances

Every functor $\mathsf{F}\colon \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ comes with a natural notion of behavioural distance on $\mathsf{F}$-coalgebras, defined in analogy to behavioural equivalence (which identifies two states if they can be identified under some coalgebra morphism) by regarding states as similar if they can be made similar under some coalgebra morphism:

▶ **Definition 11.** Let $\mathsf{F}\colon \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ be a functor. The **behavioural distance** on an $\mathsf{F}$-coalgebra $(X, a, \alpha)$, denoted by $bd^{\mathsf{F}}_{\alpha}$, is defined on all $x, y \in X$ by

$$bd^{\mathsf{F}}_{\alpha}(x, y) = \bigvee\{b(f(x), f(y)) \mid f\colon (X, a, \alpha) \to (Y, b, \beta) \in \mathsf{CoAlg}(\mathsf{F})\}. \tag{1}$$

We observe next that if $\mathsf{F}$ preserves initial morphisms, then we can restrict the supremum (1) to morphisms carried by the identity map. Given $\mathcal{V}$-category structures $a, b$ on $X$ such that $a \le b$, we write $\iota_{a,b}$ for the identity $\mathcal{V}$-functor $(X, a) \to (X, b)$. We note that $\iota_{a,b}$ is a coalgebra morphism $\iota_{a,b}\colon (X, a, \alpha) \to (X, b, \beta)$, for some $\beta$, under two conditions, the first being commutativity of the diagram

$$\begin{array}{ccc} (X,a) & \xrightarrow{\ \alpha\ } & \mathsf{F}(X,a) \\ {\scriptstyle \iota_{a,b}}\downarrow & & \downarrow{\scriptstyle \mathsf{F}\iota_{a,b}} \\ (X,b) & \xrightarrow{\ \mathsf{F}\iota_{a,b}\cdot\alpha\ } & \mathsf{F}(X,b) \end{array}$$

which just means that as a map, $\beta$ must be $\mathsf{F}\iota_{a,b} \cdot \alpha$. The second condition is that $\beta = \mathsf{F}\iota_{a,b} \cdot \alpha$ must be a $\mathcal{V}$-functor $(X, b) \to \mathsf{F}(X, b)$.

▶ **Proposition 12.** *Let $(X, a, \alpha)$ be a coalgebra for a functor $\mathsf{F}\colon \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ that preserves initial morphisms. Then, we can equivalently restrict the definition (1) of behavioural distance to morphisms of the form $f = \iota_{a,b}$:*

$$bd^{\mathsf{F}}_{\alpha}(x, y) = \bigvee\{b(x, y) \mid (X, b) \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}, a \le b, \mathsf{F}\iota_{a,b} \cdot \alpha \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}((X, b), \mathsf{F}(X, b))\}.$$

▶ Remark 13. Since the forgetful functor $|-|\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathsf{Set}$ is topological, the elements of its fiber over a set $X$ that are greater or equal than an element $(X, a)$ form a complete lattice $\{(X, b) \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \mid a \leq b\}$. Moreover, for every $\mathsf{F}$-coalgebra $(X, a, \alpha)$, the endofunction on this complete lattice that sends a $\mathcal{V}$-category $(X, b)$ to the $\mathcal{V}$-category given by the initial structure on $X$ w.r.t. the structured map $|\mathsf{F}\iota_{a,b} \cdot \alpha|\colon X \to |\mathsf{F}(X, b)|$ is monotone. Therefore, by the Knaster-Tarski fixpoint theorem this map has a greatest fixpoint. By Proposition 12, if $\mathsf{F}$ preserves initial morphisms, then this greatest fixpoint is precisely the behavioural distance on $(X, a, \alpha)$. In particular, it follows that $\beta\colon (X, bd^{\mathsf{F}}_{\alpha}) \to \mathsf{F}(X, bd^{\mathsf{F}}_{\alpha})$ is a $\mathcal{V}$-functor. Furthermore, if $\mathsf{F}$ is a lifting of a functor $\mathsf{G}\colon \mathsf{Set} \to \mathsf{Set}$, then the behavioural distance on an $\mathsf{F}$-coalgebra $(X, a, \alpha)$ is given by the greatest $\mathcal{V}$-categorical structure on $X$ that makes the $\mathsf{G}$-coalgebra $|\alpha|\colon X \to \mathsf{G}X$ an $\mathsf{F}$-coalgebra. This is in line with the notion of behavioural distance based on liftings of $\mathsf{Set}$-functors (e.g. [6, 22]).

Behavioural distance is invariant under coalgebra morphisms:

▶ **Proposition 14.** *Let $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ be a functor and $f\colon (X, a, \alpha) \to (Y, b, \beta)$ a coalgebra morphism of $\mathsf{F}$-coalgebras. Then, for all $x, y \in X$, $bd^{\mathsf{F}}_{\alpha}(x, y) = bd^{\mathsf{F}}_{\beta}(f(x), f(y))$.*

Coalgebraic modal logic complements behavioural distance with a notion of logical distance:

▶ **Definition 15.** Let $\Lambda$ be a set of predicate liftings for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$. The **logical distance** $ld^{\Lambda}_{\alpha}$ on an $\mathsf{F}$-coalgebra $(X, a, \alpha)$ is the initial structure on $X$ w.r.t. the structured cone of all maps $[\![\phi]\!]_{\alpha}\colon X \to |(\mathcal{V}, \hom_s)|$ with $\phi \in \mathcal{L}(\Lambda)$. More explicitly, for $x, y \in X$,

$$ld^{\Lambda}_{\alpha}(x, y) = \bigwedge \{\hom_s([\![\phi]\!]_{\alpha}(x), [\![\phi]\!]_{\alpha}(y)) \mid \phi \in \mathcal{L}(\Lambda)\}.$$

It is immediate from Proposition 8 that logical distance is also invariant under coalgebra morphisms. The remainder of the paper is devoted to establishing criteria under which logical distance and behavioural distance coincide. Recall that a (quantitative) coalgebraic logic is **adequate** if for every $\mathsf{F}$-coalgebra $(X, \alpha)$, $bd^{\mathsf{F}}_{\alpha} \leq ld^{\Lambda}_{\alpha}$, and **expressive** if $ld^{\Lambda}_{\alpha} \leq bd^{\mathsf{F}}_{\alpha}$, for every $\mathsf{F}$-coalgebra $(X, \alpha)$. The former property is straightforward to show:

▶ **Theorem 16** (Adequacy). *Let $\Lambda$ be a set of predicate liftings for a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$. Then, the coalgebraic logic $\mathcal{L}(\Lambda)$ is adequate.*

## 6 Expressivity of Quantitative Coalgebraic Modal Logic

We fix a functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ and a set $\Lambda$ of predicate liftings for $\mathsf{F}$ throughout this section.

The following lemma is related to the Knaster-Tarski proof principle for expressivity identified in work on codensity liftings [22, Theorem IV.5]:

▶ **Lemma 17.** *Let $(X, a, \alpha)$ be an $\mathsf{F}$-coalgebra. If the cone of all $\mathcal{V}$-functors $\lambda(f)\colon \mathsf{F}(X, ld^{\Lambda}_{\alpha}) \to \mathcal{V}_s$ with $\lambda \in \Lambda$ and $f \in [\![\mathcal{L}(\Lambda)]\!]_{\alpha}$ is initial, then $ld^{\Lambda}_{\alpha} \leq bd^{\mathsf{F}}_{\alpha}$.*

Kantorovich functors come with a natural strategy to show that the assumption of Lemma 17 holds. By definition, the cone of all $\mathcal{V}$-functors $(X, ld^{\Lambda}_{\alpha}) \to \mathcal{V}_s$ that are interpretations of formulas of $\mathcal{L}(\Lambda)$ is initial. Roughly speaking, one wishes to conclude from this fact that one can *approximate* every $\mathcal{V}$-functor $(X, ld^{\Lambda}_{\alpha}) \to \mathcal{V}_s$ by interpretations of formulas; then, to apply Lemma 17 we just need to guarantee that predicate liftings *preserve approximations* (that is, satisfy a notion of *continuity*), as, by the definition of Kantorovich functor, the cone of all $\mathcal{V}$-functors $\lambda(f)\colon \mathsf{F}(X, ld^{\Lambda}_{\alpha}) \to \mathcal{V}_s$ with $\lambda \in \Lambda$ and $f \in \mathcal{V}\text{-}\mathsf{Cat}((X, ld^{\Lambda}_{\alpha}), \mathcal{V}_s)$ is initial. We formalize this approach using closure operators. We begin by introducing some notation.

Given a $\mathcal{V}$-functor $i\colon Y \to X$ and a set $A \subseteq \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X, \mathcal{V}_s)$, we denote by $A \cdot i$ the set $\{f \cdot i \mid f \in A\}$, by $\Lambda(A)$ the set $\{\lambda(f) \mid f \in A, \lambda \in \Lambda\}$, and by $|A|$ the set $\{|f| \mid f \in A\}$ (of maps). It will be convenient to encapsulate the propositional part of the logic algebraically:

▶ **Definition 18.** Let $X$ be a $\mathcal{V}$-category. A subset $A$ of $\mathcal{V}_s^X$ is a **propositional algebra** if it contains the $\mathcal{V}$-functor that is constantly $\top$, and is closed under the operations $\wedge$, $\vee$, $\hom_s(u, -)$, and $u \otimes -$, for every $u \in \mathcal{V}$.

In particular, given an $\mathsf{F}$-coalgebra $(X, a, \alpha)$, $[\![\mathcal{L}(\Lambda)]\!]_\alpha \subseteq \mathcal{V}_s^{(X,\, \mathit{Id}_\alpha^\Lambda)}$ is a propositional algebra.

We will base the mentioned notion of continuity on a notion of *closure*, over which we parametrize the technical framework (with one intended instance being L-closure as recalled in Section 3):

▶ **Definition 19** ($\mathcal{V}_s$-closure)**.** Given a set $X$, a $\mathcal{V}_s$**-closure operator** is a family $\mathbf{C} = (\mathbf{C}_X)_{X \in \mathsf{Set}}$ of closure operators $\mathbf{C}_X$ on $\mathsf{Set}(X, \mathcal{V})$ (i.e. operators $\mathbf{C}_X\colon \mathcal{P}(\mathsf{Set}(X, \mathcal{V})) \to \mathcal{P}(\mathsf{Set}(X, \mathcal{V}))$ satisfying the standard *extensiveness*, *monotonicity* and *idempotence* laws) such that for every symmetric $\mathcal{V}$-category $(X, a)$, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}((X, a), \mathcal{V}_s) \subseteq \mathsf{Set}(X, \mathcal{V})$ is closed w.r.t. $\mathbf{C}_X$. When no ambiguities arise, we write $\mathbf{C}(A)$ instead of $\mathbf{C}_X(A)$.

A $\mathcal{V}_s$-closure operator $\mathbf{C}$ is equivalently given by a family $\overline{\mathbf{C}} = (\overline{\mathbf{C}}_X)_{X \in \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}}$ of closure operators on $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X, \mathcal{V}_s)$ such that for all $A \subseteq \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X, \mathcal{V}_s)$ and $B \subseteq \mathcal{V}\text{-}\mathsf{Cat}(Y, \mathcal{V}_s)$, where $|Y| = |X|$, if $|A| = |B|$ then $|\overline{\mathbf{C}}_X(A)| = |\overline{\mathbf{C}}_Y(B)|$. We make no distinction between $\mathbf{C}$ and $\overline{\mathbf{C}}$, e.g. we apply $\mathbf{C}$ also to $A \subseteq \mathcal{V}_s^X = \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X, \mathcal{V}_s)$, for a $\mathcal{V}$-category $X$. In particular, we say that $A \subseteq \mathcal{V}_s^X$ is **C-dense on** $X$ if $\mathbf{C}(A) = \mathcal{V}_s^X$.

▶ **Example 20.** The following closure operators on $\mathsf{Set}(X, \mathcal{V})$ are $\mathcal{V}_s$-closure operators:

1. the identity operator $\mathbf{Id}_X$;
2. the operator $\mathbf{L}_X^{\mathcal{V}_s}$ that sends every set to its L-closure in the $\mathcal{V}$-category $\mathcal{V}_s^X$;
3. the closure operator $\mathbf{cInf}^{\vee}{}_X$ that sends every set $A$ to its closure under codirected infima and finite suprema;
4. the closure operator $\mathbf{Inf}_X$ that sends every set $A$ to its closure under infima;
5. the closure operator $\mathbf{Fun}_X$ that sends every set $A$ to $|\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}(X_A, \mathcal{V}_s)|$, where $X_A$ denotes the $\mathcal{V}$-category determined by the initial structure with respected to the structured cone of all maps $f\colon X \to |\mathcal{V}_s|$ with $f \in A$ (**Fun** is in fact the closure operator of a Galois connection, relating to recent work by Beohar et al. [7]).

While **Id** is the *least* $\mathcal{V}_s$-closure operator, somewhat less trivially **Fun** is the *greatest* one (and hence induces the *weakest* notion of density).

The next result connects initiality, closure and density.

▶ **Proposition 21.** *Let* $\mathbf{C}$ *be a* $\mathcal{V}_s$*-closure operator. For every* $\mathcal{V}$*-category* $X$ *and* $A \subseteq \mathcal{V}_s^X$*,*

1. *if* $A$ *is* $\mathbf{C}$*-dense on* $X$*, then* $A$ *is initial; for* $\mathbf{C} = \mathbf{Fun}$*, the converse holds as well;*
2. *if* $\mathbf{C}(A)$ *is initial, then* $A$ *is initial.*

▶ **Definition 22.** Let $\mathbf{C}$ be a $\mathcal{V}_s$-closure operator. A predicate lifting $\lambda \in \Lambda$ is **C-continuous** if for every $\mathcal{V}$-category $X$, $\lambda_X\colon \mathcal{V}_s^X \to \mathcal{V}_s^{\mathsf{F}X}$ is continuous w.r.t. $\mathbf{C}_X$ and $\mathbf{C}_{\mathsf{F}X}$ (i.e. $f[\mathbf{C}_X(A)] \subseteq \mathbf{C}_{\mathsf{F}X}(f[A])$ for all $A \subseteq \mathcal{V}_s^X$).

▶ **Example 23.** A predicate lifting $\lambda$ is

1. always **Id**-continuous;
2. **L**-continuous iff its components are L-continuous;
3. $\mathbf{cInf}^{\vee}$-continuous iff its components preserve codirected infima and finite suprema;
4. **Inf**-continuous iff its components preserve all infima.

It is easily verified that if a predicate lifting $\lambda$ for a $\{\lambda\}$-Kantorovich functor is **Fun**-continuous, then it preserves initial cones. Thus, **Fun**-continuity is a very strong assumption, which by Lemma 17 entails that $\mathcal{L}(\{\lambda\})$ is expressive. In order to obtain expressivity results for coalgebraic logics under weaker assumptions, we will consider situations where **Fun**-density can be equivalently described as **C**-density for more suitable $\mathcal{V}_s$-closure operators **C**.

▶ **Definition 24.** Let $\mathcal{I}$ be a class of symmetric $\mathcal{V}$-categories. A $\mathcal{V}_s$-closure operator **C** **characterizes initiality** on $\mathcal{I}$ if for every $\mathcal{V}$-category $X \in \mathcal{I}$, every propositional algebra $A \subseteq \mathcal{V}_s^X$ that is **Fun**-dense on $X$ (that is, by Proposition 21, initial) is already **C**-dense on $X$ (recall that the reverse implication holds universally).

Characterization of initiality for a given class $\mathcal{I}$ depends only on the quantale and the closure operator, and may be seen as a form of Stone-Weierstraß property; we will give general Stone-Weierstraß theorems for some classes of quantales in Section 7. In most of these, $\mathcal{I}$ will be the class of finite symmetric $\mathcal{V}$-categories (and in one case, the class of totally bounded pseudometric spaces). We introduce next a key technical definition.

▶ **Definition 25.** Let **C** be a $\mathcal{V}_s$-closure operator. A cocone $(i\colon X_i \to X)_{i\in\mathcal{I}}$ of morphisms in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ **coreflects C-density** if the cone $(- \cdot i\colon \mathcal{V}_s^X \to \mathcal{V}_s^{X_i})_{i\in\mathcal{I}}$ **reflects C-density**; that is, for every $A \subseteq \mathcal{V}_s^X$, if $A \cdot i$ is **C**-dense for every $i \in \mathcal{I}$, then $A$ is **C**-dense.

Since by Proposition 21, **Fun**-density is equivalent to initiality, we refer to coreflection of **Fun**-density also as **coreflection of initiality**.

▶ **Example 26.**
1. An initial $\mathcal{V}$-functor coreflects **Id**-density iff it is L-dense.
2. A classical 1-bounded metric space $(X, d)$ is totally bounded if for every $u > 0$ there is a finite set $X_u$ such that for every $x \in X$ there is $y \in X_u$ so that $d(x, y) < u$. It can be shown that for every totally bounded metric space $(X, d)$, the cocone of embeddings of finite subspaces coreflects **L**-density.
3. It follows from [19, Lemma 1.10(4)] that every jointly L-dense directed cocone coreflects initiality. In particular, every directed colimit coreflects initiality.

Since in general, we can only replace **Fun**-density with **C**-density in a restricted class of $\mathcal{V}$-categories, our results will depend on functors that are compatible with such a class.

▶ **Definition 27.** A class $\mathcal{I}$ of symmetric $\mathcal{V}$-categories **coreflects initiality under** $\mathsf{F}$ if for every $\mathcal{V}$-category $X$, the cocone

$$(\mathsf{F}i\colon \mathsf{F}Y \to \mathsf{F}X)_{Y\in\mathcal{I},\, i\colon Y\to X \text{ initial}}$$

coreflects initiality.

▶ **Example 28.**
1. The class $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ coreflects initiality under every $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$-functor.
2. The class of all finite symmetric $\mathcal{V}$-categories coreflects initiality under Kantorovich liftings of finitary $\mathsf{Set}$-functors to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$.

The second example above indicates that coreflection of initiality relates to size bounds on the functor. The following proposition shows that such size bounds are only needed up to approximation.

▶ **Proposition 29.** *Let $j\colon \mathsf{G} \to \mathsf{F}$ be a natural transformation between $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$-functors such that each component of $j$ is initial and L-dense. If a class $\mathcal{I}$ of symmetric $\mathcal{V}$-categories coreflects initiality under $\mathsf{G}$, then $\mathcal{I}$ coreflects initiality under $\mathsf{F}$.*

For instance, combining Example 28.2 and Proposition 29, we obtain:

▶ **Corollary 30.** *If $\mathsf{G}$ is a Kantorovich lifting of a finitary set functor and $j\colon \mathsf{G} \to \mathsf{F}$ is a natural transformation between $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$-functors such that each component of $j$ is initial and L-dense, then the class of all finite symmetric $\mathcal{V}$-categories coreflects initiality under $\mathsf{F}$.*

▶ Remark 31. The previous notion of a *finitarily separable* lax extension $L$ of a set functor $\mathsf{F}_0$ [34] requires essentially that the finitary part of $\mathsf{F}_0$ is L-dense in the lifting $\mathsf{F}$ of $\mathsf{F}_0$ induced by $L$ (in a way that is immaterial here), that is, if for every $\mathcal{V}$-category $X$, the set $\{\mathsf{F}i(t) \mid Y \text{ finite}, i\colon Y \to X, t \in \mathsf{F}Y\}$ is L-dense in $\mathsf{F}X$. For instance, the standard Kantorovich lifting of the discrete distribution functor is finitarily separable [34]. For finitarily separable $\mathsf{F}$, the class of finite symmetric $\mathcal{V}$-categories coreflects initiality by Corollary 30.

We are now ready to present our main result:

▶ **Theorem 32** (Quantitative coalgebraic Hennessy-Milner theorem)**.** *Let $\mathsf{F}\colon \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ be $\Lambda$-Kantorovich, let $\mathbf{C}$ be a $\mathcal{V}_s$-closure operator, and let $\mathcal{I}$ be a class of symmetric $\mathcal{V}$-categories such that $\mathcal{I}$ coreflects initiality under $\mathsf{F}$ and $\mathbf{C}$ characterizes initiality on $\mathcal{I}$. If every predicate lifting in $\Lambda$ is $\mathbf{C}$-continuous, then $\mathcal{L}(\Lambda)$ is expressive.*

We note that there is a balance to be struck in the choice of $\mathcal{I}$: The larger $\mathcal{I}$ is, the more functors one finds under which $\mathcal{I}$ coreflects initiality, but the harder it is to establish a characterization of initiality in $\mathcal{I}$. We tackle the latter issue next.

## 7 Stone-Weierstraß-Type Theorems

We now develop some usage scenarios for Theorem 32; concrete expressivity proofs using these criteria will be given in Section 8. As a warm-up, we have

▶ **Theorem 33.** *Let $\mathcal{V}$ be a finite quantale. Then the $\mathcal{V}_s$-closure operator $\mathbf{Id}$ characterizes initiality on finite symmetric $\mathcal{V}$-categories.*

(Note that for $\mathcal{V} = 2$, this is essentially the well-known functional completeness of Boolean logic.) Hence, by instantiating Theorem 32, we obtain

▶ **Corollary 34.** *Let $\mathcal{V}$ be a finite quantale, and let $\mathsf{F}\colon \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ be a $\Lambda$-Kantorovich functor that admits as an L-dense subfunctor a lifting of a finitary $\mathsf{Set}$-functor. Then the coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive.*

Most remarkably, while Corollary 34 allows us to derive expressivity for many-valued logics, it also relates our present expressivity criterion to the known criterion for the properly qualitative case (i.e. for $\mathcal{V} = 2$) [27, 30]. To that end, recall that a set $\Lambda$ of predicate liftings for a functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$ is **separating** if for every set $X$ the cone of all maps $\lambda(f)\colon \mathsf{F}X \to 2$ with $\lambda \in \Lambda$ and $f\colon X \to 2$ is mono. For a $\mathsf{Set}$-coalgebra $(X, \alpha)$, let us denote by $beq_\alpha$ the standard notion of behavioural equivalence, as explained in preliminaries. Let $\mathsf{Equ} = 2$-$\mathsf{Cat}_{\mathsf{sym}}$ (the category of equivalence relations).

The following result generalizes [26, Theorem 11] (which applies only to functor liftings that arise from lax extensions, in particular requires modalities to be monotone):

▶ **Theorem 35.** *Let $\mathsf{F}^\Lambda\colon \mathsf{Equ} \to \mathsf{Equ}$ be a Kantorovich lifting that preserves discrete equivalence relations. Then, for every $\mathsf{F}$-coalgebra $(X, \alpha)$, $bd_\alpha^{\mathsf{F}^\Lambda} = beq_\alpha$.*

We now can recover the general expressivity result [27, 30] for $\mathcal{V} = 2$ as a direct consequence of Corollary 34 and Theorem 35.

▶ **Theorem 36** (Qualitative coalgebraic Hennessy-Milner theorem). *Let $\Lambda$ be a separating set of predicate liftings for a finitary functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$. Then the coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive; that is, if two states in an $\mathsf{F}$-coalgebra are logically indistinguishable, then they are behaviourally equivalent.*

Next, we obtain a characterization of $\mathbf{L}$-density. Recall that an element $x$ of an ordered set is *way above* an element $y$ if whenever $y \geq \bigwedge A$ for a codirected set $A$, then $x \geq a$ for some $a \in A$.

▶ **Theorem 37.** *Suppose that $\mathcal{V}$ satisfies the condition*

$$k = \bigvee \{u \otimes u \mid u \in \mathcal{V} \text{ and for all } v \in \mathcal{V},\ \hom(u, v) \text{ is way above } v\}. \tag{2}$$

*Then $\mathbf{L}$ characterizes initiality on finite symmetric $\mathcal{V}$-categories.*

We thus obtain the following quantalic generalization of the previous coalgebraic Hennessy-Milner theorem for finitarily separable $[0,1]_\oplus$-lax extensions [34, Corollary 8.6]:

▶ **Corollary 38.** *Let $\mathcal{V}$ be a quantale satisfying* (2)*, and let $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ be a $\Lambda$-Kantorovich functor that admits as an $L$-dense subfunctor a lifting of a finitary $\mathsf{Set}$-functor. If every predicate lifting in $\Lambda$ is $\mathbf{L}$-continuous, then $\mathcal{L}(\Lambda)$ is expressive.*

Specifically, besides allowing more general quantales, we drop the assumptions that the modalities in $\Lambda$ are monotone and that $\mathsf{F}$ is a lifting of a $\mathsf{Set}$-functor, and we weaken the assumption of non-expansiveness of predicate liftings to $\mathbf{L}$-continuity.

The following fact sometimes allows us to enlarge the class on which initiality is characterized:

▶ **Proposition 39.** *Let $\mathbf{C}$ be a $\mathcal{V}_s$-closure operator that characterizes initiality on $\mathcal{I}$, and let $\mathcal{J}$ be a class of symmetric $\mathcal{V}$-categories. If for every $X \in \mathcal{J}$, the cocone $(i\colon Y \to X)_{Y \in \mathcal{I},\, i \text{ initial}}$ coreflects $\mathbf{C}$-density, then $\mathbf{C}$ characterizes initiality on $\mathcal{J}$.*

In particular, we obtain by Example 26.2 that for $\mathcal{V} = [0,1]_\oplus$, $\mathbf{L}$ characterizes initiality on totally bounded pseudometric spaces, so we can, in this case, further relax the assumptions of Corollary 38 as follows.

▶ **Definition 40.** A functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$, for $\mathcal{V} = [0,1]_\oplus$, is *totally bounded* if for every symmetric $\mathcal{V}$-category $X$ and every $t \in \mathsf{F}X$, there exists a totally bounded $X_0 \subseteq X$ and $t' \in \mathsf{F}X_0$ such that $t = \mathsf{F}i(t')$, where $i$ is the inclusion $X_0 \to X$.

▶ **Corollary 41.** *Let $\mathsf{F}\colon [0,1]_\oplus\text{-}\mathsf{Cat}_{\mathsf{sym}} \to [0,1]_\oplus\text{-}\mathsf{Cat}_{\mathsf{sym}}$ be a $\Lambda$-Kantorovich functor that admits an $L$-dense totally bounded subfunctor. If every predicate lifting in $\Lambda$ is $\mathbf{L}$-continuous, then $\mathcal{L}(\Lambda)$ is expressive.*

We conclude with some variants employing order-theoretic closure operators:

▶ **Theorem 42.** *Let $\mathcal{V}$ be a quantale such that for every $u \in \mathcal{V}$ the map $u \otimes -$ preserves codirected infima. Then the closure operator $\mathbf{cInf}^\vee$ characterizes initiality on finite symmetric $\mathcal{V}$-categories.*

Notice that the assumption on $\mathcal{V}$ in the above theorem is satisfied in particular when $\mathcal{V}$ is a frame (Example 2.1).

▶ **Corollary 43.** *Let $\mathcal{V}$ be a quantale such that for every $u \in \mathcal{V}$ the map $u \otimes -$ preserves codirected infima, and let $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ be a $\Lambda$-Kantorovich functor that admits as an L-dense subfunctor a lifting of a finitary $\mathsf{Set}$-functor. If every predicate lifting in $\Lambda$ preserves codirected infima and finite suprema, then the coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive.*

▶ **Theorem 44.** *Let $\mathcal{V}$ be a completely distributive quantale such that for every $u \in \mathcal{V}$ the map $u \otimes -$ preserves codirected infima. Then the closure operator* **Inf** *characterizes initiality on finite symmetric $\mathcal{V}$-categories.*

▶ **Corollary 45.** *Let $\mathcal{V}$ be a completely distributive quantale such that for every $u \in \mathcal{V}$ the map $u \otimes -$ preserves codirected infima, and let $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$ be a $\Lambda$-Kantorovich functor that admits as an L-dense subfunctor a lifting of a finitary $\mathsf{Set}$-functor. If every predicate lifting in $\Lambda$ preserves all infima, then the coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive.*

▶ Remark 46. We note that the preservation conditions imposed on the predicate liftings in Corollary 43 and Corollary 45 (which instantiate the continuity condition from Theorem 32) are quite restrictive: Modalities must be diamond-like in Corollary 43, and box-like in Corollary 45. On the other hand, these conditions depend only on the underlying lattice of a quantale and, therefore, have the potential to be applied to multiple quantales based on the same lattice.

## 8 Examples

We apply the concretizations of the quantitiative coalgebraic Hennessy-Milner theorem (Theorem 32) proved in the previous section (Corollaries 34, 38, 41, 43, and 45) to the logics discussed in Example 1 and variants thereof.

**1. Metric transition systems.** We consider *finitely branching metric transition systems* in which every *state* is labelled with a non-negative extended real number number (while transitions are unlabelled). Such systems are coalgebras for the functor $\mathsf{F} = [0,\infty] \times \mathcal{P}_\omega$, where $\mathcal{P}_\omega$ denotes the finite powerset functor. Also, we aim for an example based on *ultrametrics*, so we we consider the quantale $[0,\infty]_{\max}$ (see Example 2(3a)). We define a nullary predicate lifting $o$ (formally accommodated as a unary predicate lifting that ignores its argument) by $o_X(r,S) = r$, and a unary predicate lifting $\Diamond$ by

$$\Diamond_X(f)(r,S) = \textstyle\bigvee_{x\in S} f(x).$$

We write $\bar{\mathsf{F}}$ for the Kantorovich lifting of $\mathsf{F}$ under the set $\Lambda$ of these modalities. We then obtain that the coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive, via Corollary 43: $\bar{\mathsf{F}}$ is itself a lifting of a finitary set functor, $o$ trivially preserves all infima and suprema, and one checks easily that $\Diamond$ preserves codirected infima and finite suprema. Similarly, by replacing the predicate lifting $\Diamond$ with the predicate lifting $\Box$ given by

$$\Box_X(f)(r,S) = \textstyle\bigwedge_{x\in S} f(x).$$

we obtain that the corresponding coalgebraic logic $\mathcal{L}(\Lambda)$ is expressive via Corollary 45. We note that these results still hold if, for example, we replace the quantale $[0,\infty]_{\max}$ with the quantale $[0,\infty]_+$ (See Example 2(3b)); that is, if we are interested in all generalized metric spaces, not only in the ultrametric ones. The metric version of these results relates to known characteristic logics for metric transition systems [10, 31]; the ultrametric versions appear to be new.

**2. $\Diamond$-valued powerset.** The lattice $\Diamond$ from Example 1.2 can be equipped with the structure of quantale by defining a commutative operator $*$, with $\mathsf{B}$ as a unit, $\bot$ as a zero and $\mathsf{N} * \mathsf{N} = \bot$, $\mathsf{N} * \top = \mathsf{N}$, $\top * \top = \top$. The resulting quantale is finite, and hence our general Corollary 34 applies to it, but not the previously existing expressivity theorem for quantale-valued distances [35], for this quantale is not a value quantale [14]. The induced logic is a paraconsistent four-valued logic with $\mathcal{L}(\Lambda)$ instantiated as follows:

$$\phi ::= \top \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid u * \phi \mid \mathrm{hom}_s(u, \phi) \mid \lambda(\phi) \qquad (u \in \mathcal{V}, \lambda \in \Lambda)$$

where $\Lambda$ can be defined in different ways, and the expressivity result remains true for any such choice (for the respective Kantorovich lifting, which also depends on $\Lambda$). A natural choice is $\Lambda = \{\Diamond\}$ where $\Diamond$ is interpreted as an instance of a generic formula: $\Diamond_X(p \in \mathcal{B}X)(q\colon X \to \mathcal{V}) = \bigvee_{x \in X} p(x) * q(x)$. This predicate lifting was previously considered by Rivieccio et al [28], who interpreted it over generalized Kripke frames, which are precisely coalgebras for the functor $\mathcal{B}$ as per Example 1.2.

**3. Discrete probabilistic transition systems.** As mentioned already in Example 10, the functor $\mathcal{D}^{\mathbb{E}}(1 + -)^A$ is the Kantorovich lifting of the finitary functor $\mathcal{D}(1 + -)^A$ w.r.t $\Lambda = \{\mathbb{E}^{a,+1} \mid a \in A\}$. Moreover, it is easy to see that every predicate lifting in $\Lambda$ gives rise to an $\mathbf{L}$-continuous predicate lifting for $\mathcal{D}^{\mathbb{E}}(1 + -)^A$. Hence, we obtain by Corollary 38 that quantitative probabilistic modal logic – the coalgebraic modal logic generated by the expectation modality – is expressive [32, 2].

**4. Weighted transition systems with negative weights.** The functor $\mathcal{W}$ defining our variant of weighted transition systems is Kantorovich for the set $\Lambda$ of (*non-monotone*) predicate liftings $\langle a \rangle^{+r}$, for $a \in A$ and $r \in \mathbb{R}$, defined by

$$[\![\langle r \rangle]\!](f)(t) = \min\Big\{1, \max\Big\{0, r + \frac{1}{2} \textstyle\sum_{x \in X} f(x)t(a)(x)\Big\}\Big\}$$

Since $\mathcal{W}$ is, moreover, a lifting of a finitary set functor, we obtain by Corollary 38 the new result that the coalgebraic modal logic $\mathcal{L}(\Lambda)$ is expressive.

**5. Continuous probabilistic transition systems.** Our variant $\mathcal{K}$ of the Kantorovich functor admits, by definition, a totally bounded L-dense subfunctor that assigns to a space $X$ the set of all Borel distributions on $X$ with totally bounded support. Hence, as $A$ is finite, it follows that the functor $\mathcal{K}(1 + -)^A$ admits a totally bounded L-dense subfunctor. Furthermore, as noted already in Example 10, the functor $\mathcal{K}(1 + -)^A$ is $\Lambda$-Kantorovich, where $\Lambda = \{\mathbb{E}^{a,+1} \mid a \in A\}$, and it is easy to see that every predicate lifting in $\Lambda$ is $\mathbf{L}$-continuous. Therefore, we obtain by Corollary 41 that the coalgebraic modal logic $\mathcal{L}(\Lambda)$ is expressive, thus essentially recovering expressivity of quantitative probabilistic modal logic on continuous probabilistic transition systems [33, 32].

## 9 Conclusions and Further Work

We have presented a quantitative Hennessy-Milner theorem in coalgebraic and quantalic generality, covering behavioural distances on a wide range of system types. Notably, our results apply to functors on metric spaces that fail to be liftings of any set functor, such as the (tight) Borel distribution functor. A key factor in the technical development was the interplay between notions of density on the one hand, and initiality of cones in the topological category of generalized metric spaces taking values in a quantale $\mathcal{V}$ ($\mathcal{V}$-*categories*) on the other hand. We have illustrated how to instantiate our results in several salient cases, in particular continuous probabilistic transition systems and weighted transition systems allowing negative weights.

For simplicity, we have worked exclusively with symmetric $\mathcal{V}$-categories throughout; nevertheless, we stress that our results carry over straightforwardly to the non-symmetric case, which covers quantitative analogues of simulation preorders (indeed, some of the existing quantitative coalgebraic Hennessy-Milner theorems already do apply to non-symmetric distances [35, 22, 36]). In fact, we expect our main expressivity theorem to be easily transported to topological categories that admit an initial dense object (which takes the role of $\mathcal{V}_s$). We leave this issue to future work. Another important direction is to develop a general coalgebraic treatment of characteristic logics for non-branching-time (e.g. linear-time) behavioural distances (e.g. [10, 12]), possibly building on recent results in this direction [7].

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