Exact Completeness of LP Hierarchies for Linear Codes

Leonardo Nagami Coregliano
Institute for Advanced Study, Princeton, NJ, USA

Fernando Granha Jeronimo
Institute for Advanced Study, Princeton, NJ, USA

Chris Jones
University of Chicago, Il, USA

Abstract

Determining the maximum size $A_2(n,d)$ of a binary code of blocklength n and distance d remains an elusive open question even when restricted to the important class of linear codes. Recently, two linear programming hierarchies extending Delsarte's LP were independently proposed to upper bound $A_2^{\mathrm{Lin}}(n,d)$ (the analogue of $A_2(n,d)$ for linear codes). One of these hierarchies, by the authors, was shown to be *approximately* complete in the sense that the hierarchy converges to $A_2^{\mathrm{Lin}}(n,d)$ as the level grows beyond n^2. Despite some structural similarities, not even approximate completeness was known for the other hierarchy by Loyfer and Linial.

In this work, we prove that both hierarchies recover the *exact* value of $A_2^{\mathrm{Lin}}(n,d)$ at level n. We also prove that at this level the polytope of Loyfer and Linial is integral. Even though these hierarchies seem less powerful than general hierarchies such as Sum-of-Squares, we show that they have enough structure to yield exact completeness via pseudoprobabilities.

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1 Introduction

A binary code is any subset of binary strings $\mathcal{C} \subseteq \mathbb{F}_2^n$. Two fundamental parameters of a code are the size $|\mathcal{C}|$ and the minimum (Hamming) distance d between pairs of distinct codewords. Determining the maximum size $A_2(n,d)$ of a binary code of blocklength n and distance d remains an elusive open problem despite much effort and interest in this fundamental question [21, 6, 20].

When the distance is $d := \lfloor \delta n \rfloor$ for some constant $\delta \in (0, 1/2)$, the growth of $A_2(n,d)$ is known to be exponential in n. It is then convenient to consider the asymptotic rate $R_2(\delta)$ defined as

$$R_2(\delta) := \limsup_{n\to\infty} \frac{1}{n} \log_2 \left(A_2(n, \lfloor \delta n \rfloor)\right).$$

Roughly speaking, the maximum size of a code grows as $2^{R_2(\delta)n}$ up to lower order terms. However, the precise asymptotic rate function $R_2(\delta)$ remains unknown, so this exponential growth is not fully understood.

The best lower bound on $R_2(\delta)$ dates back to the work of Gilbert [5] and Varshamov[1] [22]. Their bound, known as the GV bound, follows from a simple argument for the distance versus rate trade-off of random codes. The best upper bound on $R_2(\delta)$ dates back to the work of McEliece, Rodemich, Rumsey and Welch (MRRW) [13] and it is based on linear programming (LP) techniques. Specifically, $A_2(n,d)$ is upper bounded by the value of an LP of Delsarte [2], which they upper bound by constructing a dual solution using the theory of orthogonal polynomials.

Here, we will focus on the family of Delsarte's LPs used in the so-called first MRRW bound[2], which is based on the so-called Krawtchouk polynomials [21] and MacWilliams inequalities [12, 11] since this family is closer to our work. The precise details of this family of LPs are not important at this point.

Linear codes (i.e., linear subspaces) are arguably one of the most important and widely studied classes of codes [21, 6]. We denote by $A_2^{\mathrm{Lin}}(n,d)$ and $R_2^{\mathrm{Lin}}(\delta)$ the versions of $A_2(n,d)$ and $R_2(\delta)$, respectively, corresponding to linear codes. Even for this important class of codes, the known lower and upper bounds for $R_2^{\mathrm{Lin}}(\delta)$ are the same as those for $R_2(\delta)$ for general codes.

Delsarte's linear programs are a convex relaxation for $A_2(n,d)$ and there is a known gap between the value of the LP and the GV bound [16, 14]. In other words, if the GV bound is indeed tight, then Delsarte's LP is not sufficient to prove it (this would be called an integrality gap of the LP). For this reason, it is natural to look for approaches that are provably sufficient to settle the growth of $A_2(n,d)$ while having the hope of being amenable to theoretical analysis. Note that Delsarte's LPs do not distinguish between general and linear codes, hence they do not provide better bounds for $A_2^{\mathrm{Lin}}(n,d)$ nor to the asymptotic rate $R_2^{\mathrm{Lin}}(\delta)$.

There have been attempts to improve the upper bound using stronger convex relaxations of $A_2(n,d)$. The problem of computing $A_2(n,d)$ is equivalent to computing the independence number of a graph whose vertex set is $\mathbb{F}_2^n$ and pairs of vertices are adjacent if they violate the minimum distance constraint. In principle, one can employ general convex programming hierarchies such as Sum-of-Squares [8] or Sherali–Adams, which provably equal the true value $A_2(n,d)$ at a sufficiently large level. Delsarte's LP is equivalent to a convex relaxation for independent set known as Schrijver's ϑ' function [18, 8]. This is a slight strengthening of the Lovász ϑ function [9], which is equivalent to the first level of the Sum-of-Squares hierarchy for independent set. However, analyzing these general hierarchies remains elusive; in fact it even remains open to analyze an SDP proposed by Schrijver [19], which lies between Delsarte's LP and the second level of the Sum-of-Squares of hierarchy. The only convex programs we know how to analyze for this problem are Delsarte's LPs, and there are now a few different techniques for this analysis [13, 4, 14, 15, 17].

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[1] Varshamov showed Gilbert's bound for general codes remains the same for linear codes.

[2] In [13], they also analyze (in the second MRRRW bound) another family of LPs based on the Johnson association scheme.

Recently, two new convex programming hierarchies for $A_2^{\mathrm{Lin}}(n,d)$ were proposed, one by Coregliano, Jeronimo, and Jones [1] (see also [7] for an alternative exposition) and another by Loyfer and Linial [10] (in fact both hierarchies can be defined over general finite fields). In this paper, we study these hierarchies further.

The two hierarchies are similar in spirit but not exactly the same. Both hierarchies are a family of LPs (rather than SDPs) that extend Delsarte's LP into a hierarchy of tighter and tighter convex relaxations for $A_2^{\mathrm{Lin}}(n,d)$, while retaining some structural similarities with Delsarte's LP. Since Delsarte's LP is the only convex program with known theoretical analysis, there is a hope that analyzing these two new hierarchies may be possible.

The Krawtchouk hierarchy $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ of [1] was shown to be *approximately* complete beyond level n^2 in the following sense. The level of the hierarchy is ℓ, where $\ell = 1$ recovers Delsarte's LP.

▶ **Theorem 1** ([1]). *For $\ell \geq \Omega_{\varepsilon,q}(n^2)$, we have*

$$A_q^{\mathrm{Lin}}(n,d) \;\leq\; \mathrm{val}(\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell))^{1/\ell} \;\leq\; (1+\varepsilon)\cdot A_q^{\mathrm{Lin}}(n,d).$$

Our first result is the *exact* completeness at level n of the Krawtchouk hierarchy as follows.

▶ **Theorem 2.** *For $\ell \geq n$, we have $A_q^{\mathrm{Lin}}(n,d) = \mathrm{val}(\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell))^{1/\ell}$.*

Instead of relying on integrality of the feasible region (i.e., it is exactly the convex hull of true solutions corresponding to linear codes) to deduce completeness as is the case for general hierarchies such as Sherali–Adams or Sum-of-Squares, it is only possible to show that optimum solutions are integral, giving an unusual proof of completeness for a convex programming hierarchy. We also show that the polytope of $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ is *never* integral (see Proposition 13). Nonetheless, any given non-integral solution becomes infeasible as the level grows (see Proposition 14).

The partial Krawtchouk hierarchy $\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ of [10] is similar to the Krawtchouk hierarchy of [1], but it has additional constraints and a different objective function (see Section 4). Due to this different objective function and the fact that the *approximate* completeness proof of Theorem 1 crucially relies on the objective function of the Krawtchouk hierarchy being "dense", the proof of Theorem 1 did not extend to the Loyfer and Linial hierarchy. Our proof here of Theorem 2 does extend to show that $\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ is complete at level n. More precisely, our second result is the following.

▶ **Theorem 3.** *For $\ell \geq n$, we have $A_q^{\mathrm{Lin}}(n,d) = \mathrm{val}(\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell))$.*

Curiously, we show that the additional constraints of the Loyfer and Linial hierarchy make the polytope integral for $\ell \geq n$ (see Proposition 15). This integrality is not obvious from the original formulation of the hierarchy and it relies on a new perspective uncovered by this work.

The exact completeness theorems at level n (Theorems 2 and 3) improve our understanding of these hierarchies, consolidating them as provable approaches to resolve the longstanding question of improving bounds for $A_2^{\mathrm{Lin}}(n,d)$, and justifying them as natural objects in their own right. The primary open research direction is a theoretical analysis of these hierarchies to obtain tighter bounds on $R_2^{\mathrm{Lin}}(\delta)$. It is not clear which hierarchy is better suited for such a task: the Krawtchouk hierarchy may be simpler to analyze, which is of critical importance here, but the partial Krawtchouk hierarchy may provide tighter values at the same level given its additional constraints.

Proof Outline. We first briefly recall the approximate completeness proof from [1]. The hierarchy $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ can be seen as a symmetrization of the ϑ' of a graph from a carefully chosen association scheme under the actions of symbol permutation by S_n and translation by $\mathbb{F}_q^n$ (see [2, 3] and [1, §5] for more on association scheme theory). The *approximate* completeness is then obtained via a counting argument over the *unsymmetrized* ϑ' formulation, which requires level $\ell \geq n^2$ to yield non-trivial bounds.

A key insight of this work is a novel third formulation of the Krawtchouk hierarchy from which *exact* completeness can be obtained at level n. Instead of factoring symmetries that lead to variables indexed by Hamming weights, we now factor different symmetries leading to variables indexed by linear subspaces. Using a linear transformation (namely, Möbius inversion of the poset of subspaces of $\mathbb{F}_q^n$), we then rewrite the LP in terms of new variables that can interpreted as a pseudoprobability distribution over linear codes (see Section 3). In this pseudoprobability formulation, integral solutions correspond to true probability distributions and via a mass transfer argument we show that optimum solutions are integral. An interesting feature of this third formulation of the hierarchy is that the number of variables and constraints remains constant regardless of the level (see Section 3.2). Curiously, we show (Proposition 13) that the polytope of this formulation is not integral, i.e., there are non-optimum solutions that are not integral. As mentioned, these ideas also generalize to show that the partial Krawtchouk hierarchy of [10] also has *exact* completeness for $\ell \geq n$ (see Section 4).

Bibliographic Note. A preliminary version of the completeness of the Krawtchouk hierarchy $\text{KrawtchoukLP}^{\mathbb{F}_2}_{\text{Lin}}(n, d, \ell)$ (over the binary field) is included in the dissertation of one of the authors [7].

2 Preliminaries

We denote by $\mathbb{F}_q$ the finite field of size q (which must be a prime power). A *code* of blocklength $n \in \mathbb{N}^+$ (over $\mathbb{F}_q$) is a non-empty subset $\mathcal{C} \subseteq \mathbb{F}_q^n$. We denote by $\Delta(x, y) := |\{i \in [n] \mid x_i \neq y_i\}|$ the *Hamming distance* between $x, y \in \mathbb{F}_q^n$. The *minimum distance* of a code $\mathcal{C}$ is the minimum of $\Delta(x, y)$ over all distinct $x, y \in \mathcal{C}$. The *rate* $r(\mathcal{C})$ of $\mathcal{C}$ is defined as $r(\mathcal{C}) := \log_q(|\mathcal{C}|)/n$. We denote by $A_q(n, d)$ the maximum size of a code of blocklength n (over $\mathbb{F}_q$) and minimum distance at least d. We say that $\mathcal{C}$ is *linear* if it is an $\mathbb{F}_q$-linear subspace. For linear codes, we have $r(\mathcal{C}) = \dim_{\mathbb{F}_q}(\mathcal{C})$. We denote by $A_q^{\text{Lin}}(n, d)$ the analogue of $A_q(n, d)$ when codes are required to be linear.

For $\alpha \in \mathbb{F}_q^n$, we denote by $\chi_\alpha \colon \mathbb{F}_q^n \to \mathbb{C}$ the (additive) *Fourier character* associated with α and we denote by $\mathbb{1}_\alpha \colon \mathbb{F}_q^n \to \{0, 1\}$ be the *indicator function* of α. If T is a vector space (over $\mathbb{F}_q$), we will use the notation $S \leq T$ to mean that S is a subspace of T and $S < T$ to mean that S is a proper subspace of T.

The rest of this section is devoted to informal descriptions of the hierarchies from [1] and [10] in their symmetrized form. Since all arguments of this paper start from the unsymmetrized versions in Figures 3 and 6 of Sections 3 and 4 to factor different symmetries, the descriptions below serve only as guiding intuition and will not be used in any proof.

The hierarchy from [1] extends Delsarte's LPs by considering not only the Hamming weight of single codewords, but by also considering the Hamming weights of every codeword in subspaces of dimension up to a parameter $\ell \in \mathbb{N}^+$, which is the level of the hierarchy. Given an ℓ-tuple of words $(x_1, \ldots, x_\ell) \in (\mathbb{F}_q^n)^\ell$, one associates a *configuration* function mapping $c \in \mathbb{F}_q^\ell$ to the Hamming weight $|\sum_{j\in[\ell]} c_j \cdot x_j|$ (in the binary case, it is typical to naturally identify $\mathbb{F}_2^\ell$ with the set $2^{[\ell]}$ of subsets of $[\ell]$). The set of functions $\mathbb{F}_q^\ell \to \mathbb{N}$ that are configurations of some ℓ-tuple of codewords is denoted Config. If the words $x_1, \ldots, x_\ell$ belong to some linear code $\mathcal{C}$ of minimum distance d, then their configuration cannot have numbers from $[d-1] := \{1, \ldots, d-1\}$ in its image. This means that if we let a_g be the number of tuples $(x_1, \ldots, x_\ell)$ in $\mathcal{C}$ whose configuration is g, then $a_g = 0$ whenever $g \in \text{ForbConfig} := \{h \in \text{Config} \mid [d-1] \cap \text{im}(h) \neq \varnothing\}$. It is clear that $a_0 = 1$ for the zero configuration (as $0 \in \mathcal{C}$ since $\mathcal{C}$ is linear) and that $|\mathcal{C}|^\ell = \sum_{g \in \text{Config}} a_g$. Finally, by observing that the Fourier transform of the indicator $\mathbb{1}_{\mathcal{C}}$ is (up to a multiplicative constant) the indicator of the dual code $\mathcal{C}^\perp$, hence a nonnegative function, one derives the so-called (higher-order) MacWilliams inequalities based on a higher-order version of the Krawtchouk polynomials. The level ℓ of this hierarchy for codes over the field $\mathbb{F}_q$ is denoted by $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ and it is a relaxation (i.e., an upper bound) for $A_q(n, d)^\ell$. The program in Figure 1 provides an informal description of this hierarchy, where variables are indexed by configurations and K_h is the higher-order Krawtchouk polynomial associated with configuration h. In this formulation, it is immediate that the first level of this hierarchy is simply Delsarte's LP. Since we will work with a different formulation of the hierarchy (see Figure 3 in Section 3), we point the interested reader to [1, 7] for a more detailed description of this Hamming weight formulation of the hierarchy.

$$
\begin{array}{llll}
\max & \displaystyle\sum_{g \in \text{Config}} a_g & & \\
\text{s.t.} & a_0 = 1 & & \text{(Normalization)} \\
& a_g = 0 & \forall g \in \text{ForbConfig} & \text{(Distance constraints)} \\
& \displaystyle\sum_{g \in \text{Config}} K_h(g) \cdot a_g \geq 0 & \forall h \in \text{Config} & \text{(MacWilliams inequalities)} \\
& a_g \geq 0 & \forall g \in \text{Config} & \text{(Nonnegativity).}
\end{array}
$$

Figure 1 Informal description of $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$. Its optimum value is an upper bound for $A^{\text{Lin}}_q(n, d)^\ell$.

As we mentioned in the introduction, Loyfer and Linial in [10] independently proposed another linear programming hierarchy $\text{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ for linear codes that bears many structural similarities with the Krawtchouk hierarchy $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ of [1], but it is different in two important aspects. Firstly, $\text{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ uses a different objective function that sums only over configurations in $\text{Config}_1 := \{g \in \text{Config} \mid \forall c \in \text{Supp}(g), \{1\} \subseteq \text{Supp}(c)\}$ (configurations in Config_1 correspond to ℓ-tuples of codewords of the form $(x_1, 0, \ldots, 0)$); this provides an upper bound for $A^{\text{Lin}}_q(n, d)$ (as opposed to its ℓth power). Secondly, by using partial Fourier transforms as well as the usual Fourier transform (see [10] or Section 4 below for more details), the hierarchy $\text{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ also has "partial MacWilliams inequality" constraints that are not present in $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$. The program in Figure 2 provides an informal description of this hierarchy, where variables are indexed by configurations and K^S_h is the partial higher-order Krawtchouk polynomial associated with configuration h and set $S \subseteq [\ell]$.

3 Exact Completeness of the Krawtchouk LP Hierarchy

In this section, we prove the exact completeness at level n of the Krawtchouk hierarchy for linear codes, namely, we show that $A^{\text{Lin}}_q(n, d) = \text{val}(\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, n))^{1/n}$. We first give an alternative formulation of this hierarchy in terms of pseudoprobabilities in Section 3.1. Using this representation, we then show the exact completeness result in Section 3.2.

$$
\begin{array}{llll}
\max & \sum_{g \in \mathrm{Config}_1} a_g & & \\
\text{s.t.} & a_0 = 1 & & \text{(Normalization)} \\
& a_g = 0 & \forall g \in \mathrm{ForbConfig} & \text{(Distance constraints)} \\
& \sum_{g \in \mathrm{Config}} K_h^S(g) \cdot a_g \geq 0 & \forall h \in \mathrm{Config}, \forall S \subseteq [\ell] & \text{(Partial MacWilliams inequalities)} \\
& a_g \geq 0 & \forall g \in \mathrm{Config} & \text{(Nonnegativity).}
\end{array}
$$

Figure 2 Informal description of $\mathrm{PartialKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n, d, \ell)$. Its optimum value is an upper bound for $A_q^{\mathrm{Lin}}(n, d)$. The nonnegativity constraints $a_g \geq 0$ are redundant as they are also obtained as the partial MacWilliams inequalities corresponding to $S = \varnothing$. The original formulation also enforces $\mathrm{GL}_\ell(\mathbb{F}_q)$ symmetries, but these are omitted here for simplicity.

3.1 A Pseudoprobability LP Formulation

We first recall the unsymmetrized formulation of the hierarchy from [1] given in Figure 3; it corresponds to the ϑ' formulation of the Krawtchouk hierarchy expressed in "diagonalized" form using the Fourier basis. Here, we use this unsymmetrized formulation as our starting point. The interested reader is referred to [1] for more details about the connection between these equivalent formulations of the hierarchy.

$$
\begin{array}{llll}
& \text{Variables: } a_x & x \in (\mathbb{F}_q^n)^\ell & \\
\max & \sum_{x \in (\mathbb{F}_q^n)^\ell} a_x & & \\
\text{s.t.} & a_0 = 1 & & \text{(Normalization)} \\
& a_{(x_1,\ldots,x_\ell)} = 0 & \exists w \in \mathrm{span}(x_1, \ldots, x_\ell).\ |w| \in [d-1] & \text{(Distance constraints)} \\
& \sum_{x \in (\mathbb{F}_q^n)^\ell} a_x \chi_\alpha(x) \geq 0 & \forall \alpha \in (\mathbb{F}_q^n)^\ell & \text{(Fourier coefficients)} \\
& a_x = a_{-x} & \forall x \in (\mathbb{F}_q^n)^\ell & \text{(Reflection)} \\
& a_x \geq 0 & \forall x \in (\mathbb{F}_q^n)^\ell & \text{(Nonnegativity).}
\end{array}
$$

Figure 3 Unsymmetrized higher-order Krawtchouk hierarchy $\mathrm{KrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n, d, \ell)$ for $A_q(n, d)$.

To each linear code $\mathcal{C} \leq \mathbb{F}_q^n$, we have a corresponding *true solution* $a^{\mathcal{C}}$ given by

$$a_x^{\mathcal{C}} := \mathbb{1}[\forall j \in [\ell], x_j \in \mathcal{C}],$$

whose value is $|\mathcal{C}|^\ell$. Note that $a^{\mathcal{C}}$ is feasible for the program in Figure 3 if and only if $\mathcal{C}$ has minimum distance at least d.

On the other hand, the program in Figure 3 is invariant under the natural basis change action of the general linear group $\mathrm{GL}_\ell(\mathbb{F}_q)$; this means that by symmetrizing a solution a under such action, we may assume that $a_x = a_y$ whenever $\mathrm{span}(x) = \mathrm{span}(y)$; after such symmetrization, we can denote by a_S ($S \leq \mathbb{F}_q^n$) the value of a_x for any $x \in (\mathbb{F}_q^n)^\ell$ such that $\mathrm{span}(x) = S$. Note that the true solutions $a^{\mathcal{C}}$ corresponding to *linear* codes $\mathcal{C} \leq \mathbb{F}_q^n$ are already symmetrized:

$$a_x^{\mathcal{C}} = \mathbb{1}[\mathrm{span}(x) \subseteq \mathcal{C}] =: a_{\mathrm{span}(x)}^{\mathcal{C}}. \tag{1}$$

Equation (1) above suggests that we should interpret the variables a_S as the relaxation of the indicator $\mathbb{1}_{S\subseteq\mathcal{C}}$ for a code $\mathcal{C}$; or more precisely as $a_S = \widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}]$, where $\widetilde{\mathcal{C}}$ is a formal variable that represents a code drawn from a pseudodistribution of linear codes.

The next lemma uses Möbius inversion to provide a linear transformation into variables of the form $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ and shows that (symmetrized) integral solutions are precisely those in which $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ $(S \leq \mathbb{F}_q^n)$ is a (true) probability distribution (recall that a solution a is *integral* if it is a convex combination of true solutions $a^{\mathcal{C}}$).

▶ **Lemma 4.** *For every $S \leq \mathbb{F}_q^n$, let $\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}], \widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}] \in \mathbb{R}$ be real numbers. Then the following are equivalent.*

1. *For every $S \leq \mathbb{F}_q^n$, we have $\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}] = \sum_{S\leq T\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}]$.*
2. *For every $S \leq \mathbb{F}_q^n$, we have $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}] = \sum_{S\leq T\leq \mathbb{F}_q^n} \mu(S,T)\widetilde{\mathbb{P}}[T \subseteq \widetilde{\mathcal{C}}]$, where μ is the Möbius function of the poset of subspaces of $\mathbb{F}_q^n$ under inclusion.*

Furthermore, if $\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}], \widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ $(S \leq \mathbb{F}_q^n)$ satisfy the above, then for every $S \leq \mathbb{F}_q^n$, we have

$$\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}] = \sum_{T\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] \cdot a_S^T.$$

In particular, the solution $a_S := \widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}]$ $(S \leq \mathbb{F}_q^n)$ is integral if and only if $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ $(S \leq \mathbb{F}_q^n)$ is a probability distribution.

Proof. Recall that the Möbius function μ is inductively defined[3] by

$$\mu(S,T) := \begin{cases} 1, & \text{if } S = T, \\ -\displaystyle\sum_{S\leq U<T} \mu(S,U), & \text{if } S < T, \\ 0, & \text{if } S \not\leq T, \end{cases}$$

which in particular means that we have $\sum_{S\leq U\leq T} \mu(S,U) = \sum_{S\leq U\leq T} \mu(U,T) = \mathbb{1}[S = T]$ for every $S \leq T \leq \mathbb{F}_q^n$.

For the implication $1 \implies 2$, note that for every $S \leq \mathbb{F}_q^n$, we have

$$\begin{aligned} \sum_{S\leq T\leq \mathbb{F}_q^n} \mu(S,T)\widetilde{\mathbb{P}}[T \subseteq \widetilde{\mathcal{C}}] &= \sum_{S\leq T\leq \mathbb{F}_q^n} \mu(S,T) \sum_{T\leq U\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[U = \widetilde{\mathcal{C}}] \\ &= \sum_{S\leq U\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[U = \widetilde{\mathcal{C}}] \sum_{S\leq T\leq U} \mu(S,T) = \widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]. \end{aligned}$$

For the implication $2 \implies 1$, note that for every $S \leq \mathbb{F}_q^n$, we have

$$\begin{aligned} \sum_{S\leq T\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] &= \sum_{S\leq T\leq \mathbb{F}_q^n} \sum_{T\leq U\leq \mathbb{F}_q^n} \mu(T,U)\widetilde{\mathbb{P}}[U \subseteq \widetilde{\mathcal{C}}] \\ &= \sum_{S\leq U\leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[U \subseteq \widetilde{\mathcal{C}}] \sum_{S\leq T\leq U} \mu(T,U) = \widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}]. \end{aligned}$$

[3] In fact, one can show that $\mu(S,T) = (-1)^{\dim(T/S)} q^{\binom{\dim(T/S)}{2}}$ when $S \leq T$ (and 0 when $S \not\leq T$), but we will not need this explicit formula.

For the second assertion, since $a^{\mathcal{C}}_S = \mathbb{1}[S \subseteq \mathcal{C}]$, from 1, we have

$$\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}] = \sum_{T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] \cdot \mathbb{1}[S \leq T] = \sum_{T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] \cdot a^T_S,$$

that is, the solution $\widetilde{\mathbb{P}}[\,\cdot\, \subseteq \widetilde{\mathcal{C}}]$ is written as the linear combination

$$\widetilde{\mathbb{P}}[\,\cdot\, \subseteq \widetilde{\mathcal{C}}] = \sum_{T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] \cdot a^T$$

of the true solutions a^T; this linear combination is a convex combination precisely when $\widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] \geq 0$ for every $T \leq \mathbb{F}_q^n$ and $\sum_{T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T = \widetilde{\mathcal{C}}] = 1$. ◀

The idea of the proof of completeness is to rewrite the linear program in terms of the variables $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ and then argue about the program from the perspective of the pseudoprobabilities. For simplicity, let us now shorten the notation to $\widetilde{\mathbb{P}}[S] := \widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$.

	Variables: $\widetilde{\mathbb{P}}[S]$	$S \leq \mathbb{F}_q^n$	
max	$\sum_{S \leq \mathbb{F}_q^n} \lvert S\rvert^\ell \widetilde{\mathbb{P}}[S]$		
s.t.	$\sum_{S \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] = 1$		(Normalization)
	$\widetilde{\mathbb{P}}[S] = 0$	$\exists w \in S.\ \lvert w\rvert \in [d-1]$	(Distance Constraints)
	$\sum_{S \leq U} \lvert S\rvert^\ell \widetilde{\mathbb{P}}[S] \geq 0$	$\forall U \leq \mathbb{F}_q^n$	(Fourier coefficients)
	$\sum_{S \geq U} \widetilde{\mathbb{P}}[S] \geq 0$	$\forall U \leq \mathbb{F}_q^n$	(Nonnegativity).

Figure 4 $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ in terms of pseudoprobabilities for $\ell \geq n$.

▶ **Lemma 5.** *If a is a $\text{GL}_\ell(\mathbb{F}_q)$-invariant solution of the program $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ in Figure 3 and $\widetilde{\mathbb{P}}[\text{span}(x) \subseteq \widetilde{\mathcal{C}}] := a_x$ for every $x \in (\mathbb{F}_q^n)^\ell$, then $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ given by Lemma 42 is a solution of the program in Figure 4 with the same value.*

Conversely, if $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ is a solution of the program in Figure 4, then setting $a_x := \widetilde{\mathbb{P}}[\text{span}(x) \subseteq \widetilde{\mathcal{C}}]$ via Lemma 41 gives a solution of $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ with the same value.

Proof. We rewrite the ($\text{GL}_\ell(\mathbb{F}_q)$-symmetrization of the) program $\text{KrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ in Figure 3 in terms of the variables $\widetilde{\mathbb{P}}[S]$ obtained from $\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}] := a_S$ via Lemma 4.

The rewritten objective function is

$$\sum_{x \in (\mathbb{F}_q^n)^\ell} \widetilde{\mathbb{P}}[\text{span}(x) \subseteq \widetilde{\mathcal{C}}] = \sum_{x \in (\mathbb{F}_q^n)^\ell} \sum_{\text{span}(x) \leq T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T] = \sum_{S \leq \mathbb{F}_q^n} |S|^\ell \widetilde{\mathbb{P}}[S].$$

The left-hand side of the distance constraint for $S \leq \mathbb{F}_q^n$ such that there exists $w \in S$ with $|w| \in [d-1]$ is

$$\widetilde{\mathbb{P}}[S \subseteq \widetilde{\mathcal{C}}] = \sum_{T \geq S} \widetilde{\mathbb{P}}[T]$$

By induction downwards on the dimension of S, requiring the above to be equal to 0 is equivalent to the constraints

$$\widetilde{\mathbb{P}}[S] = 0 \qquad (S \leq \mathbb{F}_q^n : \exists w \in S.\ |w| \in [d-1]).$$

The left-hand side of the Fourier constraint for α is

$$\sum_{x \in (\mathbb{F}_q^n)^\ell} \widetilde{\mathbb{P}}[\operatorname{span}(x) \subseteq \widetilde{\mathcal{C}}]\chi_\alpha(x) = \sum_{x \in (\mathbb{F}_q^n)^\ell} \chi_\alpha(x) \sum_{\operatorname{span}(x) \leq T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T] = \sum_{S \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] \sum_{x \in S^\ell} \chi_\alpha(x).$$

Let us now show the following claim.

▷ Claim 6. For $\alpha \in (\mathbb{F}_q^n)^\ell, S \leq \mathbb{F}_q^n$, we have

$$\sum_{x \in S^\ell} \chi_\alpha(x) = \begin{cases} |S|^\ell, & \text{if } S \leq \operatorname{span}(\alpha)^\perp, \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Claim 6. If $S \leq \operatorname{span}(\alpha)^\perp$, then all terms of the sum are 1, so the result follows. On the other hand, if $S \not\leq \operatorname{span}(\alpha)^\perp$, then there exist $y \in S$ and $\beta \in \operatorname{span}(\alpha)$ such that $\chi_\beta(y) \neq 1$. Write $\beta = \sum_{j \in [\ell]} c_j \cdot \alpha_j$ for $c_j \in \mathbb{F}_q$ and let $z \in S^\ell$ be given by $z_j := c_j \cdot y$ ($j \in [\ell]$). Then we have

$$\sum_{x \in S^\ell} \chi_\alpha(x) = \sum_{x \in S^\ell} \chi_\alpha(x+z) = \chi_\alpha(z) \cdot \sum_{x \in S^\ell} \chi_\alpha(x) = \chi_\beta(y) \cdot \sum_{x \in S^\ell} \chi_\alpha(x),$$

and since $\chi_\beta(y) \neq 1$, we conclude that $\sum_{x \in S^\ell} \chi_\alpha(x) = 0$. ◁

From Claim 6 above, it follows that the Fourier constraint for α is equivalent to

$$\sum_{S \leq \operatorname{span}(\alpha)^\perp} |S|^\ell \widetilde{\mathbb{P}}[S] \geq 0,$$

concluding the proof. ◀

It will also be convenient to consider a weakening of this formulation that is more amenable to analysis. Let $k_0 := \log_q(A_q^{\mathrm{Lin}}(n,d))$ be the maximum dimension of a linear code of minimum distance at least d. The program of Figure 5 below is obtained from that of Figure 4 by replacing the distance constraints with the following "dimension constraints".

$a_{(x_1,\ldots,x_\ell)} = 0$	if $\dim(\operatorname{span}(x_1,\ldots,x_\ell)) > k_0$	(Dimension constraints)

▶ **Lemma 7.** *The program in Figure 5 is a relaxation of* $\mathrm{KrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$.

Proof. Since $k_0 := \log_q(A_q^{\mathrm{Lin}}(n,d))$, any subspace of dimension larger than k_0 must have minimum distance less than d, so the distance constraints imply the dimension constraints. Thus, the result follows. ◀

From Lemmas 5 and 7, to show exact completeness of $\mathrm{KrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$, it suffices to show that the weakened program of Figure 5 has optimum value $A_q^{\mathrm{Lin}}(n,d)^\ell$. The advantage of working with the formulations that use the variables $\widetilde{\mathbb{P}}[S]$ is that the Fourier constraints no longer have sign alternations. However, the challenge is now to show that optimum solutions must force $\widetilde{\mathbb{P}}[S]$ to take nonnegative values.

$$
\begin{array}{llll}
 & \text{Variables: } \widetilde{\mathbb{P}}[S] & S \leq \mathbb{F}_q^n & \\
\max & \displaystyle\sum_{S \leq \mathbb{F}_q^n} |S|^\ell \widetilde{\mathbb{P}}[S] & & \\
\text{s.t.} & \displaystyle\sum_{S \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] = 1 & & \text{(Normalization)} \\
 & \widetilde{\mathbb{P}}[S] = 0 & \text{if } \dim(S) > k_0 & \text{(Dimension constraints)} \\
 & \displaystyle\sum_{S \leq U} |S|^\ell \widetilde{\mathbb{P}}[S] \geq 0 & \forall U \leq \mathbb{F}_q^n & \text{(Fourier coefficients)} \\
 & \displaystyle\sum_{S \geq U} \widetilde{\mathbb{P}}[S] \geq 0 & \forall U \leq \mathbb{F}_q^n & \text{(Nonnegativity).}
\end{array}
$$

Figure 5 $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell)$, weakened to dimension constraints, in terms of pseudoprobabilities for $\ell \geq n$.

3.2 Exact Completeness Proof

Before we start the proof, note that by level n there is a variable for each possible basis of a subspace of $\mathbb{F}_q^n$, which means that just writing down the distance constraints of the program $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, n)$ allows one to deduce the true value of $A_q^{\mathrm{Lin}}(n, d)$. However, the LP hierarchy does not know how to use this kind of reasoning, hence our proof of completeness is more involved. On the other hand, a feature of this subspace formulation of the hierarchy is that the number of variables and constraints remains constant regardless of the level ℓ (as long as $\ell \geq n$).

Note that we *do not* show that the polytope is integral, meaning that feasible solutions are integral (i.e., convex combinations of true solutions). In fact, we will see in Proposition 13 that the polytope is not integral when $k_0 \geq 2$.

We now restate and prove our main result.

▶ **Theorem 8.** *For $\ell \geq n$, we have $A_q^{\mathrm{Lin}}(n, d) = \mathrm{val}(\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell))^{1/\ell}$. More precisely, every $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant optimum solution of $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell)$ is integral.*

Proof of Theorem 8. Since the program $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell)$ is $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant, the first assertion follows from the second assertion.

An immediate consequence of Lemmas 4 and 5 is that to show integrality of $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant optimum solutions of $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell)$, it is sufficient to prove that every optimum solution $\widetilde{\mathbb{P}}[S]$ ($S \leq \mathbb{F}_q^n$) of the program in Figure 4 is a probability distribution.

Now we claim that it is sufficient to prove that every optimum solution of the program in Figure 5 is a probability distribution. Indeed, if this is the case, then the optimum value of both programs in Figures 4 and 5 must be $|\mathbb{F}_q|^{k_0 \cdot \ell} = A_q^{\mathrm{Lin}}(n, d)^\ell$, since the definition of k_0 implies that there must be at least one true solution corresponding to a code $\mathcal{C}$ of dimension k_0 and minimum distance at least d. In particular, every optimum solution of the former program must also be an optimum solution of the latter, hence a probability distribution.

Let us then show that an optimum solution $\widetilde{\mathbb{P}}$ of the program Figure 5 is a probability distribution. Since $\sum_{S \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] = 1$ already follows from the normalization constraint, we only have to show that $\widetilde{\mathbb{P}}$ is nonnegative.

If S is a space in the support of $\widetilde{\mathbb{P}}$ of minimum dimension, then $\widetilde{\mathbb{P}}[S] \geq 0$ by the Fourier constraint on S. Thus, to show that $\widetilde{\mathbb{P}}$ is nonnegative, it suffices to show that every such space of minimum dimension has dimension exactly k_0 (note that spaces of dimension larger than k_0 are not in the support of $\widetilde{\mathbb{P}}$ due to the dimension constraints). To that end, let $S_{\min}$ be a subspace of minimum dimension in the support of $\widetilde{\mathbb{P}}$, assume for the sake of contradiction that $\dim(S_{\min}) < k_0$ and let us show that there is a way to increase the objective value of $\widetilde{\mathbb{P}}$. Indeed, we construct another solution $\widetilde{\mathbb{P}}_+$ by transferring the probability mass from $S_{\min}$ and dividing it equally among the $S > S_{\min}$ with $\dim(S) = \dim(S_{\min}) + 1$. Formally, letting $\mathcal{S} \coloneqq \{S \geq S_{\min} : \dim(S) = \dim(S_{\min}) + 1\}$ and $m \coloneqq |\mathcal{S}|$ be the number of such spaces, we define:

$$\widetilde{\mathbb{P}}_+[S] \coloneqq \begin{cases} 0, & \text{if } S = S_{\min}, \\ \widetilde{\mathbb{P}}[S] + \dfrac{\widetilde{\mathbb{P}}[S_{\min}]}{m}, & \text{if } S \geq S_{\min} \text{ and } \dim(S) = \dim(S_{\min}) + 1, \\ \widetilde{\mathbb{P}}[S], & \text{otherwise.} \end{cases}$$

Let us verify that $\widetilde{\mathbb{P}}_+$ remains a feasible solution.

- $\widetilde{\mathbb{P}}_+$ respects the normalization $\sum_{S \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}_+[S] = 1$.
- The dimension constraints are not violated since $\dim(S) = \dim(S_{\min}) + 1 \leq k_0$ for the spaces $S \in \mathcal{S}$ in the second case above.
- In Fourier constraints with $U \not\geq S_{\min}$, nothing changes. In the ones with $U = S_{\min}$, the left-hand side is 0. Finally, when $U > S_{\min}$, U contains at least one of the subspaces $S \in \mathcal{S}$ with increased mass. Therefore the change in the left-hand side is at least

  $$|S|^\ell \cdot \frac{\widetilde{\mathbb{P}}[S_{\min}]}{m} - |S_{\min}|^\ell \cdot \widetilde{\mathbb{P}}[S_{\min}].$$

  Since $m \leq |\mathbb{F}_q|^n$ while $\frac{|S|^\ell}{|S_{\min}|^\ell} = |\mathbb{F}_q|^\ell \geq |\mathbb{F}_q|^n$, this is nonnegative.
- In the nonnegativity constraints, if U is not below any space in $\mathcal{S}$, then nothing changes. If U is below $S_{\min}$, then the sum in the nonnegativity constraint is unchanged since all $S \in \mathcal{S}$ appear in the sum. Finally, if $U \in \mathcal{S}$, then the sum increased by $\widetilde{\mathbb{P}}[S_{\min}]/m$.

Finally, note that objective value of the new solution $\widetilde{\mathbb{P}}_+[S]$ is

$$\sum_{S \leq \mathbb{F}_q^n} |S|^\ell \widetilde{\mathbb{P}}_+[S] = \sum_{S \leq \mathbb{F}_q^n} |S|^\ell \widetilde{\mathbb{P}}[S] + |S_{\min}|^\ell \left(|\mathbb{F}_q|^\ell - 1\right) \widetilde{\mathbb{P}}[S_{\min}],$$

which is strictly larger than the previous objective value since $\widetilde{\mathbb{P}}[S_{\min}] > 0$, a contradiction.

Therefore, $\widetilde{\mathbb{P}}$ must be supported only on spaces of dimension exactly k_0, it is nonnegative and integral, and the proof is complete. ◀

4 Exact Completeness of the Partial Krawtchouk LP Hierarchy

The hierarchy from [10] differs from the one in the previous section in two ways. Firstly, besides the Fourier constraints, it includes the following partial Fourier constraints:

$$\sum_{x \in (\mathbb{F}_q^n)^\ell} a_x \theta_\alpha(x) \geq 0 \qquad \forall \alpha \in (\mathbb{F}_q^n)^\ell, \text{ where } \theta_\alpha \coloneqq \theta_{\alpha_1} \otimes \cdots \otimes \theta_{\alpha_\ell}, \ \theta_{\alpha_i} \in \{\chi_{\alpha_i}, \mathbb{1}_{\alpha_i}\} \qquad \text{(Partial Fourier)}$$

In the expression above, $\mathbb{1}_{\alpha_i} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is the indicator function of α_i. Secondly, its objective function is slightly different, meant to be a relaxation for the value $A_q(n,d)$ rather than $A_q(n,d)^\ell$.

We denote by $\mathrm{PartialKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$ the level ℓ of the partial Krawtchouk hierarchy for $A_q(n,d)$ from [10]. An unsymmetrized version of this hierarchy is presented in Figure 6. The exact description of the hierarchy factors $\mathrm{GL}_\ell(\mathbb{F}_q)$ and S_n symmetries (see also Figure 2).

$$
\begin{array}{llll}
 & \text{Variables: } a_x & x \in (\mathbb{F}_q^n)^\ell & \\
\max & \displaystyle\sum_{x_1 \in \mathbb{F}_q^n} a_{(x_1,0,\ldots,0)} & & \\
\text{s.t.} & a_0 = 1 & & \text{(Normalization)} \\
 & a_{(x_1,\ldots,x_\ell)} = 0 & \exists w \in \mathrm{span}(x_1,\ldots,x_\ell).\ |w| \in [d-1] & \text{(Distance constraints)} \\
 & \displaystyle\sum_{x \in (\mathbb{F}_q^n)^\ell} a_x \theta_\alpha(x) \geq 0 & \forall \alpha \in (\mathbb{F}_q^n)^\ell, \text{ where} & \text{(Partial Fourier)} \\
 & & \theta_\alpha := \theta_{\alpha_1} \otimes \cdots \otimes \theta_{\alpha_\ell}, & \\
 & & \theta_{\alpha_i} \in \{\chi_{\alpha_i}, \mathbb{1}_{\alpha_i}\} & \\
 & a_x = a_y & \forall x,y \in (\mathbb{F}_q^n)^\ell, \mathrm{span}(x) = \mathrm{span}(y) & \text{(}\mathrm{GL}_\ell(\mathbb{F}_q)\text{-symmetries)} \\
 & a_x \geq 0 & \forall x \in (\mathbb{F}_q^n)^\ell & \text{(Nonnegativity).}
\end{array}
$$

Figure 6 Unsymmetrized partial Krawtchouk hierarchy $\mathrm{PartialKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$.

To show exact completeness of the partial Krawtchouk hierarchy, we will first give an alternative description in terms of pseudoprobabilities in a similar way as done for the Krawtchouk hierarchy in Section 3.1. It is enough to show exact completeness for the following weakening given in Figure 6, where only full Fourier constraints are included. Note that $\mathrm{FullKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$ is the same as hierarchy of Figure 3 from Section 3 with a different objective function.

$$
\begin{array}{llll}
 & \text{Variables: } a_x & x \in (\mathbb{F}_q^n)^\ell & \\
\max & \displaystyle\sum_{x_1 \in \mathbb{F}_q^n} a_{(x_1,0,\ldots,0)} & & \\
\text{s.t.} & a_0 = 1 & & \text{(Normalization)} \\
 & a_{(x_1,\ldots,x_\ell)} = 0 & \exists w \in \mathrm{span}(x_1,\ldots,x_\ell).\ |w| \in [d-1] & \text{(Distance constraints)} \\
 & \displaystyle\sum_{x \in (\mathbb{F}_q^n)^\ell} a_x \chi_\alpha(x) \geq 0 & \forall \alpha \in (\mathbb{F}_q^n)^\ell & \text{(Full Fourier)} \\
 & a_x = a_y & \forall x,y \in (\mathbb{F}_q^n)^\ell, \mathrm{span}(x) = \mathrm{span}(y) & \text{(}\mathrm{GL}_\ell(\mathbb{F}_q)\text{-symmetries)} \\
 & a_x \geq 0 & \forall x \in (\mathbb{F}_q^n)^\ell & \text{(Nonnegativity).}
\end{array}
$$

Figure 7 Unsymmetrized partial Krawtchouk hierarchy $\mathrm{FullKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$, weakened to only full Fourier constraints.

4.1 A Pseudoprobability LP Formulation

Similarly to Section 3.1, we will show that the program in Figure 8 is a reformulation of the weakening of the partial Krawtchouk hierarchy $\mathrm{FullKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$.

$$\begin{array}{llll}
 & \text{Variables: } \widetilde{\mathbb{P}}[S] & S \le \mathbb{F}_q^n & \\
\max & \sum_{S \le \mathbb{F}_q^n} |S| \widetilde{\mathbb{P}}[S] & & \\
\text{s.t.} & \sum_{S \le \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] = 1 & & \text{(Normalization)} \\
 & \widetilde{\mathbb{P}}[S] = 0 & \text{if } \dim(S) > k_0 & \text{(Dimension constraints)} \\
 & \sum_{S \le U} |S|^{\ell} \widetilde{\mathbb{P}}[S] \ge 0 & \forall U \le \mathbb{F}_q^n & \text{(Fourier coefficients)} \\
 & \sum_{S \ge U} \widetilde{\mathbb{P}}[S] \ge 0 & \forall U \le \mathbb{F}_q^n & \text{(Nonnegativity).}
\end{array}$$

Figure 8 $\mathrm{FullKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$, weakened to dimension constraints and with only the full Fourier constraints, in terms of pseudoprobabilities for $\ell \ge n$.

▶ **Lemma 9.** *If a is a $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant solution of $\mathrm{FullKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ from Figure 7 and $\widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}] := a_x$ for every $x \in (\mathbb{F}_q^n)^\ell$, then $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ given by Lemma 42 is a solution of the program in Figure 8 with the same value.*

Conversely, if $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ is a solution of the program in Figure 8, then setting $a_x := \widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}]$ via Lemma 41 gives a solution of $\mathrm{FullKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ with the same value.

Proof. Since the program of Figure 6 is the same as the program of Figure 3 except for the objective function, the proof is the same as that of Lemma 5 only differing in the objective function analysis. But note that the rewritten objective function is

$$\sum_{x=(x_1,0,\ldots,0):x_1\in\mathbb{F}_q^n} \widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}] = \sum_{x=(x_1,0,\ldots,0):x_1\in\mathbb{F}_q^n} \sum_{\mathrm{span}(x)\le T\le\mathbb{F}_q^n} \widetilde{\mathbb{P}}[T] = \sum_{S\le\mathbb{F}_q^n} |S|\widetilde{\mathbb{P}}[S],$$

concluding the proof. ◀

For the exact completeness, it will be sufficient to consider the above weakened pseudoprobability formulation of Lemma 9. However, to cover some integrality properties of Section 5, it will also be useful to give a pseudoprobability formulation of $\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ that includes all partial Fourier constraints.

▶ **Lemma 10.** *If a is a $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant solution of $\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ from Figure 6 and $\widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}] := a_x$ for every $x \in (\mathbb{F}_q^n)^\ell$, then $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ given by Lemma 42 is a solution of the program in Figure 9 with the same value.*

Conversely, if $\widetilde{\mathbb{P}}[S = \widetilde{\mathcal{C}}]$ is a solution of the program in Figure 9, then setting $a_x := \widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}]$ via Lemma 41 gives a solution of $\mathrm{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n,d,\ell)$ with the same value.

$$
\begin{array}{llll}
 & \text{Variables: } \widetilde{\mathbb{P}}[S] & S \le \mathbb{F}_q^n & \\
\max & \displaystyle\sum_{S \le \mathbb{F}_q^n} |S|\widetilde{\mathbb{P}}[S] & & \\
\text{s.t.} & \displaystyle\sum_{S \le \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] = 1 & & \text{(Normalization)} \\
 & \widetilde{\mathbb{P}}[S] = 0 & \exists w \in S.\ |w| \in [d-1] & \text{(Distance Constraints)} \\
 & \displaystyle\sum_{T \le S \le U} |S|^r \widetilde{\mathbb{P}}[S] \ge 0 & \forall T \le U \le \mathbb{F}_q^n : {}^{n-\dim(U) \le r \le \ell,}_{\dim(T) \le \ell - r} & \text{(Partial Fourier coefficients)} \\
 & \displaystyle\sum_{S \ge U} \widetilde{\mathbb{P}}[S] \ge 0 & \forall U \le \mathbb{F}_q^n & \text{(Nonnegativity).}
\end{array}
$$

Figure 9 $\mathrm{KrawtchoukLP}^{\mathbb{F}_q}_{\mathrm{Lin}}(n, d, \ell)$ in terms of pseudoprobabilities for $\ell \ge n$.

Proof. The program in Figure 9 is the same as the program in Figure 8 with additional partial Fourier constraints. We can then follow the proof of Lemma 5 except for these additional constraints which we now analyze.

For $\mathcal{I} \subseteq [\ell]$ and $\alpha \in (\mathbb{F}_q^n)^\ell$, let $\theta_\alpha := \theta_{\alpha_1} \otimes \cdots \otimes \theta_{\alpha_\ell}$, where $\theta_{\alpha_i} := \chi_{\alpha_i}$ for $i \in \mathcal{I}$ and $\theta_{\alpha_i} := \mathbb{1}_{\alpha_i}$ for $i \in [\ell] \setminus \mathcal{I}$. The left-hand side of the Fourier constraint for θ_α is

$$
\sum_{x \in (\mathbb{F}_q^n)^\ell} \widetilde{\mathbb{P}}[\mathrm{span}(x) \subseteq \widetilde{\mathcal{C}}]\theta_\alpha(x) = \sum_{x \in (\mathbb{F}_q^n)^\ell} \theta_\alpha(x) \sum_{T \ge \mathrm{span}(x)} \widetilde{\mathbb{P}}[T] = \sum_{S \le \mathbb{F}_q^n} \widetilde{\mathbb{P}}[S] \sum_{x \in S^\ell} \theta_\alpha(x).
$$

We now prove the following claim.

▷ Claim 11. For $\mathcal{I} \subseteq [\ell]$, $\alpha \in (\mathbb{F}_q^n)^\ell$ and $S \le \mathbb{F}_q^n$, we have

$$
\sum_{x \in S^\ell} \theta_\alpha(x) = \begin{cases} |S|^{|\mathcal{I}|} \displaystyle\prod_{j \in [\ell] \setminus \mathcal{I}} \mathbb{1}_{\alpha_j \in S}, & \text{if } S \le \mathrm{span}(\alpha_i : i \in \mathcal{I})^\perp, \\ 0, & \text{otherwise.} \end{cases}
$$

Proof of Claim 11. Clearly, if there exists $j \in [\ell] \setminus \mathcal{I}$ such that $\alpha_j \notin S$, then the sum above is zero. Suppose then that for every $j \in [\ell] \setminus \mathcal{I}$, we have $\alpha_i \in S$. Then the sum becomes

$$
\sum_{x \in S^{\mathcal{I}}} \prod_{j \in \mathcal{I}} \chi_{\alpha_j}(x_j),
$$

and the result follows by Claim 6. ◁

Let $U := \mathrm{span}(\alpha_i : i \in \mathcal{I})^\perp$ and let $T := \mathrm{span}(\alpha_i : i \in [\ell] \setminus \mathcal{I})$. By Claim 11, the Fourier constraint corresponding to θ_α is equivalent to

$$
\sum_{T \le S \le U} |S|^{|\mathcal{I}|} \widetilde{\mathbb{P}}[S] \ge 0.
$$

Since the program in Figure 9 has the partial Fourier constraints

$$
\sum_{T \le S \le U} |S|^r \widetilde{\mathbb{P}}[S] \ge 0 \qquad \forall T \le U \le \mathbb{F}_q^n \quad : \quad {}^{n-\dim(U) \le r \le \ell,}_{\dim(T) \le \ell - r},
$$

it remains to show every U and T above can be obtained as $U = \mathrm{span}(\alpha_i : i \in \mathcal{I})^\perp$ and $T = \mathrm{span}(\alpha_i : i \in [\ell] \setminus \mathcal{I})$.

Indeed, for every such $U \leq \mathbb{F}_q^n$ with $u := \dim(U) \geq n - \ell$, we can use $r \in \{n-u, \dots, \ell\}$ entries in the vector α to specify a spanning set for $U^\perp$. These entries will correspond to some $\mathcal{I} \subseteq [\ell]$ of size r. We then use the remaining $\ell - r$ entries of α to specify a spanning set for the space $T \leq U$ of dimension at most $\ell - r$, concluding the proof. ◀

4.2 Exact Completeness Proof

We now prove the exact completeness of the partial Krawtchouk hierarchy of [10].

▶ **Theorem 12.** *For $\ell \geq n$, we have $A_q^{\mathrm{Lin}}(n,d) = \mathrm{val}(\mathrm{PartialKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell))$. More precisely, every $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant optimum solution of $\mathrm{PartialKrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$ is integral.*

Proof. By Lemma 9 and similarly to Theorem 8, it is enough to show that every optimum solution $\widetilde{\mathbb{P}}$ of the pseudoprobability program in Figure 8 is nonnegative.

Note that the feasible region of this pseudoprobability program is the same as the one of the pseudoprobability program of Figure 5, so we can follow the same completeness proof of Theorem 8, except for the objective function analysis. Inspecting that proof, we see that it only requires the property that the objective value increases if mass is moved to larger dimensional spaces. This property is also satisfied by the new objective function $\sum_{S \leq \mathbb{F}_q^n} |S| \widetilde{\mathbb{P}}[S]$ so we are done. ◀

5 On Integrality Related Properties

In this section, we discuss some properties related to the integrality of the Krawtchouk hierarchies. Recall that by the results of Sections 3 and 4, integrality of $\mathrm{GL}_\ell(\mathbb{F}_q)$-invariant solutions is equivalent to nonnegativity of solutions in the pseudoprobability formulations; as such, we will slightly abuse notation and say that the polytope of the pseudoprobability formulation is integral when all its feasible solutions are nonnegative.

We start by showing that the polytope of the pseudoprobability formulation of the program $\mathrm{KrawtchoukLP}_{\mathrm{Lin}}^{\mathbb{F}_q}(n,d,\ell)$ is not integral no matter how large is the level of the hierarchy.

▶ **Proposition 13.** *The polytope defined by the constraints of the pseudoprobability formulation from Figure 4 is not integral for any $k_0 \geq 2$.*

Proof. We construct a feasible solution to the program in Figure 5 having a negative pseudoprobability. Let $T \leq \mathbb{F}_q^n$ be any subspace of dimension k_0 of minimum distance at least d and let $T' \leq T$ be an arbitrary one dimensional space. Since $k_0 \geq 2$, we have $T' \neq T$. Let $\varepsilon \in (0,1)$. Now for each $S \leq \mathbb{F}_q^n$, we set

$$\widetilde{\mathbb{P}}[S] := \begin{cases} 1 - \left(\varepsilon - \dfrac{\varepsilon}{|S|^\ell}\right), & \text{if } S = T, \\ -\dfrac{\varepsilon}{|S|^\ell}, & \text{if } S = T', \\ \varepsilon, & \text{if } S = \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that the above is a feasible solution. The proof is a simple verification. The values $\widetilde{\mathbb{P}}[S]$ clearly sum to 1 satisfying the normalization constraint. Since T has minimum distance at least d, so does T', hence the distance constraints are satisfied. The Fourier constraint of T' is

$$|T'|^{\ell}\widetilde{\mathbb{P}}[T'] + \widetilde{\mathbb{P}}[\{0\}] = 0.$$

Since all values except from $\widetilde{\mathbb{P}}[T']$ are nonnegative and the Fourier constraint of T' is satisfied, we have that all Fourier constraints hold. The nonnegative constraint for T' is

$$\sum_{S \geq T'} \widetilde{\mathbb{P}}[S] = \widetilde{\mathbb{P}}[T] + \widetilde{\mathbb{P}}[T'] = 1 - \left(\varepsilon - \frac{\varepsilon}{|S|^{\ell}}\right) - \frac{\varepsilon}{|S|^{\ell}} = 1 - \varepsilon \geq 0,$$

where the last inequality follows from our choice of ε. All other nonnegative constraints are easily seen to hold and we conclude the proof. ◀

Despite the polytope not being integral no matter how large is the level, the following approximate integrality property holds: any given non-integral solution becomes infeasible at a sufficiently large level.

▶ **Proposition 14.** *Let $\{\widetilde{\mathbb{P}}[S]\}_{S \leq \mathbb{F}_q^n}$ be a feasible solution to level ℓ of program Figure 4. If one of the variables is negative, then there exist $\ell' \geq \ell$ large enough such that this solution is infeasible for level ℓ'.*

Proof. Let $U \leq \mathbb{F}_q^n$ be any space such that $\widetilde{\mathbb{P}}[U] < 0$ and its dimension is maximum with this property. Note that U is well-defined by assumption. We claim that the Fourier constraint

$$\sum_{S \leq U} |S|^{\ell'} \widetilde{\mathbb{P}}[S] \geq 0$$

becomes violated for a sufficiently large $\ell' \geq \ell$. By dividing this Fourier constraint by $|S|^{\ell'}$, only the coefficient of $\widetilde{\mathbb{P}}[U]$ remains 1 while all other coefficients shrink as ℓ' grows since U is the space of largest dimension appearing in the sum. ◀

Let us now show that the additional partial Fourier constraints ensure that the polytope of the pseudoprobability formulation of the hierarchy $\text{PartialKrawtchoukLP}^{\mathbb{F}_q}_{\text{Lin}}(n, d, \ell)$ is actually integral for $\ell \geq n$. Note that this provides an alternative proof of exact completeness.

▶ **Proposition 15.** *The polytope defined by the constraints of the pseudoprobability formulation from Figure 9 is integral for $\ell \geq n$.*

Proof. We will show that $\widetilde{\mathbb{P}}[T] \geq 0$ for every $T \leq \mathbb{F}_q^n$. Combined with the normalization constraint $\sum_{T \leq \mathbb{F}_q^n} \widetilde{\mathbb{P}}[T] = 1$, we will have a true probability distribution over valid codes and thus the polytope will be integral. Recall the Fourier constraints from Figure 9,

$$\sum_{T \leq S \leq U} |S|^{r} \widetilde{\mathbb{P}}[S] \geq 0 \qquad \forall T \leq U \leq \mathbb{F}_q^n \quad : \quad {\scriptstyle n - \dim(U) \leq r \leq \ell, \atop \dim(T) \leq \ell - r}.$$

Since $\ell \geq n$, by choosing $r := n - \dim(U)$, we can take $T := U$. In this case, the sum above reduces to only the term $\widetilde{\mathbb{P}}[U]$ with the coefficient $|U|^r > 0$. This readily implies $\widetilde{\mathbb{P}}[U] \geq 0$. ◀

6 Conclusion

In this paper, we proved exact completeness by level n of the LP hierarchies KrawtchoukLP and PartialKrawtchoukLP of [1] and [10], respectively. Our techniques involved passing to a formulation of these hierarchies in terms of pseudoprobabilities (after appropriate symmetrization under the natural $\mathrm{GL}_\ell(\mathbb{F}_q)$ action) and showing that optimum solutions are integral (i.e., are convex combinations of true solutions, corresponding to linear codes). We also observed two structural properties about the feasible polytopes of these hierarchies: while for KrawtchoukLP no level guarantees integrality of the polytope, for PartialKrawtchoukLP, the polytope is integral by level n.

As mentioned before, the completeness results of these hierarchies should be seen as theoretical results that can serve as basis for a theoretical analysis of the asymptotic behavior of $A_q^{\mathrm{Lin}}(n,d)$. However, neither of the hierarchies should be computationally run as high as level $\ell = n$, since even writing the constraints at this level involves checking which ℓ-dimensional subspaces satisfy the distance constraints. If $\ell > k_0 \coloneqq \log_q A_q^{\mathrm{Lin}}(n,d)$, then we would be able to deduce the value of k_0 by simply noting that no subspace of dimension $k_0 + 1$ satisfies the distance constraints. This simple observation makes plausible that the hierarchies could be complete by an earlier level, say $O(k_0)$.

References

1 Leonardo Nagami Coregliano, Fernando Granha Jeronimo, and Chris Jones. A complete linear programming hierarchy for linear codes. In *13th Innovations in Theoretical Computer Science Conference*, volume 215 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 51, 22. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022.

2 P. Delsarte. An algebraic approach to the association schemes of coding theory, 1973.

3 Philippe Delsarte and Vladimir I. Levenshtein. Association schemes and coding theory. *IEEE Trans. Inform. Theory*, 44(6):2477–2504, 1998. Information theory: 1948–1998. doi:10.1109/18.720545.

4 Joel Friedman and Jean-Pierre Tillich. Generalized Alon-Boppana theorems and error-correcting codes. *SIAM J. Discrete Math.*, 19(3):700–718, 2005. doi:10.1137/S0895480102408353.

5 E.N. Gilbert. A comparison of signalling alphabets. *Bell System Technical Journal*, 31:504–522, 1952.

6 Venkatesan Guruswami, Atri Rudra, and Madhu Sudan. Essential coding theory, 2019.

7 Chris Jones. *Symmetrized Fourier Analysis of Convex Relaxations for Combinatorial Optimization Problems.* PhD thesis, University of Chicago, 2022. Draft at http://people.cs.uchicago.edu/~csj/publications/PhDThesis.pdf.

8 Monique Laurent. Strengthened semidefinite programming bounds for codes. *Math. Program.*, 109(2-3, Ser. B):239–261, 2007. doi:10.1007/s10107-006-0030-3.

9 László Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, 25(1):1–7, 1979. doi:10.1109/TIT.1979.1055985.

10 Elyassaf Loyfer and Nati Linial. New lp-based upper bounds in the rate-vs.-distance problem for linear codes, 2022.

11 F. J. MacWilliams, N. J. A. Sloane, and J.-M. Goethals. The MacWilliams identities for nonlinear codes. *Bell System Tech. J.*, 51:803–819, 1972. doi:10.1002/j.1538-7305.1972.tb01947.x.

12 Jessie MacWilliams. A theorem on the distribution of weights in a systematic code. *Bell System Tech. J.*, 42:79–94, 1963. doi:10.1002/j.1538-7305.1963.tb04003.x.

13 Robert J. McEliece, Eugene R. Rodemich, Howard Rumsey, Jr., and Lloyd R. Welch. New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities. *IEEE Trans. Inform. Theory*, IT-23(2):157–166, 1977. doi:10.1109/tit.1977.1055688.

14 M. Navon and A. Samorodnitsky. On Delsarte's linear programming bounds for binary codes. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*, pages 327–336, 2005.

15 Michael Navon and Alex Samorodnitsky. Linear programming bounds for codes via a covering argument. *Discrete Comput. Geom.*, 41(2):199–207, 2009. doi:10.1007/s00454-008-9128-0.

16 Alex Samorodnitsky. On the optimum of Delsarte's linear program. *J. Combin. Theory Ser. A*, 96(2):261–287, 2001. doi:10.1006/jcta.2001.3176.

17 Alex Samorodnitsky. One more proof of the first linear programming bound for binary codes and two conjectures, 2021. arXiv:2104.14587.

18 Alexander Schrijver. A comparison of the Delsarte and Lovász bounds. *IEEE Trans. Inform. Theory*, 25(4):425–429, 1979. doi:10.1109/TIT.1979.1056072.

19 Alexander Schrijver. New code upper bounds from the Terwilliger algebra and semidefinite programming. *IEEE Trans. Inform. Theory*, 51(8):2859–2866, 2005. doi:10.1109/TIT.2005.851748.

20 Frank Vallentin. Semidefinite programming bounds for error-correcting codes. *CoRR*, abs/1902.01253, 2019. arXiv:1902.01253.

21 J. H. van Lint. *Introduction to coding theory*, volume 86 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 1999. doi:10.1007/978-3-642-58575-3.

22 R.R. Varshamov. Estimate of the number of signals in error correcting codes. *Doklady Akademii Nauk SSSR*, 117:739–741, 1957.