Geometric Embeddability of Complexes Is $\exists \mathbb{R}$-Complete

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Abstract

We show that the decision problem of determining whether a given (abstract simplicial) $k$-complex has a geometric embedding in $\mathbb{R}^d$ is complete for the Existential Theory of the Reals for all $d \geq 3$ and $k \in \{d-1, d\}$. Consequently, the problem is polynomial time equivalent to determining whether a polynomial equation system has a real solution and other important problems from various fields related to packing, Nash equilibria, minimum convex covers, the Art Gallery Problem, continuous constraint satisfaction problems, and training neural networks. Moreover, this implies NP-hardness and constitutes the first hardness result for the algorithmic problem of geometric embedding (abstract simplicial) complexes. This complements recent breakthroughs for the computational complexity of piece-wise linear embeddability.

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Figure 1 Illustration of different embeddings of a complex; figure taken from Bing [8, Annals of Mathematics 1959].
Introduction

For now almost 100 years, much attention has been devoted to studying embeddings of complexes [8, 21, 30, 31, 42, 54, 66, 67]. Typical types of embeddings include geometric (also referred to as linear), piecewise linear (PL), and topological embeddings, see also Figure 1. For formal definitions, we refer to Section 1.2; here we give an illustrative example. Embeddings of a 1-complex in the plane correspond to crossing-free drawings of a graph in the plane. In a topological embedding, each edge is represented by a Jordan arc, in a PL embedding it is a concatenation of a finite number of segments, and in a geometric embedding each edge is represented by a segment.

We are interested in the problem of deciding whether a given $k$-complex has a linear/piecewise linear/topological embedding in $\mathbb{R}^d$. Several necessary and sufficient conditions are easy to identify and have been known for many decades. For instance, a $k$-simplex requires $k+1$ points in general position in $\mathbb{R}^d$ and, thus, $k \leq d$ is an obvious necessary condition. Moreover, it is straightforward to verify that every set of $n$ points in $\mathbb{R}^3$ in general position allows for a geometric embedding of any 1-complex on $n$ vertices, i.e., the points are the vertices of a straight-line drawing of a (complete) graph. Indeed, this fact generalizes to higher dimensions: every $k$-complex embeds (even linearly) in $\mathbb{R}^{2k+1}$ [42]. Van Kampen and Flores [25, 57, 66] showed that this bound is tight by providing $k$-complexes that do not topologically embed into $\mathbb{R}^{2k}$. For some time, it was believed that the existence of a topological embedding also implies the existence of a geometric embedding, e.g., Grünbaum conjectured that if a $k$-complex topologically embeds in $\mathbb{R}^{2k}$, then it also geometrically embeds in $\mathbb{R}^{2k}$ [30]. In $\mathbb{R}^2$, this is in fact true: For 1-complexes this is commonly known as Fáry’s theorem [35] but it also follows from Steinitz’ earlier theorem [62]; for 2-complexes one needs a few additional arguments [32]. In higher dimensions, however, the conjecture was disproven. In particular, for every $k, d \geq 2$ with $k + 1 \leq d \leq 2k$, there exist $k$-complexes that have a PL embedding in $\mathbb{R}^d$, but no geometric embedding in $\mathbb{R}^d$ [9, 10, 11]. In contrast, PL and topological embeddability coincides in many cases, e.g., if $d \leq 3$ [8, 48] or $d - k \geq 3$ [12]. Very recently, Frick, Hu, Scheel, and Simon [27] characterized when a complex on $d + 3$ vertices embeds into the $d$-sphere, namely, if and only if its non-faces do not form an intersecting family. Additionally, they showed that if a complex on $d + 3$ vertices embeds topologically into $\mathbb{R}^d$ then it also embeds linearly into $\mathbb{R}^d$. There are many further necessary and sufficient conditions known for geometric embeddings [6, 46, 47, 57, 63, 64] and PL or/and topological embeddings [20, 26, 49, 54, 65, 61].

In recent years, the algorithmic complexity of deciding whether or not a given complex is embeddable gained attention. In the absence of a complete characterization, an efficient algorithm is the best tool to decide embeddability. For instance, deciding whether a 1-complex embeds in the plane corresponds to testing graph planarity and is thus polynomial time decidable [33]. Similarly, Gross and Rosen [29] present a linear time planarity algorithm for 2-complexes in the plane. On the other hand, PL embeddability is sometimes even algorithmically undecidable. To give a concrete example, let $\text{Embed}_{k \rightarrow d}$ denote the algorithmic problem of determining whether a given $k$-complex has a PL embedding in $\mathbb{R}^d$. Because $\text{Embed}_{4 \rightarrow 5}$ has been shown to be algorithmically undecidable [40], there is no algorithm to decide the problem (never mind an efficient one). This provides strong evidence that PL embeddability for these parameters does not allow a reasonable characterization.

More recently, there have been several breakthroughs concerning the PL embeddability. For an overview of the state of the art, consider Table 1. In dimensions $d \geq 4$, the decision problem $\text{Embed}_{k \rightarrow d}$ is polynomial-time decidable for $k < \frac{2}{3} \cdot (d - 1)$ [16, 13, 15, 36] and
Table 1 Overview of the complexity of Embed$_{k,d}$.

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✓: always yes
X: always no
P: polynomial-time
D: decidable
U: undecidable
NP-hard

NP-hard for all remaining non-trivial cases [40], i.e., for all $k$ with $2/3 \cdot (d−1) ≤ k ≤ 2d$. For $d \geq 5$ and $k \in \{d−1,d\}$, Embed$_{k,d}$ is even known to be undecidable [40]. For all other NP-hard cases and $d \geq 4$ decidability is unknown; we note that the proof for undecidability in the case of codimension $> 1$ in [24] has an error [58]. For the case $d = 3$, Matoušek, Sedgwick, Tancer, and Wagner proved decidability of Embed$_{2+3}$ and Embed$_{3+3}$ [39] and de Mesmay, Rieck, Sedgwick, and Tancer proved NP-hardness [43].

Building upon [40], Skopenkov and Tancer [60] proved NP-hardness for a relaxed notion called almost (PL/topological) embeddability where it is only required that disjoint sets are mapped to disjoint objects, i.e., two edges incident to a common vertex may cross in an interior point. More precisely, they showed that recognizing almost embeddability of $k$-complexes in $\mathbb{R}^d$ is NP-hard for all $d,k ≥ 2$ with $d$ (mod 3) = 1 and $2/3 \cdot (d−1) ≤ k ≤ d$.

The analogous questions for geometric embeddings are wide open. Let GEM$_{k,d}$ denote the algorithmic problem of determining whether a given $k$-complex has a geometric embedding in $\mathbb{R}^d$. In contrast to PL embeddability, however, it is easy to see that GEM$_{k,d}$ is decidable for all $k,d$, since every instance can be expressed as a sentence in the first order theory of the reals, which is decidable; for more details see Section 1.1.

The question of whether GEM$_{k,d}$ is complete for $\exists \mathbb{R}$ is a well-known open problem, mentioned for example by Cardinal [18, Section 4].

**Our Results.** In this work, we present the first results concerning open problem for any non-trivial entry with $d \geq 3$. More precisely, we establish the exact computational complexity of GEM$_{k,d}$ for all values $d ≥ 3$ and $k ∈ \{d−1,d\}$. This includes a complete understanding of the most intriguing entries with $d = 3$.

**Theorem 1.** For every $d ≥ 3$ and each $k ∈ \{d−1,d\}$, the decision problem GEM$_{k,d}$ is $\exists \mathbb{R}$-complete. Moreover, the statement remains true even if a PL embedding is given.

Table 2 summarizes the current knowledge on the computational complexity of GEM$_{k,d}$. Our proof implies that distinguishing between $k$-complexes with PL and geometric embeddings in $\mathbb{R}^d$ is complete for $\exists \mathbb{R}$. Because NP ⊆ $\exists \mathbb{R}$, our result yields NP-hardness for $d ≥ 3$ and each $k ∈ \{d−1,d\}$. This confirms the conjecture by Skopenkov that GEM$_{k,d}$ is NP-hard for all $k,d$ with $\max\{3,k\} \leq d ≤ 3/2 \cdot k + 1$ for the corresponding values of $k$ and $d$ [59, Conjecture 3.2.2]. Moreover, if NP ⊄ $\exists \mathbb{R}$, the problem GEM$_{k,d}$ cannot be tackled with well developed tools for NP-complete problems such as SAT and ILP solvers. For more details, we refer to Section 1.1.
The closely related question of polyhedral complexes (generalizing simplicial complexes because each simplex is a basic polyhedron), posed in the Handbook of Discrete and Computational Geometry, reads as follows: When is a given finite poset isomorphic to the face poset of some polyhedral complex in a given space $\mathbb{R}^d$? [53, Problem 20.1.1]. The recognition of polyhedral complexes (with triangles and quadrangles) in $\mathbb{R}^3$ has been claimed to be $\exists\exists$-complete [18, Theorem 5]. Focussing on convex polytopes, Richter-Gebert proved that recognizing convex polytopes in $\mathbb{R}^4$ is $\exists\exists$-complete [50, 51]. Our result settles the computational aspects of the question, even for the special case of simplicial complexes.

A geometric embedding of a complex can also be viewed as a *simplicial representation* of a hypergraph, i.e., a representation in which every hyperedge is represented by a simplex. Of particular interest is the case of uniform hypergraphs where all hyperedges have the same number of elements. Thus, in the language of hypergraphs, our result reads as follows.

**Corollary 2.** For all $d \geq 3$ and every $k \in \{d - 1, d\}$, deciding whether a $(k + 1)$-uniform hypergraph has a simplicial representation in $\mathbb{R}^d$ is $\exists\exists$-complete.

**Outline and techniques.** Our proof of Theorem 1 consists of three steps: Establishing $\forall\exists$-membership, showing $\exists\forall$-hardness in $\mathbb{R}^3$, i.e., of $\text{GEM}_{2 \to 3}$ and $\text{GEM}_{3 \to 3}$, and reducing $\text{GEM}_{k \to d}$ to $\text{GEM}_{k+1 \to d+1}$. The core of the proof lies in establishing hardness of $\text{GEM}_{2 \to 3}$.

The main idea to prove hardness of $\text{GEM}_{2 \to 3}$ is to reduce from the problem Stretchability. In Stretchability, we are given an arrangement of pseudolines (curves) in the plane and we are asked to decide whether there exists a set of straight lines that has the same combinatorial pattern as the pseudoline arrangement, see Figure 2(a) for an illustration and Section 1.2 for a formal definition. Given a pseudoline arrangement $L$, we construct a 2-complex $C$ which has a geometric embedding in $\mathbb{R}^3$ if and only if $L$ is stretchable. On a high level, our construction of $C$ goes along the following lines: We add a helper triangle that contains all intersections of the pseudolines, see Figure 2(b). We place each pseudoline in $\mathbb{R}^3$ and replace it by a special edge of the complex $C$; these will not be part of any triangle of $C$. We surround the special edges by so called tunnels, which are tubes formed by triangular sections, see Figure 2(c) and (d). One side of the tunnel defines its bottom, while the other two span its roof. For each crossing in $L$, we glue the corresponding tunnel sections together, see Figure 2(e). At last, we insert an apex $u$ high above that is connected to all visible tunnel parts, see Figure 2(f) and we insert additional objects in order to ensure that the neighborhood of $u$ is an essentially 3-connected graph, Figure 2(i). The objects incident to the apex will also ensure that the special edges actually lie inside the tunnel.
Figure 2 (a) We start with a pseudoline arrangement $L$. (b) We add three segments forming a triangle that contains all intersections of $L$. (c) Each pseudoline is represented by a special edge that is surrounded by a tunnel. (d) Each tunnel consists of tunnel sections. (e) For the crossings of the special edges, we identify parts of the tunnels. (f) We add an apex $u$ and insert triangles to the visible parts of the construction; we enhance the neighborhood of the apex to an essentially 3-connected graph depicted in (i). (g) In the correctness proof, we use a small sphere around the apex and the projection of each special edge onto the sphere. (h) We argue that the combinatorics of the projected special edges on the sphere are equivalent to $L$ and then project the special edges onto a plane. This will yield a stretched arrangement. (i) The neighborhood graph of the apex $u$. 
It is relatively straightforward to verify that if \( L \) is stretchable, then the complex \( C \) embeds geometrically into \( \mathbb{R}^3 \). The other direction requires more care and work: We show that a geometric embedding of \( C \) induces a line arrangement with the same combinatorics as \( L \). The idea of the proof is to consider a small sphere around the apex \( u \) and to project its neighborhood and the special edges onto the sphere, see Figure 2(g). Because the neighborhood graph of \( u \) is essentially 3-connected by construction, all its crossing-free drawings on the sphere are equivalent. This is a crucial property to show that each special edge lies in the projection of its tunnel roof (when restricting the attention to an interesting part within the helper triangle). We remark that our proof does not show this explicitly. Instead, we establish some even stronger properties. As a consequence, the projection of the tunnels have the intended combinatorics and thus also the special edges which represent the pseudolines. At last, we project the arcs from the sphere onto a plane, see Figure 2(h). In this way, we obtain a line arrangement with the same combinatorics as \( L \).

In order to show hardness of \( \text{GEM}_{k+3} \), we use a similar construction, in which we “fatten” each triangle to a tetrahedron, by adding extra vertices.

We finally present a dimension reduction, i.e., we reduce \( \text{GEM}_{k+d} \) to \( \text{GEM}_{k+1-d+d+1} \). Given a \( k \)-complex \( C \), we create a \((k+1)\)-complex \( C' \) that contains \( C \) and has two additional vertices \( a \) and \( b \). Moreover, for each subset \( e \) of \( C \), \( C' \) has the additional subsets \( e \cup \{a\} \) and \( e \cup \{b\} \). We prove that \( C \) geometrically embeds in \( \mathbb{R}^d \) if and only if \( C' \) geometrically embeds in \( \mathbb{R}^{d+1} \). In this way, we show that distinguishing PL embeddable and geometrically embeddable complexes is \( \exists \mathbb{R} \)-complete.

### 1.1 Existential Theory of the Reals

The class of the existential theory of the reals \( \exists \mathbb{R} \) (pronounced as is a complexity class which has gained a lot of interest in recent years, specifically in the computational geometry community. To define this class, we first consider the algorithmic problem \textit{Existential Theory of the Reals} (ETR). An instance of this problem consists of a sentence of the form

\[
\exists x_1, \ldots, x_n \in \mathbb{R} : \Phi(x_1, \ldots, x_n),
\]

where \( \Phi \) is a well-formed quantifier-free formula in the variables and the alphabet \{0, 1, +, , \geq, >, \land, \lor, \neg\}, and the goal is to check whether this sentence is true. As an example of an ETR-instance, consider \( \exists x, y \in \mathbb{R} : \Phi(x, y) = (x \cdot y^2 + x \geq 0) \land \neg(y < x) \), for which the goal is to determine whether there exist real numbers \( x \) and \( y \) satisfying the formula \( \Phi(x, y) \).

The \textit{complexity class} \( \exists \mathbb{R} \) is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that \( \text{NP} \subseteq \exists \mathbb{R} \subseteq \text{PSPACE} \). The first inclusion follows from the definition of \( \exists \mathbb{R} \). Showing the second inclusion was first established by Canny in his seminal paper [17]. The complexity class \( \exists \mathbb{R} \) gains its significance because a number of well-studied problems from different areas of theoretical computer science have been shown to be complete for this class.

Famous examples from discrete geometry are the recognition of geometric structures, such as unit disk graphs [41], segment intersection graphs [38], \textit{Stretchability} [45, 56], and order type realizability [38]. Other \( \exists \mathbb{R} \)-complete problems are related to graph drawing [37], Nash-Equilibria [7, 28], geometric packing [5], the art gallery problem [3], non-negative matrix factorization [55], polytopes [22, 51], geometric linkage constructions [1], training neural networks [4], visibility graphs [19], continuous constraint satisfaction problems [44], and convex covers [2]. The fascination for the complexity class stems not merely from the number of \( \exists \mathbb{R} \)-complete problems but from the large scope of seemingly unrelated \( \exists \mathbb{R} \)-complete problems. We refer the reader to the lecture notes by Matoušek [38] and surveys by Schaefer [52] and Cardinal [18] for more information on the complexity class \( \exists \mathbb{R} \).
1.2 Definitions

**Simplex.** A $k$-simplex $\sigma$ is a $k$-dimensional polytope which is the convex hull of its $k+1$ vertices $V$, which are not contained in the same $(k-1)$-dimensional hyperplane. Hence, a 0-simplex corresponds to a point, a 1-simplex to a segment, and a 2-simplex to a triangle etc. The convex hull of any nonempty proper subset of $V$ is called a face of $\sigma$. A simplicial complex $K$ is a set of simplices satisfying the following two conditions: (i) Every face of a simplex from $K$ is also in $K$. (ii) For any two simplices $\sigma_1, \sigma_2 \in K$ with a non-empty intersection, the intersection $\sigma_1 \cap \sigma_2$ is a face of both simplices $\sigma_1$ and $\sigma_2$. The purely combinatorial counterpart to a simplicial complex is an abstract simplicial complex, which we refer to simply as a complex.

**Complex.** A complex $C = (V, E)$ is a finite set $V$ together with a collection of subsets $E \subseteq 2^V$ which is closed under taking subsets, i.e., $e \in E$ and $e' \subseteq e$ imply that $e' \in E$. A $k$-complex is a complex where the largest subset contains exactly $k+1$ elements. We call a complex pure if all (inclusion-wise) maximal elements in $E$ have the same cardinality. For any vertex $v \in V$ in a $k$-complex $C = (V, E)$, the neighbourhood of $v$ gives rise to a lower dimensional complex $C_v := (V', E')$, where $E' := \{ e \setminus \{ v \} \mid v \in e \in E \}$ and $V' := N(v) = \bigcup_{e \in E, v \in e} e$ are the neighbors of $v$. Complexes are in close relation to Hypergraphs.

**Hypergraphs.** Hypergraphs generalize graphs by allowing edges to contain any number of vertices. Formally, a hypergraph $H$ is a pair $H = (V, E)$ where $V$ is a set of vertices, and $E$ is a set of non-empty subsets of $V$ called hyperedges (or edges). A $k$-uniform hypergraph is a hypergraph such that all its hyperedges contain exactly $k$ elements. Note that the maximal sets of a pure $k$-complex yield a $(k+1)$-uniform hypergraph and vice versa. Hence, $(k+1)$-uniform hypergraphs and pure $k$-complexes are in a straightforward one-to-one correspondence. A simplicial representation of a $(k+1)$-uniform hypergraph is a geometric embedding of the corresponding complex.

**Geometric embeddings.** A geometric embedding of a complex $C = (V, E)$ in $\mathbb{R}^d$ is a function $\varphi : V \to \mathbb{R}^d$ fulfilling the following two properties: (i) for every $e \in E$, $\overline{\varphi(e)} := \text{conv}(\{ \varphi(v) : v \in e \})$ is a simplex of dimension $|e| - 1$ and (ii) for every pair $e, e' \in E$, it holds that

$$\overline{\varphi(e)} \cap \overline{\varphi(e')} = \overline{\varphi(e \cap e')}.$$  

Note that if $\varphi$ is a geometric embedding, then $\{ \overline{\varphi(e)} : e \in E \}$ is a simplicial complex. The problem GEM$_{k\to d}$ asks whether a given $k$-complex has a geometric embedding in $\mathbb{R}^d$.

**Topological and PL embeddings.** Consider a complex $C = (V, E)$. In contrast to geometric embeddings, for PL or topological embeddings it is not sufficient to describe the mapping of the vertices $V$. Choose $d'$ so large that $C$ admits a geometric embedding $\varphi' : V \to \mathbb{R}^{d'}$, and define $S = \bigcup_{e \in E} \overline{\varphi'(e)}$. We then say that an injective and continuous function $\varphi : S \to \mathbb{R}^d$ is a topological embedding of $C$ in $\mathbb{R}^d$. If furthermore for each $e \in E$, the image $\varphi(\overline{\varphi'(e)})$ is a finite union of connected subsets of $(|e| - 1)$-dimensional hyperplanes, then $\varphi$ is a piecewise linear (PL) embedding. The problem EMBED$_{k\to d}$ asks whether a given $k$-complex has a PL embedding in $\mathbb{R}^d$. 

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Graph Drawings. A graph is a 1-complex. A graph is planar if there exists a crossing-free drawing in the plane, i.e., a (topological) embedding in $\mathbb{R}^2$. As mentioned above, a graph has a topological embedding in $\mathbb{R}^2$ if and only if it has a geometric embedding in $\mathbb{R}^2$. A plane graph is a planar graph together with a rotation system, i.e., a cyclic ordering of the incident edges around each vertex that comes from a crossing-free drawing. By means of stereographic projection, any graph that has a crossing-free drawing in the plane also has a crossing-free drawing on the sphere and vice versa. Two crossing-free drawings of a graph (in the plane or on the sphere) are equivalent if they can be transformed into one another by a homeomorphism (of the plane or the sphere); note that the homeomorphism could be orientation reversing. In particular, two equivalent drawings have the same rotation system; two equivalent drawings in the plane additionally have the same outer face. When talking about an arbitrary drawing $D$ of a plane graph $G$, we mean a crossing-free drawing with the same rotation system.

Stretchability. A pseudoline arrangement is a family of curves that apart from “straightness” share similar properties with a line arrangement. More formally, a (Euclidean) pseudoline arrangement is a set of labeled $x$-monotone curves in the Euclidean plane such that any two meet in exactly one point. A curve in $\mathbb{R}^2$ is $x$-monotone if it is the image of a continuous function $f: \mathbb{R} \to \mathbb{R}$. In fact, each pseudoline arrangement can be encoded by a wiring diagram; see also Figure 4. A pseudoline arrangement is stretchable if it is combinatorially equivalent to an arrangement of straight-lines, i.e., if the arrangements can be transformed into one another by a homeomorphism of the plane. STRETCHABILITY denotes the algorithmic problem of deciding whether a given pseudoline arrangement is stretchable. In a seminal paper, Shor [56] proved that STRETCHABILITY is NP-hard. Shor points out that Mnëv’s proof implies that stretchability is complete for the existential theory of the reals. For a stream-line exposition of this result see the expository paper by Matoušek [38].

1.3 Pitfalls

While the general proof ideas are fairly straightforward, our arguments in Section 2 may at first glance appear a bit tedious. In the following, we highlight one of the appearing challenges. It is easy to see that each special edge lies inside its tunnel in any geometric embedding. It follows that the projection of the special edge lies also inside the projection of the tunnel on the sphere centered at the apex. Furthermore, we know that the roofs of the tunnels are seen by the apex. One may be tempted to (directly) conclude that the projection of the special edge is thus also contained in the projection of the roof; the underlying thought being that the projection of the tunnel bottom lies below the tunnel roof in the geometric representation and thus the projection of the tunnel bottom is contained in the projection of the tunnel roof. Yet, the latter is not true in general, as can be seen in Figure 3. In the

![Figure 3](image-url) From the perspective of $u$, the tunnel bottom is not always hidden below the tunnel roof: From the three sections displayed, the bottom (yellow) of the middle one is partially visible.
figure, the tunnel bottom is not covered by the roof. We (implicitly) show that the projection of the special edge lies inside the projection of the roof by establishing some even stronger topological and geometric properties.

2 The Proof

In this section, we prove Theorem 1. Our proof consists of the following three parts.

a) Establishing $∃R$-membership (Section 2.1: Lemma 3).

b) Showing $∃R$-hardness in $R^d$, i.e., of GEM$_{2→3}$ and GEM$_{3→3}$ (Section 2.2: Theorem 4 and Corollary 9).

c) Reducing GEM$_{k→d}$ to GEM$_{k+1→d+1}$ (Section 2.3: Lemma 10).

Together Lemmas 3 and 10, Theorem 4 and Corollary 9 prove Theorem 1.

2.1 Membership

In this subsection, we show $∃R$-membership of GEM$_{k→d}$. Note that this is essentially folklore [14]. We present a proof for the sake of completeness.

Lemma 3. For all $k, d ∈ \mathbb{N}$, the decision problem GEM$_{k→d}$ is contained in $∃R$.

Proof. In order to show membership in $∃R$, we use the following characterization by Erickson, Hoog and Miltzow [23]: A problem $P$ lies in $∃R$ if and only if there exists a verification algorithm $A$ for $P$ that runs in polynomial time on the real RAM, which we refer to as a real verification algorithm. In particular, for every yes-instance $I$ of $P$ there exists a polynomial sized witness $w$ such that $A(I, w)$ returns yes, and for every no-instance $I$ of $P$ and any witness $w$, $A(I, w)$ returns no. In contrast to the definition of the complexity class NP, we also allow witnesses that consist of real numbers. Consequently, we execute $A$ on the real RAM as well.

It remains to present a real verification algorithm for GEM$_{k→d}$. While the witness describes the coordinates of the vertices, the algorithm checks for intersections between any two simplices. Note that each simplex is a convex set and the intersection of convex sets is a convex set as well. For any simplex $S$ with $n$ vertices, we can efficiently determine $n$ linear inequalities and at most one linear equality that together describe $S$: the inequalities may describe the $n$ facets and the equality describes the subspace in case $S$ is not $d$-dimensional. Then checking for intersections can be reduced to a linear program, which is polynomial time solvable in any fixed dimension. This finishes the description of the real verification algorithm.

We note that one does not need to resort to the characterization of $∃R$ with verifiers as in [23]. It is possible to directly construct a polynomial system of polynomial size (in fixed dimension) in the coordinates of the vertices of the given complex in order to encode its geometric realizability. It may appear to be overly complicated to use the tools from [23], if you do not know this tool. However, if you know this tool it appears strange not to use it.
2.2 Hardness in three dimensions

This section is dedicated to proving Theorem 1 for $d = 3$ and $k \in \{2, 3\}$. The crucial part lies in the case $k = 2$.

> **Theorem 4.** The decision problem $\text{GEM}_{2 \to 3}$ is $\exists R$-hard.

**Proof.** We reduce from the $\exists R$-hard problem $\text{Stretchability}$, as described in Section 1.2. In particular, for each pseudoline arrangement $L$, we construct a 2-dimensional complex $C$ in time polynomial in $L$ such that $C$ geometrically embeds in $\mathbb{R}^3$ if and only if $L$ is stretchable.

Let $L$ be an arrangement of $n$ pseudolines in the plane. Every pseudoline arrangement has a representation as a wiring diagram in which each pseudoline is given by a monotone curve consisting of $2n - 1$ sections. For an illustration consider Figure 4; each section could be represented by a segment, however for a visual appealing display, the bend points are rounded. We add a pseudoline $\ell_0$ that intersects all pseudolines in the beginning, see Figure 4, and call the resulting pseudoline arrangement $L^*$. Note that $L^*$ is stretchable if and only if $L$ is stretchable. For later reference, we endow a natural orientation upon each pseudoline from left to right. In the following, we construct a 2-complex $C = (V, E)$ that allows for a geometric realization if and only if $L^*$ (and thus $L$) is stretchable. In order to define $C$, we add a helper triangle $\triangle$ (consisting of three segments!) to our arrangement that intersects the pseudolines of $L^*$ as illustrated in Figure 4. In particular, the helper triangle contains all intersection points of $L^*$.

![Figure 4](image_url)

**Construction of the 2-complex.** In order to define $C$, we associate an almost geometric embedding of already defined parts along the way; where only a set of special edges is represented in a PL fashion, all other elements are already geometrically embedded. We will refer to the subsets in $C$ as vertices, edges, and triangles depending on whether they contain one, two or three elements. The construction has five steps.

**In the first step,** we place the pseudolines and the helper triangle $\triangle$ in 3-space. Each pseudoline $\ell_i$ lies in the plane $z = i$ such that an observer high above (at infinity) sees the wiring diagram. Similarly, we place the segments of the helper triangle $\triangle$ in 3-space such that it lies in the plane $z = n + 1$. Note that no two pseudolines intersect. Therefore, we can surround each lifted pseudoline by a triangulated sphere which we call a *tunnel*; see also Figure 5. The tunnel $T_i^+$ of $\ell_i$ is formed by $2n + 3 + i$ sections; later, we will be particularly interested in a part of a tunnel, denoted by $T_i$, in which the first two and last two sections are removed. Each section consists of six triangles forming a triangulated triangular prism as illustrated in Figure 5. We close the tunnel with triangles at the ends and think of the *bottom* side of the prism to lie in the plane $z = i - 1/2$ (for now). The remaining part of the tunnel, i.e., the tunnel without its bottom, constitutes the *roof*, see Figure 5. The roof...
contains three disjoint paths on $2n + 4 + i$ vertices. The edges and vertices on the boundary of both the bottom and the roof form the left and right roof path; the edges of the closing triangles on either end do not belong to either path. The remaining vertices induce the central roof path. The three roof paths are thickened in Figure 5.

![Figure 5](image-url) First step in the construction of the complex $C$ – tunnel construction. (left) A tunnel viewed from side. (middle) A section of a tunnel. (right) A tunnel viewed from above.

Note that we do not add a tunnel for the helper triangle. We distribute the sections along $T^+_i$ to edges and crossings of the crossing diagram as follows: Generally, we associate one section per edge and one section per crossing of two pseudolines. Moreover, we associate one extra section of $T^+_j$ to a crossing of $\ell_i$ and $\ell_j$ whenever $i < j$. In order to represent the pseudoline $\ell_i$, we insert a special edge $e_i$ between the two top vertices on either end of the tunnel; for later reference, we denote the start vertex by $s_i$ and the end vertex by $t_i$. In the associated almost geometric embedding, $e_i$ is represented inside the tunnel by a concatenation of segments, one for each tunnel section. We aim for the fact that the special edge $e_i$ lies inside the tunnel in every geometric embedding (if one exists).

In the second step, we identify parts of the tunnels. To this end, consider the tunnel sections assigned to a crossing of a pseudoline $\ell_i$ with $\ell_j$, $i < j$. Recall that we assigned one section of $T^+_i$ and two sections of $T^+_j$ to the crossing. We identify the four triangles in the bottom of the two sections of $T^+_j$ with the four triangles in the roof of one section of $T^+_i$ as indicated in Figure 6. Note that we hereby identify six times two vertices, four of which belong to a left or right roof path of both tunnels, $T^+_i$ and $T^+_j$.

![Figure 6](image-url) Second step in the construction of the complex $C$: (left) Gluing of tunnel parts viewed from above. (right) During the identification process, the vertices of the top tunnel are moved to the vertices of the bottom tunnel.

For the associated almost geometric embedding, we shortly explain here how to geometrically embed the tunnels. To this end, we may easily distribute the sections of the tunnels such that the six vertices of both tunnels (which will be pairwise identified) have the same $x,y$-coordinates. Then, during the identification process, we move the vertices of the top tunnel to the vertices of the lower tunnel.

In the third step, we add a new vertex to the construction that we call the apex and which we denote by $u$. We think of $u$ as the observer high above (at infinity) and insert a triangle defined by $u$ and the vertices of every edge that is visible from $u$. Clearly, every edge of the helper triangle $\triangle$ is visible. Moreover, note that every roof section that is neither glued in a crossing nor hidden by the helper triangle is visible. In contrast, no bottom of any tunnel is visible in the almost geometric embedding.
In the fourth step, we enhance the 1-complex induced by the neighborhood \( N(u) \) of the apex \( u \) such that it corresponds to an essentially 3-connected planar graph \( G^+ \). We call a graph essentially 3-connected if it is a subdivision of a 3-connected graph. With the description so far, the 1-complex corresponds to the graph \( H \) depicted in black in Figure 7.

To construct \( G^+ \), we make use of the following fact. We define the degree of a face in a potentially disconnected plane graph as the number of edges in the face boundary (counted with multiplicity), plus 1 for each but one component incident to the face. Note that the degree of a face is thus lower bounded by the number of incident vertices and upper bounded by twice the number of incident vertices.

\[ \text{Claim 5.} \quad \text{For every plane graph } G_1 = (V_1, E_1), \text{ there exists an essentially 3-connected plane graph } G_2 = (V_2, E_2) \text{ such that } G_1 \text{ is a subgraph of } G_2 \text{ and any straight-line drawing } D_1 \text{ of } G_1 \text{ in the plane can be extended to a straight-line drawing of } G_2. \text{ Moreover, if the maximum face degree of } G_1 \text{ is } k, \text{ then the size of } G_2 \text{ can be bounded by } |V_2| + |E_2| \leq O(k|V_1|). \]

Let \( G^+ := G_2 \) be an essentially 3-connected plane graph guaranteed by Claim 5 for the case that \( G_1 = H \). Note that \( G_1 \) has \( O(n^2) \) vertices and edges, and every face has degree \( O(n) \). Hence, the size of \( G_2 \) is in \( O(n^3) \). We denote the outer face of \( G_2 \) by \( f_0 \). The reader is invited to think about the far more sparse graph depicted in Figure 7, which also serves as a candidate for \( G^+ \). Indeed, the depicted graph also fulfills all properties necessary for our construction; however, not all properties of Claim 5. For example, the depicted graph is even 3-connected. The proof of this is straightforward, but a bit tedious. Thus, we leave it as an exercise to the interested reader to check that the graph remains connected even after the deletion of any two vertices or alternatively, that any pair of vertices is connected by three disjoint paths.
Later, the subgraph $G$ of $G^+$ that is induced by all vertices of $\bigcup_i T_i$ will be of particular interest; in Figure 7, these vertices (and their convex hull) lie inside the helper triangle $\Delta$. Recall that $T_i$ denotes the part of the tunnel $T_i^+$ obtained by deleting the first two and last two sections.

It is a well-known fact that all (straight-line or topological) planar drawings of a 3-connected planar graph on the sphere are equivalent [34]; for a definition of equivalent drawings consult Section 1.2. Consequently, the result extends to essentially 3-connected graphs as it also holds for topological drawings. For later reference, we note the following.

▷ Claim 6. The planar graph $G^+$ is essentially 3-connected. Therefore, all crossing-free drawings of $G^+$ on a sphere are equivalent. Furthermore, any straight-line drawing of $H$ in the plane can be extended to a straight-line drawing of $G^+$.

We ensure that the neighborhood complex of $u$ is the underlying planar graph of $G^+$, i.e., for each edge of $G^+$ not present in $H$, we insert a triangle formed by the vertices of this edge together with $u$ and call the resulting complex $C$.

In the fifth and last step, our final complex $C$ consist of two copies of $C$ in which the apex vertices are identified. We use these two copies in order to guarantee that in any geometric embedding the apex lies outside of all tunnels for one copy of $C$. This finishes the construction of the abstract complex $C$.

It remains to show that our construction runs in polynomial time and fulfills the claimed properties.

**Time Complexity.** In order to verify that the construction shows $\exists \mathbb{R}$-hardness, we argue that it has a running time that is polynomial in the size of the input. To this end, note that a pseudoline arrangement with $n$ pseudolines can be described by the sequence of crossings along each pseudoline, i.e., by the $O(n^2)$ crossings. Thus, the input size is $N = O(n^2)$. After adding the helper triangle and $\ell_0$, the crossing diagram still has a size in $O(n^2)$. It is easy to see that our construction has a size proportional to $N^{3/2}$: For each segment and crossing of the diagram, we insert a constant number of objects. Moreover, we add a triangle for every (additional) edge in $G^+$; recall that $G^+$ has size $O(n^3)$. Consequently, the total construction has size $O(n^3) = O(N^{3/2})$. We remark, that a more careful choice of $G^+$, as in Figure 7, yields a construction that is linear in $N$.

It remains to show that the pseudoline arrangement $L$ is stretchable if and only if $C$ has a geometric embedding in $\mathbb{R}^3$.

**Correctness.** If $L$ is stretchable, it is relatively straight-forward to construct a geometric embedding of $C$.

▷ Claim 7. If $L$ is stretchable, then $C$ has a geometric embedding.

The reverse direction is more involved and the interesting challenge.

▷ Claim 8. If $C$ has a geometric embedding, then $L$ is stretchable.

This finishes the proof of Theorem 4.
Fattening the Complex. In the following, we present a simple modification for the proof of Theorem 4 to obtain hardness for pure 2- and 3-complexes.

**Corollary 9.** The decision problems GEM$_{2\to 3}$ and GEM$_{3\to 3}$ are $\exists \mathbb{R}$-hard, even when restricting to pure complexes.

**Proof.** The constructed 2-complex $C$ in the proof of Theorem 4 was not pure because the special edges are not contained in any triangle. We obtain a pure 2-complex $\hat{C}$ by adding one new vertex to each special edge such that it forms a special triangle. On the one hand, given a geometric embedding of $C$ in $\mathbb{R}^3$, the new vertices can easily be added close enough to their defining set in $C$. On the other hand, any geometric embedding of $\hat{C}$ induces an embedding of $C$. Hence, $C$ has a geometric embedding if and only if $\hat{C}$ has a geometric embedding in $\mathbb{R}^3$.

Analogously, we can add a private vertex to each triangle of $\hat{C}$ to form a pure 3-complex which has a geometric embedding if and only if $C$ has a geometric embedding in $\mathbb{R}^3$.  

Alternatively for showing hardness of GEM$_{3\to 3}$, we remark that hardness of GEM$_{k\to d}$ for $k < d$ easily implies hardness of GEM$_{\ell\to d}$ for all $k \leq \ell \leq d$ by adding a disjoint $\ell$-simplex to the construction which always has a geometric embedding in $\mathbb{R}^d$.

### 2.3 Dimension Reduction

In order to show hardness for all remaining cases of Theorem 1, we establish the following dimension reduction.

**Lemma 10.** The decision problem GEM$_{k\to d}$ reduces to GEM$_{k+1\to d+1}$.

The idea is to add two apices to a $k$-complex $C$ in order to obtain a $(k+1)$-complex $C^+$. We then argue that $C$ has a geometric embedding in $\mathbb{R}^d$ if and only if $C^+$ has a geometric embedding in $\mathbb{R}^{d+1}$. More formally, for a complex $C = (V, E)$ and a disjoint vertex set $U$, $C \ast U$ denotes the join complex $(V \cup U, E')$ where $E' := \{ e \cup u \mid e \in E, u \in U \}$. The following claim immediately implies Lemma 10.

**Claim 11.** Let $C = (V, E)$ be a complex, $a, b \notin V$ two new vertices, and $C^+ := C \ast \{ a, b \}$ their join complex. Then $C$ has a geometric embedding in $\mathbb{R}^d$ if and only if $C^+$ has a geometric embedding in $\mathbb{R}^{d+1}$.

**Proof.** Let $\varphi$ be a geometric embedding of $C$ in $\mathbb{R}^d$. Then, we define for $v \in V \cup \{ a, b \}$,

$$\varphi'(v) = \begin{cases} (\varphi(v) , 0) & \text{if } v \in V, \\ (0, \ldots, 0, +1) & \text{if } v = a, \\ (0, \ldots, 0, -1) & \text{if } v = b. \end{cases}$$

It is easy to check that $\varphi'$ is a geometric embedding of $C^+$ in $\mathbb{R}^{d+1}$. By definition of the last coordinate, any two simplices where one possibly contains $a$ and the other possibly contains $b$ can only intersect in the subspace induced by the first $d$ coordinates. Consequently, all (interesting) potential intersections happen in the $d$-dimensional subspace induced by the first $d$ coordinates. Hence $\varphi$ implies the correctness of the geometric embedding.

For the reverse direction, consider a geometric embedding $\varphi$ of $C^+$ in $\mathbb{R}^{d+1}$. Let $\varphi_a := \varphi(a)$ and $\varphi_b := \varphi(b)$. Without loss of generality, we assume that $\varphi_a - \varphi_b$ is orthogonal to the first $d$ coordinates, i.e., $\varphi_a - \varphi_b$ is parallel to the $(d+1)$-st coordinate axis. Let $\overline{\varphi}(C) := \bigcup_{e \in E} \overline{\varphi}(e)$
denote the induced geometric subrepresentation of $C$. We claim that the orthogonal projection $f : \mathfrak{v}(C) \rightarrow \mathbb{R}^d$ to the first $d$ coordinates (i.e., the effect of the function is the one of restricting to the first $d$ coordinates, :) is injective. Thus $\varphi' := f \circ \varphi$ yields a representation of $C$ in $\mathbb{R}^d$.

For the purpose of a contradiction, suppose that $f$ is not injective. Then there exist two distinct points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ with $p, q \in \mathfrak{v}(C)$ such that $(p_1, \ldots, p_d) = (q_1, \ldots, q_d)$ and $p_d \neq q_d + 1$. Without loss of generality, we may assume that $p_d > q_d + 1$. Consider the plane $P$ spanned by $\varphi_a, \varphi_b, p$. Note that $q \not\in P$ because $\varphi_a - \varphi_b$ and $p - q$ are parallel (to the $(d + 1)$st coordinate axis). For an illustration, see Figure 8.

Let us denote with $e_p \in E$ and $e_q \in E$ any choice of hyperedges such that $p \in \mathfrak{v}(e_p)$ and $q \in \mathfrak{v}(e_q)$. Consider the two open segments $\text{seg}^\circ(\varphi_a, q) \in \mathfrak{v}(e_q \cup a)$ and $\text{seg}^\circ(\varphi_b, p) \in \mathfrak{v}(e_p \cup b)$. Clearly, these open segments intersect in a point $x$, as illustrated in Figure 8. Because $\varphi$ is a geometric embedding, it holds that

$$x \in \mathfrak{v}(e_q \cup a) \cap \mathfrak{v}(e_p \cup b) = \mathfrak{v}(e_q \cup e_p) = \mathfrak{v}(e_q) \cap \mathfrak{v}(e_p).$$

In particular, this implies that $x \in \mathfrak{v}(e_q)$ and thus that $x \in \text{seg}^\circ(\varphi_a, q) \cap \mathfrak{v}(e_q)$. However, because $\mathfrak{v}(e_q \cup a)$ is a simplex, $\varphi_a$ does not lie in $\text{span}(\mathfrak{v}(e_q))$ and thus $\text{seg}^\circ(\varphi_a, q) \cap \mathfrak{v}(e_q) = \emptyset$. A contradiction.

3 Conclusion

We established the computational complexity of $\text{GEM}_{k \rightarrow d}$ for all $d \geq 3$ and $k \in \{d-1, d\}$. In particular, we showed that for these values it is complete for $\exists \mathbb{R}$ to distinguish PL embeddable $k$-complexes in $\mathbb{R}^d$ from geometrically embeddable ones. Arguably, $\text{GEM}_{2 \rightarrow 3}$ is the most interesting case.

Investigating the computational complexity for the remaining open entries in Table 2 remains for future work. We strengthen the conjecture of Skopenkov [59] as follows.

We refer to the online appendix for the proof of Claim 11.

- **Conjecture.** The problem $\text{GEM}_{k \rightarrow d}$ is $\exists \mathbb{R}$-complete for all $k, d$ such that $\max\{3, k\} \leq d \leq 2k$.

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### References


Geometric Embeddability of Complexes Is \( \exists R \)-Complete


Geometric Embeddability of Complexes Is $\mathbb{R}$-Complete


