Reconfiguration of Colorings in Triangulations of the Sphere

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Abstract

In 1973, Fisk proved that any 4-coloring of a 3-colorable triangulation of the 2-sphere can be obtained from any 3-coloring by a sequence of Kempe-changes. On the other hand, in the case where we are only allowed to recolor a single vertex in each step, which is a special case of a Kempe-change, there exists a 4-coloring that cannot be obtained from any 3-coloring.

In this paper, we present a linear-time checkable characterization of a 4-coloring of a 3-colorable triangulation of the 2-sphere that can be obtained from a 3-coloring by a sequence of recoloring operations at single vertices. In addition, we develop a quadratic-time algorithm to find such a recoloring sequence if it exists; our proof implies that we can always obtain a quadratic length recoloring sequence. We also present a linear-time checkable criterion for a 3-colorable triangulation of the 2-sphere that all 4-colorings can be obtained from a 3-coloring by such a sequence. Moreover, we consider a high-dimensional setting. As a natural generalization of our first result, we obtain a polynomial-time checkable characterization of a k-coloring of a $(k-1)$-colorable triangulation of the $(k-2)$-sphere that can be obtained from a $(k-1)$-coloring by a sequence of recoloring operations at single vertices and the corresponding algorithmic result. Furthermore, we show that the problem of deciding whether, for given two $(k+1)$-colorings of a $(k-1)$-colorable triangulation of the $(k-2)$-sphere, one can be obtained from the other by such a sequence is PSPACE-complete for any fixed $k \geq 4$. Our results above can be rephrased as new results on the computational problems named k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph, which are fundamental problems in the field of combinatorial reconfiguration.

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1 Introduction

In 1973, Fisk [14] proved that all 4-colorings of a 3-colorable triangulation of the 2-sphere are *Kempe-equivalent*, that is, for any two 4-colorings of the graph, one is obtained from the other by a sequence of *Kempe-changes*. The method of Kempe-changes is known as a powerful tool for coloring of graphs (see e.g., [19, 10]), and has been intensively studied in graph theory (see e.g., [26, 24, 27, 28, 13, 1, 12, 2]). In particular, Mohar [27] proved that all 4-colorings of a 3-colorable planar graph are Kempe-equivalent using Fisk's result, and then Feghali [12] improved this for 4-critical planar graphs. Mohar and Salas [28] extended Fisk's result to toroidal triangulations.

The formal definitions of Kempe-change and Kempe-equivalence are given as follows. Let $\alpha\colon V(G) \to \{0, 1, \dots, k-1\}$ be a k-coloring of a graph G, let a, b be two distinct colors in $\{0, 1, \dots, k-1\}$, and let C be a connected component of the subgraph of G induced by the vertices colored with either a or b. Then, a *Kempe-change* of α (at C) is an operation to give rise to a new k-coloring by exchanging the colors a and b on all vertices in C. In particular, if C consists of a single vertex, then we refer to such a Kempe-change at C as a *single-change*. Two k-colorings of G are *Kempe-equivalent* if one is obtained from the other by a sequence of Kempe-changes, and *single-equivalent* if one is obtained from the other by a sequence of single-changes.

Let us return to Fisk's result for the Kempe-equivalence. Let G be a 3-colorable triangulation of the 2-sphere. The proof consists of the following two statements: All 3-colorings of G are Kempe-equivalent under 4-colorings, and any 4-coloring of G is Kempe-equivalent to a 3-coloring. Here, a 3-coloring means that a coloring uses only three colors in $\{0, 1, 2, 3\}$. The first statement, which is folklore, can be easily obtained as follows. Since G is a 3-colorable triangulation, for any two 3-colorings α, β of G there uniquely exists a permutation π on $\{0, 1, 2, 3\}$ such that $\beta = \pi \circ \alpha$. Then, according to π, we can obtain β from α by a sequence of Kempe-changes (under 4-colorings) each of which changes a color at only one vertex, namely, a sequence of single-changes, by using the fourth color not appearing in α; see Figure 1 for example. Therefore, the nontrivial and crucial part in Fisk's result is to show the second statement.

The above observation for the first statement says that all 3-colorings of G are single-equivalent under 4-colorings. On the other hand, in general, some 4-coloring is not single-equivalent to any of 3-colorings (Figure 2). Here natural questions arise: *What 4-colorings are single-equivalent to some 3-coloring?* and *which 3-colorable triangulations of the 2-sphere have the property that all 4-colorings are single-equivalent?*

In this paper, we resolve these questions in the following sense, where n denotes the number of vertices of G.

1. We present an $O(n)$-time checkable characterization for a 4-coloring of G to be single-equivalent to some 3-coloring (Theorem 2). In addition, we show that, for any 4-colorings α, β of G single-equivalent to some 3-coloring, there exists a sequence of single-changes of length $O(n^2)$ from α to β and we can obtain it in $O(n^2)$ time (Theorem 6).

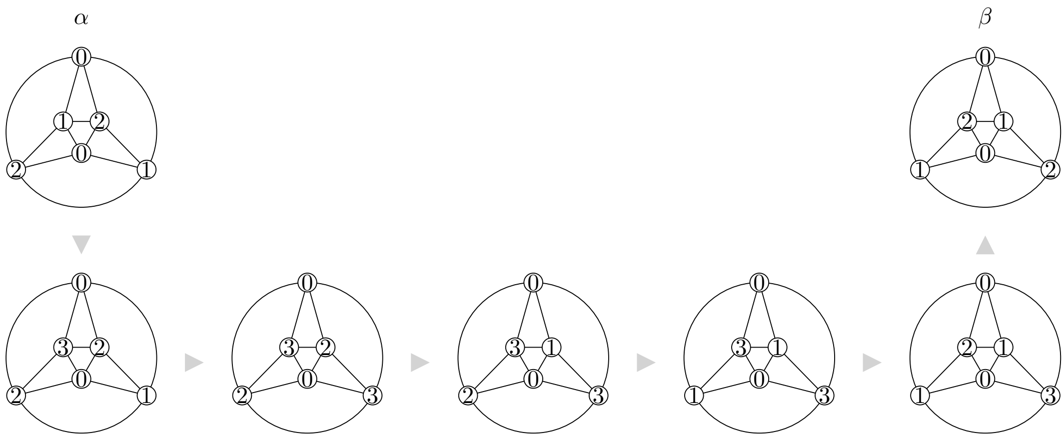

Figure 1 Single-equivalence of two 3-colorings of a 3-colorable triangulation of the 2-sphere.

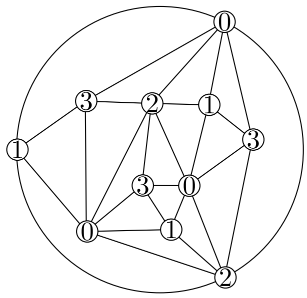

Figure 2 A 4-coloring of a 3-colorable triangulation of the 2-sphere such that it is not single-equivalent to any 3-coloring; no vertex can be recolored by a single-change. This coloring is “frozen”.

2. We provide an $O(n)$-time checkable criterion for a 3-colorable triangulation of the 2-sphere that all 4-colorings are single-equivalent (Theorem 8).

Furthermore, we consider a triangulation of a high-dimensional sphere. Let G be a $(k-1)$-colorable triangulation of the $(k-2)$-sphere for some positive integer $k \geq 4$. Then, by the same argument as in the case of $k = 4$ above, all $(k-1)$-colorings of G are single-equivalent under k-colorings. The following is a generalization of our first results (Theorem 2 and Theorem 6):

3. We present a characterization for a k-coloring of a $(k-1)$-colorable triangulation G of the $(k-2)$-sphere to be single-equivalent to some $(k-1)$-coloring. In addition, we show that, for any k-colorings α, β of G single-equivalent to some $(k-1)$-coloring, there exists a sequence of single-changes of length $O(n^{2\lfloor (k-1)/2 \rfloor})$ from α to β and we can obtain it in $O(n^{2\lfloor (k-1)/2 \rfloor})$ time.

In fact, the third result can be further generalized to $(k-1)$-colorable triangulations of connected closed $(k-2)$-manifolds satisfying a certain condition. This result is omitted in this paper, and given in the full version [21].

Our results are deeply related to the computational problems named k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph, which ask the connectedness of a k-coloring reconfiguration graph. Here, the *k-coloring reconfiguration graph* of a k-colorable graph G, denoted by $\mathcal{R}_k(G)$, is a graph such that its vertex set consists of all k-colorings of G and there is an edge between two k-colorings α and β of G if and only if β is obtained from α by recoloring only a single vertex in G, i.e., by a single-change. Thus, two k-colorings of G are single-equivalent if and only if they are connected in $\mathcal{R}_k(G)$. Then k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph are defined as follows.

k-Recoloring
Input: A k-colorable graph G and k-colorings α and β of G.
Output: YES if α and β are connected in $\mathcal{R}_k(G)$, and NO otherwise.

Connectedness of k-Coloring Reconfiguration Graph
Input: A k-colorable graph G.
Output: YES if $\mathcal{R}_k(G)$ is connected, and NO otherwise.

The problems k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph are fundamental in the recently emerging field of *combinatorial reconfiguration* (see [35, 30] for surveys and [22] for a general solver), which are extensively studied. It is shown that k-Recoloring is polynomial-time solvable if $k \leq 3$ [6], while PSPACE-complete if $k \geq 4$ [3]. According to [35, Section 3.2], the situation is very different from that for Kempe-equivalence, whose complexity is widely open. Bonsma and Cereceda [3] considered k-Recoloring for (bipartite) planar graphs; k-Recoloring for planar graphs is PSPACE-complete if $4 \leq k \leq 6$ and that for bipartite planar graphs is PSPACE-complete if $k = 4$. Cereceda, van den Heuvel, and Johnson [4] showed that $\mathcal{R}_k(G)$ is connected for any $(k-2)$-degenerate graph. By combining it with the fact that any planar graph is 5-degenerate and any bipartite planar graph is 3-degenerate, we see that k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph are in P (all instances are YES-instances) for any planar graph with $k \geq 7$ and for any bipartite planar graph with $k \geq 5$. In another paper [5], Cereceda, van den Heuvel, and Johnson also showed that Connectedness of 3-Coloring Reconfiguration Graph is coNP-complete in general and is in P for bipartite planar graphs.

The problem Connectedness of k-Coloring Reconfiguration Graph is also fundamental in the studies of the Glauber dynamics (a class of Markov chains) for k-colorings of a graph, which are used for random sampling and approximate counting. In each step of the Glauber dynamics of k-colorings, we are given a k-coloring of a graph. Then, we pick a vertex v and a color c uniformly at random, and change the color of v to c when the neighbors of v are not colored by c. Hence, one step of this Markov chain is exactly a single-exchange as long as we move to another coloring, and the state space is identical to the k-coloring reconfiguration graph. The connectedness of the k-coloring reconfiguration graph ensures that the Markov chain is irreducible. For the Glauber dynamics, the mixing property is one of the main concerns. It is an open question whether the Glauber dynamics of k-colorings has polynomial mixing time when $k \geq \Delta + 2$, where Δ is the maximum degree of a graph [23]. From continuing work in the literature, we know that the Glauber dynamics mixes quickly when $k > 2\Delta$ [23], $k > \frac{6}{11}\Delta$ [36], and finally $k > (\frac{6}{11} - \varepsilon)\Delta$ for a small absolute constant $\varepsilon > 0$ [7]. Results on restricted classes of graphs have also been known. For example, Hayes, Vera and Vigoda [17] proved that the Glauber dynamics mixes fast for planar graphs when $k = \Omega(\Delta/\log\Delta)$.

Our proofs provide algorithms for special cases of k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph. Here, we are supposed to be given a simplicial complex K whose geometric realization is homeomorphic to the $(k-2)$-sphere such that its 1-skeleton G is $(k-1)$-colorable. As we have seen, all $(k-1)$-colorings of G belong to the same connected component of $\mathcal{R}_k(G)$; we refer to it as the $(k-1)$-*coloring component* of $\mathcal{R}_k(G)$. Our third result (including the first) implies that, provided one of the input k-colorings α

and β belongs to the $(k-1)$-coloring component of $\mathcal{R}_k(G)$, the problem k-Recoloring for G can be solved in linear time in the size $\#K$ of the input simplicial complex K. In particular, if k is fixed, then our result says that it can be solved in polynomial time in n. Our second result implies that Connectedness of 4-Coloring Reconfiguration Graph for a 3-colorable triangulation of the 2-sphere can be solved in linear time in n.

We further investigate the computational complexity of the recoloring problem for a $(k-1)$-colorable triangulation G of the $(k-2)$-sphere. It is still open whether k-Recoloring for G can be solved in polynomial time, although we prove the polynomial-time solvability of the special case where one of the input k-colorings α and β belongs to the $(k-1)$-coloring component of $\mathcal{R}_k(G)$. In this paper, we additionally show that, if the number of colors which we can use increases by one, then it is difficult to check the single-equivalence between given two colorings:

4. For any fixed $k \geq 4$, the problem $(k+1)$-Recoloring is PSPACE-complete for $(k-1)$-colorable triangulations of the $(k-2)$-sphere (Theorem 13).

In the case of $k = 4$, our result is stronger than the PSPACE-completeness of 5-Recoloring for planar graphs, which is known in the literature [3].

We here emphasize that, for our algorithmic results, we are given a triangulation of a sphere, but not only its 1-skeleton. We need this assumption since our algorithm uses the triangulation and obtaining the triangulation from the 1-skeleton is hard. Indeed, for each fixed $d \geq 5$, the sphere recognition problem is undecidable [37, 8]: Namely it is undecidable whether a given simplicial complex is a triangulation of the d-sphere. This implies that it is also undecidable whether a given graph is the 1-skeleton of some triangulation of the d-sphere. When $d = 3$, the sphere recognition is decidable [31, 34], but not known to be solved in polynomial time (while it is known to be in NP [33]); the decidability is open when $d = 4$. Therefore, when $d \geq 3$, to filter out the intrinsic intractability of sphere recognition, we assume that a triangulation is also given along with a graph. On the other hand, when $d = 2$, we can decide whether a graph is the 1-skeleton of some triangulation in linear time [20]. In this case, the size of a triangulation is the same as the size of its 1-skeleton in the order of magnitude by Euler's formula, and therefore, the assumption that a triangulation is also given is not relevant.

Organization

This paper is organized as follows. In Section 2, we introduce notation. We provide a linear-time checkable characterization of the 3-coloring component of a 3-colorable triangulation of the 2-sphere in Section 3, which answers the first question. Section 4 is devoted to resolving the second question: We present a linear-time checkable criterion for a 3-colorable triangulation of the 2-sphere that any two 4-colorings are single-equivalent. In Section 5, we show the PSPACE-completeness of $(k+1)$-Recoloring for $(k-1)$-colorable triangulations of the $(k-2)$-sphere for $k \geq 4$. The arguments on our third result (a high-dimensional generalization of our first result) and several proofs of the statements marked with $\star$ are omitted. They are given in the full version of this paper [21].

2 Preliminaries

For a set A, we denote by $\#A$ the cardinality of A.

Let $G = (V, E)$ be a graph. For $v \in V$, we denote by $N_G(v)$ the set of neighbors of v and by $\delta_G(v)$ the set of edges incident to v; we simply write $N(v)$ and $\delta(v)$ if G is clear from the context. A map $\alpha\colon V \to \{0, 1, \ldots, k-1\}$ is called a *k-coloring* if $\alpha(u) \neq \alpha(v)$ for each edge

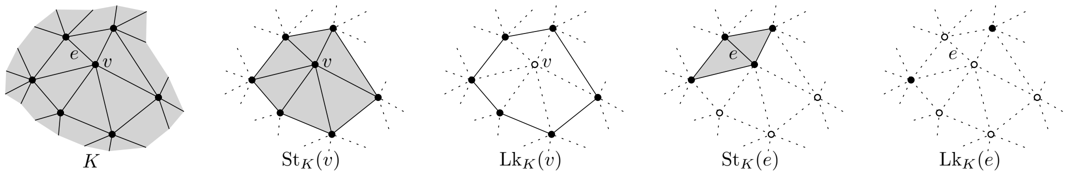

Figure 3 An example of the star complexes and the link complexes of a 2-dimensional simplicial complex K.

$\{u, v\} \in E$. A vertex $v \in V$ is said to be *recolorable* with respect to a k-coloring α if there is a k-coloring α' such that $\alpha'(u) = \alpha(u)$ for $u \in V \setminus \{v\}$ and $\alpha'(v) \neq \alpha(v)$, i.e., we can change the color $\alpha(v)$ of v.

Let S^d denote the d-sphere. A *triangulation* of S^d is a pair of a simplicial complex K and a homeomorphism $h\colon |K| \to S^d$, where $|K|$ denotes the geometric realization of K. See, for instance, Munkres [29] for fundamental terminology in simplicial complexes. Throughout this paper, we identify $|K|$ with S^d and omit to write h. For a simplex $\sigma \in K$, its *star complex* $\mathrm{St}_K(\sigma)$ and *link complex* $\mathrm{Lk}_K(\sigma)$ are defined by

$$\begin{aligned}\mathrm{St}_K(\sigma) &:= \{\tau \in K \mid \sigma \text{ and } \tau \text{ are faces of a common simplex in } K\},\\ \mathrm{Lk}_K(\sigma) &:= \{\tau \in K \mid \sigma \cap \tau = \emptyset,\ \sigma * \tau \in K\},\end{aligned}$$

where $\sigma * \tau$ denotes the join of σ and τ (see [29, Section 62]). Figure 3 shows examples. Also, let $\mathrm{St}^d_K(\sigma)$ denote the d-simplices in $\mathrm{St}_K(\sigma)$. For a subset $K' \subseteq K$, we define $|K'| \subseteq S^d$ by $|K'| := \bigcup_{\sigma \in K'} \sigma$. For instance, if v is a vertex of a triangulation of a surface without boundary, then $|\,\mathrm{St}_K(v)|$ and $|\,\mathrm{Lk}_K(v)|$ are homeomorphic to a closed disk and a circle, respectively. In this paper, we specify a triangulation by an embedded graph G in S^d, which is actually the 1-skeleton of a triangulation K. Also, we suppose that the input of k-Recoloring and Connectedness of k-Coloring Reconfiguration Graph is the simplicial complex K; for example, we are given the set of faces of a triangulation of the 2-sphere. We use $\mathrm{St}_G(\sigma)$ instead of $\mathrm{St}_K(\sigma)$ by abuse of notation. For example, $\mathrm{St}^0_G(v) \setminus \{v\} = N_G(v)$ and $\mathrm{St}^1_G(v) \setminus \mathrm{Lk}_G(v) = \delta_G(v)$. Also, we simply write $\mathrm{St}(\sigma)$ and $\mathrm{Lk}(\sigma)$ if G or K is clear from the context.

It is well-known that a triangulation of the 2-sphere is 3-colorable if and only if every vertex has an even degree (i.e., Eulerian). In this sense, a 3-colorable triangulation is said to be *even*. More generally, a triangulation K of a closed d-manifold is *even* if $\#\,\mathrm{St}^d(\sigma^{d-2})$ is even for every $(d-2)$-simplex $\sigma^{d-2} \in K$, where $d \geq 2$. If the 1-skeleton of K is $(d+1)$-colorable, then K is even. By [16, Sections I.4 and VI.2], the converse is also true for S^d, more generally, for simply-connected manifolds. Hence, it is easy to check whether a given triangulation of S^d is $(d+1)$-colorable.

3 A characterization of the $(k-1)$-coloring component

In this section, we resolve the first question posed in Section 1: *In a 3-colorable triangulation G of the 2-sphere, what 4-colorings are single-equivalent to some 3-coloring?* A characterization for high-dimensional cases can be obtained by a similar argument, which is omitted and given in the full version of this paper [21].

Let $G = (V, E)$ be a 3-colorable triangulation of the 2-sphere. Recall that all 3-colorings of G belong to the same connected component of $\mathcal{R}_4(G)$; we refer to it as the 3-coloring component of $\mathcal{R}_4(G)$. Let n denote the number $\#V$ of vertices of G.

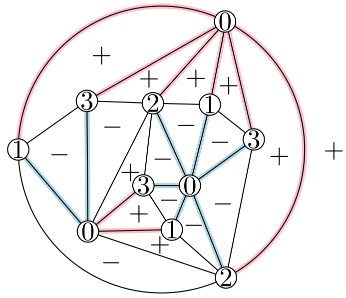

Figure 4 An example of the signature assignment to the faces. Red edges depict the +-nonsingular edges and blue edges depict the −-nonsingular edges.

Let F be the set of faces of G. We first define the signature on a face in F with respect to a 4-coloring of G and its related concepts, which were originally introduced in [18] (see also [32, Section 8 of Chapter 2]). These play an important role in our characterization.

Let $\alpha\colon V \to \{0,1,2,3\}$ be a 4-coloring of G. We assign a signature $+1/-1$ to each face $f \in F$ so that, for every pair of adjacent faces f, f' with $f = \{u, v, w\}$ and $f' = \{u', v, w\}$, they have the same signature if and only if $\alpha(u) \neq \alpha(u')$, where two faces $f, f' \in F$ are said to be *adjacent* if f and f' share an edge, i.e., $\#(f \cap f') = 2$. Such an assignment can be obtained as follows. For each face $f = \{u, v, w\} \in F$, we denote by $[\alpha(f)]$ the cyclically ordered set $[\alpha(u)\alpha(v)\alpha(w)]$ on $\{\alpha(u), \alpha(v), \alpha(w)\}$, where u, v, w are arranged in counterclockwise order in G if we see it from the outside of the 2-sphere. We define $\varepsilon_\alpha\colon F \to \{+1, -1\}$ by

$$\varepsilon_\alpha(f) := \begin{cases} +1 & \text{if } [\alpha(f)] \in \{[123], -[023], [013], -[012]\}, \\ -1 & \text{if } [\alpha(f)] \in \{-[123], [023], -[013], [012]\}, \end{cases}$$

where the minus sign $-$ indicates the opposite order, that is, $-[ijk] = [jik]$. We note here that when we regard $[123], -[023], [013], -[012]$ as oriented 2-simplices, they appear in the boundary of an oriented 3-simplex $[0123]$: $\partial[0123] = [123] \cup -[023] \cup [013] \cup -[012]$. A face $f \in F$ with $\varepsilon_\alpha(f) = +1$ (resp. $\varepsilon_\alpha(f) = -1$) is called a *+-face* (resp. *−-face*) with respect to α. Figure 4 shows an example. Recall that, for $v \in V$, the set of faces containing v is denoted as $\mathrm{St}^2(v)$. For a 4-coloring α, let $F^+_\alpha(v)$ (resp. $F^-_\alpha(v)$) denote the set of +-faces (resp. −-faces) in $\mathrm{St}^2(v)$.

An edge $e \in E$ is said to be *singular* with respect to α if the two adjacent faces $f, f' \in F$ sharing e have different signatures, i.e., $\varepsilon_\alpha(f) \neq \varepsilon_\alpha(f')$, and to be *nonsingular* if it is not singular [14, 15]. A nonsingular edge is particularly said to be *+-nonsingular* (resp. *−-nonsingular*) if $\varepsilon_\alpha(f) = \varepsilon_\alpha(f') = +1$ (resp. $\varepsilon_\alpha(f) = \varepsilon_\alpha(f') = -1$). Figure 4 also illustrates the +- and −-nonsingular edges. For $v \in V$, we denote by $\mathrm{NS}_\alpha(v)$, $\mathrm{NS}^+_\alpha(v)$, and $\mathrm{NS}^-_\alpha(v)$ the set of nonsingular, +-nonsingular, and −-nonsingular edges incident to v, respectively. Also, the set of nonsingular edges is denoted as NS_α. The following are obtained by direct observations.

▶ **Lemma 1.** *Let α be any 4-coloring of a 3-colorable triangulation G of the 2-sphere.*

(1) *A vertex $v \in V$ is recolorable with respect to α if and only if all edges incident to v are singular, i.e., $\mathrm{NS}_\alpha(v) = \emptyset$.*

(2) *The coloring α is a 3-coloring if and only if all edges are singular, i.e., $\mathrm{NS}_\alpha = \emptyset$.*

We can derive a necessary condition for a 4-coloring α of G to belong to the 3-coloring component of $\mathcal{R}_4(G)$ as follows. Let α' be a 4-coloring obtained from α by changing the color of v, i.e., $\alpha'(v) \neq \alpha(v)$ and $\alpha'(u) = \alpha(u)$ for all $u \in V \setminus \{v\}$. Then the signatures of all faces in $\mathrm{St}^2(v)$ are inverted (see also Figure 5):

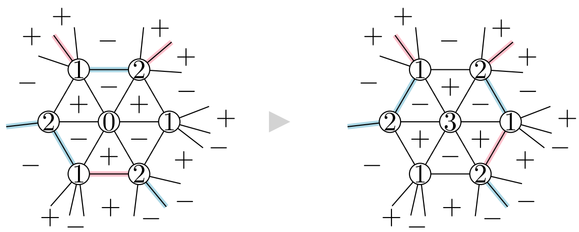

Figure 5 An example of the change of the signatures by a single-change. As in Figure 4, red and blue edges depict the +- and −-nonsingular edges, respectively.

$$\varepsilon_{\alpha'}(f) = \begin{cases} -\varepsilon_\alpha(f) & \text{if } f \in \mathrm{St}^2(v), \\ \varepsilon_\alpha(f) & \text{if } f \notin \mathrm{St}^2(v). \end{cases} \tag{1}$$

This implies that, if α and α' belong to the same connected component in $\mathcal{R}_4(G)$, then we have $\#F^+_\alpha(v) = \#F^+_{\alpha'}(v)$ and $\#F^-_\alpha(v) = \#F^-_{\alpha'}(v)$ for all $v \in V$. Furthermore, Lemma 1 (2) implies that, if α is a 3-coloring of G, then we have $\varepsilon_\alpha(f) \neq \varepsilon_\alpha(f')$ for all $v \in V$ and all adjacent $f, f' \in \mathrm{St}^2(v)$. Therefore, it follows from the above equation and Lemma 1 (2) that the following *balanced condition* holds if α belongs to the 3-coloring component of $\mathcal{R}_4(G)$:

(B) For each $v \in V$,

$$\#F^+_\alpha(v) = \#F^-_\alpha(v). \tag{2}$$

Our main result in this subsection is showing that the balanced condition (B) is also sufficient, that is, condition (B) characterizes the 3-coloring component of $\mathcal{R}_4(G)$.

▶ **Theorem 2.** *Let $\alpha\colon V \to \{0,1,2,3\}$ be a 4-coloring of a 3-colorable triangulation G of the 2-sphere. Then, α belongs to the 3-coloring component of $\mathcal{R}_4(G)$ if and only if it satisfies the balanced condition* (B).

For the proof of Theorem 2, we observe the behavior of NS_α when we recolor a vertex from a 4-coloring α. If we change the color $\alpha(v)$ of a vertex v, then it follows from the equation (1) that all singular edges in $\mathrm{Lk}(v)$ will be nonsingular and vice versa (see Figure 5). Thus, the following holds, where, for sets A and B, let $A \triangle B$ denote the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B.

▶ **Lemma 3.** *Let α be a 4-coloring of a 3-colorable triangulation G of the 2-sphere and α' a 4-coloring obtained from α by changing the color of a vertex v. Then*

$$\mathrm{NS}_{\alpha'}(u) = \begin{cases} \mathrm{NS}_\alpha(u) & \text{if } u \notin N(v), \\ \mathrm{NS}_\alpha(u) \triangle (\mathrm{Lk}(v) \cap \delta(u)) & \text{if } u \in N(v). \end{cases}$$

In particular, $\mathrm{NS}_{\alpha'} = \mathrm{NS}_\alpha \triangle (\mathrm{Lk}(v) \cap E)$.

In our proof, the set NS_α of nonsingular edges is viewed as the disjoint union of noncrossing closed trails in G. Here, a *closed trail* is a closed walk such that all edges are distinct. For a closed trail C of G and a vertex $v \in V$, we denote by C_v the set of subpaths of C obtained from the restriction of C to $\delta(v)$, i.e., $C_v := \{\{e, e'\} \mid \{e, e'\}$ is a subpath of C such that $e, e' \in \delta(v)\}$. A closed trail C of G is said to be *noncrossing* if for any vertex v, no pair of subpaths $P, P' \in C_v$ crosses in S^2, i.e., P' is contained in the closure of a connected component of $|\mathrm{St}(v)| \setminus P$ in S^2, where P is viewed as a curve in

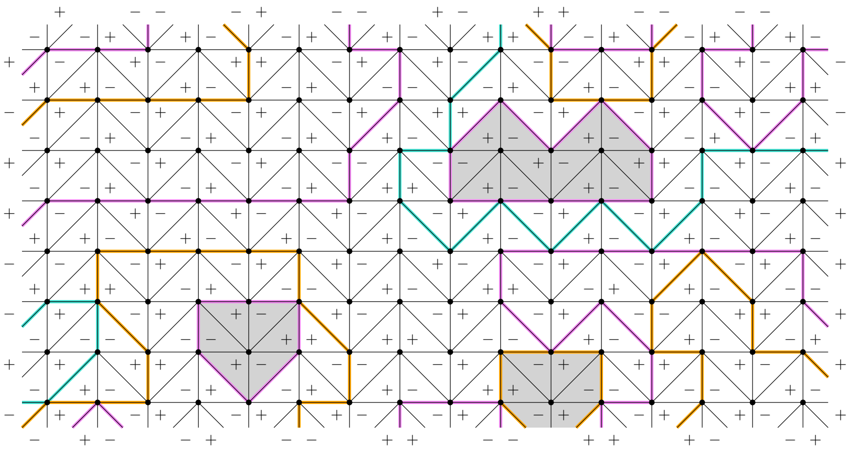

Figure 6 An example of NS-pairings. A part of a triangulation of a 2-sphere is depicted. Colors show closed trails. This NS-pairing is admissible. The gray areas show innermost closed trails.

S^2. For a noncrossing closed trail C with a fixed orientation, we define L_C by the union of connected components of $S^2 \setminus C$ such that it lies on the left side of some edge in C. Similarly, we define R_C by the union of connected components of $S^2 \setminus C$ such that it lies on the right side of some edge in C. Since C is noncrossing, the family $\{\mathrm{L}_C, \mathrm{R}_C\}$ forms a bipartition of $S^2 \setminus C$.

By fixing a certain face $f_{\mathrm{out}} \in F$ as the *outer face* of G, we can define the *volume* of a set of noncrossing closed trails in G as follows. We say that one of L_C and R_C is the *outside* of C if it contains the outer face f_{out}. The other is called the *inside* of C. Let $F_C \subseteq F$ denote the set of faces in the inside of C. Then, for a set $\mathcal{C}$ of noncrossing closed trails in G, its *volume*, denoted as $\mathrm{vol}(\mathcal{C})$, is the sum of the number of faces contained in the inside of C over all $C \in \mathcal{C}$, i.e., $\mathrm{vol}(\mathcal{C}) := \sum_{C \in \mathcal{C}} \# F_C$. It is clear that $\mathrm{vol}(\mathcal{C}) = 0$ if and only if $\mathcal{C} = \emptyset$. We will prove that any 4-coloring satisfying the balanced condition (B) has a recolorable vertex v such that, by changing the color of v, the volume of a set of noncrossing closed trails corresponding to the resulting 4-coloring strictly decreases from that of the original one. This implies that, by repeating this, we can obtain a 4-coloring such that its volume is zero, i.e., a 3-coloring.

We here see how NS_α corresponds to a set of noncrossing closed trails in G. It is known that, for any 4-coloring α of G and $v \in V$, the number $\# \mathrm{NS}_\alpha(v)$ of nonsingular edges incident to v is even (see e.g., [14, Lemma 5]). For $v \in V$, let π_v be a partition of $\mathrm{NS}_\alpha(v)$ such that each member of π_v is of size two (such a partition exists since $\# \mathrm{NS}_\alpha(v)$ is even), and define $\pi := \bigcup_{v \in V} \pi_v$. We refer to π as an *NS-pairing* (with respect to α). An NS-pairing $\pi = \bigcup_{v \in V} \pi_v$ uniquely determines a family $\mathcal{C}_\pi$ of closed trails in G satisfying that all closed trails in $\mathcal{C}_\pi$ are disjoint and $\pi_v = \bigcup_{C \in \mathcal{C}_\pi} C_v$ for all $v \in V$. Note that NS_α equals the disjoint union of all closed trails $C \in \mathcal{C}_\pi$. Figure 6 provides an example of the set of closed trails induced by an NS-pairing.

An NS-pairing $\pi = \bigcup_{v \in V} \pi_v$ is said to be *admissible* if the following hold for any $v \in V$:

(A1) All members of π_v consist of one $+$-nonsingular edge and one $-$-nonsingular edge;

(A2) No two pairs $P, P' \in \pi_v$ cross in $|\mathrm{St}(v)| \subseteq S^2$.

Let π be an admissible NS-pairing. Since each $C \in \mathcal{C}_\pi$ is noncrossing by (A2), the inside of C, and hence F_C, are well-defined. We define the face set family $\mathcal{L}_\pi \subseteq 2^F$ by $\mathcal{L}_\pi := \{F_C \mid C \in \mathcal{C}_\pi\}$.

The admissibility of π induces interesting properties on $\mathcal{C}_\pi$ and $\mathcal{L}_\pi$ as follows. Here, a set family $\mathcal{F} \subseteq 2^A$ is said to be *laminar* if, for any $X, Y \in \mathcal{F}$, we have $X \subseteq Y$, $X \supseteq Y$, or $X \cap Y = \emptyset$.

▶ **Lemma 4.** *Let π be an admissible NS-pairing with respect to a 4-coloring α.*
(1) *The restriction of α to C is a 2-coloring.*
(2) *The family $\mathcal{L}_\pi$ is laminar.*

Proof. (1). Take any member $\{\{u,v\},\{v,w\}\}$ of π_v, which forms a subpath of some $C \in \mathcal{C}_\pi$. It suffices to show that $\alpha(u) = \alpha(w)$. We may assume that $\alpha(v) = 3$.

Let n_+ (resp. n_-) denote the number of $+$-faces (resp. $-$-faces) in $\mathrm{St}^2(v) \cap F_C$. By the definition of the signature map ε_α, we have $\alpha(w) \equiv \alpha(u) + (n_+ - n_-) \pmod 3$ or $\alpha(w) \equiv \alpha(u) - (n_+ - n_-) \pmod 3$. Since π_v is noncrossing, the set of nonsingular edges incident to v in the inside of C is of the form of the union of a subset of π_v. Moreover, since all members of π_v consist of one $+$-nonsingular edge and one $-$-nonsingular edge, the number of $+$-nonsingular edges incident to v in the inside of C equals that of $-$-nonsingular edges. This implies that $n_+ = n_-$. Thus $\alpha(u) = \alpha(w)$ follows, as required.

(2). Take any two closed trails $C, C' \in \mathcal{C}_\pi$. Since π is admissible, in particular, no pair of members in π_v crosses in $|\mathrm{St}(v)|$ for any $v \in V$, the closed trail C' is contained in either the inside or the outside of C. Thus, in the former case we have $F_{C'} \subseteq F_C$, and in the latter case we have $F_C \subseteq F_{C'}$ or $F_C \cap F_{C'} = \emptyset$, which implies that $\mathcal{L}_\pi$ is laminar. ◀

Lemma 4 (2) implies that $\mathcal{C}_\pi$ has an innermost closed trail in S^2, which corresponds to a minimal set in $\mathcal{L}_\pi$.

We are ready to prove Theorem 2.

Proof of Theorem 2. We have already seen the only-if part. In the following, we show the if part. Let $\alpha\colon V \to \{0,1,2,3\}$ be a 4-coloring of G satisfying the balanced condition (B) but not a 3-coloring, i.e., $\mathrm{NS}_\alpha \neq \emptyset$ by Lemma 1 (2).

We first see that α has an admissible NS-pairing. Since $2 \cdot \# F^+_\alpha(v) = 2 \cdot \#\,\mathrm{NS}^+_\alpha(v) + \#(\delta(v) \setminus \mathrm{NS}_\alpha(v))$ and $2 \cdot \# F^-_\alpha(v) = 2 \cdot \#\,\mathrm{NS}^-_\alpha(v) + \#(\delta(v) \setminus \mathrm{NS}_\alpha(v))$, we have $\#\,\mathrm{NS}^+_\alpha(v) = \#\,\mathrm{NS}^-_\alpha(v)$ by (B). We construct an admissible NS-pairing as follows. For $v \in V$, let $\pi' := \emptyset$, $N^+_v := \mathrm{NS}^+_\alpha(v)$, and $N^-_v := \mathrm{NS}^-_\alpha(v)$. While $N^+_v \neq \emptyset$ and $N^-_v \neq \emptyset$, we take $e^+ \in N^+_v$ and $e^- \in N^-_v$ such that one of the connected components of $|\mathrm{St}(v)| \setminus \{e^+, e^-\}$ contains no edges in $N^+_v \cup N^-_v$ (such a pair (e^+, e^-) always exists) and update $\pi' \leftarrow \pi' \cup \{\{e^+, e^-\}\}$, $N^+_v \leftarrow N^+_v \setminus \{e^+\}$, and $N^-_v \leftarrow N^-_v \setminus \{e^-\}$. After the above procedure stops, we define π_v as the resulting π'. Then, we can see that π_v satisfies (A1) and (A2). Therefore, $\pi := \bigcup_{v \in V} \pi_v$ is an admissible NS-pairing.

The following claim is crucial for the proof of Theorem 2.

▷ Claim 5. There exists a recolorable vertex $v_0 \in V$ such that the 4-coloring α' obtained from α by recoloring v_0 has an admissible NS-pairing π' satisfying $\mathrm{vol}(\mathcal{C}_{\pi'}) < \mathrm{vol}(\mathcal{C}_\pi)$.

If this claim is true, then by recoloring such v_0 repeatedly, we finally obtain a 4-coloring α^* and an admissible NS-pairing π^* with respect to α^* such that $\mathrm{vol}(\mathcal{C}_{\pi^*}) = 0$. The equality $\mathrm{vol}(\mathcal{C}_{\pi^*}) = 0$ implies $\mathrm{NS}_{\alpha^*} = \emptyset$, i.e., α^* is actually a 3-coloring by Lemma 1 (2). Therefore, α belongs to the 3-coloring component of $\mathcal{R}_4(G)$, as required.

In the following, we prove Claim 5. Take an arbitrary innermost closed trail $C \in \mathcal{C}_\pi$, the existence of which is guaranteed by Lemma 4 (2), and an edge $e = \{v_1, v_2\} \in C$. Let $\{v_0, v_1, v_2\}$ be the face in the inside of C, or in F_C. Since α is a 4-coloring, the color $\alpha(v_0)$ is different from both $\alpha(v_1)$ and $\alpha(v_2)$. Therefore v_0 does not belong to C by Lemma 4 (1),

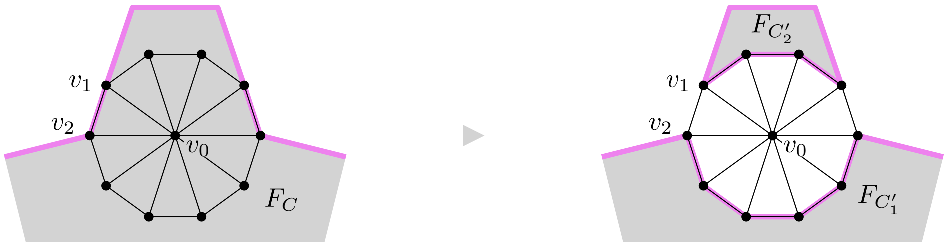

Figure 7 Reducing the volume. In this example, $\mathcal{C}' = \{C'_1, C'_2\}$.

implying that $\mathrm{St}^2(v_0) \subseteq F_C$. Since C is an innermost closed trail, no edge incident to the vertex v_0 is nonsingular with respect to α. Thus, by Lemma 1 (1), we can change the color of v_0.

Let α' be the 4-coloring obtained from α by changing the color of v_0. For each $v \in N(v_0)$, we have $\#(\delta(v) \cap \mathrm{Lk}(v_0)) = 2$, and denote $\delta(v) \cap \mathrm{Lk}(v_0)$ by P_v. We define $\pi' = \bigcup_{v \in V} \pi'_v$ by

$$\pi'_v := \begin{cases} \pi_v & \text{if } v \notin N(v_0), \\ \pi_v \cup \{P_v\} & \text{if } v \in N(v_0) \text{ and } \mathrm{NS}_\alpha(v) \cap \mathrm{Lk}(v_0) = \emptyset, \\ (\pi_v \setminus \{P\}) \cup \{P \triangle P_v\} & \text{if } v \in N(v_0) \text{ and } \pi_v \text{ contains } P \text{ with } |P \cap P_v| = 1, \\ \pi_v \setminus \{P_v\} & \text{if } v \in N(v_0) \text{ and } \pi_v \text{ contains } P_v. \end{cases}$$

See also Figure 5. Then π' is an NS-pairing with respect to α' by Lemma 3.

Moreover, we can see that π' is admissible as follows. It is clear that $\pi'_v \cap \pi_v$ satisfies (A1) and (A2) for each $v \in V$, implying that π'_v satisfies (A1) and (A2) if $v \notin N(v_0)$, or $v \in N(v_0)$ and π_v contains P_v. Since the edge $\{v_0, v\}$ is singular with respect to α', the path P_v (resp. $P \triangle P_v$) does not cross any $P' \in \pi_v$ (resp. $P' \in \pi_v \setminus \{P\}$); π'_v satisfies (A2) even for other v. Suppose that $P_v = \{\{u,v\},\{v,w\}\}$. Then, we have $\varepsilon_{\alpha'}(\{u,v,v_0\}) = \varepsilon_\alpha(\{w,v,v_0\})$ and $\varepsilon_{\alpha'}(\{w,v,v_0\}) = \varepsilon_\alpha(\{u,v,v_0\})$. This implies that, if $v \in N(v_0)$ and π_v contains P with $|P \cap P_v| = 1$, then $P \triangle P_v$ consists of one $+$-nonsingular edge and one $-$-nonsingular edge with respect to α', and if $v \in N(v_0)$ and $\mathrm{NS}_\alpha(v) \cap \mathrm{Lk}(v_0) = \emptyset$, then P_v consists of one $+$-nonsingular edge and one $-$-nonsingular edge with respect to α'. Thus π'_v satisfies (A1) for other v.

Let $\mathcal{C}'$ be the set of the closed trails in $\mathcal{C}_{\pi'}$ containing some $e \in \mathrm{Lk}(v_0) \cap \mathrm{NS}_{\alpha'}$. Then, we have $F_C = \bigcup_{C' \in \mathcal{C}'} F_{C'} \cup \mathrm{St}^2(v_0)$ and $\mathcal{C}_{\pi'} = \mathcal{C}_\pi \setminus \{C\} \cup \mathcal{C}'$. Therefore, we obtain $\mathrm{vol}(\mathcal{C}_{\pi'}) = \mathrm{vol}(\mathcal{C}_\pi) - \#\,\mathrm{St}^2(v_0) < \mathrm{vol}(\mathcal{C}_\pi)$; see also Figure 7

This completes the proof of the claim (and hence that of Theorem 2). ◀

Our proof of Theorem 2 is constructive; for a 4-coloring α satisfying the balanced condition (B), we explicitly construct a sequence of single-changes from α to a certain 3-coloring α^*. This leads to the following.

▶ **Theorem 6** ($\star$). *Let G be a 3-colorable triangulation of the 2-sphere. For any α and β belonging to the 3-coloring component of G, we can obtain in $O(n^2)$ time a sequence of single-changes of length $O(n^2)$ from α to β. In particular, the diameter of the 3-coloring component of G is $O(n^2)$.*

Theorems 2 and 6 immediately imply the polynomial-time solvability of 4-Recoloring for G if one of the given α or β belongs to the 3-coloring component. We here note that, for a 4-coloring α of G, we can check if it satisfies the balanced condition (B) in $O(\#F) = O(n)$ time.

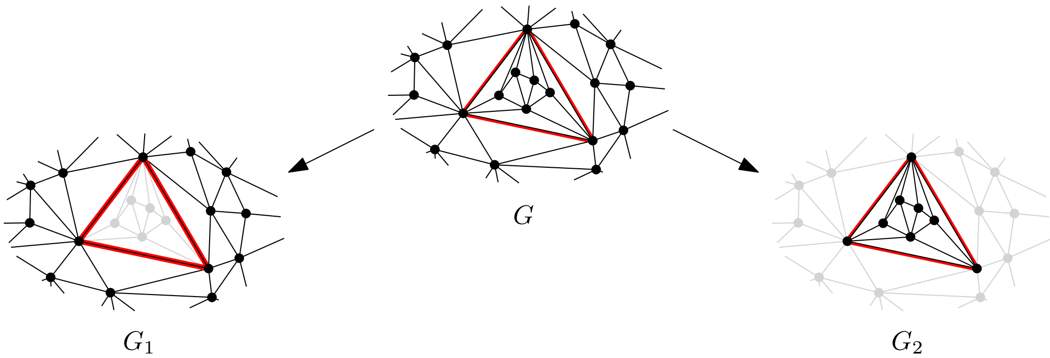

Figure 8 A triangulation is split by the separating triangle C highlighted by red lines.

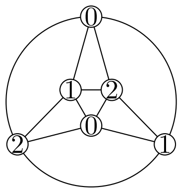

Figure 9 The octahedral graph with a 3-coloring.

▶ **Corollary 7.** *Let G be a 3-colorable triangulation of the 2-sphere.* 4-Recoloring *for G can be solved in $O(n)$ time, provided one of the input 4-colorings α and β belongs to the 3-coloring component of $\mathcal{R}_4(G)$. In addition, if both α and β belong to the 3-coloring component, then we can obtain a reconfiguration sequence from α to β in $O(n^2)$ time.*

4 Connectedness of the 4-coloring reconfiguration graph

In this section, we solve the second question posed in Introduction: *In what 3-colorable triangulation of the 2-sphere all 4-colorings are single-equivalent?* To explain the answer, we introduce some notation. Since we deal with only the case of the 2-sphere in this section, we simply use the term a *triangulation* instead of a triangulation of the 2-sphere.

A *separating triangle* in a triangulation is a cycle of length 3 that does not bound a face. Note that a triangulation with at least five vertices is 4-connected if and only if it has no separating triangles. A triangulation with a separating triangle C can be split into two triangulations, the subgraph induced by the inside of C and that by the outside of C, respectively (Figure 8). Note that they share C. By iteratively applying this procedure to a triangulation G with k separating triangles, we obtain a collection of $k+1$ triangulations without separating triangles. We call the $k+1$ triangulations 4-*connected pieces* of G. It is known [11] that the collection of the 4-connected pieces is uniquely determined. It is easy to see that G is 3-colorable if and only if every 4-connected piece of G is 3-colorable.

The *octahedral graph* is the 1-skeleton of the octahedron (Figure 9), which has six vertices, twelve edges, and eight faces, and is 3-colorable. A triangulation is said to be *octahedron-stacked* if every 4-connected piece of G is isomorphic to the octahedral graph. The following is the main theorem in this section.

▶ **Theorem 8** ($\star$). *Let G be a 3-colorable triangulation. Then, $\mathcal{R}_4(G)$ is connected if and only if G is octahedron-stacked.*

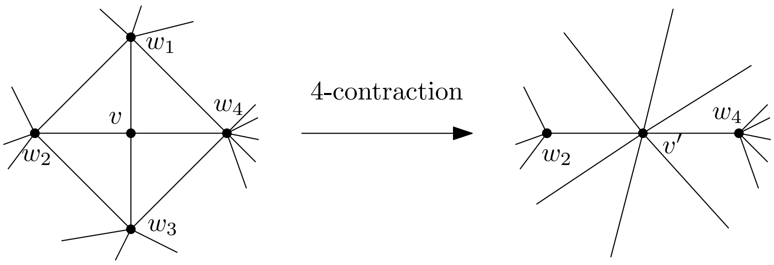

Figure 10 The 4-contraction of v at $\{w_1, w_3\}$.

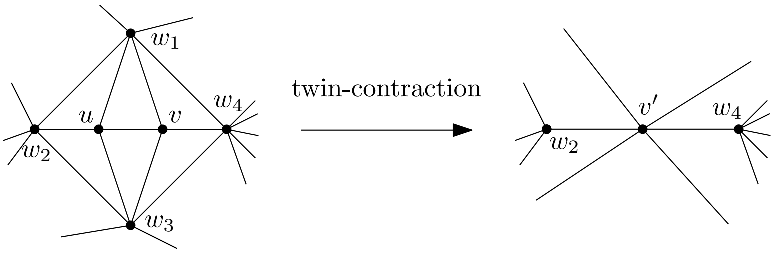

Figure 11 The twin-contraction of $\{u, v\}$ at $\{w_1, w_3\}$.

Since we can enumerate all separating triangles in linear time [9], the criterion in Theorem 8 can be used to obtain a linear-time algorithm for Connectedness of 4-Coloring Reconfiguration Graph for a 3-colorable triangulation of the 2-sphere, as follows.

▶ **Corollary 9.** Connectedness of 4-Coloring Reconfiguration Graph *for a* 3-*colorable triangulation* G *of the* 2-*sphere is solvable in* $O(n)$ *time.*

We prove Theorem 8 by combining some lemmas together with the so-called generating theorem. The following lemma deals with splitting a triangulation to obtain a 4-connected piece, and allows us to focus on 4-connected 3-colorable triangulations. Due to space limit, we leave a proof to the readers.

▶ **Lemma 10.** *Let* G *be a* 3-*colorable triangulation with a separating triangle* C*, and let* G_1 *and* G_2 *be the two triangulations obtained by splitting along* C*. Then* $\mathcal{R}_4(G)$ *is connected if and only if both* $\mathcal{R}_4(G_1)$ *and* $\mathcal{R}_4(G_2)$ *are connected.*

The if part of Theorem 8 is easily proven by Lemma 10 and the fact that $\mathcal{R}_4(G)$ is connected, where G is the octahedral graph. To prove the only if part, we now define two operations to reduce a 3-colorable triangulation G to a smaller triangulation G' as follows. Let v be a vertex of degree four in G and let $\{w_1, w_2, w_3, w_4, w_1\}$ be the cycle that forms the link of v. The 4-*contraction* of v at $\{w_1, w_3\}$, illustrated in Figure 10, is to remove v, identify the vertices w_1 and w_3, and replace the two pairs of multiple edges obtained from $\{\{w_1, w_2\}, \{w_2, w_3\}\}$ and $\{\{w_1, w_4\}, \{w_3, w_4\}\}$ with two single edges, respectively. Let u and v be adjacent vertices of degree four, where $\{w_1, w_2, w_3, v, w_1\}$ and $\{w_1, u, w_3, w_4, w_1\}$ are the cycles that form the links of u and v, respectively. The *twin-contraction* of $\{u, v\}$ at $\{w_1, w_3\}$, illustrated in Figure 11, is to remove u and v, identify the vertices w_1 and w_3, and replace the two pairs of multiple edges obtained from $\{\{w_1, w_2\}, \{w_2, w_3\}\}$ and $\{\{w_1, w_4\}, \{w_3, w_4\}\}$ with two single edges, respectively.

Notice that we do not perform these operations if they give rise to multiple edges. Matsumoto and Nakamoto proved the following generating theorem.

▶ **Theorem 11** ([25]). *For every 4-connected 3-colorable triangulation G, there exists a sequence $G_0, G_1, \dots, G_\ell$ from $G_0 := G$ such that G_ℓ is the octahedral graph, G_i is a 4-connected 3-colorable triangulation for $0 \le i \le \ell$, and G_i is obtained from G_{i-1} by either a 4-contraction or a twin-contraction for $1 \le i \le \ell$.*

For a 4-contraction and a twin-contraction, we need the following lemma.

▶ **Lemma 12.** *Let G be a 4-connected 3-colorable triangulation, and let G' be a 4-connected 3-colorable triangulation obtained from G by either a 4-contraction or a twin-contraction. If $\mathcal{R}_4(G')$ is disconnected, then so is $\mathcal{R}_4(G)$.*

It is not difficult to see that if the octahedral graph is obtained from a 3-colorable triangulation G by a 4-contraction, then $\mathcal{R}_4(G)$ is disconnected, and if the octahedral graph is obtained from a 3-colorable triangulation G by a twin-contraction, then G has a separating triangle, i.e. G is not 4-connected. Therefore, it follows from Theorem 11 and Lemma 12 that for a 4-connected 3-colorable triangulation G, $\mathcal{R}_4(G)$ is disconnected, unless G is the octahedral graph. By Lemma 10, this completes the proof of the only if part of Theorem 8.

The proof of Theorem 8 implies that if the answer to Connectedness of 4-Coloring Reconfiguration Graph is NO, then in a given 3-colorable triangulation G, we can find in polynomial time a 4-coloring that does not belong to the 3-coloring component of $\mathcal{R}_4(G)$. This would be a certificate for being a NO-instance.

5 PSPACE-completeness

As in Section 1, we show the following result in this section.

▶ **Theorem 13** ($\star$). *For $k \ge 4$, the problem $(k+1)$-Recoloring for $(k-1)$-colorable triangulations of the $(k-2)$-sphere is PSPACE-complete.*

When restricted to the case $k = 4$, Theorem 13 implies that 5-Recoloring is PSPACE-complete even for planar 3-colorable triangulations (i.e., even triangulations).

In order to prove Theorem 13, we introduce a new recoloring problem. For a list coloring, we associate a *list assignment* $L = (L(v))_{v \in V(G)}$ with a graph G such that each $v \in V(G)$ is assigned a list $L(v)$ of colors. For a list assignment L of a graph G, a map α on $V(G)$ is an *L-coloring* if $\alpha(v) \in L(v)$ for every $v \in V(G)$ and $\alpha(u) \neq \alpha(v)$ for every $\{u, v\} \in E(G)$. For a graph G and a list assignment L of G, the *L-coloring reconfiguration graph*, denoted by $\mathcal{R}(G, L)$, is defined as follows: Its vertex set consists of all L-colorings of G and there is an edge between two L-colorings α and β of G if and only if β is obtained from α by recoloring only a single vertex in G. We consider the following reconfiguration problem named List-Recoloring.

List-Recoloring
Input: A graph G, a list assignment L of G, and two L-colorings α and β of G.
Output: YES if α and β are connected in $\mathcal{R}(G, L)$, and NO otherwise.
Bonsma and Cereceda [3] proved that List-Recoloring is PSPACE-complete for particularly restricted graphs and list assignments.

We give a brief outline of the reduction from List-Recoloring to 5-Recoloring. In [3], restricted graphs are planar (not necessarily even triangulations) and a list of restricted list assignments is $\{0, 1, 2\}$ or $\{0,1\}$. We construct an even triangulation graph from a restricted graph used in [3] by inserting some vertices and graphs into faces and consider a 5-coloring by using colors $0, 1, 2, 3, 4$. Then, inserted graphs have a 5-coloring such that for each vertex v, all colors except for the color assigned to v appear in the neighbor of v. Such a 5-coloring is called a frozen 5-coloring. We insert new graphs in such a way that their frozen 5-colorings do not conflict. Since the coloring in the inserted graphs are frozen, for each vertex v not contained in the original graph, all colors except for the color assigned to v appear in the neighbor of v, i.e., all vertices not contained in the original graph have the property being "frozen." Therefore, the vertices contained in the original graph can only use colors in a restricted list assignment used in [3]. Consequently, 5-Recoloring in our even triangulation is the same as List-Recoloring in a restricted graph in [3].

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