On Higher Dimensional Point Sets in General Position

Andrew Suk
Department of Mathematics, University of California San Diego, La Jolla, CA, USA

Ji Zeng
Department of Mathematics, University of California San Diego, La Jolla, CA, USA

Abstract

A finite point set in $\mathbb{R}^d$ is in general position if no $d+1$ points lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of $N$ points in $\mathbb{R}^d$ with no $d+2$ members on a common hyperplane, contains a subset of size $\alpha_d(N)$ in general position. Using the method of hypergraph containers, Balogh and Solymosi showed that $\alpha_2(N) < N^{5/6+o(1)}$. In this paper, we also use the container method to obtain new upper bounds for $\alpha_d(N)$ when $d \geq 3$.

More precisely, we show that if $d$ is odd, then $\alpha_d(N) < N^{1/2 + 1/(2d+1) + o(1)}$, and if $d$ is even, we have $\alpha_d(N) < N^{1/2 + 1/d - 1 + o(1)}$.

We also study the classical problem of determining the maximum number $a(d, k, n)$ of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a $k$-flat. For fixed $d$ and $k$, we show that

$$a(d, k, n) \leq O\left(n^{d(d+k+2)/4} (1 - \frac{1}{d+k+2}) \right),$$

which improves the previously best known bound of $O\left(n^{d(d+k+2)/2} \right)$ due to Lefmann when $k+2$ is congruent to 0 or 1 mod 4.

2012 ACM Subject Classification Mathematics of computing → Combinatorics

Keywords and phrases independent sets, hypergraph container method, generalised Sidon sets

Digital Object Identifier 10.4230/LIPIcs.SoCG.2023.59


Funding Andrew Suk: Supported by NSF CAREER award DMS-1800746 and NSF award DMS-1952786.
Ji Zeng: Supported by NSF grant DMS-1800746.

1 Introduction

A finite point set in $\mathbb{R}^d$ is said to be in general position if no $d+1$ members lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of $N$ points in $\mathbb{R}^d$ with no $d+2$ members on a hyperplane, contains $\alpha_d(N)$ points in general position.

In 1986, Erdős [8] proposed the problem of determining $\alpha_2(N)$ and observed that a simple greedy algorithm shows $\alpha_2(N) \geq \Omega(\sqrt{N})$. A few years later, Füredi [10] showed that

$$\Omega(\sqrt{N} \log N) < \alpha_2(N) < o(N),$$

where the lower bound uses a result of Phelps and Rödl [20] on partial Steiner systems, and the upper bound relies on the density Hales-Jewett theorem [11, 12]. In 2018, a breakthrough was made by Balogh and Solymosi [3], who showed that $\alpha_2(N) < N^{5/6+o(1)}$. Their proof was based on the method of hypergraph containers, a powerful technique introduced independently by Balogh, Morris, and Samotij [1] and by Saxton and Thomason [24], that reveals an underlying structure of the independent sets in a hypergraph. We refer interested readers to [2] for a survey of results based on this method.
In higher dimensions, the best lower bound for $\alpha_d(N)$ is due to Cardinal, Tóth, and Wood [5], who showed that $\alpha_d(N) \geq \Omega((N \log N)^{1/d})$, for every fixed $d \geq 2$. For upper bounds, Milićević [18] used the density Hales-Jewett theorem to show that $\alpha_d(N) = O(N)$ for every fixed $d \geq 2$. However, these upper bounds in [18], just like that in [10], are still almost linear in $N$. Our main result is the following.

**Theorem 1.** Let $d \geq 3$ be a fixed integer. If $d$ is odd, then $\alpha_d(N) < N^{1/2 + \frac{1}{2d} + o(1)}$. If $d$ is even, then $\alpha_d(N) < N^{1/2 + \frac{1}{4d} + o(1)}$.

Our proof of Theorem 1 is also based on the hypergraph container method. A key ingredient in the proof is a new supersaturation lemma for $(k + 2)$-tuples of the grid $[n]^d$ that lie on a $k$-flat, which we shall discuss in the next section. Here, by a $k$-flat we mean a $k$-dimensional affine subspace of $\mathbb{R}^d$.

We also study the classical problem of determining the maximum number of points selected from the grid $[n]^d$ such that no $k + 2$ members lie on a $k$-flat. The key ingredient of Theorem 1 mentioned above can be seen as a supersaturation version of this Turán-type problem. When $k = 1$, this is the famous no-three-in-line problem raised by Dudeney [7] in 1917: Is it true that one can select all logarithms are in base two. For example, we have

$$\alpha_2(n) = \Theta(n^{1/2})$$

For integer $n > 0$, we let $[n] = \{1, \ldots, n\}$, and $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$. We systematically omit floors and ceilings whenever they are not crucial for the sake of clarity in our presentation. All logarithms are in base two.
2 \((k + 2)\)-tuples of \([n]^d\) on a \(k\)-flat

In this section, we establish two lemmas that will be used in the proof of Theorem 1.

Given a set \(T\) of \(k + 2\) points in \(\mathbb{R}^d\) that lie on a \(k\)-flat, we say that \(T\) is degenerate if there is a subset \(S \subset T\) of size \(j\), where \(3 \leq j \leq k + 1\), such that \(S\) lies on a \((j - 2)\)-flat. Otherwise, we say that \(T\) is non-degenerate. We establish a supersaturation lemma for non-degenerate \((k + 2)\)-tuples of \([n]^d\).

\textbf{Lemma 3.} For real number \(\gamma > 0\) and fixed positive integers \(d, k\), such that \(k\) is even and \(d - 2\gamma > (k - 1)(k + 2)\), any subset \(V \subset [n]^d\) of size \(n^{d - \gamma}\) spans at least \(\Omega(n^{(k+1)d - (k+2)\gamma})\) non-degenerate \((k + 2)\)-tuples that lie on a \(k\)-flat.

\textbf{Proof.} Let \(V \subset [n]^d\) such that \(|V| = n^{d - \gamma}\). Set \(r = \frac{k}{2} + 1\) and \(E_r = \binom{V}{r}\) to be the collection of \(r\)-tuples of \(V\). Notice that the sum of a \(r\)-tuple from \(V\) belongs to \([rn]^d\). For each \(v \in [rn]^d\), we define

\[E_r(v) = \{v_1, \ldots, v_r\} \in E_r : v_1 + \cdots + v_r = v\].

Then for \(T_1, T_2 \in E_r(v)\), where \(T_1 = \{v_1, \ldots, v_r\}\) and \(T_2 = \{u_1, \ldots, u_r\}\), we have

\[v_1 + \cdots + v_r = v + u_1 + \cdots + u_r\],

which implies that \(T_1 \cup T_2\) lies on a common \(k\)-flat. Let

\[E_{2r} = \bigcup_{v \in [rn]^d} \bigcup_{T_1, T_2 \in E_r(v)} \{T_1, T_2\}\].

Hence, for each \(\{T_1, T_2\} \in E_{2r}\), \(T_1 \cup T_2\) lies on a \(k\)-flat. Moreover, by Jensen’s inequality, we have

\[|E_{2r}| = \sum_{v \in [rn]^d} \left(\frac{|E_r(v)|}{2}\right) \geq (rn)^d \left(\sum_{v \in [rn]^d} \frac{|E_r(v)|}{2}\right) = (rn)^d \left(\frac{|E_r|}{(rn)^d}\right) \geq \frac{|E_r|^2}{4(rn)^d}\].

Since \(k\) and \(d\) are fixed and \(r = \frac{k}{2} + 1\) and \(|V| = n^{d - \gamma}\),

\[|E_r|^2 = \left(\frac{|V|}{r}\right)^2 = \left(\frac{|V|}{(k/2) + 1}\right)^2 \geq \Omega(n^{(k+2)(d-\gamma)})\].

Combining the two inequalities above gives

\[|E_{2r}| \geq \Omega(n^{(k+1)d - (k+2)\gamma})\].

We say that \(\{T_1, T_2\} \in E_{2r}\) is good if \(T_1 \cap T_2 = \emptyset\), and the \((k + 2)\)-tuple \((T_1 \cup T_2)\) is non-degenerate. Otherwise, we say that \(\{T_1, T_2\}\) is bad. In what follows, we will show that at least half of the pairs (i.e. elements) in \(E_{2r}\) are good. To this end, we will need the following claim.

\textbf{Claim 4.} If \(\{T_1, T_2\} \in E_{2r}\) is bad, then \(T_1 \cup T_2\) lies on a \((k-1)\)-flat.

\textbf{Proof.} Write \(T_1 = \{v_1, \ldots, v_r\}\) and \(T_2 = \{u_1, \ldots, u_r\}\). Let us consider the following cases.

\textbf{Case 1.} Suppose \(T_1 \cap T_2 \neq \emptyset\). Then, without loss of generality, there is an integer \(j < r\) such that

\[v_1 + \cdots + v_j = u_1 + \cdots + u_j\].

We say that \(\{T_1, T_2\} \in E_{2r}\) is good if \(T_1 \cap T_2 = \emptyset\), and the \((k + 2)\)-tuple \((T_1 \cup T_2)\) is non-degenerate. Otherwise, we say that \(\{T_1, T_2\}\) is bad. In what follows, we will show that at least half of the pairs (i.e. elements) in \(E_{2r}\) are good. To this end, we will need the following claim.

\textbf{Claim 4.} If \(\{T_1, T_2\} \in E_{2r}\) is bad, then \(T_1 \cup T_2\) lies on a \((k-1)\)-flat.

\textbf{Proof.} Write \(T_1 = \{v_1, \ldots, v_r\}\) and \(T_2 = \{u_1, \ldots, u_r\}\). Let us consider the following cases.
where \( v_1, \ldots, v_j, u_1, \ldots, u_j \) are all distinct elements, and \( v_t = u_t \) for \( t > j \). Thus \( |T_1 \cup T_2| = 2j + (r - j) \). The \( 2j \) elements above lie on a \((2j-2)\)-flat. Adding the remaining \( r - j \) points implies that \( T_1 \cup T_2 \) lies on a \((j-2+r)\)-flat. Since \( r = \frac{k}{2} + 1 \) and \( j \leq \frac{k}{2} \), \( T_1 \cup T_2 \) lies on a \((k-1)\)-flat.

**Case 2.** Suppose \( T_1 \cap T_2 = \emptyset \). Then \( T_1 \cup T_2 \) must be degenerate, which means there is a subset \( S \subset T_1 \cup T_2 \) of \( j \) elements such that \( S \) lies on a \((j-2)\)-flat, for some \( 3 \leq j \leq k + 1 \).

Without loss of generality, we can assume that \( v_1 \notin S \). Hence, \( (T_1 \cup T_2) \setminus \{v_1\} \) lies on a \((k-1)\)-flat. On the other hand, we have

\[
v_1 = u_1 + \cdots + u_r - v_2 - \cdots - v_r.
\]

Hence, \( v_1 \) is in the affine hull of \( (T_1 \cup T_2) \setminus \{v_1\} \) which implies that \( T_1 \cup T_2 \) lies on a \((k-1)\)-flat.

We are now ready to prove the following claim.

**Claim 5.** At least half of the pairs in \( E_{2r} \) are good.

**Proof.** For the sake of contradiction, suppose at least half of the pairs in \( E_{2r} \) are bad. Let \( H \) be the collection of all the \( j \)-flats spanned by subsets of \( V \) for all \( j \leq k - 1 \). Notice that if \( S \subset V \) spans a \( j \)-flat \( h \), then \( h \) is also spanned by only \( j + 1 \) elements from \( S \). So we have

\[
|H| \leq \sum_{j=0}^{k-1} |V|^j + 1 \leq k n^k (d-\gamma).
\]

For each bad pair \( \{T_1, T_2\} \in E_{2r} \), \( T_1 \cup T_2 \) lies on a \( j \)-flat from \( H \) by Claim 4. By the pigeonhole principle, there is a \( j \)-flat \( h \) with \( j \leq k - 1 \) such that at least

\[
\frac{|E_{2r}|/2}{|H|} \geq \frac{\Omega(n^{(k+1)d-(k+2)\gamma})}{2kn^{k(d-\gamma)}} = \Omega(n^{d-2\gamma})
\]

bad pairs from \( E_{2r} \) have the property that their union lies in \( h \). On the other hand, since \( h \) contains at most \( n^{k-1} \) points from \([n]^d\), \( h \) can correspond to at most \( O(n (k-1)(k+2)) \) bad pairs from \( E_{2r} \). Since we assumed \( d - 2\gamma > (k-1)(k+2) \), we have a contradiction for \( n \) sufficiently large.

Each good pair \( \{T_1, T_2\} \in E_{2r} \) gives rise to a non-degenerate \((k+2)\)-tuple \( T_1 \cup T_2 \) that lies on a \( k \)-flat. On the other hand, any such \((k+2)\)-tuple in \( V \) will correspond to at most \( \binom{k+2}{r} \) good pairs in \( E_{2r} \). Hence, by Claim 5, there are at least

\[
\frac{|E_{2r}|}{2} \left( \frac{k+2}{r} \right) = \Omega(n^{(k+1)d-(k+2)\gamma})
\]

non-degenerate \((k+2)\)-tuples that lie on a \( k \)-flat, concluding the proof.

In the other direction, we will use the following upper bound.

**Lemma 6.** For real number \( \gamma > 0 \) and fixed positive integers \( d, k, \ell \), such that \( \ell < k + 2 \), suppose \( U, V \subset [n]^d \) satisfy \( |U| = \ell \) and \( |V| = n^{d-\gamma} \), then \( V \) contains at most \( n^{(k+1-\ell)(d-\gamma)+k} \) non-degenerate \((k+2)\)-tuples that lie on a \( k \)-flat and contain \( U \).
Proof. If $U$ spans a $j$-flat for some $j < \ell - 1$, then by definition no non-degenerate $(k + 2)$-tuple contains $U$. Hence we can assume $U$ spans a $(\ell - 1)$-flat. Observe that a non-degenerate $(k + 2)$-tuple $T$, which lies on a $k$-flat and contains $U$, must contain a $(k + 1)$-tuple $T' \subset T$ such that $T'$ spans a $k$-flat and $U \subset T'$. Then there are at most $n^{(k+1-\ell)(d-\ell)}$ ways to add $k + 1 - \ell$ points to $U$ from $V$ to obtain such $T'$. After $T'$ is determined, there are at most $n^k$ ways to add a final point from the affine hull of $T'$ to obtain $T$. So we conclude the proof by multiplication. \hfill \blacksquare

\section{The container method: Proof of Theorem 1}

In this section, we use the hypergraph container method to prove Theorem 1. We follow the method outlined in [3]. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ denote a $(k + 2)$-uniform hypergraph. For any $U \subset V(\mathcal{H})$, its degree $d(U)$ is the number of edges containing $U$. For each $\ell \in [k + 2]$, we use $\Delta(\ell, \mathcal{H})$ to denote the maximum $d(U)$ among all $U$ of size $\ell$. For parameter $\tau > 0$, we define the following quantity

$$\Delta(\mathcal{H}, \tau) = \frac{2^{(k+2)/2} - 1}{(k + 2)!} \sum_{\ell=2}^{k+2} \Delta(\ell, \mathcal{H}) \frac{1}{\ell^{\tau-1} \binom{k+2}{\ell}}.$$ 

Then we have the following hypergraph container lemma from [3], which is a restatement of Corollary 3.6 in [24].

\begin{lemma}
Let $\mathcal{H}$ be a $(k + 2)$-uniform hypergraph and $0 < \epsilon, \tau < 1/2$. Suppose that $\tau < 1/(200 \cdot (k + 2) \cdot (k + 2)!)$ and $\Delta(\mathcal{H}, \tau) \leq \epsilon/(12 \cdot (k + 2)!)$.
Then there exists a collection $C$ of subsets (containers) of $V(\mathcal{H})$ such that
1. Every independent set in $\mathcal{H}$ is a subset of some $C \in C$;
2. $\log |C| \leq 1000 \cdot (k + 2) \cdot ((k + 2)!)^3 \cdot |V(\mathcal{H})| \cdot \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau)$;
3. For every $C \in C$, the induced subgraph $\mathcal{H}[C]$ has at most $\epsilon |E(\mathcal{H})|$ many edges.
\end{lemma}

The main result in this section is the following theorem.

\begin{theorem}
Let $k, r$ be fixed integers such that $r \geq k \geq 2$ and $k$ is even. Then for any $0 < \alpha < 1$, there are constants $c = c(\alpha, k, r)$ and $d = d(\alpha, k, r)$ such that the following holds. For infinitely many values of $N$, there is a set $V$ of $N$ points in $\mathbb{R}^d$ such that no $r + 3$ members of $V$ lie on an $r$-flat, and every subset of $V$ of size $cN^{\frac{k+2}{k+1} + \alpha}$ contains $k + 2$ members on a $k$-flat.
\end{theorem}

Before we prove Theorem 8, let us show that it implies Theorem 1. In dimensions $d_0 \geq 3$ where $d_0$ is odd, we apply Theorem 8 with $k = r = d_0 - 1$ to obtain a point set $V$ in $\mathbb{R}^d$ with the property that no $d_0 + 2$ members lie on a $(d_0 - 1)$-flat, and every subset of size $cN^{\frac{k+2}{k+1} + \alpha}$ contains $d_0 + 1$ members on a $(d_0 - 1)$-flat. By projecting $V$ to a generic $d_0$-dimensional subspace of $\mathbb{R}^d$, we obtain $N$ points in $\mathbb{R}^{d_0}$ with no $d_0 + 2$ members on a common hyperplane, and no $cN^{\frac{k+2}{k+1} + \alpha}$ members in general position.

In dimensions $d_0 \geq 4$ where $d_0$ is even, we apply Theorem 8 with $k = d_0 - 2$ and $r = d_0 - 1$ to obtain a point set $V$ in $\mathbb{R}^d$ with the property that no $d_0 + 2$ members on a $(d_0 - 1)$-flat, and every subset of size $cN^{\frac{k+2}{k+1} + \alpha}$ contains $d_0$ members on a $(d_0 - 2)$-flat. By adding another point from this subset, we obtain $d_0 + 1$ members on a $(d_0 - 1)$-flat. Hence, by projecting to $V$ a generic $d_0$-dimensional subspace of $\mathbb{R}^d$, we obtain $N$ points in $\mathbb{R}^{d_0}$ with no $d_0 + 2$ members on a common hyperplane, and no $cN^{\frac{k+2}{k+1} + \alpha}$ members in general position. This completes the proof of Theorem 1.
On Higher Dimensional Point Sets in General Position

Proof of Theorem 8. We set \( d = d(\alpha, k, r) \) to be a sufficiently large integer depending on \( \alpha, k, \) and \( r \). Let \( H \) be the hypergraph with \( V(H) = [n]^d \) and \( E(H) \) consists of non-degenerate \( (k+2) \)-tuples \( T \) such that \( T \) lies on a \( k \)-flat. Let \( C^0 = [n]^d, C^0 = \{C^0\}, \) and \( H^0 = H \). In what follows, we will apply the hypergraph container lemma to \( H^0 \) to obtain a family of containers \( C^1 \). For each \( C^1 \) of \( C^1 \), we consider the induced hypergraph \( H^1 = H[C^1] \), and we apply the hypergraph container lemma to it. The collection of containers obtained from all \( H^1 \) will form another collection of containers \( C^2 \). We iterate this process until each container in \( C^1 \) is sufficiently small, and moreover, we will only produce a small number of containers. As a final step, we apply the probabilistic method to show the existence of the desired point set. We now flesh out the details of this process.

We start by setting \( C^0 = [n]^d, C^0 = \{C^0\}, \) and set \( H^0 = H[C^0] = H \). Having obtained a collection of containers \( C^1 \), for each container \( C^1 \in C^1 \) with \( |C^1| \geq n^{\frac{d}{k+1} + k} \), we set \( H^1 = H[C^1] \). Let \( \gamma = \gamma(i, j) \) be defined by \( |V(H^1)| = n^{d-\gamma} \). So, \( \gamma \leq \frac{d}{k+1} - k \). We set \( \tau = \tau(i, j) = n^{-\frac{k+2}{k+1} + \gamma + \alpha} \) and \( \epsilon = \epsilon(i, j) = c_1 n^{-\alpha} \), where \( c_1 = c_1(d, k) \) is a sufficiently large constant depending on \( d \) and \( k \). Then we can verify the following condition.

\[ \Delta(H^1, \tau) \leq \epsilon/(12 \cdot (k + 2)!). \]

**Proof.** Since \( |V(H^1)| = n^{d-\gamma}, \gamma \leq \frac{d}{k+1} - k, \) and \( d \) is sufficiently large, Lemma 3 implies that \( |E(H^1)| \geq c_2 n^{(k+1)d-(k+2)\gamma} \) for some constant \( c_2 = c_2(d, k) \). Hence, we have

\[ \frac{|V(H^1)|}{|E(H^1)|} \leq \frac{n^{d-\gamma}}{c_2 n^{(k+1)d-(k+2)\gamma}} = \frac{1}{c_2 n^{kd-(k+1)\gamma}}. \]

On the other hand, by Lemma 6, we have

\[ \Delta(H^1) \leq n^{(d-\gamma)(k+1-\ell)+k} \]

and obviously \( \Delta_{k+2}(H^1) \leq 1 \).

Applying these inequalities together with the definition of \( \Delta \), we obtain

\[ \Delta(H^1, \tau) = \frac{2\left(\frac{k+2}{k+1}\right)-1 |V(H^1)| \sum_{\ell=2}^{k+2} \Delta(H^1, \tau)}{(k+2)|E(H^1)|} \leq \frac{c_3}{n^{kd-(k+1)\gamma}} \left( \sum_{\ell=2}^{k+1} \frac{n^{(k+1-\ell)(d-\gamma)+k}}{\tau^{\ell-1}} + \frac{1}{\tau^{k+1}} \right) \]

for some constant \( c_3 = c_3(d, k) \). Let us remark that the summation above is where we determined our \( \tau \) and \( \gamma \). In order to make the last term small, we choose \( \tau = n^{-\frac{k+2}{k+1} + \gamma + \alpha} \).

Having determined \( \tau \), in order for the first term in the summation to be small, we choose \( \gamma \leq \frac{d}{k+1} - k \).

By setting \( \epsilon = c_1 n^{-\alpha} \) with \( c_1 = c_1(d, k) \) sufficiently large, we have

\[ \Delta(H^1, \tau) \leq c_3 \left( \sum_{\ell=2}^{k+1} n^{-\frac{k+2}{k+1} + \gamma + k - (\ell-1) \alpha} + n^{-(k+1)\alpha} \right) \]

\[ \leq c_3 n^{-\alpha} + c_3 n^{-(k+1)\alpha} \]

\[ \leq \epsilon \frac{c_3 n^{-(k+1)\alpha}}{12(k+2)!}. \]

This verifies the claimed condition.
Given the condition above, we can apply Lemma 7 to $\mathcal{H}_j^i$ with chosen parameters $\tau$ and $\epsilon$. Hence we obtain a family of containers $\mathcal{C}_j^{i+1}$ such that

$$|\mathcal{C}_j^{i+1}| \leq 2^{10^9(k+2)((k+2)!)^3|V(H)|_{\tau \log(1/\epsilon)} \log(1/\tau)}$$

$$\leq 2c_4 n^{d+\alpha} \log^2 n,$$

for some constant $c_4 = c_4(d, k)$. In the other case where $|C_j^i| < n^{1/\alpha} d + k$, we just define $C_j^{i+1} = \{C_j^i\}$. Then, for each container $C \in C_j^{i+1}$, we have either $|C| < n^{1/\alpha} d + k$ or $|E(H[C])| \leq \epsilon|E(H)| \leq \epsilon^4|E(H)|$. After applying this procedure for each container in $\mathcal{C}^i$, we obtain a new family of containers $\mathcal{C}^{i+1} = \bigcup C_j^i$ such that

$$|\mathcal{C}^{i+1}| \leq |\mathcal{C}^i| 2^{c_4 n^{1/\alpha} \log^2 n} \leq 2^{(i+1)c_4 n^{1/\alpha} \log^2 n} n.$$

Notice that the number of edges in $\mathcal{H}_j^i$ shrinks by a factor of $c_1 n^{-\alpha}$ whenever $i$ increases by one, while on the other hand, Lemma 3 tells us that every large subset $C \subset [n]^d$ induces many edges in $\mathcal{H}$. Hence, after at most $t \leq c_5/\alpha$ iterations, for some constant $c_5 = c_5(d, k)$, we obtain a collection of containers $\mathcal{C} = \mathcal{C}^t$ such that: each container $C \in \mathcal{C}$ satisfies $|C| < n^{1/\alpha} d + k$; every independent set of $\mathcal{H}$ is a subset of some $C \in \mathcal{C}$; and

$$|\mathcal{C}| \leq 2^{(c_5/\alpha)c_4 n^{1/\alpha} \log^2 n}.$$

Before we construct the desired point set, we make the following crude estimate.

**Claim 10.** The grid $[n]^d$ contains at most $O(n^{(r+1)d+2r})$ many $(r+3)$-tuples that lie on a $r$-flat.

**Proof.** Let $T$ be an arbitrary $(r+3)$-tuple that spans a $j$-flat. There are at most $n^{(j+1)d}$ ways to choose a subset $T' \subset T$ of size $j + 1$ that spans the affine hull of $T$. After this $T'$ is determined, there are at most $n^{(r+2-j)}$ ways to add the remaining $r + 2 - j$ points from the $j$-flat spanned by $T'$. Then the total number of $(r + 3)$-tuples that lie on a $r$-flat is at most

$$\sum_{j=1}^{r} n^{(j+1)d+(r+2-j)} \leq \sum_{j=1}^{r} n^{(j+1)d+(r+2-j)r} \leq n^{(r+1)d+2r},$$

since we can assume $d > r$.

Now, we randomly select a subset of $[n]^d$ by keeping each point independently with probability $p$. Let $S$ be the set of selected elements. Then for each $(r+3)$-tuple $T$ in $S$ that lies on an $r$-flat, we delete one point from $T$. We denote the resulting set of points by $S'$. By the claim above, the number of $(r + 3)$-tuples in $[n]^d$ that lie on a $r$-flat is at most $c_6 n^{(r+1)d+2r}$ for some constant $c_6 = c_6(r)$. Therefore,

$$\mathbb{E}[|S'|] \geq pn^d - c_6 n^{r+3} n^{(r+1)d+2r}.$$ 

By setting $p = (2c_6)^{-1/\alpha} n^{-1/\alpha} (d+2)$, we have

$$\mathbb{E}[|S'|] \geq \frac{pn^d}{2} = \Omega(n^{2(d-r)/\alpha}).$$

Finally, we set $m = (c_7/\alpha)n^{1/\alpha + 2\alpha}$ for some sufficiently large constant $c_7 = c_7(d, k, r)$. Let $X$ denote the number of independent sets of size $m$ in $S'$. Using the family of containers
On Higher Dimensional Point Sets in General Position

\[ \mathcal{C}, \text{ we have} \]

\[ \mathbb{E}[X] \leq |\mathcal{C}| \left( \frac{n^k}{m} \right)^{d + k} \]

\[ \leq \left( 2^{(c_5/\alpha)c_4n^d / n^2 + \alpha \log^2 n} \right) \left( \frac{c_7 n^{d+k} p}{m} \right)^{m} \]

\[ \leq \left( 2^{(c_5/\alpha)c_4n^d / n^2 + \alpha \log^2 n} \right) \left( \frac{c_8 \alpha_{n^{d+k} \alpha^{-2} (d+2)}}{n^{d+k+2}} \right) \]

\[ \leq \left( 2^{(c_5/\alpha)c_4n^d / n^2 + \alpha \log^2 n} \right) \left( c_8 \alpha^{-n^{d+k} \alpha^{-2} (d+2)} \right) \]

\[ \leq 2^{(c_5/\alpha)c_4n^d / n^2 + \alpha \log^2 n} \left( c_8 \alpha^{-n^{d+k} \alpha^{-2} (d+2)} \right) \]

\[ \leq \frac{2(k-r+1)d}{c \alpha n^{d-1} (d-r)} + \frac{r+2}{d} 2\alpha \leq \alpha. \]

This completes the proof. ▶

4 Avoiding non-trivial solutions: Proof of Theorem 2

In this section, we will give a proof of Theorem 2. Let \( V \subset \mathbb{Z}^d \) such that there are no \( k+2 \) points that lie on a \( k \)-flat. In [17], Lefmann showed that \( |V| \leq O \left( \frac{n^{d-1} + \alpha}{d-1} \right) \). To see this, assume that \( k \) is even and consider all elements of the form \( v_1 + \cdots + v_{k+1} \), where \( v_i \neq v_j \) and \( v_i \in V \). All of these elements are distinct, since otherwise we would have \( k+2 \) points on a \( k \)-flat. In other words, the equation

\[ (x_1 + \cdots + x_{k+1}) - \left( x_{k+2} + \cdots + x_{k+2} \right) = 0, \]

does not have a solution with \( \{x_1, \ldots, x_{k+1}\} \) and \( \{x_{k+2}, \ldots, x_{k+2}\} \) being two different \( (\frac{k}{2} + 1) \)-tuples of \( V \). Therefore, we have \( \left( \frac{|V|}{\frac{k}{2} + 1} \right) \leq (kn)^2 \), and this implies Lefmann’s bound.

More generally, let us consider the equation

\[ c_1 x_1 + c_2 x_2 + \cdots + c_r x_r = 0, \quad (1) \]

with constant coefficients \( c_i \in \mathbb{Z} \) and \( \sum_i c_i = 0 \). Here, the variables \( x_i \) takes value in \( \mathbb{Z}^l \). A solution \( (x_1, \ldots, x_r) \) to equation (1) is called trivial if there is a partition \( P : [r] = I_1 \cup \cdots \cup I_s \), such that \( x_j = x_{j+1} \) if and only if \( j, j+1 \in I_i \), and \( \sum_{j \in I_i} c_j = 0 \) for all \( i \in [s] \). In other words,
being trivial means that, after combining like terms, the coefficient of each $x_i$ becomes zero. Otherwise, we say that the solution $(x_1, \ldots, x_r)$ is \textit{non-trivial}. A natural extremal problem is to determine the maximum size of a set $A \subset [n]^d$ with only trivial solutions to (1). When $d = 1$, this is a classical problem in additive number theory, and we refer the interested reader to [23, 19, 15, 6].

By combining the arguments of Cilleruelo and Timmons [6] and Jia [14], we establish the following theorem.

\textbf{Theorem 11.} Let $d, r$ be fixed positive integers. Suppose $V \subset [n]^d$ has only trivial solutions to each equation of the form

$$c_1 (x_1 + \cdots + x_r) = c_2 (x_{2r} + \cdots + x_{3r}),$$

for integers $c_1, c_2$ such that $1 \leq c_1, c_2 \leq n^{\frac{d+1}{2d}}$. Then we have

$$|V| \leq O \left( n^{\frac{d+1}{2}} \left( 1 - \frac{1}{2d} \right) \right).$$

Notice that Theorem 2 follows from Theorem 11. Indeed, when $k + 2$ is divisible by 4, we set $r = (k + 2)/4$. If $V \subset [n]^d$ contains $k + 2$ points $\{v_1, \ldots, v_{k+2}\}$ that is a non-trivial solution to (2) with $x_i = v_i$, then $\{v_1, \ldots, v_{k+2}\}$ must lie on a $k$-flat. Hence, when $k + 2$ is divisible by 4, we have

$$a(d, k, n) \leq O \left( n^{\frac{d+1}{2d}} \left( 1 - \frac{1}{2d} \right) \right).$$

Since we have $a(d, k, n) < a(d, k - 1, n)$, this implies that for all $k \geq 2$, we have

$$a(d, k, n) \leq O \left( n^{\frac{d+1}{2d}} \left( 1 - \frac{1}{2d} \right) \right).$$

In the proof of Theorem 11, we need the following well-known lemma (see e.g. [6]Lemma 2.1 and [23]Theorem 4.1). For $U, T \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we define

$$\Phi_{U-T}(x) = \{(u, t) : u - t = x, u \in U, t \in T\}.$$
On Higher Dimensional Point Sets in General Position

for two different r-tuples in V. Then by setting \((x_1, \ldots, x_r) = (v_1, \ldots, v_r), (x_{r+1}, \ldots, x_{2r}) = (v'_1, \ldots, v'_r), (x_{2r+1}, \ldots, x_{3r}) = (x_{3r+1}, \ldots, x_{4r})\) arbitrarily, and \(c_1 = c_2 = 1\), we obtain a non-trivial solution to (2), which is a contradiction. In particular, we have \(|S_r| = \binom{|V|}{r}\).

For \(j \in [m]\) and \(w \in \mathbb{Z}_d^d\), we let

\[ U_{j,w} = \{ u \in \mathbb{Z}^d : ju + w \in S_r \}. \]

Notice that for fixed \(j \in [m]\), we have

\[ \sum_{w \in \mathbb{Z}_d^d} |U_{j,w}| = \sum_{w \in \mathbb{Z}_d^d} |\{ v \in S_r : v \equiv w \mod j \}| = |S_r|. \]

Applying Jensen’s inequality to above, we have

\[ \sum_{w \in \mathbb{Z}_d^d} |U_{j,w}|^2 \geq |S_r|^2 / j^d. \] (3)

For \(i \geq 0\), we define

\[ \Phi_{U_{j,w} - U_{j,w}}^i(x) = \{ (u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}(x) : |\sigma^{-1}(ju_1 + w) \cap \sigma^{-1}(ju_2 + w)| = i \}. \]

It’s obvious that these sets form a partition of \(\Phi_{U_{j,w} - U_{j,w}}(x)\). We also make the following claims.

\> Claim 13. For a fixed \(x \in \mathbb{Z}^d\), we have

\[ \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^d} |\Phi_{U_{j,w} - U_{j,w}}^i(x)| \leq 1, \]

\textbf{Proof.} For the sake of contradiction, suppose the summation above is at least two, then we have \((u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}^i(x)\) and \((u_3, u_4) \in \Phi_{U_{j',w'} - U_{j',w'}}^i(x)\) such that either \((u_1, u_2) \neq (u_3, u_4)\) or \((j, w) \neq (j', w')\).

Let \(s_1, s_2, s_3, s_4 \in S_r\) such that \(s_1 = ju_1 + w, s_2 = ju_2 + w, s_3 = j'u_3 + w', s_4 = j'u_4 + w'\) and write \(\sigma^{-1}(s_1) = \{ v_1, \ldots, v_4, r \}\). Notice that \(u_1 - u_2 = x = u_3 - u_4\). Putting these equations together gives us

\[ j'((v_{1,1} + \cdots + v_{1,r}) - (v_{2,1} + \cdots + v_{2,r})) = j((v_{3,1} + \cdots + v_{3,r}) - (v_{4,1} + \cdots + v_{4,r})). \] (4)

It suffices to show that (4) can be seen as a non-trivial solution to (2). The proof now falls into the following cases.

\textbf{Case 1.} Suppose \(j 
eq j'\). Without loss of generality we can assume \(j' > j\). Notice that \((u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}^i(x)\) implies

\[ \{ v_{1,1}, \ldots, v_{1,r} \} \cap \{ v_{2,1}, \ldots, v_{2,r} \} = \emptyset. \]

Then after combining like terms in (4), the coefficient of \(v_{1}^j\) is at least \(j' - j\), which means this is indeed a non-trivial solution to (2).

\textbf{Case 2.} Suppose \(j = j'\), then we must have \(s_1 \neq s_3\). Indeed, if \(s_1 = s_3\), we must have \(w = w'\) (as \(s_1\) modulo \(j\) equals \(s_3\) modulo \(j'\)) and \(s_2 = s_4\) (as \(j'(s_4 - s_2) = j(s_3 - s_4)\)). This is a contradiction to either \((u_1, u_2) \neq (u_3, u_4)\) or \((j, w) \neq (j', w')\).

Given \(s_1 \neq s_3\), we can assume, without loss of generality, \(v_{1,1} \notin \{ v_{3,1}, \ldots, v_{3,r} \}\). Again, we have \(\{ v_{1,1}, \ldots, v_{1,r} \} \cap \{ v_{2,1}, \ldots, v_{2,r} \} = \emptyset\). Hence, after combining like terms in (4), the coefficient of \(v_{1}^j\) is positive and we have a non-trivial solution to (2).
Claim 14. For a finite set $T \subset \mathbb{Z}^d$, and fixed integers $i, j \geq 1$, we have
\[
\sum_{w \in \mathbb{Z}_j^d} \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)| \leq |V|^{2r-i}|T|.
\]

Proof. The summation on the left-hand side counts all (ordered) quadruples $(u_1, u_2, t_1, t_2)$ such that $(u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}(t_1 - t_2)$. For each such a quadruple, let $s_1, s_2 \in S_r$ such that 
\[
s_1 = j u_1 + w \quad \text{and} \quad s_2 = j u_2 + w.
\]
There are at most $|V|^{2r-i}$ ways to choose a pair $(s_1, s_2)$ satisfying $|\sigma^{-1}(s_1) \cap \sigma^{-1}(s_2)| = i$. Such a pair $(s_1, s_2)$ determines $(u_1, u_2)$ uniquely. Moreover, $(s_1, s_2)$ also determines the quantity
\[
t_1 - t_2 = u_1 - u_2 = \frac{s_1 - w}{j} - \frac{s_2 - w}{j} = \frac{1}{j}(s_1 - s_2).
\]
After such a pair $(s_1, s_2)$ is chosen, there are at most $|T|$ ways to choose $t_1$ and this will also determine $t_2$. So we conclude the claim by multiplication.

Now, we set $T = \mathbb{Z}_j^d$ for some integer $\ell$ to be determined later. Notice that $U_{j,w} + T \subset \{0, 1, \ldots, |rn/j| + \ell - 1\}^d$, which implies
\[
|U_{j,w} + T| \leq (rn/j + \ell)^d.
\]

By Lemma 12, we have
\[
\frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} \leq \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)|.
\]

Summing over all $j \in [m]$ and $w \in \mathbb{Z}_j^d$, and using Claims 13 and 14, we can compute
\[
\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} \frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} \leq \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)|
\]
\[
= \sum_{x \in \mathbb{Z}^d} \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} \left( |\Phi_{U_{j,w} - U_{j,w}}^0(x)| + \sum_{i=1}^r |\Phi_{U_{j,w} - U_{j,w}}^i(x)| \right) |\Phi_{T-T}(x)|
\]
\[
\leq \sum_{x \in \mathbb{Z}^d} |\Phi_{T-T}(x)| \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} |\Phi_{U_{j,w} - U_{j,w}}^0(x)| + \sum_{j \in [m]} \sum_{i=1}^r |V|^{2r-i}\ell^d
\]
\[
\leq \sum_{x \in \mathbb{Z}^d} |\Phi_{T-T}(x)| + \sum_{j \in [m]} \sum_{i=1}^{r-1} |V|^{2r-i}\ell^d
\]
\[
\leq \ell^d + rm|V|^{2r-1}\ell^d.
\]
On the other hand, using (3) and (5), we can compute
\[\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} |U_{j,w}|^2 |T|^2 \geq \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} |U_{j,w}|^2 \ell^{2d} \]
\[\geq \sum_{j \in [m]} \frac{|S_r| \ell^{2d}}{(jn/j + \ell)^d} \]
\[= \sum_{j \in [m]} \frac{|S_r| \ell^{2d}}{(jn + j \ell)^d} \geq \frac{m|S_r| \ell^{2d}}{(rn + m\ell)^d}.\]
Combining the two inequalities above gives us
\[\frac{m|S_r| \ell^{2d}}{(rn + m\ell)^d} \leq \ell^{2d} + rm|V|^{2r-1} \ell^d\]
\[\Rightarrow |S_r|^2 \leq \frac{(rn + m\ell)^d}{m} + r|V|^{2r-1} (rn + m\ell)^d \ell^d.\]
By setting \(m = \frac{n}{d^{2/d-1}}\) and \(\ell = n^{1-\frac{2}{d^{2/d-1}}},\) we get
\[\left(\frac{|V|}{r}\right)^2 = |S_r|^2 \leq cn^{d-\frac{2}{d^{2/d-1}}} + c|V|^{2r-1} n^{\frac{2}{d^{2/d-1}}},\]
for some constant \(c\) depending only on \(d\) and \(r\). We can solve from this inequality that
\[|V| = O\left(n^{\frac{d}{d^2} (1 - \frac{2}{d^{2/d-1}})}\right),\]
completing the proof.

5 Concluding remarks

1. One can consider a generalization of the quantity \(\alpha_d(N)\). We let \(\alpha_{d,s}(N)\) be the largest integer such that any set of \(N\) points in \(\mathbb{R}^d\) with no \(d + s\) members on a hyperplane, contains \(\alpha_{d,s}(N)\) points in general position. Hence, \(\alpha_d(N) = \alpha_{d,2}(N)\). Following the arguments in our proof of Theorem 1 with a slight modification, we show the following.

\textbf{Theorem 15.} Let \(d, s \geq 3\) be fixed integers. If \(d\) is odd and \(\frac{2d+s-2}{d^{2/d-1}-2} < \frac{d-1}{d}\), then \(\alpha_{d,s}(N) \leq N^{\frac{1}{d} + o(1)}\). If \(d\) is even and \(\frac{2d+s-2}{d^{2/d-1}-2} < \frac{d-2}{d}\), then \(\alpha_{d,s}(N) \leq N^{\frac{1}{d} + o(1)}\).

For example, when we fix \(d = 3\) and \(s \geq 5\), we have \(\alpha_{d,s}(N) \leq N^{\frac{1}{3} + o(1)}\). In the other direction, it is easy to show that \(\alpha_{d,s}(N) \geq \Omega(N^{1/d})\) for any fixed \(d, s \geq 2\) (see [8]).

\textbf{Problem 16.} Are there fixed integers \(d, s \geq 3\) such that \(\alpha_{d,s}(N) \leq o(N^{\frac{1}{d}})\) ?

2. We call a subset \(V \subset [n]^d\) an \(m\)-fold \(B_g\)-set if \(V\) only contains trivial solutions to the equations
\[c_1x_1 + c_2x_2 + \cdots + c_gx_g = c_1x_1' + c_2x_2' + \cdots + c_gx_g',\]
with constant coefficients \(c_i \in [m]\). We call 1-fold \(B_g\)-sets simply \(B_g\)-sets. By counting distinct sums, we have an upper bound \(|V| \leq O(n^{\frac{1}{d}})\) for any \(B_g\)-set \(V \subset [n]^d\).

Our Theorem 11 can be interpreted as the following phenomenon: by letting \(m\) grow as some proper polynomial in \(n\), we have an upper bound for \(m\)-fold \(B_g\)-sets, where \(g\) is even, which gives a polynomial-saving improvement from the trivial \(O(n^{\frac{1}{d}})\) bound. We believe this phenomenon should also hold without the parity condition on \(g\).