Quotients and Extensionality in Relational Doctrines

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Abstract

Taking a quotient roughly means changing the notion of equality on a given object, set or type. In a quantitative setting, equality naturally generalises to a distance, measuring how much elements are similar instead of just stating their equivalence. Hence, quotients can be understood quantitatively as a change of distance. Quotients are crucial in many constructions both in mathematics and computer science and have been widely studied using categorical tools. Among them, Lawvere's doctrines stand out, providing a fairly simple functorial framework capable to unify many notions of quotient and related constructions. However, abstracting usual predicate logics, they cannot easily deal with quantitative settings. In this paper, we show how, combining doctrines and the calculus of relations, one can unify quantitative and usual quotients in a common picture. More in detail, we introduce relational doctrines as a functorial description of (the core of) the calculus of relations. Then, we define quotients and a universal construction adding them to any relational doctrine, generalising the quotient completion of existential elementary doctrine and also recovering many quantitative examples. This construction deals with an intensional notion of quotient and breaks extensional equality of morphisms. Then, we describe another construction forcing extensionality, showing how it abstracts several notions of separation in metric and topological structures.

2012 ACM Subject Classification Theory of computation → Logic; Theory of computation → Categorical semantics

Keywords and phrases Quantitative Reasoning, Calculus of Relations, Hyperdoctrines, Metric Spaces

Digital Object Identifier 10.4230/LIPIcs.FSCD.2023.25

Acknowledgements We are grateful to the anonymous referees for their many useful comments. They have been very helpful in improving the paper.

1 Introduction

Quotients are pervasive both in mathematic and computer science, as they are crucial in carrying out many fundamental arguments. Quotients have been widely studied and several constructions have been refined to allow one to work with quotients even though they are not natively available in the setting in which one is reasoning (such as within a type theory, where usually quotients are not a primitive concept). The intuition behind these constructions is that taking a quotient changes the notion of equality on an object to a given equivalence relation. Then, to work with (formal) quotients, one just endows each object (set, type, space, ...) with an (abstract) equivalence relation and forces the object to "believe" that that equivalence relation is the equality. This idea underlies the construction of setoids in type theories [7, 30], which are the common solution to work with quotients in that setting and underlies also the exact completion of a category with weak finite limits [14, 15].

Quantitative methods are increasingly used in many different domains, such as differential privacy [51, 9, 59, 8], denotational semantics [5, 21], algebraic theories [45, 46, 47, 1, 4], program/behavioural metrics [17, 19, 20, 22, 26, 57], and rewriting [27]. This is mainly due to the fact that these methods better deal with the imprecision arising when one reasons about the behaviour of complex software systems, especially when interacting with physical processes. In a quantitative setting, equivalence relations naturally generalise to distances, which measure how much two elements are similar instead of just saying whether they are equivalent or not. Hence, quotients could be seen quantitatively as a change of distance. Indeed, this operation is often used when dealing with metric structures, see for instance the construction of monads associated with quantitative equational theories [1, 45, 46].

A unified view of quotients covering both usual and quantitative settings is missing. The aim of this paper is to develop a notion of quotient, related concepts and constructions extending known results and incorporating new quantitative examples.

Many mathematical tools have been adopted to study quotients. Among them, Lawvere's doctrines [36, 37] stand out as a simple and powerful framework capable to cope with a large variety of situations (see [31, 50, 58] and references therein). Doctrines provide a functorial description of logical theories, abstracting the essential algebraic structure shared by both syntax and semantics of logics.

In particular, Maietti and Rosolini [42, 41] identified doctrines modelling the conjunctive fragment of first order logic with equality as the minimal setting where to define equivalence relations and quotients. Then they defined a universal construction, named elementary quotient completion, that freely adds quotients to such doctrines, showing that it subsumes many others, such as setoids and the exact completion.

In order to move this machinery to a quantitative setting, one may try to work with doctrines where the usual conjunction is replaced by its linear counterpart. In this way, equivalence relations becomes distances as transitivity becomes a triangular inequality. However smooth, this transition is less innocent than it appears. As shown in [18], to properly deal with a quantitative notion of equality one needs a more sophisticated structure, which however fails to capture important examples like the category of metric spaces and non-expansive maps. The main difficulty in working with Lawvere's doctrines is that doctrines, modelling usual predicate logic, take care of variables. This is problematic in a quantitative setting as the use of variables usually has an impact on the considered distances.

For these reasons, in this paper we take a different approach: we work with doctrines abstracting the *calculus of relations* [3, 49, 55] which is a variable-free alternative to first order logic. Here one takes as primitive concept (binary) relations instead of (unary) predicates, together with some basic operations, such as relational identities, composition and the converse of a relation. Even though in general it is less expressive than first order logic,[1] it is still quite expressive, for instance, one can axiomatise set theory in it [54]. Moreover, being variable-free, it scales well to quantitative settings, as witnessed by the fruitful adoption of relational techniques to develop quantitative methods [19, 26, 27].

Then, in this paper, we introduce *relational doctrines*, as a functorial description of the calculus of relations. Relying on this structure, we define a notion of quotient capable to deal with also quantitative settings. We present a universal construction to add such quotients to any relational doctrine. The construction extends the one in [42, 41] and can also capture quantitative instances such as the category of metric spaces and non-expansive maps.

Furthermore, related to quotients, we study the notion of extensional equality. Roughly, two functions or morphisms are extensionally equal if their outputs coincide on equal inputs. Even if quotients and extensionality are independent concepts, several known constructions that add quotients often force extensionality (see e.g., Bishop's sets, setoids over a type theory

[1] The calculus of relations is equivalent to first order logic with three variables [28].

or the ex/lex completion). Therefore the study of extensionality is essential to cover these well-known examples. We show that the relational quotient completion, changing the notion of equality on objects without affecting plain equality on arrows in the base category, may break this property. Thus, we define another universal construction that forces extensionality. We show also how this logical principle captures many notions of separation in metric and topological structures.

These results are developed using the language of 2-categories [33]. To this end, we organise relational doctrines in a suitable 2-category where morphisms abstract the usual notion of relation lifting [32]. Since many categorical concepts can be defined internally to any 2-category, in this way we get them for free also for relational doctrines. For instance, following [53], we can define (co)monads on relational doctrines, which nicely corresponds to (co)monadic relation liftings used to reason about (co)effectful programs [26, 19]. The universality of our constructions is then expressed in terms of (lax) 2-adjunctions [10], thus describing their action not only on relational doctrines, but on their morphisms as well.

The paper is organised as follows. In Section 2 we introduce relational doctrines with their basic properties, presenting several examples. In Section 3 we define quotients and the relational quotient completion, proving it is universal. In Section 4 we discuss extensionality, its connection with separation and the universal construction forcing it, showing also how it interacts with quotients. In Section 5 we compare our approach with two important classes of examples: ordered categories with involution [35], which are a generalisation of both allegories and cartesian bicategories, and elementary existential doctrines [42, 41]. Finally, Section 6 summarises our contributions and discusses directions for future work.

2 Relational Doctrines: Definition and First Properties

Doctrines are a simple and powerful framework introduced by Lawvere [36, 37] to study several kinds of logics using categorical tools. A *doctrine* P on $\mathcal{C}$ is a contravariant functor $P : \mathcal{C}^{\mathrm{op}} \to \mathcal{Pos}$, where $\mathcal{Pos}$ denotes the category of posets and monotone functions. The category $\mathcal{C}$ is named the *base* of the doctrine and, for X in $\mathcal{C}$, the poset $P(X)$ is called *fibre over* X. For $f : X \to Y$ an arrow in $\mathcal{C}$, the monotone function $P_f : P(Y) \to P(X)$ is called *reindexing along* f. Roughly, the base category collects the objects one is interested in with their transformations, a fibre $P(X)$ collects predicates over the object X ordered by logical entailment and reindexing allows to transport predicates between objects according to their transformations. An archetypal example of a doctrine is the contravariant powerset functor $\mathcal{P} : \mathcal{Set}^{\mathrm{op}} \to \mathcal{Pos}$, where predicates are represented by subsets ordered by set inclusion.

Doctrines capture the essence of predicate logic. In this section, we will introduce *relational doctrines* as a functorial description of the essential structure of relational logics. To this end, since binary relations can be seen as predicates over a pair of objects, we will need to index posets over pairs of objects, that is, to consider functors $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{Pos}$, where each fibre $R(X, Y)$ collects relations from X to Y. Here the reference example are set-theoretic relations: they can be organised into a functor $\mathsf{Rel} : (\mathcal{Set} \times \mathcal{Set})^{\mathrm{op}} \to \mathcal{Pos}$ where $\mathsf{Rel}(X, Y) = \mathcal{P}(X \times Y)$ and sending f, g to the inverse image $(f \times g)^{-1}$.

We endow these functors with a structure modelling a core fragment of the calculus of relations given by relational identities, composition and converse [3, 49, 55]. For set-theoretic relations, the identity relation on a set X is the diagonal $\mathsf{d}_X = \{\langle x, x'\rangle \in X \times X \mid x = x'\}$, the composition of $\alpha \in \mathsf{Rel}(X, Y)$ with $\beta \in \mathsf{Rel}(Y, Z)$ is the set $\alpha \,;\beta = \{\langle x, z\rangle \in X \times Z \mid \langle x, y\rangle \in \alpha, \langle y, z\rangle \in \beta$ for some $y \in Y\}$, and the converse of $\alpha \in \mathsf{Rel}(X, Y)$ is the set $\alpha^{\perp} = \{\langle y, x\rangle \in Y \times X \mid \langle x, y\rangle \in \alpha\}$. These operations interact with reindexing, *i.e.* inverse images, by the following inclusions: $\mathsf{d}_X \subseteq (f \times f)^{-1}(\mathsf{d}_Y)$ and $(f \times g)^{-1}(\alpha)\,;(g \times h)^{-1}(\beta) \subseteq (f \times h)^{-1}(\alpha\,;\beta)$ and also $((f \times g)^{-1}(\alpha))^{\perp} \subseteq (g \times f)^{-1}(\alpha^{\perp})$. The first two inclusions are not equalities in general: the former is an equality when f is injective, while the latter is an equality when g is surjective. These observations lead us to the following definition.

▶ **Definition 1.** *A* relational doctrine *consists of the following data:*

- *a base category $\mathcal{C}$,*
- *a functor $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{Pos}$,*
- *an element $\mathsf{d}_X \in R(X,X)$, for every object X in $\mathcal{C}$, such that $\mathsf{d}_X \leqslant R_{f,f}(\mathsf{d}_Y)$, for every arrow $f : X \to Y$ in $\mathcal{C}$,*
- *a monotone function $-;- : R(X,Y) \times R(Y,Z) \to R(X,Z)$, for every triple of objects X, Y, Z in $\mathcal{C}$, such that $R_{f,g}(\alpha)\,; R_{g,h}(\beta) \leqslant R_{f,h}(\alpha\,;\beta)$, for all $\alpha \in R(A,B)$, $\beta \in R(B,C)$ and $f : X \to A$, $g : Y \to B$ and $h : Z \to C$ arrows in $\mathcal{C}$,*
- *a monotone function $(-)^{\perp} : R(X,Y) \to R(Y,X)$, for every pair of objects X, Y in $\mathcal{C}$, such that $(R_{f,g}(\alpha))^{\perp} \leqslant R_{g,f}(\alpha^{\perp})$, for all $\alpha \in R(A,B)$ and $f : X \to A$ and $g : Y \to B$,*

satisfying the following equations for all $\alpha \in R(X,Y)$, $\beta \in R(Y,Z)$ and $\gamma \in R(Z,W)$

$$\alpha\,;(\beta\,;\gamma) = (\alpha\,;\beta)\,;\gamma \qquad \mathsf{d}_X\,;\alpha = \alpha \qquad \alpha\,;\mathsf{d}_Y = \alpha$$
$$(\alpha\,;\beta)^{\perp} = \beta^{\perp}\,;\alpha^{\perp} \qquad \mathsf{d}_X^{\perp} = \mathsf{d}_X \qquad \alpha^{\perp\perp} = \alpha$$

The element d_X is the *identity* or *diagonal* relation on X, $\alpha\,;\beta$ is the *relational composition* of α followed by β, and $\alpha^{\perp}$ is the *converse* of the relation α. Note that all relational operations are lax natural transformations, but the operation of taking the converse, being an involution, is actually strictly natural. Also, each one of the two axioms stating that d is the neutral element of the composition, together with the other axioms, implies the other.

▶ Remark 2. The data defining a relational doctrine $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{Pos}$ determine the following diagram in the category of doctrines and lax natural transformations, describing an *internal dagger category*:

$$\begin{tikzcd}
R^2 \arrow[r, "\langle\langle\pi_1,\pi_3\rangle,-\,;-\rangle"] & R \arrow[loop above, "\langle\langle\pi_2,\pi_1\rangle,(-)^{\perp}\rangle"] \arrow[r, bend left, "\langle\pi_2,\zeta\rangle"] \arrow[r, bend right, "\langle\pi_1,\zeta\rangle"'] & \mathbf{1}_{\mathcal{C}} \arrow[l, "\langle\Delta,\mathsf{d}\rangle"']
\end{tikzcd}$$

Here, $\mathbf{1}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{Pos}$ is the trivial doctrine, mapping every object of $\mathcal{C}$ to the singleton poset, ζ is the natural transformation whose components are the unique maps into the singleton poset, and R^2 is the pullback of $\langle\pi_1,\zeta\rangle$ against $\langle\pi_2,\zeta\rangle$, that is, the functor $R^2 : (\mathcal{C} \times \mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{Pos}$ defined by $R^2(X,Y,Z) = R(X,Y) \times R(Y,Z)$ and $R^2_{f,g,h} = R_{f,g} \times R_{g,h}$.

The following list of examples is meant to give a broad range of situations that can be described by relational doctrines. Order categories and existential elementary doctrines provide two large classes of examples which are intentionally omitted as, due to their relevance, they will be discussed separately in Section 5.

▶ **Example 3.**

1. Let $V = \langle |V|, \preceq, \cdot, 1\rangle$ be a commutative quantale. A *V-relation* [29] between sets X and Y is a function $\alpha : X \times Y \to |V|$, where $\alpha(x,y) \in |V|$ intuitively measures how much elements x and y are related by α. Then, we consider the functor $V\text{-}\mathsf{Rel} : (\mathcal{Set} \times \mathcal{Set})^{\mathrm{op}} \to \mathcal{Pos}$

where $V\text{-}\mathsf{Rel}(X,Y) = |V|^{X\times Y}$ is the set of V-relations from X to Y with the pointwise order, $V\text{-}\mathsf{Rel}_{f,g}$ is precomposition with $f \times g$ and The identity relation, composition and converse are defined as follows:

$$\mathsf{d}_X(x,x') = \begin{cases} 1 & x = x' \\ \bot & x \neq x' \end{cases} \qquad (\alpha\,;\beta)(x,z) = \bigvee_{y\in Y}(\alpha(x,y)\cdot\beta(y,z)) \qquad \alpha^{\perp}(y,x) = \alpha(x,y)$$

where $\alpha \in V\text{-}\mathsf{Rel}(X,Y)$ and $\beta \in V\text{-}\mathsf{Rel}(Y,Z)$. Special cases of this doctrine are $\mathsf{Rel} : (\mathcal{S}et \times \mathcal{S}et)^{\mathrm{op}} \to \mathcal{P}os$, when the quantale is $\mathbb{B} = \langle\{0,1\},\leqslant,\wedge,1\rangle$, and metric relations, when one considers the Lawvere's quantale $\mathbb{R}_{\geqslant 0} = \langle[0,\infty],\geqslant,+,0\rangle$ as in [38].

2. Let $R = \langle|R|,\preceq,+,\cdot,0,1\rangle$ be a continuous semiring [34, 48], that is, an ordered semiring where $\langle|R|,\preceq\rangle$ is a directed complete partial order (DCPO), 0 is the least element and $+$ and $\cdot$ are Scott-continuous functions. In this setting, we can compute sums of arbitrary arity. For a function $f : X \to |R|$, we can define its sum $\sum f$, also denoted by $\sum_{x\in X} f(x)$, as

$$\sum f = \bigvee_{I\in\mathcal{P}_\omega(X)} \sum_{i\in I} f(i)$$

where $\mathcal{P}_\omega(X)$ is the finite powerset of X. Consider $R\text{-}\mathsf{Mat} : (\mathcal{S}et \times \mathcal{S}et)^{\mathrm{op}} \to \mathcal{P}os$ where $R\text{-}\mathsf{Mat}(X,Y)$ is the set of functions $X \times Y \to |R|$ with the pointwise order, $\mathbb{R}\text{-}\mathsf{Mat}_{f,g}$ is precomposition with $f \times g$. Elements in $R\text{-}\mathsf{Mat}(X,Y)$ are a matrices with entries in $|R|$ and indices for rows and columns taken from X and Y. The identity relation, composition and converse are given by the Kronecker's delta (*i.e.* the identity matrix), matrix multiplication and transpose, defined as follows:

$$\mathsf{d}_X(x,x') = \begin{cases} 1 & x = x' \\ 0 & x \neq x' \end{cases} \qquad (\alpha\,;\beta)(x,z) = \sum_{y\in Y}(\alpha(x,y)\cdot\beta(y,z)) \qquad \alpha^{\perp}(y,x) = \alpha(x,y)$$

where $\alpha \in R\text{-}\mathsf{Mat}(X,Y)$ and $\beta \in R\text{-}\mathsf{Mat}(Y,Z)$. This relational doctrine generalises V-relations since any quantale is a continuous semiring (binary/arbitrary joins give addition/infinite sum). The paradigmatic example of a continuous semiring which is not a quantale is that of extended non-negative real numbers $[0,\infty]$, with the usual order, addition and multiplication. Restricting the base to finite sets all sums become finite, hence the definition works also for a plain ordered semiring.

3. Let $\mathcal{C}$ be a category with weak pullbacks. Denote by $\mathsf{Spn}^{\mathcal{C}}(X,Y)$ the poset reflection of the preorder whose objects are spans in $\mathcal{C}$ between X and Y and $X \xleftarrow{p_1} A \xrightarrow{p_2} Y \leqslant X \xleftarrow{q_1} B \xrightarrow{q_2} Y$ iff there is an arrow $f : A \to B$ such that $p_1 = q_1 \circ f$ and $p_2 = q_2 \circ f$. Given a span $\alpha = X \xleftarrow{p_1} A \xrightarrow{p_2} Y$ and arrows $f : X' \to X$ and $g : Y' \to Y$ in $\mathcal{C}$, define $\mathsf{Spn}_{f,g}(\alpha) \in \mathsf{Spn}^{\mathcal{C}}(X',Y')$ by one of the following equivalent diagrams:

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[ddrr] & & \\
 & W' \arrow[dl] \arrow[dr] \arrow[drrr, phantom, "wpb"] & & & \\
X' \arrow[dr, "f"'] \arrow[rr, phantom, "wpb"] & & A \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & & Y' \arrow[dl, "g"] \\
 & X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & W \arrow[ddll] \arrow[dr] & & \\
 & & & W' \arrow[dl] \arrow[dr] \arrow[dlll, phantom, "wpb"] & \\
X' \arrow[dr, "f"'] & & A \arrow[dl, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, phantom, "wpb"] & & Y' \arrow[dl, "g"] \\
 & X & & Y &
\end{tikzcd}$$

The functor $\mathsf{Spn}^{\mathcal{C}} : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ is a relational doctrine where, for $\alpha = X \xleftarrow{p_1} A \xrightarrow{p_2} Y$ and $\beta = X \xleftarrow{q_1} B \xrightarrow{q_2} Y$ it is

$$\mathsf{d}_X = \begin{tikzcd} & X \arrow[dl, "\mathsf{id}_X"'] \arrow[dr, "\mathsf{id}_X"] & \\ X & & X \end{tikzcd} \qquad \alpha \mathbin{;} \beta = \begin{tikzcd} & & W \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, phantom, "wpb"] & & B \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & Y & & Z \end{tikzcd} \qquad \alpha^{\perp} = \begin{tikzcd} & A \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ Y & & X \end{tikzcd}$$

One can do a similar construction for jointly monic spans, provided that the category $\mathcal{C}$ has strong pullbacks and a proper factorisation system. In particular, the relational doctrine of jointly monic spans over $\mathcal{S}et$ is the relational doctrine Rel of set-based relations already mentioned in Item 1.

4. Let $\mathcal{V}ec$ the category of vector spaces over real numbers and linear maps. Write $|X|$ for the underlying set of the vector space X. The functor $\mathsf{Vec} : (\mathcal{V}ec \times \mathcal{V}ec)^{\mathrm{op}} \to \mathcal{P}os$ sends X, Y to the suborder of $\mathbb{R}_{\geqslant 0}$-$\mathsf{Rel}(|X|, |Y|)$ on those α that are subadditive functions, *i.e.* $\alpha(\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}') \geqslant \alpha(\mathbf{x}, \mathbf{y}) + \alpha(\mathbf{x}', \mathbf{y}')$ and homogeneous, $\alpha(a\mathbf{x}, a\mathbf{y}) = |a|\alpha(\mathbf{x}, \mathbf{y})$. The functor Vec is a relational doctrine where

$$\mathsf{d}_X(\mathbf{x}, \mathbf{x}') = \begin{cases} 0 & \mathbf{x} = \mathbf{x}' \\ \infty & \mathbf{x} \neq \mathbf{x}' \end{cases} \quad (\alpha \mathbin{;} \beta)(\mathbf{x}, \mathbf{z}) = \inf_{\mathbf{y} \in |Y|} (\alpha(\mathbf{x}, \mathbf{y}) + \beta(\mathbf{y}, \mathbf{z})) \quad \alpha^{\perp}(\mathbf{y}, \mathbf{x}) = \alpha(\mathbf{x}, \mathbf{y})$$

5. Let $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ be a relational doctrine and $F : \mathcal{D} \to \mathcal{C}$ a functor. The change-of-base of R along F is the relational doctrine $F^{\star}R : (\mathcal{D} \times \mathcal{D})^{\mathrm{op}} \to \mathcal{P}os$ obtained precomposing R with $(F \times F)^{\mathrm{op}}$. The change of base allows to use relations of R to reason about the category $\mathcal{D}$. For example the forgetful functor $U : \mathcal{C} \to \mathcal{S}et$ of a concrete category $\mathcal{C}$ allows the use of set-theoretic relations to reason about $\mathcal{C}$, considering the doctrine $U^{\star}\mathsf{Rel}$ which maps a pair of objects X, Y in $\mathcal{C}$ to $\mathcal{P}(UX \times UY)$.

Let R be a relational doctrine on $\mathcal{C}$ and $\alpha \in R(X, Y)$ a relation, α is *functional* if $\alpha^{\perp} \mathbin{;} \alpha \leqslant \mathsf{d}_Y$, *total* if $\mathsf{d}_X \leqslant \alpha \mathbin{;} \alpha^{\perp}$, *injective* if $\alpha \mathbin{;} \alpha^{\perp} \leqslant \mathsf{d}_X$, and *surjective* if $\mathsf{d}_Y \leqslant \alpha^{\perp} \mathbin{;} \alpha$. The next proposition shows that functional and total relations are discretely ordered.

▶ **Proposition 4.** *For functional and total relations $\alpha, \beta \in R(X, Y)$ if $\alpha \leqslant \beta$, then $\alpha = \beta$.*

Every arrow $f : X \to Y$ defines a relation $\Gamma_f = R_{f, \mathsf{id}_Y}(\mathsf{d}_Y) \in R(X, Y)$, called the *graph* of f whose converse is given by $\Gamma_f^{\perp} = R_{f, \mathsf{id}_Y}(\mathsf{d}_Y)^{\perp} = R_{\mathsf{id}_Y, f}(\mathsf{d}_Y^{\perp}) = R_{\mathsf{id}_Y, f}(\mathsf{d}_Y)$.

▶ **Proposition 5.** *Let $f : X \to Y$ an arrow in $\mathcal{C}$. Then, Γ_f is functional and total.*

Relational composition allows us to express reindexing in relational terms and to show it has left adjoints, as proved below. Recall that in $\mathcal{P}os$ a left adjoint of a monotone function $g : K \to H$ is a monotone function $f : H \to K$ such that for every x in K and y in H, both $y \leqslant gf(y)$ and $fg(x) \leqslant x$ hold, or, equivalently, $y \leqslant g(x)$ if and only if $f(y) \leqslant x$.

▶ **Proposition 6.** *For $f : A \to X$ and $g : B \to Y$ in $\mathcal{C}$ the reindexing $R_{f,g} : R(X, Y) \to R(A, B)$ has a left adjoint $\exists^R_{f,g} : R(A, B) \to R(X, Y)$ and for $\alpha \in R(X, Y)$ and $\beta \in R(A, B)$*

$$R_{f,g}(\alpha) = \Gamma_f \mathbin{;} \alpha \mathbin{;} \Gamma_g^{\perp} \qquad \exists^R_{f,g}(\beta) = \Gamma_f^{\perp} \mathbin{;} \beta \mathbin{;} \Gamma_g$$

We conclude the section describing the 2-category **RD** of relational doctrines. Objects are relational doctrines, while a 1-arrow $F : R \to S$, where $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ and $S : (\mathcal{D} \times \mathcal{D})^{\mathrm{op}} \to \mathcal{P}os$, is a pair $\langle \widehat{F}, \overline{F} \rangle$ consisting of a functor $\widehat{F} : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\overline{F} : R \xrightarrow{\cdot} S \circ (\widehat{F} \times \widehat{F})^{\mathrm{op}}$, laxly preserving relational identities, composition and converse, that is, satisfying $\mathsf{d}_{\widehat{F}X} \leqslant \overline{F}_{X,X}(\mathsf{d}_X)$ and $\overline{F}_{X,Y}(\alpha)\,;\overline{F}_{Y,Z}(\beta) \leqslant \overline{F}_{X,Z}(\alpha\,;\beta)$ and $(\overline{F}_{X,Y}(\alpha))^{\perp} \leqslant \overline{F}_{Y,X}(\alpha^{\perp})$, for $\alpha \in R(X,Y)$ and $\beta \in R(Y,Z)$. A 2-arrow $\theta : F \Rightarrow G$ is a natural transformation $\theta : \widehat{F} \xrightarrow{\cdot} \widehat{G}$ such that $\overline{F}_{X,Y} \leqslant S_{\theta_X,\theta_Y} \circ \overline{G}_{X,Y}$, for all objects X, Y in the base of R. By Propositions 5 and 6 the condition of a 2-arrow $\theta : F \Rightarrow G$ is equivalent to both $\overline{F}_{X,Y}(\alpha) \leqslant \Gamma_{\theta_X}\,;\overline{G}_{X,Y}(\alpha)\,;\Gamma^{\perp}_{\theta_Y}$ and $\overline{F}_{X,Y}(\alpha)\,;\Gamma_{\theta_Y} \leqslant \Gamma_{\theta_X}\,;\overline{G}_{X,Y}(\alpha)$, for $\alpha \in R(X,Y)$.

It is easy to see that 1-arrows actually strictly preserve the converse, since it is an involution, and laxly preserve graphs of arrows, that is, $\Gamma_{\widehat{F}f} \leqslant \overline{F}_{X,Y}(\Gamma_f)$ and $\Gamma^{\perp}_{\widehat{F}f} \leqslant \overline{F}_{Y,X}(\Gamma^{\perp}_f)$, for every arrow $f : X \to Y$ in the base of R. A 1-arrow is called *strict* if it strictly preserves relational identities and composition. In this case, it also strictly preserves graphs of arrows. We denote by $\mathbf{RD_s}$ tns the 2-full 2-subcategory of $\mathbf{RD}$ where 1-arrows are strict.

▶ **Example 7** (Relation lifting)**.** A key notion used in relational methods is that of *relation lifting* or *lax extension* or *relator* [6, 32, 56]. It can be used to formulate bisimulation for coalgebras or other notions of program equivalence. A (conversive) relation lifting of a functor $F : \mathcal{S}et \to \mathcal{S}et$ is a family of monotonic maps $\overline{F}_{X,Y} : \mathsf{Rel}(X,Y) \to \mathsf{Rel}(FX,FY)$, indexed by sets X and Y, such that $\overline{F}_{X;Y}(\alpha)^{\perp} \subseteq \overline{F}_{Y,X}(\alpha^{\perp})$, $\overline{F}_{X,Y}(\alpha)\,;\overline{F}_{Y,Z}(\beta) \subseteq \overline{F}_{X,Z}(\alpha\,;\beta)$ and $Ff \subseteq \overline{F}_{X,Y}(f)$, where α and β are relations and $f : X \to Y$ is a function. Note that in the last condition we are using the function to denote its graph, which is perfectly fine since set-theoretic functions coincide with their graph. It is easy to see that these requirements ensure that $\langle F, \overline{F}\rangle : \mathsf{Rel} \to \mathsf{Rel}$ is a 1-arrow in $\mathbf{RD}$. Conversely any 1-arrow $G : \mathsf{Rel} \to \mathsf{Rel}$ is such that $\overline{G}$ is a relation lifting of $\widehat{G}$, showing that 1-arrows between Rel and Rel are exactly the relation liftings. Hence, 1-arrows of the form $F : R \to R$ in $\mathbf{RD}$ can be regarded as a generalisation of relation lifting to an arbitrary relational doctrine R.

Finally, relying on the 2-categorical structure of $\mathbf{RD}$, we get for free a notion of monad on a relational doctrine. A monad consists of a 1-arrow $T : R \to R$ together with 2-arrows $\eta : \mathsf{Id}_R \Rightarrow T$ and $\mu : T \circ T \Rightarrow T$ satisfying usual diagrams:

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathsf{id}"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathsf{id}"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

Thanks to the conditions that 2-arrows in $\mathbf{RD}$ have to satisfy, such monads capture precisely the notion of monadic relation lifting used to reason about effectful programs [26]. Similarly, comonads in $\mathbf{RD}$ abstracts comonadic relation liftings [19].

▶ **Example 8.** Recall V-$\mathsf{Rel} : (\mathcal{S}et \times \mathcal{S}et)^{\mathrm{op}} \to \mathcal{P}os$ the doctrine of V-relations from Example 3(1). Consider the 1-arrow $P : V\text{-}\mathsf{Rel} \to V\text{-}\mathsf{Rel}$ where $\widehat{P} : \mathcal{S}et \to \mathcal{S}et$ is the covariant powerset functor and $\overline{P}_{X,Y} : V\text{-}\mathsf{Rel}(X,Y) \to V\text{-}\mathsf{Rel}(\widehat{P}X, \widehat{P}Y)$ maps a V-relation α to the function $\overline{P}_{X,Y}(\alpha)(A,B) = h_\alpha(A,B) \wedge h_{\alpha^{\perp}}(B,A)$ where $\wedge$ denotes the binary meet operation in V and for every $\beta : Z \times W \to |V|$, we set

$$h_\beta(A,B) = \bigwedge_{x \in A} \bigvee_{y \in B} \beta(x,y) \qquad\qquad \text{for } A \subseteq Z \text{ and } B \subseteq W$$

It is easy to check that this is indeed a 1-arrow. In particular, when considering the boolean quantale $\mathbb{B}$, given $\alpha : X \times Y \to \{0,1\}$ we have that $\overline{P}_{X,Y}(\alpha)$ relates A and B iff for all $x \in A$, there is $y \in B$ s.t. $\alpha(x,y) = 1$ and viceversa; considering instead Lawvere's quantale $\mathbb{R}_{\geqslant 0}$, $\overline{P}_{X,Y}(\alpha)$ is a generalisation to arbitrary $\mathbb{R}_{\geqslant 0}$-relations of the Hausdorff pseudometric on subsets of (pseudo)metric spaces.

▶ **Example 9.** Let Ω be a signature of function symbols with finite arity. Denote by Ω_X the signature obtained from Ω by adding a constant symbol for every element in X. Write $\widehat{T_\Omega}X$ for the set of closed Ω_X-terms. It is known that $\widehat{T_\Omega}$ extends to a monad on $\mathcal{S}et$. Consider the doctrine V-Rel of V-relations (cf. Example 3(1)). Every V-relation $\alpha \in V\text{-}\mathsf{Rel}(X,Y)$ can be extended to a V-relation $\alpha^\star \in V\text{-}\mathsf{Rel}(\widehat{T_\Omega}X, \widehat{T_\Omega}Y)$ by induction on the structure of terms: $\alpha^\star(x,) = \alpha(x,y)$, if $x \in X$ and $y \in Y$, $\alpha^\star(f(t_1,\ldots,t_n), f(s_1,\ldots,s_n)) = \bigwedge_{i \in 1..n} \alpha^\star(t_i,s_i)$, if f is an n-ary symbol of Ω, and $\alpha^\star(t,s) = \bot$, otherwise. We set $\overline{T_\Omega}_{X,Y}(\alpha) = \alpha^\star$. Then, it is not difficult to see that $T_\Omega : \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel} \to \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}$ is a monad in $\mathbf{RD}$.

▶ **Example 10** (Bisimulations)**.** We can express the notion of bisimulation for coalgebras in an arbitrary relational doctrine, thus covering both usual and quantitative versions of bisimulation. If $F : R \to R$ is a 1-arrow in $\mathbf{RD}$ and $\langle X, c\rangle$ and $\langle Y, d\rangle$ two $\widehat{F}$-coalgebras, then a relation $\alpha \in R(X,Y)$ is a F-bisimulation from $\langle X, c\rangle$ to $\langle Y, d\rangle$ if $\alpha \leqslant \Gamma_c \,;\, \overline{F}_{X,Y}(\alpha)\,;\, \Gamma_d^\perp$ or, equivalently, $\alpha\,;\,\Gamma_d \leqslant \Gamma_c\,;\,\overline{F}_{X,Y}(\alpha)$. This means that α has to agree with the dynamics of the two coalgebras. Indeed, if R is Rel (the doctrine of set-theoretic relations), this condition states that, if $x \in X$ is related to $y \in Y$ by α and y evolves to $B \in \widehat{F}Y$ through d, then x evolves to some $A \in \widehat{F}X$ through c and A is related to B by the lifted relation $\overline{F}_{X,Y}(\alpha)$. This definition looks very much like that of simulation, but, since 1-arrows preserve the converse, it is easy to check that, if α is a bisimulation, then $\alpha^\perp$ is a bisimulation as well, thus justifying the name. Furthermore, one can easily check that F-bisimulations are closed under relational identities and composition. Then, the category of $\widehat{F}$-coalgebras is the base of a relational doctrine bisim^F where relations in $\mathsf{bisim}^F(\langle X,c\rangle, \langle Y,d\rangle)$ are F-bisimulations between coalgebras $\langle X, c\rangle$ and $\langle Y, d\rangle$.

As a concrete example, let us consider the 1-arrow $P : V\text{-}\mathsf{Rel} \to V\text{-}\mathsf{Rel}$ of Example 8. A $\widehat{P}$-coalgebra is a usual (non-deterministic) transition system and a P-bisimulation from $\langle X, c\rangle$ to $\langle Y, d\rangle$ is a V-relation $\alpha : X \times Y \to |V|$ such that $\alpha(x,y) \preceq h_\alpha(c(x), d(y)) \wedge h_\alpha(d(y), c(x))$, for all $x \in X$ and $y \in Y$. Roughly, this means that similar states reduce to similar states. When considering the boolean quantale $\mathbb{B}$, we get the usual notion of bisimulation, while considering Lawvere's quantale $\mathbb{R}_{\geqslant 0}$ we get a form of metric bisimulation.

▶ **Example 11** (Barr lifting)**.** Let $\mathcal{C}$ be a category with weak pullbacks and $F : \mathcal{C} \to \mathcal{C}$ a weak pullbacks preserving functor. It induces a strict 1-arrow $\langle F, \overline{F}\rangle : \mathsf{Spn}^{\mathcal{C}} \to \mathsf{Spn}^{\mathcal{C}}$ mapping a span $X \xleftarrow{p_1} A \xrightarrow{o_2} Y$ to $FX \xleftarrow{Fp_1} FA \xrightarrow{Fp_2} FY$. This provides an abstract version of the well-known Barr lifting for set-theoretic relations. It is easy to see that this construction extends to a 2-functor Spn^- from the 2-category of categories with weak pullbacks, functor preserving them and natural transformations to the 2-category $\mathbf{RD}$. Hence, every weak pullbacks preserving monad on a category $\mathcal{C}$ with weak pullbacks, induces a monad on $\mathsf{Spn}^{\mathcal{C}}$.

3 The Relational Quotient Completion

Here we show how one can deal with quotients in relational doctrines extending the quotient completion in [41, 42] which we used as inspiration for many notions and constructions. We present instances having a quantitative flavour that usual doctrines do not cover, showing that quotients are the key structure characterising them.

In a relational doctrine $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ an R-*equivalence relation* on an object X in $\mathcal{C}$ is a relation $\rho \in R(X,X)$ satisfying the following properties:

$$\text{reflexivity: } \mathsf{d}_X \leqslant \rho \qquad \text{symmetry: } \rho^\perp \leqslant \rho \qquad \text{transitivity: } \rho\,;\rho \leqslant \rho$$

▶ **Example 12.**

1. In the doctrine of V-relations V-Rel (cf. Example 3(1)), an equivalence relation $\rho : X \times X \to |V|$ on a set X is a (symmetric) V-metric [29]: reflexivity is $1 \preceq \rho(x,x)$, for all $x \in X$, symmetry is $\rho(x,y) \preceq \rho(y,x)$, for all $x,y \in X$, and transitivity is $\bigvee_{y\in X} \rho(x,y) \cdot \rho(y,z) \preceq \rho(x,z)$, which is equivalent to $\rho(x,y) \cdot \rho(y,z) \preceq \rho(x,z)$, for all $x,y,z \in X$, by properties of suprema. For the boolean quantale $\mathbb{B}$ these are usual equivalence relations, while for the Lawvere's quantale $\mathbb{R}_{\geqslant 0}$ these are the so-called pseudometrics as the transitivity property is exactly the triangular inequality.
2. In the doctrine $\mathsf{Spn}^{\mathcal{C}}$ (cf. Example 3(3)) of spans in a category with weak pullbacks, an equivalence relation on X is a pair of parallels arrows $r_1, r_2 : A \to X$ such that there are arrows $r : X \to A$ with $r_1 r = r_2 r = \mathsf{id}_X$ (reflexivity), $s : A \to A$ with $r_1 s = r_2$ and $r_2 s = r_1$ (symmetry), and $t : W \to A$ with $r_1 t = r_1 d_1$ and $r_2 t = r_2 d_2$ where

$$\begin{tikzcd} & W \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ A \arrow[dr, "r_2"'] & wpb & A \arrow[dl, "r_1"] \\ & X & \end{tikzcd}$$

   is a weak pullback. These spans are the pseudo-equivalence relations of [14, 15].
3. In the relational doctrine $\mathsf{Vec} : (\mathcal{Vec} \times \mathcal{Vec})^{\mathrm{op}} \to \mathcal{Pos}$ (cf. Example 3(4)) an equivalence relation over a vector space X is a subadditive and homogeneous function $\rho : |X| \times |X| \to [0,\infty]$ such that $\rho(\mathbf{x},\mathbf{x}) = 0$ as reflexivity suffices to get symmetry and transitivity. Indeed one can prove that $\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{0},\mathbf{y}-\mathbf{x})$. Symmetry follows from $\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{0},\mathbf{y}-\mathbf{x}) = |-1|\rho(\mathbf{0},\mathbf{x}-\mathbf{y}) = \rho(\mathbf{y},\mathbf{x})$ and transitivity from $\rho(\mathbf{x},\mathbf{y})+\rho(\mathbf{y},\mathbf{z}) = \rho(\mathbf{0},\mathbf{y}-\mathbf{x})+\rho(\mathbf{0},\mathbf{z}-\mathbf{y}) \geqslant \rho(\mathbf{0}+\mathbf{0},(\mathbf{y}-\mathbf{x})+(\mathbf{z}-\mathbf{y})) = \rho(\mathbf{0},\mathbf{z}-\mathbf{x}) = \rho(\mathbf{x},\mathbf{z})$. Hence a Vec-equivalence on a vector space X is a subadditive and homogeneous pseudometric on it.

Every arrow $f : X \to Y$ in $\mathcal{C}$ induces a R-equivalence relation on X, dubbed *kernel* of f, given by $\Gamma_f \,;\Gamma_f^{\perp}$. The fact that this is an equivalence follows immediately since Γ_f is a total and functional relation. Roughly, the kernel of f relates those elements which are identified by f; indeed, for the relational doctrine $\mathsf{Rel} : (\mathcal{Set} \times \mathcal{Set})^{\mathrm{op}} \to \mathcal{Pos}$ it is defined exactly in this way. Kernels are crucial to talk about quotients as the following definition shows.

▶ **Definition 13.** *Let $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{Pos}$ be a relational doctrine on $\mathcal{C}$ and ρ a R-equivalence relation on an object X in $\mathcal{C}$. A* quotient arrow *of ρ is an arrow $q : X \to W$ in $\mathcal{C}$ such that $\rho \leqslant \Gamma_q \,;\Gamma_q^{\perp}$ and, for every arrow $f : X \to Z$ with $\rho \leqslant \Gamma_f \,;\Gamma_f^{\perp}$, there is a unique arrow $h : W \to Z$ such that $f = h \circ q$. The quotient arrow q is* effective *if $\rho = \Gamma_q \,;\Gamma_q^{\perp}$ and it is* descent *if $\mathsf{d}_W \leqslant \Gamma_q^{\perp} \,;\Gamma_q$.*

We say that R has quotients *if every R-equivalence relation admits an effective descent quotient arrow.*

Intuitively, a quotient of ρ is the "smallest" arrow q which transforms the equivalence ρ into the relational identity, that is, such that ρ is smaller than the kernel of q. The quotient q is effective when its kernel $\Gamma_q \,;\Gamma_q^{\perp}$ coincides with the equivalence relation ρ and it is descent when its graph is surjective.

▶ **Example 14.** To exemplify the definition above, let us unfold it for the relational doctrine $\mathsf{Rel} : (\mathcal{Set} \times \mathcal{Set})^{\mathrm{op}} \to \mathcal{Pos}$, which has quotients. Recall from Example 12(1) that a Rel-equivalence is just a usual equivalence relation. Here, a quotient arrow for an equivalence relation ρ on a set X is a function $q : X \to W$ which is universal among those functions f

whose kernel includes the equivalence ρ, that is, such that $\rho(x,x')$ implies $f(x) = f(x')$. Effectiveness requires the converse inclusion, *i.e.* $q(x) = q(x')$ implies $\rho(x,x')$. Finally, the descent condition amounts to requiring q to be surjective in the usual sense. A choice for such a function q is the usual quotient projection from X to the set X/ρ of ρ-equivalence classes, which maps $x \in X$ to its equivalence class $[x]$. Indeed, by definition this function is surjective and $\rho(x,x')$ holds iff $[x] = [x']$. Moreover, for every function f such that $\rho(x,x')$ implies $f(x) = f(x')$, the function $[x] \mapsto f(x)$ turns out to be well-defined, proving that the quotient projection is universal.

▶ **Example 15.** Consider the relational doctrine $\mathbb{R}_{\geqslant 0}$-Rel of $\mathbb{R}_{\geqslant 0}$-relations where $\mathbb{R}_{\geqslant 0}$ is the Lawvere's quantale $\langle [0,\infty], \geqslant, +, 0\rangle$ (cf. Example 3(1)) and suppose $\rho : X \times X \to [0,\infty]$ is a $\mathbb{R}_{\geqslant 0}$-Rel-equivalence relation , *i.e.* a pseudometric on X (cf. Example 12(1)). Define an equivalence relation on X setting $x \sim_\rho y$ whenever $\rho(x,x') \neq \infty$, that is, when x and x' are connected. The canonical surjection $q : X \to X/\sim$ mapping x to $q(x) = [x]$ is a quotient arrow for ρ. It is immediate to see that $\rho(x,x') \geqslant \mathsf{d}_{X/\sim_\rho}([x],[x'])$ as $\mathsf{d}_X([x],[x'])$ is either 0 or ∞ and $\mathsf{d}_{X/\sim_\rho}([x],[x']) = \infty$ precisely when x and x' are not connected, that is, when $\rho(x,x') = \infty$. The universality of q easily follows from its universal property as a quotient of $\sim_\rho$ in Rel (cf. Example 14). This shows that $\mathbb{R}_{\geqslant 0}$-Rel has quotient arrows for all pseudometrics, which are descent: for $q : X \to X/\sim_\rho$ a quotient of ρ, the descent condition becomes $\mathsf{d}_{X/\sim_\rho}(y,y') \geqslant \inf_{x\in X}\big(\mathsf{d}_{X/\sim_\rho}(y,q(x)) + \mathsf{d}_{X/\sim_\rho}(q(x),y')\big)$, which trivially holds since q is surjective and $\mathsf{d}_{X/\sim_\rho}$ is either 0 or ∞. However, such quotients arrows cannot be effective. Indeed, if $f : X \to Y$ is a function, since the relational identity d_Y is only either 0 or ∞, the kernel of f is given by $x,x' \mapsto \mathsf{d}_Y(f(x), f(x'))$, thus it takes values in $\{0,\infty\}$. Hence, if a quotient arrow q for a pseudometric ρ was effective, then ρ would be either 0 or ∞, as it would coincide with the kernel of q and clearly this is not the case in general. This shows that $\mathbb{R}_{\geqslant 0}$-Rel has not quotients in the sense of Definition 13.

Example 15 shows that relational doctrines need not have quotients in general. Hence, we now describe a free construction that takes a relational doctrine $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ and builds a new one $(R)^q : (\mathcal{Q}_R \times \mathcal{Q}_R)^{\mathrm{op}} \to \mathcal{P}os$ which has (effective descent) quotients for all equivalence relations. The construction is inspired by the quotient completion in [41, 42] and a comparison with it is delayed to Section 5.

The category $\mathcal{Q}_R$ is defined as follows:

- an object is a pair $\langle X, \rho\rangle$, where X is an object in $\mathcal{C}$ and ρ is a R-equivalence relation on X,
- an arrow $f : \langle X,\rho\rangle \to \langle Y,\sigma\rangle$ is an arrow $f : X \to Y$ in $\mathcal{C}$ such that $\rho \leqslant R_{f,f}(\sigma)$, and
- composition and identities are those of $\mathcal{C}$.

By Proposition 6 the condition $\rho \leqslant R_{f,f}(\sigma)$ is equivalent to both $\rho \leqslant \Gamma_f \,;\sigma\,;\Gamma_f^\perp$ and $\Gamma_f^\perp\,;\rho\,;\Gamma_f \leqslant \sigma$.

Given R-equivalence relations ρ and σ over X and Y the suborder $\mathcal{D}es_{\rho,\sigma}(X,Y)$ of $R(X,Y)$ of *descent data* with respect to ρ and σ is defined by

$$\mathcal{D}es_{\rho,\sigma}(X,Y) = \{\alpha \in R(X,Y) \mid \rho^\perp\,;\alpha\,;\sigma \leqslant \alpha\}$$

Roughly, a descent datum is a relation which is closed w.r.t. ρ on the left and σ on the right. For every arrow $f : \langle X,\rho\rangle \to \langle X',\rho'\rangle$ and $g : \langle Y,\sigma\rangle \to \langle Y',\sigma'\rangle$ in $\mathcal{Q}_R$, the monotone function $R_{f,g} : R(X',Y') \to R(X,Y)$ applies $\mathcal{D}es_{\rho',\sigma'}(X',Y')$ into $\mathcal{D}es_{\rho,\sigma}(X,Y)$ as Indeed, for $\alpha \in \mathcal{D}es_{\rho',\sigma'}(X',Y')$, we have

$$\rho^\perp\,;R_{f,g}(\alpha)\,;\sigma \leqslant R_{f,f}(\rho'^\perp)\,;R_{f,g}(\alpha)\,;R_{g,g}(\sigma') \leqslant R_{f,g}(\rho'^\perp\,;\alpha\,;\sigma') \leqslant R_{f,g}(\alpha)$$

Therefore the assignments $(R)^q(\langle X,\rho\rangle,\langle Y,\sigma\rangle) = \mathcal{Des}_{\rho,\sigma}(X,Y)$ and $(R)^q_{f,g} = R_{f,g}$ determine a functor $(R)^q : (\mathcal{Q}_R \times \mathcal{Q}_R)^{\mathrm{op}} \to \mathcal{Pos}$.

▶ **Proposition 16.** *The functor $(R)^q : (\mathcal{Q}_R \times \mathcal{Q}_R)^{\mathrm{op}} \to \mathcal{Pos}$ is a relational doctrine, where composition and converse are those of R and $\mathsf{d}_{\langle X,\rho\rangle} = \rho$.*

A $(R)^q$-equivalence relation over an object $\langle X,\rho\rangle$ is a R-equivalence σ over X such that $\rho \leqslant \sigma$. Note that these conditions imply that σ is a descent datum in $\mathcal{Des}_{\rho,\rho}(X,X)$. Then, $\langle X,\sigma\rangle$ is an object of $\mathcal{Q}_R$ and $\mathsf{id}_X : \langle X,\rho\rangle \to \langle X,\sigma\rangle$ is a well-defined arrow in $\mathcal{Q}_R$, which turns out to be an effective descent quotient arrow for σ. In this way we construct quotient arrows for all $(R)^q$-equivalence relations, thus obtaining the following result.

▶ **Proposition 17.** *The relational doctrine $(R)^q$ over $\mathcal{Q}_R$ has effective descent quotients.*

▶ **Example 18.**

1. For the doctrine V-Rel of V-relations, the category $\mathcal{Q}_{V\text{-}\mathsf{Rel}}$ is the category of V-metric spaces with non-expansive maps. By Example 12(1), an object $\langle X,\rho\rangle$ is a V-metric space and $f : \langle X,\rho\rangle \to \langle Y,\sigma\rangle$ has to satisfy $\rho(x,x') \le \sigma(f(x),f(x'))$.
2. For the relational doctrine Vec over the category of real vector spaces, $\mathcal{Q}_{\mathsf{Vec}}$ is the category of semi-normed vector spaces with short maps. An object $\langle X,\rho\rangle$ in $\mathcal{Q}_{\mathsf{Vec}}$ is a vector space with a subadditive and homogeneous pseudometric on it. Such a pseudometric satisfies $\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{0},\mathbf{y}-\mathbf{x})$ (see Example 12(3)), so $\|\mathbf{x}\| = \rho(\mathbf{0},\mathbf{x})$ defines a semi-norm on X.

Following Lawvere's structural approach to logic, we can characterise the property of having effective descent quotients by an adjunction in $\mathbf{RD}$. First observe that the doctrine R is embedded into $(R)^q$ by the 1-arrow $E^R : R \to (R)^q$ in $\mathbf{RD}$ defined as follows: the functor $\widehat{E^R} : \mathcal{C} \to \mathcal{Q}_R$ maps $f : X \to Y$ in $\mathcal{C}$ to $f : \langle X,\mathsf{d}_X\rangle \to \langle Y,\mathsf{d}_Y\rangle$; the natural transformation $\overline{E^R} : R \xrightarrow{\cdot} (R)^q \circ (\widehat{E^R} \times \widehat{E^R})^{\mathrm{op}}$ is the family of identities $R(X,Y) = \mathcal{Des}_{\mathsf{d}_X,\mathsf{d}_Y}(X,Y)$. The 1-arrow E^R shows that constructing $(R)^q$ "extends" R adding (effective descent) quotients for any equivalence relation.

▶ **Lemma 19.** *A relational doctrine R has effective descent quotients if and only if E^R has a strict reflection left adjoint $F : (R)^q \to R$.*

This means that the 1-arrow $F : (R)^q \to R$ is strict and it is a left adjoint of E^R in $\mathbf{RD}$ and the counit of this adjunction is an isomorphism, hence $F \circ E^R \cong \mathsf{Id}_R$. Intuitively, the 1-arrow $F : (R)^q \to R$ computes quotients of R-equivalence relations: the object $\widehat{F}\langle X,\rho\rangle$ is the codomain of a quotient arrow obtained by applying $\widehat{F}$ to $\mathsf{id}_X : \langle X,\mathsf{d}_X\rangle \to \langle X,\rho\rangle$ which is the quotient arrow of ρ in $(R)^q$ viewed as a $(R)^q$-equivalence over $\langle X,\mathsf{d}_X\rangle$.

The construction of $(R)^q$ is universal as it is part of a lax 2-adjunction [10]. To show this, we first introduce the 2-category $\mathbf{QRD}$ as the 2-full 2-subcategory of $\mathbf{RD}$ whose objects are relational doctrines with quotients and whose 1-arrows are those of $\mathbf{RD}$ that preserve quotient arrows, *i.e.* 1-arrows $F : R \to S$ in $\mathbf{RD}$ mapping a quotient arrow for a R-equivalence ρ over X to a quotient arrow for $\overline{F}_{X,X}(\rho)$, which can be easily proved to be a S-equivalence over $\widehat{F}X$. There is an obvious inclusion 2-functor $\mathrm{U_q} : \mathbf{QRD} \to \mathbf{RD}$ which simply forgets quotients. Moreover, the construction above determines a 2-functor $\mathrm{Q} : \mathbf{RD} \to \mathbf{QRD}$, defined as follows: for a 1-arrow $F : R \to S$ in $\mathbf{RD}$, the 1-arrow $\mathrm{Q}(F) = (F)^q : (R)^q \to (S)^q$ is given by $\widehat{(F)^q}\langle X,\rho\rangle = \langle \widehat{F}X, \overline{F}_{X,X}(\rho)\rangle$ and $\widehat{(F)^q}f = \widehat{F}f$ and $\overline{(F)^q}_{\langle X,\rho\rangle,\langle Y,\sigma\rangle}(\alpha) = \overline{F}_{X,Y}(\alpha)$, and for a 2-arrow $\theta : F \Rightarrow G$ in $\mathbf{RD}$, the 2-arrow $\mathrm{Q}(\theta) = (\theta)^q : (F)^q \Rightarrow (G)^q$ is given by $(\theta)^q_{\langle X\rho\rangle} = \theta_X$.

▶ **Theorem 20.** *The 2-functors Q and $\mathrm{U_q}$ are such that $\mathrm{Q} \dashv_l \mathrm{U_q}$ is a lax 2-adjunction.*

This means that, for every relational doctrine R and every relational doctrine with quotients S, the functor

$$\mathrm{U_q}(-) \circ E^R : \mathbf{QRD}((R)^q, S) \to \mathbf{RD}(R, \mathrm{U_q}(S)) \tag{1}$$

has a left adjoint.

▶ **Example 21.** Let R be a relational doctrine with quotients and $F : R \to R$ be a 1-arrow in $\mathbf{QRD}$, that is, it preserves quotient arrows. Recall from Example 10 the doctrine bisim^F on the category $\mathcal{CoAlg}(\widehat{F})$ of $\widehat{F}$-coalgebras, where relations between coalgebras are F-bisimulations. It is easy to see that bisim^F has quotients. Indeed, a bisim^F-equivalence relation ρ on a $\widehat{F}$-coalgebra $\langle X, c\rangle$ is an F-bisimulation which is also a R-equivalence relaiton on X. Since R has quotients, ρ admits an effective descent quotient arrow $q : X \to W$ in the base of R. To conclude, it suffices to endow W with an $\widehat{F}$-coalgebra structure, making q an $\widehat{F}$-coalgebra homomorphism. To this end, note that, since ρ is a F-bisimulation and $\widehat{F}q$ is a quotient arrow for $\overline{F}_{X,X}(\rho)$, we get $\rho \leqslant \Gamma_{\widehat{F}q\circ c} ; \Gamma^{\perp}_{\widehat{F}q\circ c}$. Thus by the universal property of quotients, we get a unique arrow $c_\rho : W \to \widehat{F}W$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "c"'] & W \arrow[d, dotted, "c_\rho"] \\
\widehat{F}X \arrow[r, "\widehat{F}q"] & \widehat{F}W
\end{tikzcd}$$

This shows that the doctrine of F bisimulations inherits quotients, provided that F preserves them. If however quotients are not available in R and/or F does not preserve them, we can use the relational quotient completion to freely add them to bisim^F. In this way, we get the doctrine $(\mathsf{bisim}^F)^q$ whose base category has as objects triple $\langle X, c, \rho\rangle$ where $\langle X, c\rangle$ is an $\widehat{F}$-coalgebra and ρ is an F-bisimulation equivalence on it. Notice that, applying Q to the 1-arrow F, we get a 1-arrow $(F)^q : (R)^q \to (R)^q$. Then, we can construct the doctrine $\mathsf{bisim}^{(F)^q}$ of $(F)^q$-bisimulations. It is easy to check that $(\mathsf{bisim}^F)^q$ is isomorphic to $\mathsf{bisim}^{(F)^q}$, that is, the costruction of coalgebras commutes with the quotient completion.

▶ **Example 22.** Let Ω be a signature of function symbols with finite arity. Recall from Example 9 that we have the monad $T_\Omega : V\text{-}\mathsf{Rel} \to V\text{-}\mathsf{Rel}$ of terms over Ω. Applying the relational quotient completion, since it is a 2-functor, we get a monad $(T_\Omega)^q : (V\text{-}\mathsf{Rel})^q \to (V\text{-}\mathsf{Rel})^q$. In particular, we get a monad $\widehat{(T_\Omega)^q} : \mathcal{Q}_{V\text{-}\mathsf{Rel}} \to \mathcal{Q}_{V\text{-}\mathsf{Rel}}$ on the category of V-metric spaces (the base of $(V\text{-}\mathsf{Rel})^q$), which is a slight generalisation of the free monad for quantitative algebras over Ω described in [1, 2].

We conjecture that a similar construction should be possible also when considering a Quantitative Equational Theory [45, 46, 4, 1] over Ω, extending the construction in that papers to V-Rel. However, this is still an open problem, we leave for future work.

The 2-adjunction of Theorem 20, being lax, establishes a weak correspondence between $\mathbf{RD}$ and $\mathbf{QRD}$: between their hom-categories there is neither an isomorphism, nor an equivalence, but just an adjunction. Moreover, the family of 1-arrows E^R is only a lax natural transformation. This is essentially due to the fact that 1-arrows of $\mathbf{RD}$ and $\mathbf{QRD}$ laxly preserve relational operations, in particular, relational identities. Hence, a way to recover a stronger correspondence may be to restrict to strict 1-arrows.

Denote by $\mathbf{QRD_s}$ the 2-full 2-subcategory of $\mathbf{QRD}$ whose 1-arrows are strict. Then, it is easy to see that Q applies $\mathbf{RD_s}$ into $\mathbf{QRD_s}$, obtaining the following result.

▶ **Theorem 23.** *The lax 2-adjunction* $\mathrm{Q} \dashv_l \mathrm{U_q}$ *restricts to a (pseudo) 2-adjunction between* $\mathbf{QRD_s}$ *and* $\mathbf{RD_s}$*.*

This means that the family of 1-arrows E^R becomes a strict 2-natural transformation and the functor in Equation (1) becomes an equivalence of categories when restricted to $\mathbf{QRD_s}$ and $\mathbf{RD_s}$.

4 Extensionality and separation

An important logical principle commonly assumed is the *extensionality* of equality. Intuitively, it means that two functions f and g are equal exactly when their outputs coincide on equal inputs, that is, whenever $x = y$ implies $f(x) = g(y)$. This is the usual notion of equality for set-theoretic functions, however, if we move to more constructive settings such as Type Theory, it is not necessarily the case that extensionality holds. Relational doctrines are able to distinguish the two notions of equality of arrows.

▶ **Definition 24.** *Let* $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ *be a relational doctrine and* $f, g : X \to Y$ *two parallel arrows in* $\mathcal{C}$*. We say that* f *and* g *are* R-equal*, notation* $f \approx g$*, if* $\mathsf{d}_X \leqslant R_{f,g}(\mathsf{d}_Y)$*. We say that* R *is* extensional *if for every* f, g *in* $\mathcal{C}$*,* $f \approx g$ *implies* $f = g$*.*

That is, R is extensional if R-equality implies equality of arrows. The other implication always holds, therefore in an extensional relational doctrine $f \approx g$ if and only if $f = g$.

▶ **Proposition 25.** *Let* $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ *be a relational doctrine and* $f, g : X \to Y$ *two parallel arrows in* $\mathcal{C}$*. Then,* $f \approx g$ *iff* $\Gamma_f = \Gamma_g$*.*

Proposition 25 with Proposition 6 mean that R-equal arrows cannot be distinguished by the logic of R since they behave in the same way w.r.t. reindexing. Indeed given $f, f' : X \to A$ and $g, g' : Y \to B$ in the base, $f \approx f'$ and $g \approx g'$ imply $R_{f,g} = R_{f',g'}$.

From a quantitative or topological perspective, extensional equality is related to various notions of *separation*. Take for example the doctrine $(\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel})^q$ over the category $\mathcal{Q}_{\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}}$ of pseudometric spaces and non-expansive maps (cf. Example 18(1)). Functions $f, g : \langle X, \rho\rangle \to \langle Y, \sigma\rangle$ are $(\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel})^q$-equal iff $\sigma(f(x), g(x)) = 0$, which implies $f = g$ exactly when $\langle Y, \sigma\rangle$ satisfies the identity of indiscernibles, *i.e.* the axiom stating that $\sigma(x, y) = 0$ implies $x = y$. This requirement turns a pseudometric space into a usual metric space and forces a strong separation property: the topology associated with the metric space is Hausdorff.

This observation shows that the relational quotient completion does not preserve extensionality. Indeed the relational doctrine $\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}$ on $\mathcal{S}et$ is extensional, while $(\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel})^q$ is not as not all pseudometric spaces are separated. This is due to the fact that the relational quotient completion changes equality, as it modifies identity relations, while the equality between arrows of the base category remains unchanged. The following completion forces extensionality or, in quantitative terms, separation. As for the relational quotient completion, it is inspired by the extensional collapse of an elementary doctrines introduced in [42].

▶ **Proposition 26.** *Let* R *be a relational doctrine and* $f, f' : X \to Y$ *and* $g, g' : Y \to Z$ *are arrows in the base* $\mathcal{C}$*. Then* $f \approx f'$ *and* $g \approx g'$ *imply* $g \circ f \approx g' \circ f'$*.*

It shows that $\approx$ is a congruence on $\mathcal{C}$. Let $\mathcal{E}_R$ be the quotient of $\mathcal{C}$ modulo $\approx$, notably, objects are those of $\mathcal{C}$ and arrows are equivalence classes of arrows in $\mathcal{C}$ modulo $\approx$, denoted by $[f]$. Define a functor $(R)^e : (\mathcal{E}_R \times \mathcal{E}_R)^{\mathrm{op}} \to \mathcal{P}os$ by $(R)^e(X, Y) = R(X, Y)$ and $(R)^e_{[f],[g]}(\alpha) = R_{f,g}(\alpha)$. It is well-defined on arrows by Propositions 6 and 25.

▶ **Lemma 27.** *The functor* $(R)^e : (\mathcal{E}_R \times \mathcal{E}_R)^{\mathrm{op}} \to \mathcal{Pos}$ *together with relational operations of* R *is an extensional relational doctrine.*

Taking terminology from [42], the doctrine $(R)^e$ is the *extensional collapse* of R. The following examples show some connections between the extensional collapse and notions of separation in metric and topological structures.

▶ **Example 28.**

1. Let $V = \langle |V|, \preceq, \cdot, 1\rangle$ be a commutative quantale. Recall from Example 18(1) that the category $\mathcal{Q}_{V\text{-}\mathsf{Rel}}$ is the category of V-metric spaces and non-expansive maps. It is the base of the doctrine $(V\text{-}\mathsf{Rel})^q$, whose identity relation is given by $\mathsf{d}_{\langle X,\rho\rangle} = \rho$ for every V-metric space $\langle X, \rho\rangle$. A V-metric sapce $\langle X, \rho\rangle$ is *separated* if $1 \preceq \rho(x, y)$ implies $x = y$. Notice that a separated $\mathbb{R}_{\geqslant 0}$-metric space is the usual notion of metric space. Denote by $V\text{-}\mathcal{Met}_{\mathsf{s}}$ the full subcategory of $\mathcal{Q}_{V\text{-}\mathsf{Rel}}$ of separated V-metric spaces. Applying the extensional collapse to $(V\text{-}\mathsf{Rel})^q$ we get $((V\text{-}\mathsf{Rel})^q)^e$ where two arrows $[f], [g] : \langle X, \rho\rangle \to \langle Y, \sigma\rangle$ of its base $\mathcal{E}_{(V\text{-}\mathsf{Rel})^q}$ are equal when $\rho(x, y) \preceq \sigma(f(x), g(y))$. The fully faithful inclusion of $V\text{-}\mathcal{Met}_{\mathsf{s}}$ into $\mathcal{E}_{(V\text{-}\mathsf{Rel})^q}$ is an equivalence: for any V-metric space $\langle X, \rho\rangle$, write $x \sim y$ when $1 \preceq \rho(x, y)$ and take the quotient space $\langle X/\sim, \rho_\sim\rangle$, where $\rho_\sim([x], [y]) = \rho(x, y)$, is separated. The projection map $[q] : \langle X, \rho\rangle \to \langle X/\sim, \rho_\sim\rangle$ is an isomorphism whose inverse is represented by any chosen section $s : X/\sim \to X$ of q.
2. Recall from Example 18(2) that the base $\mathcal{Q}_{\mathsf{Vec}}$ of the relational doctrine $(\mathsf{Vec})^q$ is the category of semi-normed real vector spaces and short linear maps: an object $\langle X, \rho\rangle$ is a real vector space X with a subadditive and homogeneous pseudometric ρ that gives a semi-norm $\|\mathbf{x}\| = \rho(\mathbf{0}, \mathbf{x})$. A semi-norm is a norm when $\|\mathbf{x}\| = 0$ implies $\mathbf{x} = \mathbf{0}$, which is equivalent to ρ being separated. The category $\mathcal{NVec}$ of normed vector spaces is equivalent to the base category $\mathcal{E}_{(\mathsf{Vec})^q}$ of the extensional collapse of $(\mathsf{Vec})^q$. The proof of the essential surjectivity of the obvious inclusion of $\mathcal{NVec}$ into $\mathcal{E}_{(\mathsf{Vec})^q}$ uses arguments similar to those used in Example 28(1). In particular it relies on the axiom of choice. There is only a little care in taking sections $s : X/\sim \to X$ of a quotient map q in $\mathcal{Vec}$ as these have to be linear. But from a section s one cane take its values on the vectors of a chosen base of $X/\sim$ and generate from this assignment a linear map $s' : X/\sim \to X$ which is easily proved to be a section of q.

▶ **Example 29.** Let $\mathcal{Top}$ be the category of topological spaces and continuous functions and $\mathsf{TRel} : (\mathcal{Top} \times \mathcal{Top})^{\mathrm{op}} \to \mathcal{Pos}$ be the change-of-base $U^\star\mathsf{Rel}$ along the forgetful functor $U : \mathcal{Top} \to \mathcal{Set}$ as in Example 3(5). The base $\mathcal{Q}_{\mathsf{TRel}}$ of the relational quotient completion of TRel provides an "intensional" version of Scott's equilogical spaces[2] [52]. Objects of $\mathcal{Q}_{\mathsf{TRel}}$ are pairs $\langle X, \rho\rangle$ of a topological space X and an equivalence relation ρ on the underlying set of X and arrows are continuous maps preserving the equivalences. Any section $S : \mathcal{Top} \to \mathcal{Q}_{\mathsf{TRel}}$ of the forgetful functor $\mathcal{Q}_{\mathsf{TRel}} \to \mathcal{Top}$ picks an equivalence relation over every space in a way that relations are compatible with continuous maps. The change-of-base $S^\star(\mathsf{TRel})^q$ provides a new logic on $\mathcal{Top}$ where identity relations are changed according to S. For a space X, the doctrine $S^\star\mathsf{TRel}$ can not distinguish points which are related by ρ_X, while such points may differ in the base. The extensional collapse makes such points indistinguishable in the base as well. Instances of this construction are the category $\mathcal{Top}_0$ of T_0-spaces and the homotopy category $h\mathcal{Top}$. The former is given by defining ρ_X as follows: $\langle x, y\rangle \in \rho_X$ iff x and y are topologically indistinguishable, that is, for every open subset $U \subseteq X$, $x \in U$ iff $y \in U$. The latter is given by defining ρ_X as follows: $\langle x, y\rangle \in \rho_X$ iff there is a continuous path from x to y, that is, there is a continuous function $h : [0, 1] \to X$ such that $h(0) = x$ and $h(1) = y$.

[2] Applying the extensional collapse we get exactly the category of equilogical spaces.

The relational doctrine $(R)^e$ comes together with a 1-arrow $C^R : R \to (R)^e$ where $\widehat{C^R} : \mathcal{C} \to \mathcal{E}_R$ maps $f : A \to B$ to $[f] : A \to B$ and $\overline{C^R}_{X,Y}$ maps $R(X,Y)$ to itself. The extensional collapse is universal if we restrict to strct 1-arrows. Let $\mathbf{ERD}_s$ denote the full 2-subcategory of $\mathbf{RD_s}$ whose objects are extensional relational doctrines and $\mathrm{U_e} : \mathbf{ERD}_s \to \mathbf{RD_s}$ the obvious inclusion 2-functor.

▶ **Theorem 30.** *The 2-functor* $\mathrm{U_e} : \mathbf{ERD}_s \to \mathbf{RD_s}$ *has a left 2-adjoint* $\mathrm{E} : \mathbf{RD_s} \to \mathbf{ERD}_s$ *such that* $\mathrm{E}(R) = (R)^e$.

The extensional collapse interacts well with quotients. Indeed, if R has (effective descent) quotients, its extensional collapse $(R)^e$ has (effective descent) quotients as well. More precisely, let us denote by $\mathbf{EQRD}_s$ the full 2-subcategory of $\mathbf{QRD_s}$ whose objects are extensional relational doctrines with quotients. We get two obvious inclusion 2-functors $\mathrm{U_q}' : \mathbf{EQRD}_s \to \mathbf{QRD_s}$ and $\mathbf{EQRD}_s : \mathbf{ERD}_s \to$ which respectively forget extensionality and quotients.[3] Then, we get the following result.

▶ **Theorem 31.** *The 2-adjunction* $\mathrm{E} \dashv \mathrm{U_e}$ *restricts to a 2-adjunction between* $\mathbf{EQRD}_s$ *and* $\mathbf{QRD_s}$.

In summary, by Theorems 23, 30, and 31, we get the following diagram

$$\begin{tikzcd}
\mathbf{EQRD}_s \arrow[r, bend right, "\mathrm{U_q}'"'] \arrow[r, phantom, "\perp"] \arrow[d, "\mathrm{U_e}'"'] & \mathbf{QRD_s} \arrow[l, bend right, "\mathrm{E}'"'] \arrow[d, bend right, "\mathrm{U_q}"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{ERD}_s \arrow[r, bend left, "\mathrm{U_e}"] \arrow[r, phantom, "\top"] & \mathbf{RD_s} \arrow[l, bend left, "\mathrm{E}"] \arrow[u, bend right, "\mathrm{Q}"']
\end{tikzcd}$$

where the external square commutes and E' is a lifting of E, that is, $\mathrm{U_e}' \circ \mathrm{E}' = \mathrm{E} \circ \mathrm{U_q}$. The composite $\mathrm{E}' \circ \mathrm{Q} : \mathbf{RD_s} \to \mathbf{EQRD}_s$ gives a universal construction adding (effective descent) quotients and forcing extensionality. Finally note that the relational quotient completion does not preserve extensionality. Therefore the restriction of Q to $\mathbf{ERD}_s$, namely, the composite $\mathrm{Q} \circ \mathrm{U_e}$, may not provide a left 2-adjoint to $\mathrm{U_e}'$. To get such a left 2-adjoint, we need to force extensionality again, that is, we need the 2-functor $\mathrm{E}' \circ \mathrm{Q} \circ \mathrm{U_e}$, which however is not the lifting of Q (in other words, the diagram of left 2-adjoints would not commute).

▶ **Example 32.**

1. Recall from [11, 12] that a Bishop's set, or setoid, is a pair $\langle A, \rho \rangle$ of a set A and an equivalence relation $\rho \subseteq A \times A$. A Bishop's function from the setoid $\langle A, \rho \rangle$ to the setoid $\langle B, \sigma \rangle$ is an equivalence class of functions $f : A \to B$ preserving the equivalence relations, where f and g belong to the same equivalence class if $f(a)\sigma g(a)$ for all $a \in A$. A relation from $\langle A, \rho \rangle$ to $\langle B, \sigma \rangle$ is a subset $U \subset A \times B$ such that $(a, b) \in U$, $a\rho a'$, $b\sigma b'$ imply $(a', b') \in U$. Call $\mathcal{BSet}$ the category of Bishop's sets and functions and $\mathsf{BRel} : (\mathcal{BSet} \times \mathcal{BSet})^{\mathrm{op}} \to \mathcal{Pos}$ the relational doctrine that maps two setoids to the collection of relations between them. The relational doctrine $((\mathsf{Rel})^q)^e$ (obtained completing $\mathsf{Rel} : (\mathcal{Set} \times \mathcal{Set})^{\mathrm{op}} \to \mathcal{Pos}$ first with quotients and then forcing extensionality) is BRel.

[3] $\mathbf{EQRD}_s$ can be seen as the pullback of $\mathrm{U_e}$ against $\mathrm{U_q}$.

2. One of the most widely used constructions to complete a category with quotients is the exact completion of a weakly lex category presented in $\mathcal{C}$ [14, 15]. This is an instance of our constructions. Recall the relational doctrine $\mathsf{Spn}^{\mathcal{C}}$ from Example 3(3). Complete it first with quotients and then force extensionality. One get the relational doctrine $((\mathsf{Spn}^{\mathcal{C}})^q)^e$ whose base is $\mathcal{C}_{\mathrm{ex/wlex}}$. If products of $\mathcal{C}$ are strong, the construction coincides with the elementary quotient completion of the doctrines of weak subobjects of $\mathcal{C}$ shown in [41, 42]. A comparison between these two constructions is in Section 5.

5 Related Structures

There are many categorical models abstracting the essence of the calculus of relations, such as cartesian bicategories [16] or allegories [25] which are both special cases of ordered categories with involution [35]. Also existential and elementary doctrines, *i.e.* those doctrines that model $(\exists, \wedge, \top, =)$-fragment of first order logic, encode a calculus of relations. A natural question is how relational doctrines differ from these models.

We show that when working with an ordered category, one implicitly accepts two logical principles, which are not necessarily there in a relational doctrine, and we show that when working with existential elementary doctrines, one implicitly accepts to work with variables, which are not necessarily there in relational doctrines. These comparison are carried out restricting to the 2-category $\mathbf{RD_s}$ where 1-arrows are strict.

Ordered categories with involution. An ordered category with involution [35] is a $\mathcal{P}os$-enriched category $\mathcal{C}$ together with an identity-on-objects and self inverse $\mathcal{P}os$-functor $(-)^{\perp}$: $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. Intuitively, arrows can be seen as relations whose inverse is given by the involution.

A relational doctrine $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ defines an ordered category with involution $\mathcal{O}_R$ as follows: objects are those of $\mathcal{C}$, the poset of arrows between X and Y is the fibre $R(X, Y)$, composition and identities are given by relational ones and the involution is given by the converse operation. The assignment extends to a 2-functor $\mathcal{O} : \mathbf{RD_s} \to \mathbf{OCI}$, where $\mathbf{OCI}$ is the 2-category of ordered categories with involution whose 1-arrows $F : \mathcal{C} \to \mathcal{D}$ are ordered functors preserving involution and a 2-arrows $\theta : F \Rightarrow G$ are lax natural transformations.

To see how to obtain a relational doctrine from an ordered category, first note that any ordered category with involution $\mathcal{C}$ induces a category $\mathcal{M}ap(\mathcal{C})$, called the category of maps in $\mathcal{C}$, whose objects are those of $\mathcal{C}$ and an arrow $f : X \to Y$ is an arrow in $\mathcal{C}$ such that $f^{\perp} : Y \to X$ is its right adjoint, that is $f \circ f^{\perp} \leqslant \mathsf{id}_Y$ and $\mathsf{id}_X \leqslant f^{\perp} \circ f$. We define a relational doctrine $\mathsf{Map}^{\mathcal{C}} : (\mathcal{M}ap(\mathcal{C}) \times \mathcal{M}ap(\mathcal{C}))^{\mathrm{op}} \to \mathcal{P}os$ where $\mathsf{Map}^{\mathcal{C}}(X, Y) = \mathcal{C}(X, Y)$ is the poset of all arrows in $\mathcal{C}$ from X to Y and, for $f : A \to X$ and $g : B \to Y$ arrows in $\mathcal{M}ap(\mathcal{C})$, the map $\mathsf{Map}^{\mathcal{C}}_{f,g} : \mathsf{Map}^{\mathcal{C}}(X, Y) \to \mathsf{Map}^{\mathcal{C}}(A, B)$ sends α to the composition $g^{\perp} \circ\ \alpha \circ f$. Relational composition and identities are composition and identities of $\mathcal{C}$ and the relational converse is given by the involution $(-)^{\perp}$. The assignment extends to a 2-functor $\mathrm{Map} : \mathbf{OCI} \to \mathbf{RD_s}$.

Relational doctrines of the form $\mathsf{Map}^{\mathcal{C}}$ have extensional equality. They also validates the rule of unique choice which says that whenever a relation is functional and total, there is a function that for every x in the domain picks the unique y related to x. Formally $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ satisfies the *rule of unique choice*, (RUC) for short, if for every $\alpha \in R(X, Y)$ such that $\mathsf{d}_X \leqslant \alpha ; \alpha^{\perp}$ and $\alpha^{\perp} ; \alpha \leqslant \mathsf{d}_Y$, there is $f : X \to Y$ in $\mathcal{C}$ with $\Gamma_f \leqslant \alpha$.

Next theorem shows that extensionality and (RUC) are exactly the two logical principles that a relational doctrine needs to coincide with an ordered category. Indeed the essential image of 2-functor Map is $\mathbf{ERD}_{(\mathrm{RUC})}$, the full 2-subcategory of $\mathbf{RD_s}$ on extensional doctrines satisfying (RUC) and its inverse is the restriction of $\mathcal{O} : \mathbf{RD_s} \to \mathbf{OCI}$ to $\mathbf{ERD}_{(\mathrm{RUC})}$.

▶ **Theorem 33.** *The 2-categories* **OCI** *and* $\mathbf{ERD}_{(\text{RUC})}$ *are 2-equivalent.*

The equivalence stated in Theorem 33 generalises a similar result proved in [13], which compares cartesian bicategories and existential elementary doctrines. The way we built the two functors of the equivalence shows also that **OCI** is a 1-full subcategory of $\mathbf{ERD}_{(\text{RUC})}$. Examples of relational doctrines that are not in **OCI** because they are not extensional were given in Section 4. The following example presents a relational doctrine outside **OCI** because it does not satisfy (RUC).

▶ **Example 34.** Take a set A with more than one element. The set $\mathcal{P}(A)$ of subsets of A is a complete Heyting algebra, therefore a commutative quantale. Recall from Item 1 that in the relational doctrine $\mathcal{P}(A)\text{-}\mathsf{Rel} : (\mathcal{S}et \times \mathcal{S}et)^{\text{op}} \to \mathcal{P}os$ of $\mathcal{P}(A)$-relations, for every set X the relation d_X maps (x, x') to A if $x = x'$ and to $\varnothing$ if $x \neq x'$. This relational doctrine does not satisfy the (RUC). Consider $\alpha \in \mathcal{P}(A)\text{-}\mathsf{Rel}(1, A)$ given by $\alpha(*, a) = \{a\}$, it holds

$$\mathsf{d}_1 = A = \bigcup_{a \in A} \{a\} = \alpha\,;\alpha^\perp \quad \text{and} \quad (\alpha^\perp\,;\alpha)(a, a') = \{a\} \cap \{a'\} \subseteq \mathsf{d}_A$$

Suppose $f : 1 \to A$ is such that $\Gamma_f \subseteq \alpha$, *i.e.* $\mathsf{d}_A(f(*), a) \subseteq \alpha(*, a) = \{a\}$. Then $A = \mathsf{d}_A(f(*), f(*)) \subseteq \{f(*)\}$, but this inclusion is contradictory with the assumption that A has more than one element.

Existential elementary doctrines. Doctrines $P : \mathcal{C}^{\text{op}} \to \mathcal{P}os$ are algebraic representations of fragment of first order predicate logic, where objects and arrows of $\mathcal{C}$ are contexts and terms and fibres $P(X)$ collect the predicates with free variables over X ordered by logical entailment. To sustain this intuition in practice the base category $\mathcal{C}$ needs finite products to model context concatenation (see also [50]). Once $\mathcal{C}$ is assumed to have finite products, an easy way to extract a relational doctrine out of P is to consider the functor $\mathsf{Rel}^P : (\mathcal{C} \times \mathcal{C})^{\text{op}} \to \mathcal{P}os$ mapping $\langle X, Y\rangle$ to $P(X \times Y)$ and $\langle f, g\rangle$ to $P_{f\times g}$. To define relational composition mimicking the standard definition one needs to restrict to those doctrines that models at least and the $(\exists, \wedge, \top, =)$-fragment of first order logic. These are called elementary existential doctrines.

A doctrine $P : \mathcal{C}^{\text{op}} \to \mathcal{P}os$ is existential elementary if all the following hold: $\mathcal{C}$ has finite products; every fibre has finite meets and these are preserved by reindexing; for every $f : X \to Y$ in $\mathcal{C}$ the reindexing P_f has a left adjoint $\mathit{\exists}_f : P(X) \to P(Y)$ such that for every $\phi \in P(X)$ and every $\psi \in P(Y)$ it holds that $\mathit{\exists}_f(\phi) \wedge \psi = \mathit{\exists}_f(\phi \wedge P_f\psi)$ (*Frobenius reciprocity*); for every arrow $f : A \to B$ in $\mathcal{C}$ and every object X in $\mathcal{C}$ it holds that $P_f \mathit{\exists}_{\pi_B} = \mathit{\exists}_{\pi_A} P_{\mathsf{id}_X \times f}$, where $\pi_A : X \times A \to A$ and $\pi_B : X \times B \to B$ are projections (*Beck-Chevalley condition*).

▶ **Example 35.** An archetypal example of existential elementary doctrine is the contravariant powerset functor $\mathcal{P} : \mathcal{S}et^{\text{op}} \to \mathcal{P}os$. For a function $f : X \to Y$, the left adjoint $\mathit{\exists}_f$ is the direct image mapping. Two instances are of interest. The first is when f is the diagonal $\Delta_X : X \to X \times X$. In this case the direct image evaluated on the the top element (*i.e.* the whole X) is the diagonal relation, that is $\mathit{\exists}_{\Delta_X}(\top_X) = \{(x, x') \in X \times X \mid x = x'\}$. The other is when f is a projection $\pi_2 : X \times Y \to Y$. In this case $\mathit{\exists}_{\pi_2}(\phi) = \{y \in Y \mid \exists_{x \in X}\, (x, y) \in \phi\}$.

The previous example shows the underling idea that, in an existential elementary doctrine, left adjoints along diagonals compute diagonal relations, lefts adjoints along projections compute existential quantifications. So every existential elementary doctrine $P : \mathcal{C}^{\text{op}} \to \mathcal{P}os$ generates a relational doctrine $\mathsf{Rel}^P : (\mathcal{C} \times \mathcal{C})^{\text{op}} \to \mathcal{P}os$ setting $\mathsf{Rel}^P(X, Y) = P(X \times Y)$ and $\mathsf{Rel}^P_{f,g} = P_{f\times g}$ and

$$\mathsf{d}_X = \mathit{\exists}_{\Delta_X}(\top) \qquad \alpha\,;\beta = \mathit{\exists}_{\langle \pi_1, \pi_3\rangle}(P_{\langle \pi_1, \pi_2\rangle}(\alpha) \wedge P_{\langle \pi_2, \pi_3\rangle}(\beta)) \qquad \alpha^\perp = P_{\langle \pi_2, \pi_1\rangle}(\alpha)$$

for $\alpha \in P(X \times Y)$ and $\beta \in P(Y \times Z)$. The assignment extends to a 2-functor $\mathsf{Rel} : \mathbf{EED} \to \mathbf{RD_s}$ where $\mathbf{EED}$ denotes the 2-category whose objects are existential elementary doctrines, 1-arrows $F : P \to Q$ are pairs $\langle \widehat{F}, \overline{F} \rangle$ where tha functor $\widehat{F} : \mathcal{C} \to \mathcal{D}$ preserves finite products and $\overline{F} : P \stackrel{\cdot}{\to} Q \circ \widehat{F}$ preserves finite meets and commutes with left adjoints.

▶ **Example 36.** Consider the powerset functor as an existential and elementary doctrine as in Example 35. It is immediate to see that $\mathsf{Rel}^{\mathcal{P}}$ is Rel.

From a relational doctrine of the form Rel^P one recovers P mapping A to $\mathsf{Rel}^P(A, 1) = P(A \times 1) \simeq P(A)$. This suggests where to look for the inverse of Rel.

First of all note that existential elementary doctrines have finite products in the base, finite meets on all fibres preserved by reindexing, while relational doctrines need not have. These structures have a neat algebraic description that uses the finite products in 2-category $\mathbf{Dtn}$: a doctrine P is based on a category with finite products, has finite meets on each fibre and these are stable under reindexing if and only if both the unique arrow $!_P$ and the diagonal Δ_P have a right adjoint in $\mathbf{Dtn}$. Since the 2-category $\mathbf{RD_s}$ of relational doctrines has finite products too, we take advantage of this characterisation and we say that a relational doctrine R is *cartesian* if the 1-arrows $!_R$ and Δ_R have right adjoints in $\mathbf{RD_s}$.

▶ **Example 37.** The doctrine $\mathsf{Rel}^{\mathcal{P}} = \mathsf{Rel} : (\mathcal{S}et \times \mathcal{S}et)^{\mathrm{op}} \to \mathcal{P}os$ is cartesian. The right adjoint to Δ_{Rel} is given using products. Indeed for $\langle\langle A, B\rangle, \langle X, Y\rangle\rangle$ the base of $\mathsf{Rel} \times \mathsf{Rel}$ the natural transformation $\mathsf{Rel}(A, B) \times \mathsf{Rel}(X, Y) \stackrel{\cdot}{\to} \mathsf{Rel}(A \times X, B \times Y)$ maps $\alpha \in \mathsf{Rel}(A, B)$ and $\beta \in \mathsf{Rel}(X, Y)$ to $\{\langle\langle a, x\rangle, \langle b, y\rangle\rangle \mid \langle a, b\rangle \in \alpha \text{ and } \langle x, y\rangle \in \beta\}$.

For a cartesian relational doctrine R denote by Doc^R the doctrine obtained by the composition of R with $\langle -, 1\rangle : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C} \times \mathcal{C})^{\mathrm{op}}$. Proposition 6 shows that Doc^R has left adjoints to all reindexing maps. One can also show that the left adjoints along projections satisfies the Beck-Chevalley condition, the only missing ingredient is the Frobenius reciprocity. We say that a relational doctrine R is *cartesian Frobenius* if it is cartesian and for every X and Y in $\mathcal{C}$ and every $\alpha \in R(X, Y)$ it holds

$$\Gamma^{\perp}_{\Delta_X} \,;\Gamma_{\Delta_X} = \Gamma_{\Delta_X \times \mathsf{id}_X} \,;\Gamma^{\perp}_{\mathsf{id}_X \times \Delta_X}$$

$$\Gamma_{\Delta_X \times \mathsf{id}_Y} \,;(\mathsf{d}_X \bullet \alpha \bullet \mathsf{d}_Y) \,;\Gamma^{\perp}_{\mathsf{id}_X \times \Delta_Y} = (\Gamma_{\Delta_X \times \mathsf{id}_Y} \,;(\mathsf{d}_X \bullet \alpha \bullet \mathsf{d}_Y) \,;\Gamma^{\perp}_{\mathsf{id}_X \times \Delta_Y})^{\perp}$$

where the first condition is inspired by [13]. In general relational doctrines need not be cartesian Frobenius as shown by the following example.

▶ **Example 38.** Consider the relational doctrine $\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel} : (\mathcal{S}et \times \mathcal{S}et)^{\mathrm{op}} \to \mathcal{P}os$ of metric relations, where $\mathbb{R}_{\geqslant 0} = \langle [0, \infty], \geqslant, +, 0\rangle$ is the Lawvere's quantale as in Example 3. This doctrine is not cartesian. Indeed to be cartesian would imply the existence of a 1-arrow $\times : \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel} \times \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel} \to \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}$ which is right adjoint to $\Delta_{\mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}}$. The right adjoint should be a pair $\langle \widehat{\times}, \overline{\times}\rangle$ where $\overline{\times}$ commutes with relational composition, that is it satisfies equations of the form $(\alpha \overline{\times} \beta) \,;(\alpha' \overline{\times} \beta') = (\alpha \,; \alpha') \overline{\times} (\beta \,; \beta')$. For $\alpha \in \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}(A, B)$ and $\beta \in \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}(X, Y)$ the relation $\alpha \overline{\times} \beta$ is computed as follows

$$(\alpha \overline{\times} \beta)(a, x, b, y) = \sup\{\alpha(a, b), \beta(x, y)\}$$

Suppose α and β are constant functions, and take two other constant functions for $\alpha' \in \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}(B, C)$ and $\beta' \in \mathbb{R}_{\geqslant 0}\text{-}\mathsf{Rel}(Y, Z)$. The equation $(\alpha \overline{\times} \beta) \,;(\alpha' \overline{\times} \beta') = (\alpha \,; \alpha') \overline{\times} (\beta \,; \beta')$ reduces to $\sup\{\alpha, \beta\} + \sup\{\alpha', \beta'\} = \sup\{\alpha + \alpha', \beta + \beta'\}$ that need not hold (take $\alpha = \beta' = 0$ and $\alpha' = \beta = 1$).

Relational doctrines of the form Rel^P are cartesian Frobenius, therefore the essential image of Rel is $\mathbf{FRD}$, the 2-full 2-subcategory of $\mathbf{RD_s}$ on cartesian Frobenius relational doctrines and 1-arrows that preserve the cartesian structure. Moreover doctrines of the form Doc^R are existential elementary if and only if R is cartesian Frobenius. This determines a 2-functor $\mathrm{Doc} : \mathbf{FRD} \to \mathbf{EED}$ that, together with $\mathsf{Rel} : \mathbf{EED} \to \mathbf{FRD}$, determine the equivalence stated by the following theorem.

▶ **Theorem 39.** *The 2-categories* $\mathbf{FRD}$ *and* $\mathbf{ERD}_{(\text{RUC})}$ *are 2-equivalent.*

Relying on the equivalence proved in Theorem 39, the completion of an elementary existential doctrine with quotients introduced in [42] is equivalent to the relational quotient completion of a cartesian Frobenius relational doctrine. The elementary quotient completion of an existential elementary doctrine introduced in [41] is equivalent to the extensional collapse of the relational quotient completion of cartesian Frobenius relational doctrines. This results in a wide range of examples of relational doctrines such as realisability doctrines, doctrines of (strong/weak) subobjects and syntactic doctrines [31, 50, 58]. Also dependent Types Theories give rise to existential elementary doctrines whose elementary quotient completion is the category of setoids [41, 42].

We proved that existential elementary doctrines and cartesian Frobenius relational doctrines are equivalent, and the completions introduced in this paper coincide with the corresponding ones introduced by Maietti and Rosolini. Both of them work on larger classes of doctrines. More specifically, the completions proposed by Maietti and Rosolini can be applied to doctrines that need not be existential in the sense that they need not have left adjoints to all the reindexing maps, but they need finite products in the base and finite meets in the fibres. On the other hand relational doctrines intrinsically have left adjoints to all reindexing maps, but the completions described in this paper work also on relational doctrines that need not be cartesian and need not have a base with finite products. This is a crucial ingredient to cover the quantitative examples.

6 Conclusions

We introduced relational doctrines as a functorial description of the essence of the calculus of relations. Relying on this structure, we defined quotients and a universal construction adding them capable to cover quantitative settings as well. Then, we studied extensional equality in relational doctrines, showing it captures various notion of separation in metric and topological structures. Moreover, we described a universal construction forcing extensionality, thus separation, analysing how it interacts with quotients. Finally, we compared relational doctrines with two important classes of examples: ordered categories with involution, proving these correspond to relational doctrines having both extensional equality and the rule of unique choice, and existential elementary doctrines, showing they correspond to cartesian relational doctrines satisfying suitable Frobenius rules.

There are many directions for future work. The first one is the study of choice rules in the framework of relational doctrines, extending known results for doctrines [40, 44], giving them a quantitative interpretation, for instance in terms of completeness, following the connection between (RUC) and Cauchy completeness pointed out in [39]. Moreover, this could lead us to the definition of a quantitative counterpart of the tripos-to-topos construction, generalising known results [24, 43], which could generate categories of complete (partial) metric spaces.

We also plan to bring the study of relations to the proof-relevant setting of type theories. Algebraically this can be done moving from doctrines to arbitrary fibrations as it is common practice. On the syntactic side, instead, things are much less clear: developing a proper syntax and rules for a "relational type theory" is something interesting per se. Actually, we do not even have a syntactic calculus behind relational doctrines. Then, another interesting direction is to design it, possibly in a diagrammatic way, for instance in the style of string diagrams.

Another interesting direction is to study relational doctrines with tools coming from the theory of double categories. Indeed, from Remark 2 one can easily read a relational doctrine as a special double category and equivalence relations looks similar to monads in such a double category [23]. Thus, it would be interesting applying general results for monads in double categories to this specific case, possibly deriving properties and constructions related to equivalences and quotients.

Finally, a promising direction would be the use of relational doctrines as an abstract framework where to formulate and develop relational techniques used in the study of programming languages and software systems, such as (coalgebraic) bisimulation, program equivalence or operational semantics, as well as, quantitative equational theories and rewriting.

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