Witnessed Symmetric Choice and Interpretations in Fixed-Point Logic with Counting

Moritz Lichter
TU Darmstadt, Germany

Abstract

At the core of the quest for a logic for PTIME is a mismatch between algorithms making arbitrary choices and isomorphism-invariant logics. One approach to tackle this problem is witnessed symmetric choice. It allows for choices from definable orbits certified by definable witnessing automorphisms.

We consider the extension of fixed-point logic with counting (IFPC) with witnessed symmetric choice (IFPC+WSC) and a further extension with an interpretation operator (IFPC+WSC+I). The latter operator evaluates a subformula in the structure defined by an interpretation. When similarly extending pure fixed-point logic (IFP), IFP+WSC+I simulates counting which IFP+WSC fails to do. For IFPC+WSC, it is unknown whether the interpretation operator increases expressiveness and thus allows studying the relation between WSC and interpretations beyond counting.

In this paper, we separate IFPC+WSC from IFPC+WSC+I by showing that IFPC+WSC is not closed under FO-interpretations. By the same argument, we answer an open question of Dawar and Richerby regarding non-witnessed symmetric choice in IFP. Additionally, we prove that nesting WSC-operators increases the expressiveness using the so-called CFI graphs. We show that if IFPC+WSC+I canonizes a particular class of base graphs, then it also canonizes the corresponding CFI graphs. This differs from various other logics, where CFI graphs provide difficult instances.

2012 ACM Subject Classification Theory of computation → Finite Model Theory; Theory of computation → Complexity theory and logic

Keywords and phrases witnessed symmetric choice, interpretation, fixed-point logic, counting, CFI graphs, logic for PTime

Digital Object Identifier 10.4230/LIPIcs.ICALP.2023.133

Category Track B: Automata, Logic, Semantics, and Theory of Programming

Related Version *Full Version*: `https://arxiv.org/abs/2210.07869` [30]

Funding This work received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (EngageS: agreement No. 820148).

1 Introduction

The quest for a logic for PTIME is one of the prominent open questions in finite model theory [6, 19]. It asks whether there is a logic defining exactly all polynomial-time decidable properties of finite structures. While Fagin's theorem [14] initiated descriptive complexity theory by showing that there is a logic capturing NPTIME, the question for PTIME is still open. One problem at the core of the question is a mismatch between logics and algorithms. For algorithms, it is common to make arbitrary choices as long as the output is still isomorphism-invariant. In general, it is undecidable whether an algorithm is isomorphism-invariant. Showing this is usually part of the proof that the algorithm is correct. On the other hand, every reasonable logic is required to be isomorphism-invariant by design [22, 13], so in contrast to algorithms we cannot define something non-isomorphism-invariant. That is, a logic has to enforce isomorphism-invariance syntactically and it is generally not clear how algorithms making choices can be implemented in a logic.

For totally ordered structures, inflationary fixed-point logic (IFP) captures PTIME due to the Immerman-Vardi Theorem [27]. On ordered structures, no arbitrary choices are needed and the total order is used to "choose" the unique minimal element. Thus, the lack of making choices is crucial on unordered structures. We therefore would like to support choices in a logic while still guaranteeing isomorphism-invariance. There are logics in which arbitrary choices can be made [2, 10], but for these it is undecidable whether a formula is isomorphism-invariant [2]. In particular, such logics fail to be reasonable in the sense of Gurevich [22]. Similarly, when extending structures by an arbitrary order it is undecidable whether a formula is order-invariant, i.e., it evaluates equally for all such orders (see [16]).

One approach to overcome the lack of choices in logics is to support a restricted form of choice. If only choices from definable orbits (of the automorphism group of the input structure) are allowed, that is, from sets of definable objects related by an automorphism of the input structure, the output is guaranteed to be still isomorphism-invariant [9, 15]. This form of choice is called *symmetric choice* (SC). However, it is unknown whether orbits can be computed in PTIME. So it is unknown whether a logic with symmetric choice can be evaluated in PTIME because during the evaluation we have to verify that the choice-sets are indeed orbits. This is solved by handing over the obligation to check whether the choice-sets are orbits from the evaluation to the formulas themselves. To make a choice, not only a choice-set but also a set of witnessing automorphisms has to be defined. These automorphisms certify that the choice-set is indeed an orbit in the following way: For every pair of elements a and b in the choice-set, an automorphism mapping a to b has to be provided. This condition guarantees evaluation in PTIME. We call this restricted form of choice *witnessed symmetric choice* (WSC).

Besides witnessed symmetric choice, other operators were proposed to extend the expressiveness of logics not capturing PTIME including a counting operator (see [35]) and an operator based on logical interpretations [15]. It was shown that witnessed symmetric choice increases the expressiveness of IFP [15] and that counting operators increase the expressiveness of IFP and the logic of Choiceless Polynomial Time (CPT) [4] (one usually refers with CPT to its extension with counting, which we also do from now). However, for the combination of counting and choices not much is known. In this paper, we investigate the relation of counting, witnessed symmetric choice, and interpretations to better understand their expressive power.

Extending IFP with symmetric and witnessed symmetric choice was first studied by Gire and Hoang [15]. They extend IFP with symmetric choice (which we denote IFP+SC) and which witnessed symmetric choice (which we denote IFP+WSC). The authors show that IFP+WSC distinguishes CFI graphs over ordered base graphs, which IFP (and even fixed-point logic with counting IFPC) fails to do [5]. Afterwards IFP+SC was studied by Dawar and Richerby [9]. They allowed for nested symmetric choice operators, proved that parameters of choice operators increase the expressiveness, showed that nested symmetric choice operators are more expressive than a single one, and conjectured that with additional nesting depth the expressiveness increases.

Recently, extending CPT with witnessed symmetric choice (CPT+WSC) was studied by Lichter and Schweitzer [32]. CPT+WSC has the interesting property that a CPT+WSC-definable isomorphism test on a class of structures implies a CPT+WSC-definable canonization for this class. Canonization is the task of defining an isomorphic but totally ordered copy. The only requirement is that the class is closed under individualization, so under assigning unique colors to vertices. This is often unproblematic [28, 33]. Individualization is natural in the context of choices because a choice is, in some sense, an individualization. The concept of canonization is essential in the quest for a logic for PTIME. It provides the routinely employed approach to capture PTIME on a class of structures: Define canonization, obtain isomorphic and ordered structures, and apply the Immerman-Vardi Theorem (e.g. [42, 20, 21, 31]). While in CPT+WSC defining isomorphism implies canonization, we do not know whether the same holds for CPT or whether CPT+WSC is more expressive than CPT. Proving this requires separating CPT from PTIME, which has been open for a long time.

(Witnessed) symmetric choice has the drawback that it can only choose from orbits of the input structure. This structure might have complicated orbits that we cannot define or witness in the logic. However, there could be a reduction to a different structure with easier orbits exploitable by witnessed symmetric choice. For logics, the natural concept of a reduction is an interpretation, i.e., defining a structure in terms of another one. Interpretations are in some sense incompatible with (witnessed) symmetric choice because we always have to choose from orbits of the input structure. Orbits of the interpreted structure are always unions of orbits of the input structure, i.e., an interpretation may add more automorphisms but never removes one. To exploit a combination of choices and interpretations, Gire and Hoang proposed an interpretation operator [15]. It evaluates a formula in the image of an interpretation. For logics closed under interpretations (e.g. IFP, IFPC, and CPT) such an interpretation operator does not increase expressiveness. However, for the extension with witnessed symmetric choice this is different: IFP+WSC is less expressive than the extension of IFP+WSC with the interpretation operator. The interpretation operator in combination with WSC simulates counting, which for WSC alone is indicated not to be the case [15].

We are interested in the relation between witnessed symmetric choice and the interpretation operator not specifically for IFP but more generally. Most of the existing results in [15, 9] showing that (witnessed) symmetric choice or the interpretation operator increases in some way the expressiveness of IFP are based on counting. However, counting is not the actual reason for using witnessed symmetric choice. Counting can be achieved more naturally in IFPC. Thus, it is unknown whether the interpretation operator increases expressiveness of IFPC+WSC. In CPT, it is not possible to show that witnessed symmetric choice or the interpretation operator increases expressiveness without separating CPT from PTIME [32].

Overall, a natural base logic for studying the interplay of witnessed symmetric choice and the interpretation operator is IFPC. In IFPC, separation results based on counting are not applicable. But there are known IFPC-undefinable PTIME properties, namely the already mentioned CFI query, which can be used to separate extensions of IFPC. The CFI construction assigns to a connected graph, the so-called base graph, two non-isomorphic CFI graphs: One is called even and the other one is called odd. The CFI query is to define whether a given CFI graph is even.

Results. We define the logics IFPC+WSC and IFPC+WSC+I, which extend IFPC by a fixed-point operator featuring witnessed symmetric choice and the latter additionally by an interpretation operator. We show that the interpretation operator increases expressiveness:

▶ **Theorem 1.** IFPC+WSC $<$ IFPC+WSC+I $\leq$ *PTIME.*

In particular, this separates IFPC+WSC from PTIME. Such a result does not follow from existing techniques because separating IFP+WSC from PTIME is based on counting in [15]. Moreover, we show that both IFPC+WSC and IFP+SC are not even closed under FO-interpretations. This answers an open question of Dawar and Richerby [9]. Proving Theorem 1 relies on the CFI construction and on defining the CFI query for certain classes of base graphs. To show this, we show that IFPC+WSC+I-distinguishable orbits imply an IFPC+WSC+I-definable canonization (similarly to [32]). We apply this to CFI graphs:

▶ **Theorem 2.** *If* IFPC+WSC+I *canonizes a class of colored base graphs $\mathcal{K}$ (closed under individualization), then* IFPC+WSC+I *canonizes the class of CFI graphs* $\mathsf{CFI}(\mathcal{K})$ *over* $\mathcal{K}$*.*

The conclusion is that for IFPC+WSC+I canonization of a class of CFI graphs is not more difficult than canonization of the corresponding class of base graphs, which is different in many other logics [5, 17, 29, 7]. However, to canonize the CFI graphs in our proof, the nesting depth of WSC-fixed-point operators and interpretation operators increases. We show that this increase is unavoidable: For $L \subseteq$ IFPC+WSC+I, we denote by WSCI(L) the IFPC+WSC+I-fragment which uses IFPC-formula-formation-rules to compose L-formulas and an additional interpretation operator nested inside a WSC-fixed-point operator.

▶ **Theorem 3.** *There is a class of base graphs $\mathcal{K}$, for which* WSCI(IFPC) *defines a canonization but does not define the CFI query for* $\mathsf{CFI}(\mathcal{K})$ *and* WSCI(WSCI(IFPC)) *canonizes* $\mathsf{CFI}(\mathcal{K})$*.*

Theorem 3 provides a first step towards an operator nesting hierarchy for IFPC+WSC+I.

Our Techniques. We adapt the techniques of [32] from CPT to IFPC to define a WSC-fixed-point operator. It has some small but essential differences to [15, 9]. Similar to [32] for CPT, Gurevich's canonization algorithm [23] is expressible in IFPC+WSC: It suffices to distinguish orbits of a class of individualization-closed structures to define a canonization.

To prove Theorem 2, we use the interpretation operator to show that if IFPC+WSC+I distinguishes orbits of the base graphs, then IFPC+WSC+I distinguishes also orbits of the CFI graphs and thus canonizes the CFI graphs. The CFI-graph-canonizing formula nests one WSC-fixed-point operator (for Gurevich's algorithm) and one interpretation operator (to distinguish orbits) more than the orbit-distinguishing formula of the base graphs. To show that this increase in nesting depth is necessary, we construct double CFI graphs. We start with a class of CFI graphs $\mathsf{CFI}(\mathcal{K}')$ canonized in WSCI(IFPC). We create new base graphs $\mathcal{K}$ from the $\mathsf{CFI}(\mathcal{K}')$-graphs. Applying the CFI construction once more, $\mathsf{CFI}(\mathcal{K})$ is canonized in WSCI(WSCI(IFPC)) but not in WSCI(IFPC): To define orbits of $\mathsf{CFI}(\mathcal{K})$, we have to define orbits of the base graph, for which we need to define the CFI query for $\mathsf{CFI}(\mathcal{K}')$.

To prove IFPC+WSC $<$ IFPC+WSC+I, we construct a class of asymmetric structures, i.e., structures without non-trivial automorphisms, for which isomorphism is not IFPC-definable. Because asymmetric structures have only singleton orbits, witnessed symmetric choice is not beneficial, thus IFPC+WSC = IFPC, and isomorphism is not IFPC+WSC-definable. These structures combine CFI graphs and the so-called multipedes [24], which are asymmetric and for which IFPC fails to distinguish orbits. An interpretation removes the multipedes and reduces the isomorphism problem to the ones of CFI graphs. Thus, isomorphism of this class of structures is IFPC+WSC+I-definable.

Related Work. The logic IFPC was separated from PTIME using the CFI graphs [5]. CFI graphs not only turned out to be difficult for IFPC but variants of them were also used to separate rank logic [29] and the more general linear-algebraic logic [7] from PTIME. CPT was shown to define the so-called CFI query for ordered base graphs [12] and base graphs of maximal logarithmic color class size [37]. Defining the CFI query for these graphs in CPT turned out to be comparatively more complicated than in IFP+WSC for ordered base graphs in [15]. In general, it is still open whether CPT defines the CFI query for all base graphs.

The definitions of the (witnessed) symmetric choice operator in [15, 9] differ at crucial points form the one in [32] and in this paper: The formula defining the witnessing automorphism has access to the obtained fixed-point. This is essential to implement Gurevich's canonization algorithm but has the drawback to impose (possibly) stronger orbit conditions. Although it is unknown whether these two variants of witnessed symmetric choice have the same expressiveness, existing results [15, 9] transfer to the variant used in this paper. We expect that the results of this paper also hold for the other variant. However, proving Theorem 2 will be more effort because Gurevich's canonization algorithm cannot be used.

CPT+WSC in [32] is a three-valued logic using, beside true and false, an error marker for non-witnessed choices. This is needed for CPT because fixed-point computations in CPT do not necessarily terminate in a polynomial number of steps. Instead, computation is aborted (and orbits cannot be witnessed). For other approaches to integrate choice in first-order logic, but which are no PTIME-logic candidates, see e.g. [3, 36, 10]. We refer to [38] for an overview.

Logical interpretations can also be used to characterize CPT. The logic CPT as the same expressive power as polynomial-time interpretation logic [18]. This logic essentially applies a first-order interpretation (with an appropriate counting extension) iteratively. The iterative application of an interpretation and thereby the ability to iteratively take quotients strictly increases the expressive power. The interpretation operator that we consider evaluates a formula in the interpreted structure, that is, it evaluates an interpretation only once. By nesting this operator, constantly many interpretations can be nested. Hence, for logics closed under interpretations, the interpretation operator does not increase the expressive power.

Multipedes [24] are a class of asymmetric structures not characterized up to isomorphism in k-variable counting logic for every fixed k. Asymmetry turns multipedes to hard examples for graph isomorphism algorithms in the individualization-refinement framework [34, 1]. The size of a multipede not identifiable in k-variable counting logic is large with respect to k. There also exists a class of asymmetric graphs [8] for which the number of variables needed for identification is linear. Both classes are based on the CFI construction.

There is another remarkable but not directly-connected coincidence to lengths of resolution proofs. Resolution proofs for non-isomorphism of CFI-graphs have exponential size [40]. When adding a global symmetry rule (SRC-I), which exploits automorphisms of the formula (so akin to symmetric choice), the length becomes polynomial [39]. For asymmetric multipedes the length in the SRC-I system is still exponential [41]. But when considering the local symmetry rule (SRC-II) exploiting local automorphisms (so somewhat akin to symmetric choice after restricting to a substructure with an interpretation) the length becomes polynomial again [39].

2 Preliminaries

We set $[k] := \{1, \dots, k\}$. The i-th entry of a k-tuple $\bar{t} \in N^k$ is denoted by t_i and its length by $|\bar{t}| = k$. The set of all tuples of length at most k is $N^{\leq k}$ and the set of all finite tuples is N^*.

A *relational signature* τ is a set of relation symbols $\{R_1, \dots, R_\ell\}$ of *arities* $\mathrm{ar}(R_i)$. We use letters τ and σ for signatures. A *τ-structure* is a tuple $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \dots, R_\ell^{\mathfrak{A}})$ where $R_i^{\mathfrak{A}} \subseteq A^{\mathrm{ar}(R_i)}$. The set A is called *universe* and its elements *vertices*. We use letters $\mathfrak{A}$ and $\mathfrak{B}$ for structures, A and B for their universes, and u, v, and w for vertices. The *reduct* $\mathfrak{A} \restriction \sigma$ is the restriction of $\mathfrak{A}$ to the relations in $\sigma \subseteq \tau$. This paper considers finite structures.

A *colored graph* is an $\{E, \preceq\}$-structure $G = (V, E^G, \preceq^G)$. The relation E is the edge relation and the relation $\preceq$ is a total preorder. Its equivalence classes are the *color classes* or *colors*. We often write $G = (V, E, \preceq)$ for a colored graph. The *neighborhood* of a vertex $u \in V$ is $N_G(u)$. The *induced subgraph* of G by $W \subseteq V$ is $G[W]$. The graph G is *k-connected* if $|V| > k$ and, for every $V' \subseteq V$ of size at most $k - 1$, the graph $G \setminus V'$ is connected. The *treewidth* $\mathrm{tw}(G)$ of G measures how close G is to being a tree. We omit a definition (see [11]) and only use the fact that if G is a *minor* of H, i.e., G is obtained from H by deleting vertices or edges and contracting edges, then $\mathrm{tw}(G) \leq \mathrm{tw}(H)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures and $\bar{u} \in A^k$ and $\bar{v} \in B^k$. We write $(\mathfrak{A}, \bar{u}) \cong (\mathfrak{B}, \bar{v})$ if there is a *isomorphism* $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ satisfying $\varphi(\bar{u}) = \bar{v}$. An *automorphism* φ of $(\mathfrak{A}, \bar{u})$ is an isomorphism $(\mathfrak{A}, \bar{u}) \to (\mathfrak{A}, \bar{u})$. We say that φ *fixes* $\bar{u}$ and write $\mathsf{Aut}((\mathfrak{A}, \bar{u}))$ for the set of all automorphisms fixing $\bar{u}$. We will use the same notation also for other objects, e.g., for automorphisms fixing relations. A *k-orbit* of $(\mathfrak{A}, \bar{u})$ is a maximal set of k-tuples $O \subseteq A^k$ such that for every $\bar{v}, \bar{w} \in O$, there is an automorphism $\varphi \in \mathsf{Aut}((\mathfrak{A}, \bar{u}))$ satisfying $\varphi(\bar{v}) = \bar{w}$.

Fixed-Point Logic with Counting. We recall fixed-point logic with counting IFPC (proposed in [26], see [35]). Let τ be a signature and $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \dots, R_\ell^{\mathfrak{A}})$ be a τ-structure. We extend τ and $\mathfrak{A}$ with counting. Set $\tau^{\#} := \tau \uplus \{\cdot, +, 0, 1\}$ and $\mathfrak{A}^{\#} := (A, R_1^{\mathfrak{A}}, \dots, R_\ell^{\mathfrak{A}}, \mathbb{N}, \cdot, +, 0, 1)$ to be the two-sorted $\tau^{\#}$-structure that is the disjoint union of $\mathfrak{A}$ and $\mathbb{N}$. IFPC$[\tau]$ is a two-sorted logic using the signature $\tau^{\#}$. *Element variables* range over vertices and *numeric variables* range over $\mathbb{N}$. The letters x, y, and z are used for element variables, ν and μ for numeric variables, and s and t for numeric terms. IFPC-formulas are built from first-order formulas, a fixed-point operator, and counting terms. The range of numeric variables needs to be bounded to ensure PTIME-evaluation: For an IFPC-formula Φ, a closed numeric IFPC-term s, a numeric variable ν, and a quantifier $Q \in \{\forall, \exists\}$, the formula $Q\nu \leq s.\, \Phi$ is an IFPC-formula. An *inflationary fixed-point operator* defines a relation R. For an IFPC$[\tau, R]$-formula Φ and variables $\bar{x}\bar{\mu}$, the fixed-point operator $[\mathsf{ifp} R\bar{x}\bar{\mu} \leq \bar{s}.\, \Phi]\,(\bar{\nu}\bar{\mu})$ is an IFPC$[\tau]$-formula. The tuple $\bar{s}$ of $|\bar{\mu}|$ closed numeric terms bounds the values of $\bar{\mu}$. The crucial element of IFPC are *counting terms*. For an IFPC-formula Φ, variables $\bar{x}\bar{\nu}$, and $|\bar{\nu}|$ closed numeric IFPC-terms $\bar{s}$, $\#\bar{x}\bar{\nu} \leq \bar{s}.\, \Phi$ is a numeric IFPC-term.

IFPC-formulas (or terms) are evaluated over $\mathfrak{A}^{\#}$. For a numeric term $s(\bar{x}\bar{\nu})$, the function $s^{\mathfrak{A}}\colon A^{|\bar{x}|} \times \mathbb{N}^{|\bar{\nu}|} \to \mathbb{N}$ maps the values for the free variables of s to the value that s takes in $\mathfrak{A}^{\#}$. Likewise, for a formula $\Phi(\bar{x}\bar{\nu})$, we write $\Phi^{\mathfrak{A}} \subseteq A^{|\bar{x}|} \times \mathbb{N}^{|\bar{\nu}|}$ for the set of values for the free variables satisfying Φ. Evaluating a counting term for a formula $\Phi(\bar{y}\bar{x}\bar{\mu}\bar{\nu})$ is defined as follows: $(\#\bar{x}\bar{\nu} \leq s.\, \Phi)^{\mathfrak{A}}(\bar{u}\bar{m}) := |\{\bar{w}\bar{n} \in A^{|\bar{x}|} \times \mathbb{N}^{|\bar{\nu}|} \mid n_i \leq s_i^{\mathfrak{A}} \text{ for all } i \in [|\bar{\nu}|], \bar{u}\bar{w}\bar{m}\bar{n} \in \Phi^{\mathfrak{A}}\}|$.

Finite Variable Counting Logic. The k-variable logic with counting $\mathcal{C}_k$ extends the k-variable fragment of first-order logic (FO) with counting quantifiers $\exists^{\geq j}x.\, \Phi$ ("at least j distinct vertices satisfy Φ"). For every fixed $n \in \mathbb{N}$, every k-variable IFPC-formula is equivalent to a $\mathcal{C}_{\mathcal{O}(k)}$-formula [35] on structures of order up to n. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures and $\bar{u} \in A^\ell$ and $\bar{v} \in B^\ell$. A logic L *distinguishes* $(\mathfrak{A}, \bar{u})$ from $(\mathfrak{B}, \bar{v})$ if there is an L-formula Φ with ℓ free variables such that $\bar{u} \in \Phi^{\mathfrak{A}}$ and $\bar{v} \notin \Phi^{\mathfrak{B}}$. Otherwise, the structures are *L-equivalent*. We write $(\mathfrak{A}, \bar{u}) \simeq_{\mathcal{C}}^k (\mathfrak{B}, \bar{v})$ if $(\mathfrak{A}, \bar{u})$ and $(\mathfrak{B}, \bar{v})$ are $\mathcal{C}_k$-equivalent. The logics $\mathcal{C}_k$ are used to prove IFPC-undefinability: Let $(\mathfrak{A}_k, \mathfrak{B}_k)$ be a sequence of finite structures for every $k \in \mathbb{N}$ such that $\mathfrak{A}_k$ has a property P but $\mathfrak{B}_k$ does not. If $\mathfrak{A}_k \simeq_{\mathcal{C}}^k \mathfrak{B}_k$ for every k, then IFPC does not define P. The logic $\mathcal{C}_k$ is characterized by the *bijective k-pebble game* [25]. The game is played on two structures $\mathfrak{A}$ and $\mathfrak{B}$ by two players called Spoiler and Duplicator. There are pebble pairs (p_i, q_i) for every $i \in [k]$. Positions in the game are tuples $(\mathfrak{A}, \bar{u}; \mathfrak{B}, \bar{v})$ for tuples $\bar{u} \in A^{\leq k}$ and $\bar{v} \in B^{\leq k}$ of the same length. A pebble p_j is placed on u_i and q_j is placed on v_i for some $j \in [k]$. No pebbles are placed initially. If $|A| \neq |B|$, then Spoiler wins. If not, Spoiler picks up a pair of pebbles (p_i, q_i). Duplicator answers with a bijection $\lambda\colon A \to B$. Spoiler places p_i on $w \in A$ and q_i on $\lambda(w) \in B$. If in the resulting position $(\mathfrak{A}, \bar{u}; \mathfrak{B}, \bar{v})$ the map $u_i \mapsto v_i$ is not a *pebble-respecting* local isomorphism $(\mathfrak{A}[\bar{u}], \bar{u}) \to (\mathfrak{B}[\bar{v}], \bar{v})$, then Spoiler wins. Otherwise, the game continues with the next round. Duplicator wins if Spoiler never wins. Spoiler/Duplicator has a *winning strategy* in position $(\mathfrak{A}, \bar{u}; \mathfrak{B}, \bar{v})$ if they can always win the game. Spoiler has a winning strategy in position $(\mathfrak{A}, \bar{u}; \mathfrak{B}, \bar{v})$ if and only if $(\mathfrak{A}, \bar{u}) \not\simeq_{\mathcal{C}}^k (\mathfrak{B}, \bar{v})$ [25].

Logical Interpretations. Interpretations define maps between structures via formulas evaluated on tuples containing vertices and numbers. We use $\bar{x}$, $\bar{y}$, and $\bar{z}$ for a tuple of element and numeric variables and $\bar{u}$ and $\bar{v}$ for a tuple of vertices and numbers in the following.

Let $\sigma := \{R_1, \ldots, R_\ell\}$. A *d-dimensional* IFPC$[\tau,\sigma]$*-interpretation* $\Theta(\bar{z})$ with parameters $\bar{z}$ is a tuple

$$\Theta(\bar{z}) = \big(\Phi_{\mathrm{dom}}(\bar{z}\bar{x}), \Phi_{\cong}(\bar{z}\bar{x}\bar{y}), \Phi_{R_1}(\bar{z}\bar{x}_1 \ldots \bar{x}_{\mathrm{ar}(R_1)}), \ldots, \Phi_{R_\ell}(\bar{z}\bar{x}_1 \ldots \bar{x}_{\mathrm{ar}(R_\ell)}), \bar{s}\big)$$

of IFPC$[\tau]$-formulas and a j-tuple $\bar{s}$ of closed numeric IFPC$[\tau]$-terms, where j is the number of numeric variables in $\bar{x}$. The tuples of variables $\bar{x}$, $\bar{y}$, and the $\bar{x}_i$ are all of length d and agree on whether the k-th variable is an element variable or not. Let $\mathfrak{A}$ be a τ-structure and $\bar{u} \in (A \cup \mathbb{N})^{|\bar{z}|}$ match the parameter variables (element or numeric). Assume that the first j variables in $\bar{x}$ are numeric variables and set $D := \{0, ..., s_1^{\mathfrak{A}}\} \times \cdots \times \{0, ..., s_j^{\mathfrak{A}}\} \times A^{d-j}$. We define the σ-structure $\mathfrak{B} = (B, R_1^{\mathfrak{B}}, \ldots, R_\ell^{\mathfrak{B}})$ via

$$B := \left\{ \bar{v} \in D \;\middle|\; \bar{u}\bar{v} \in \Phi_{\mathrm{dom}}^{\mathfrak{A}} \right\} \text{ and}$$
$$R_i^{\mathfrak{B}} := \left\{ (\bar{v}_1, \ldots, \bar{v}_{\mathrm{ar}(R_i)}) \in B^{\mathrm{ar}(R_i)} \;\middle|\; \bar{u}\bar{v}_1 \ldots \bar{v}_{\mathrm{ar}(R_i)} \in \Phi_{R_i}^{\mathfrak{A}} \right\}$$

and the relation $E := \{(\bar{v}_1, \bar{v}_2) \in B^2 \mid \bar{u}\bar{v}_1\bar{v}_2 \in \Phi_{\cong}^{\mathfrak{A}}\}$. Set $\Theta(\mathfrak{A}, \bar{u}) := \mathfrak{B}/E$ if E is a congruence relation on $\mathfrak{B}$ and otherwise leave $\Theta(\mathfrak{A}, \bar{u})$ undefined. An interpretation is called *equivalence-free* if $\Phi_{\cong}(\bar{z}\bar{x}\bar{y})$ is the formula $\bar{x} = \bar{y}$. For extensions L of IFPC, the notion of an $L[\tau,\sigma]$-interpretation is defined similarly. For logics without numeric variables like FO or IFP, interpretations are defined analogously omitting the numeric parts.

A property P of τ-structures is *L-reducible* to a property Q of σ-structures if there is a parameter-free $L[\tau,\sigma]$-interpretation Θ such that $\mathfrak{A} \in P$ if and only if $\Theta(\mathfrak{A}) \in Q$ for every τ-structure $\mathfrak{A}$. A logic L' is *closed under L-interpretations* if for every property P that is L-reducible to an L'-definable property Q, the property P itself is L'-definable (cf. [13, 35]).

3 Witnessed Symmetric Choice

We extend IFPC with an inflationary fixed-point operator with witnessed symmetric choice. Let τ be a relational signature and R, R^*, and S be relation symbols not in τ satisfying $k = \mathrm{ar}(R) = \mathrm{ar}(R^*)$. The relation R will be used for stages in the fixed-point computation, R^* for a fixed-point, and S will be a singleton relation containing a chosen tuple.

We define the WSC-fixed-point operator with parameters $\bar{p}\bar{\nu}$. If $\Phi_{\mathrm{step}}(\bar{p}\bar{x}\bar{\nu})$ is an IFPC + WSC$[\tau, R, S]$-formula such that $|\bar{x}| = \mathrm{ar}(R)$, $\Phi_{\mathrm{choice}}(\bar{p}\bar{y}\bar{\nu})$ is a IFPC + WSC$[\tau, R]$-formula such that $|\bar{y}| = \mathrm{ar}(S)$, $\Phi_{\mathrm{wit}}(\bar{p}\bar{y}\bar{y}'z_1z_2\bar{\nu})$ is an IFPC + WSC$[\tau, R, R^*]$-formula where $|\bar{y}| = |\bar{y}'| = \mathrm{ar}(S)$, and $\Phi_{\mathrm{out}}(\bar{p}\bar{\nu})$ is an IFPC + WSC$[\tau, R^*]$-formula, then

$$\Phi(\bar{p}\bar{\nu}) = \mathsf{ifp\text{-}wsc}_{R,\bar{x};R^*;S,\bar{y},\bar{y}';z_1z_2}.\ \big(\Phi_{\mathrm{step}}(\bar{p}\bar{x}\bar{\nu}), \Phi_{\mathrm{choice}}(\bar{p}\bar{y}\bar{\nu}), \Phi_{\mathrm{wit}}(\bar{p}\bar{y}\bar{y}'z_1z_2\bar{\nu}), \Phi_{\mathrm{out}}(\bar{p}\bar{\nu})\big)$$

is an IFPC + WSC$[\tau]$-formula. The formulas Φ_{step}, Φ_{choice}, Φ_{wit}, and Φ_{out} are called *step formula*, *choice formula*, *witnessing formula*, and *output formula* respectively. Note that only element variables are used for defining the fixed-point in the WSC-fixed-point operator. This suffices for our purpose in this paper. We expect that our arguments also work with numeric variables. We omit the free numeric variables $\bar{\nu}$ when defining the semantics of the WSC-fixed-point operator because numeric parameters do not change automorphisms.

Intuitively, Φ is evaluated as follows. The formula Φ defines the set of vertex-tuples $\bar{u}$ that, when interpreting $\bar{p}$ with $\bar{u}$, satisfy the following process: Initialize R as the empty relation. Define a choice-set using the choice formula. Pick an arbitrary tuple of this set and let S

only contain this tuple. Define the next stage from R and S using the step formula. Then set R to be this next stage and repeat until a fixed-point R^* is reached. If the witnessing formula certifies that all choice-set were indeed orbits and the output formula is satisfied by R^*, then $\bar{u}$ satisfies Φ.

We now consider the evaluation in more detail. Let $\mathfrak{A}$ be a τ-structure and $\bar{u} \in A^{|\bar{p}|}$. We define a sequence of relations called *stages* $\emptyset =: R_1^{\mathfrak{A}}, \dots, R_\ell^{\mathfrak{A}} = R_{\ell+1}^{\mathfrak{A}} =: (R^*)^{\mathfrak{A}}$. Given the relation $R_i^{\mathfrak{A}}$, the choice formula defines the choice-set

$$T_{i+1}^{\mathfrak{A}} := \left\{ \bar{v} \;\middle|\; \bar{u}\bar{v} \in \Phi_{\text{choice}}^{\mathfrak{A}, R_i^{\mathfrak{A}}} \right\}.$$

We pick an arbitrary tuple $\bar{v} \in T_{i+1}^{\mathfrak{A}}$ and set $S_{i+1}^{\mathfrak{A}} := \{\bar{v}\}$ or $S_{i+1}^{\mathfrak{A}} := \emptyset$ if $T_{i+1}^{\mathfrak{A}} = \emptyset$. The step formula defines the next stage on the structure $(\mathfrak{A}, R_i^{\mathfrak{A}}, S_{i+1}^{\mathfrak{A}})$:

$$R_{i+1}^{\mathfrak{A}} := R_i^{\mathfrak{A}} \cup \left\{ \bar{w} \;\middle|\; \bar{u}\bar{w} \in \Phi_{\text{step}}^{(\mathfrak{A}, R_i^{\mathfrak{A}}, S_{i+1}^{\mathfrak{A}})} \right\}.$$

We proceed until a fixed-point $(R^*)^{\mathfrak{A}}$ is reached. This fixed-point is in general not isomorphism-invariant, i.e., not invariant under automorphisms of $(\mathfrak{A}, \bar{u})$. Isomorphism-invariance of Φ is ensured as follows: Choices are only allowed from orbits, which is certified by the witnessing formula. A set $N \subseteq \mathsf{Aut}((\mathfrak{A}, \bar{u}))$ *witnesses a relation* $R \subseteq A^k$ *as* $(\mathfrak{A}, \bar{u})$*-orbit*, if for very $\bar{v}, \bar{v}' \in R$ there is a $\varphi \in N$ with $\bar{v} = \varphi(\bar{v}')$. Because we consider isomorphism-invariant sets, the relation R is never a proper subset of an orbit. We require that Φ_{wit} defines a set of automorphisms. For $\bar{v}, \bar{v}' \in T_{i+1}^{\mathfrak{A}}$, a map $\varphi_{\bar{v},\bar{v}'}$ is defined by

$$w \mapsto w' \text{ whenever } \bar{u}\bar{v}\bar{v}'ww' \in \Phi_{\text{wit}}^{(\mathfrak{A}, R_i^{\mathfrak{A}}, (R^*)^{\mathfrak{A}})}.$$

The set $\{\varphi_{\bar{v},\bar{v}'} \mid \bar{v}, \bar{v}' \in T_{i+1}^{\mathfrak{A}}\}$ has to witness $T_{i+1}^{\mathfrak{A}}$ as $(\mathfrak{A}, \bar{u}, R_1^{\mathfrak{A}}, \dots, R_i^{\mathfrak{A}})$-orbit. The witnessing formula always has access to the fixed-point (note that orbits are witnessed after the fixed-point is computed). This is possible because $T_{i+1}^{\mathfrak{A}}$ is an $(\mathfrak{A}, \bar{u}, R_1^{\mathfrak{A}}, \dots, R_i^{\mathfrak{A}})$-orbit and not just an $(\mathfrak{A}, \bar{u}, R_i^{\mathfrak{A}})$-orbit. If some choice is not witnessed, then $\bar{u} \notin \Phi^{\mathfrak{A}}$. Otherwise, the output formula is evaluated on the fixed-point:

$$\Phi^{\mathfrak{A}} := \Phi_{\text{out}}^{(\mathfrak{A}, (R^*)^{\mathfrak{A}})}.$$

Because all choices are witnessed, all possible fixed-points (for different choices) are related by an automorphism of $(\mathfrak{A}, \bar{u})$ and thus either all or none satisfy the output formula.

An Example. We give an illustrating example (an adaption of [32]). We show that the class of threshold graphs (i.e., graphs which can be reduced to the empty graph by iteratively deleting universal or isolated vertices) is IFPC+WSC-definable (it is actually IFP-definable). The set of all isolated or universal vertices of a graph forms a 1-orbit (there cannot be an isolated and a universal vertex in a nontrivial graph). We chose one such vertex, collect it in a unary relation R, and repeat as follows: The choice formula

$$\Phi_{\text{choice}}(y) := \neg R(y) \land \big((\forall z.\, \neg R(y) \Rightarrow E(y,z)) \lor (\forall z.\, \neg R(y) \Rightarrow \neg E(y,z))\big)$$

defines the set of all isolated or universal vertices after deleting the vertices in R. The step formula $\Phi_{\text{step}}(x) := R(x) \lor S(x)$ adds the chosen vertex contained in the relation S to R. The output formula $\Phi_{\text{out}} := \forall x.\, R^*(x)$ defines whether it was possible to delete all vertices. Witnessing orbits is easy: To show that two isolated (or universal) vertices y and y' are related by an automorphism, it suffices to define their transposition via

$$\Phi_{\text{wit}}(y, y', z_1, z_2) := (z_1 = y \land z_2 = y') \lor (z_2 = y \land z_1 = y') \lor (y \neq z_1 = z_2 \neq y').$$

The formula $\mathsf{ifp\text{-}wsc}_{R,x;R^*;S,y,y';z_1z_2}.\,(\Phi_{\text{step}}, \Phi_{\text{choice}}, \Phi_{\text{wit}}, \Phi_{\text{out}})$ defines the class of threshold graphs.

Reduct Semantics. The semantics is defined formally using the WSC*-operator from [32]. It formalizes the former evaluation strategy. The WSC-fixed-point operator Φ is evaluated in the structure $\mathfrak{A} \upharpoonright \mathrm{sig}(\Phi)$, where $\mathrm{sig}(\Phi)$ are all relations symbols used in Φ. So adding a relation to $\mathfrak{A}$ that is not mentioned in Φ but possibly changes the orbits of $\mathfrak{A}$ does not change $\Phi^{\mathfrak{A}}$. This is a desirable property of a logic [13]. The *reduct semantics* of a choice operator can also be found in [9].

Extension with an Operator for Logical Interpretations. We extend IFPC+WSC with another operator using interpretations. Every IFPC+WSC-formula is an IFPC+WSC+I-formula. If $\Theta(\bar{p}\bar{\nu})$ is an IFPC+WSC+I$[\tau,\sigma]$-interpretation with parameters $\bar{p}\bar{\nu}$ and Φ is an IFPC+WSC+I$[\sigma]$-sentence, then the *interpretation operator*

$$\Psi(\bar{p}\bar{\nu}) := \mathsf{I}(\Theta(\bar{p}\bar{\nu}); \Phi)$$

is an IFPC+WSC+I$[\tau]$-formula with free variables $\bar{p}\bar{\nu}$. The semantics is defined by

$$\mathsf{I}(\Theta(\bar{p}\bar{\nu}); \Phi)^{\mathfrak{A}} := \left\{ \bar{u}\bar{n} \in A^{|\bar{p}|} \times \mathbb{N}^{|\bar{\nu}|} \;\middle|\; \Phi^{\Theta(\mathfrak{A},\bar{u}\bar{n})} \neq \emptyset \right\}.$$

Note that $\Phi^{\Theta(\mathfrak{A},\bar{u}\bar{n})} = \{()\}$ if Φ is satisfied and $\Phi^{\Theta(\mathfrak{A},\bar{u}\bar{n})} = \emptyset$ otherwise. The interpretation operator evaluates a subformula in the image of an interpretation. For IFPC, such an operator does not increase expressiveness because IFPC is closed under interpretations [35]. For IFPC+WSC this is not clear: Because $\Theta(\mathfrak{A}, \bar{u}\bar{n})$ may have a different automorphism structure, Φ possibly can exploit the WSC-fixed-point operator in a way impossible on $\mathfrak{A}$. Indeed, we will see that IFPC+WSC is not even closed under FO-interpretations. We now study the properties of IFPC+WSC+I and its relation to IFPC+WSC.

4 The CFI Construction

We give an overview of the CFI construction. For more details, we refer to [5]. The *degree-d CFI gadget* consists of d pairs of *edge vertices* $\{a_{i0}, a_{i1}\}$ for every $i \in [d]$ and the set of *gadget vertices* $\{\bar{b} \in \mathbb{F}_2^d \mid b_1 + \dots + b_d = 0\}$. There is an edge $\{a_{ij}, \bar{b}\}$ if and only if $b_i = j$. If we use d colors to additionally color the vertices $\{a_{i0}, a_{i1}\}$ for every $i \in [d]$, then the CFI gadget realizes precisely the automorphisms exchanging the vertices $\{a_{i0}, a_{i1}\}$ for an even number of $i \in [d]$. We later need a relational variant of the CFI gadgets in which every gadget vertex $\bar{b}$ is replaced by the d-tuple $(a_{1b_1}, \dots, a_{db_d})$. This gadget has the same automorphisms [24]. A *base graph* is a simple connected graph. Let $G = (V, E, \preceq)$ be a colored base graph. We call V *base vertices* and E *base edges* and use fraktur letters for base vertices or edges. For $f \colon E \to \mathbb{F}_2$, we construct the *CFI graph* $\mathsf{CFI}(G, f)$ as follows: Replace every base vertex by a CFI gadget of the same degree. For every base edge $\{\mathfrak{u}, \mathfrak{v}\} \in E$, we obtain two edge-vertex-pairs $\{a_{i0}, a_{i1}\}$ and $\{a'_{j0}, a'_{j1}\}$. The first one is given by the gadget of $\mathfrak{u}$ and the second one by the gadget of $\mathfrak{v}$. Now add the edges $\{a_{ik}, a'_{j\ell}\}$ satisfying $k + \ell = f(\{\mathfrak{u}, \mathfrak{v}\})$. The gadget vertices of the gadget for a base vertex $\mathfrak{u}$ *originate* from $\mathfrak{u}$, the edge vertices $\{a_{i0}, a_{i1}\}$ originate from $(\mathfrak{u}, \mathfrak{v})$, and the edge vertices $\{a'_{j0}, a'_{j1}\}$ originate from $(\mathfrak{v}, \mathfrak{u})$. The color of edge and gadget vertices is obtained from the color of its origin.

It is known [5] that $\mathsf{CFI}(G, g) \cong \mathsf{CFI}(G, f)$ if and only if $\sum g := \sum_{\mathfrak{e} \in E} g(\mathfrak{e}) = \sum f$. If we are interested in the graph up to isomorphism, we write $\mathsf{CFI}(G, 0)$ and $\mathsf{CFI}(G, 1)$. CFI graphs with $\sum g = 0$ are called *even* and the others *odd*. A base edge $\mathfrak{e} \in E$ is *twisted* by f and g if $g(\mathfrak{e}) \neq f(\mathfrak{e})$. Twisted edges can be moved by path isomorphisms [29]: If $\mathfrak{u}_1, \dots, \mathfrak{u}_\ell$ is a path in G, then there is an isomorphism $\varphi \colon \mathsf{CFI}(G, g) \to \mathsf{CFI}(G, g')$, where $g'(\mathfrak{e}) = g(\mathfrak{e})$ apart from

$\mathfrak{e}_1 := \{\mathfrak{u}_1, \mathfrak{u}_2\}$ and $\mathfrak{e}_2 := \{\mathfrak{u}_{\ell-1}, \mathfrak{u}_\ell\}$ satisfying $g'(\mathfrak{e}_i) = g(\mathfrak{e}_i) + 1$. If G is totally ordered, then every isomorphism is composed of path-isomorphisms. When considering a cycle instead of a path, we obtain an automorphism of $\mathsf{CFI}(G, g)$.

For a class of base graphs $\mathcal{K}$, set $\mathsf{CFI}(\mathcal{K}) := \{\mathsf{CFI}(G, g) \mid G = (V, E) \in \mathcal{K}, g \colon E \to \mathbb{F}_2\}$. The *CFI query* for $\mathsf{CFI}(\mathcal{K})$ is to decide whether a given CFI graph in $\mathsf{CFI}(\mathcal{K})$ is even.

▶ **Lemma 4** ([11]). *If G is of minimum degree 2 and has treewidth at least k, in particular if G is k-connected, then $\mathsf{CFI}(G, 0) \simeq^k_{\mathcal{C}} \mathsf{CFI}(G, 1)$.*

▶ **Lemma 5.** *Let $G = (V, E, \preceq)$ be $(k+2)$-connected and $\mathfrak{A} = \mathsf{CFI}(G, f)$ for some $f \colon E \to \mathbb{F}_2$. Let $\bar{u} \in A^{\leq k}$ and $\{\mathfrak{u}, \mathfrak{v}\} \in E$ be a base edge such that no vertex in $\bar{u}$ has origin $\mathfrak{u}$, $\mathfrak{v}$, $(\mathfrak{u}, \mathfrak{v})$, or $(\mathfrak{v}, \mathfrak{u})$. Then the two edge vertices with origin $(\mathfrak{u}, \mathfrak{v})$ are contained in the same orbit of $(\mathfrak{A}, \bar{u})$.*

5 Canonization of CFI Graphs in IFPC+WSC+I

We show that canonizing CFI graphs in IFPC+WSC+I is not harder than canonizing the base graphs. We work with a class of base graphs closed under individualization, i.e., intuitively closed under assigning new unique colors to some vertices.

▶ **Definition 6** (Individualization of Vertices). *Let $\mathfrak{A}$ be a τ-structure. A binary relation $\trianglelefteq^{\mathfrak{A}} \subseteq A^2$ is an* individualization *of $V \subseteq A$ if $\trianglelefteq^{\mathfrak{A}}$ is a total order on V and $\trianglelefteq^{\mathfrak{A}} \subseteq V^2$. We say that $\trianglelefteq^{\mathfrak{A}}$ is an individualization if it is an individualization of some $V \subseteq A$ and that a vertex $u \in A$ is* individualized *by $\trianglelefteq^{\mathfrak{A}}$ if $u \in V$.*

The *closure under individualization* of a class of τ-structures $\mathcal{K}$ is the class $\mathcal{K}^{\trianglelefteq}$ of $(\tau \uplus \{\trianglelefteq\})$-structures such that $(\mathfrak{A}, \trianglelefteq^{\mathfrak{A}}) \in \mathcal{K}^{\trianglelefteq}$ for every $\mathfrak{A} \in \mathcal{K}$ and every individualization $\trianglelefteq^{\mathfrak{A}}$. Instead of $(\mathfrak{A}, \trianglelefteq^{\mathfrak{A}})$, one can think of $(\mathfrak{A}, \bar{u})$ where $u_1 \trianglelefteq^{\mathfrak{A}} \cdots \trianglelefteq^{\mathfrak{A}} u_{|\bar{u}|}$ are the $\trianglelefteq^{\mathfrak{A}}$-individualized vertices. In the following, let L be one of the logics IFPC, IFPC+WSC, or IFPC+WSC+I. We adapt some notions related to canonization from [32] to our first-order setting. Note that all definitions that follow implicitly include the closure under individualization.

▶ **Definition 7** (Canonization). *Let $\mathcal{K}$ be a class of τ-structures. An L-*canonization *for $\mathcal{K}$ is an $L[\tau \uplus \{\trianglelefteq\}, \tau \uplus \{\trianglelefteq, \leq\}]$-interpretation Θ such that $\leq^{\Theta(\mathfrak{A})}$ is a total order on $\Theta(\mathfrak{A})$ for every $\mathfrak{A} \in \mathcal{K}^{\trianglelefteq}$, $\mathfrak{A} \cong \Theta(\mathfrak{A}) \restriction (\tau \uplus \{\trianglelefteq\})$ for every $\mathfrak{A} \in \mathcal{K}^{\trianglelefteq}$, and $\Theta(\mathfrak{A}) \cong \Theta(\mathfrak{B})$ if and only if $\mathfrak{A} \cong \mathfrak{B}$ for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}^{\trianglelefteq}$. L* canonizes *$\mathcal{K}$ if there is an L-canonization for $\mathcal{K}$.*

▶ **Definition 8** (Distinguishable Orbits). *The logic L* distinguishes k-orbits *for a class of τ-structures $\mathcal{K}$ if some $L[\tau \uplus \{\trianglelefteq\}]$-formula $\Phi(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}| = k$ defines, for every $\mathfrak{A} \in \mathcal{K}^{\trianglelefteq}$, a total preorder on A^k whose equivalence classes form the k-orbit partition of $\mathfrak{A}$, i.e., Φ orders the k-orbits.*

▶ **Definition 9** (Ready for Individualization). *A class of τ-structures $\mathcal{K}$ is* ready for individualization in *L if there is an $L[\tau \uplus \{\trianglelefteq\}]$-formula $\Phi(x)$ defining for every $\mathfrak{A} \in \mathcal{K}^{\trianglelefteq}$ a set of vertices $O = \Phi^{\mathfrak{A}}$ such that O is a 1-orbit of $\mathfrak{A}$, $|O| > 1$ if $\mathfrak{A}$ has a non-trivial 1-orbit, and if $O = \{u\}$ is a singleton set, then u is not individualized by $\trianglelefteq^{\mathfrak{A}}$ unless $\trianglelefteq^{\mathfrak{A}}$ individualizes A.*

Let L be one logic of IFPC+WSC and IFPC+WSC+I. The following lemma is similar to [32] for CPT+WSC and the proof is analogous (we do not include definable isomorphism):

▶ **Lemma 10.** *Let $\mathcal{K}$ be a class of τ-structures. Then the following are equivalent:*

1. *L defines a canonization for $\mathcal{K}$.*
2. *L distinguishes the k-orbits of $\mathcal{K}$ for every $k \in \mathbb{N}$.*
3. *$\mathcal{K}$ is ready for individualization in L.*

Gurevich's canonization algorithm [23] is used to define the canonization. It requires the WSC-fixed-point operator. When canonizing using Lemma 10, defining witnessing automorphisms is hidden in Gurevich's algorithm and they do not have to be defined explicitly.

▶ **Lemma 11.** *Let $\mathcal{K}$ be a class of colored base graphs. If* IFPC+WSC+I *distinguishes* 2*-orbits of $\mathcal{K}$, then* $\mathsf{CFI}(\mathcal{K})$ *is ready for individualization in* IFPC+WSC+I*.*

Proof Sketch. For a CFI graph $\mathfrak{A}$ over $G \in \mathcal{K}^{\trianglelefteq}$, the base graph G is definable by an IFPC-interpretation. With an interpretation operator, we evaluate a 2-orbit-defining formula on G.

a) If there is a 2-orbit O of G such that for all edges in O there is a cycle not using the origins of individualized vertices, we define the non-trivial orbit of the CFI graph containing the edge-vertex-pairs with origin in O.

b) Otherwise, we can order each edge-vertex-pair. If there is a non-trivial 2-orbit O of G, we define the non-trivial orbit of the CFI graph containing the greater edge vertex of each edge-vertex-pair (with respect to the edge-vertex-pair-order) with origin in O.

c) Otherwise, the edge-vertex-pair-order extends to a total order. ◀

Proof of Theorem 2. The claim follows immediately from Lemmas 10 and 11. ◀

▶ **Corollary 12.** *If $\mathcal{K}$ is a class of base graphs of bounded degree, then* IFPC+WSC+I *defines canonization for $\mathcal{K}$ if and only if* IFPC+WSC+I *defines canonization for* $\mathsf{CFI}(\mathcal{K})$*.*

Proof. One direction is by Theorem 2. For the other direction, a base graph G of bounded degree is canonized by defining $\mathsf{CFI}(G, 0)$ (which is possible because G is of bounded degree), canonizing $\mathsf{CFI}(G, 0)$, and defining the base graph of the ordered copy of $\mathsf{CFI}(G, 0)$. ◀

Theorem 2 can be applied iteratively: If IFPC+WSC+I canonizes $\mathcal{K}$, then IFPC+WSC+I canonizes $\mathsf{CFI}(\mathcal{K})$, and so IFPC+WSC+I canonizes $\mathsf{CFI}(\mathsf{CFI}(\mathcal{K}))$. Every iteration adds one WSC-fixed-point operator (Gurevich's algorithm in Lemma 10) and one interpretation operator (define the base graph in Lemma 11), i.e., the nesting depth of these operators increases.

6 The CFI Query and Nesting of Operators

We show that the increased nesting depth of operators in Theorem 2 is unavoidable. If IFPC distinguishes orbits of the base graphs, then the nesting depth of WSC-fixed-point operators has to increase because IFPC does not define the CFI query. To show this for IFPC+WSC+I-distinguishable orbits, we combine non-isomorphic CFI graphs into a new base graph and apply the CFI construction again. To define orbits of these double CFI graphs, one has to define the CFI query for the base CFI graphs, which requires a WSC-fixed-point operator. However, parameters to WSC-fixed-point operators will complicate matters.

Nested WSC-Fixed-Point and Interpretation Operators. Let IFPC $\subseteq L \subseteq$ IFPC+WSC+I. We write WSC(L) for formulas composed by IFPC-formula-formation rules from L-formulas and WSC-fixed-point-operators, for which all subformulas are L-formulas. We define I(L) similarly: One can use interpretation operators $I(\Theta, \Psi)$ where Θ is an L-interpretation and Ψ is an L-formula. We set $\mathrm{WSCI}(L) := \mathrm{WSC}(\mathrm{I}(L))$ and $\mathrm{WSCI}^{k+1}(L) := \mathrm{WSCI}(\mathrm{WSCI}^{k}(L))$. Note the construction in Lemmas 10 and 11:

▶ **Corollary 13.** *Let $\mathcal{K}$ be a class of base graphs.*

1. *If L distinguishes* 2*-orbits of $\mathcal{K}$, then* $\mathsf{CFI}(\mathcal{K})$ *is ready for individualization in* I(L)*.*
2. *If* $\mathsf{CFI}(\mathcal{K})$ *is ready for individualization in L, then* WSC(L) *canonizes* $\mathsf{CFI}(\mathcal{K})$*.*
3. *If L distinguishes* 2*-orbits of $\mathcal{K}$, then* WSCI(L) *canonizes* $\mathsf{CFI}(\mathcal{K})$*.*

Color Class Joins. Let $G_1, \dots, G_\ell$ be connected colored graphs such that all G_i have c colors. The *color class join* $J_{\mathrm{cc}}(G_1, \dots, G_\ell)$ is the following graph: Start with the disjoint union of the G_i and add c additional vertices $u_1, \dots, u_c$. Add, for every $i \in [c]$, edges between u_i and every vertex v in the i-th color class of all G_j. The resulting colored graph $J_{\mathrm{cc}}(G_1, \dots, G_\ell)$ has $2c$ color classes: For every $i \in [c]$, the vertex u_i forms a singleton color class and the union of the i-th color classes of every G_j forms a color class. The G_j are the *parts* of $J_{\mathrm{cc}}(G_1, \dots, G_\ell)$. The u_j are the *join vertices* and the others the *part vertices.* The *part* of a part vertex v is the G_j containing v. Defining orbits of $J_{\mathrm{cc}}(G_1, \dots, G_\ell)$ is at least as hard as defining isomorphism of the G_j.

▶ **Lemma 14.** *If two part vertices v and v' are in the same orbit of $J_{cc}(G_1, \dots, G_\ell)$, then the part of v is isomorphic to the one of v'.*

We set $J^k_{\mathrm{cc}}(G, H, K) := J_{\mathrm{cc}}(G, \dots, G, H, \dots, H, K, \dots, K)$, where G, H, and K are repeated k times. To consider color class joins of CFI graphs, let $\mathcal{K}$ be a class of colored base graphs. For $G \in \mathcal{K}$ and $g \in \mathbb{F}_2$, we set

$$\begin{aligned} \mathsf{CFI}^k(G, g) &:= J^k_{\mathrm{cc}}(\mathsf{CFI}(G, 0), \mathsf{CFI}(G, g), \mathsf{CFI}(G, 1)), \\ \mathsf{CFI}^k(\mathcal{K}) &:= \left\{ \mathsf{CFI}^k(G, g) \;\middle|\; G \in \mathcal{K}, g \in \mathbb{F}_2 \right\}, \\ \mathsf{CFI}^\omega(\mathcal{K}) &:= \bigcup_{k \in \mathbb{N}} \mathsf{CFI}^k(\mathcal{K}). \end{aligned}$$

▶ **Lemma 15.** *If L canonizes $\mathsf{CFI}(\mathcal{K})$, then L canonizes $\mathsf{CFI}^\omega(\mathcal{K})$.*

Let $G_1, \dots, G_\ell$ be colored base graphs and $h \in \mathbb{F}_2$. We transfer the notion of part and join vertices from $H := J_{\mathrm{cc}}(G_1, \dots, G_\ell)$ to $\mathfrak{A} := \mathsf{CFI}(H, h)$. The *$G_i$-part* of $\mathfrak{A}$ is the set of vertices originating from G_i in H. These vertices are called *part vertices of* G_i. A vertex is a part vertex, if it is a part vertex of some G_i. The remaining vertices are the *join vertices.*

We consider a special class of individualizations of $\mathfrak{A}$. Let $\bar{u} \in A^*$. A part of $\mathfrak{A}$ is *pebbled by* $\bar{u}$ if the part contains u_i for some i. The set of *$\bar{u}$-pebbled-part vertices* $V_{\bar{u}}(\mathfrak{A})$ is the set of all join vertices and all part vertices of a part pebbled by $\bar{u}$. The set of *$\bar{u}$-pebbled-part individualizations* $P_{\bar{u}}(\mathfrak{A})$ is the set of all individualizations of $V_{\bar{u}}(\mathfrak{A})$.

▶ **Definition 16** (Unpebbled-Part-Distinguishing)**.** *For a tuple $\bar{u} \in A^\ell$, a relation $R \subseteq A^k$ is $\bar{u}$-*unpebbled-part-distinguishing *if there are $m \in [k]$ and $i \neq j \in [\ell]$ such that both the G_i-part and the G_j-part of $\mathfrak{A}$ are $\bar{u}$-unpebbled, there is a $\bar{v} \in R$ such that v_m is a part vertex of G_i, and for every $\bar{w} \in R$, the vertex w_m is not a part vertex of G_j.*

If $G_i \not\cong G_j$ are not $\bar{u}$-pebbled, then every k-orbit O of $(\mathfrak{A}, \bar{u})$ satisfies $O \subseteq V_{\bar{u}}(\mathfrak{A})^k$ or is $\bar{u}$-unpebbled-part-distinguishing because G_i- and G_j-part vertices are not in the same orbit.

Quantifying over Pebbled-Part Individualizations. We now define an extension of $\mathcal{C}_k$ which allows for quantifying over pebbled-part individualizations. This (unnatural) extension can only be evaluated on CFI graphs over color class joins. We use this logic for proving WSCI(IFPC)-undefinability. If $\Phi(\bar{x})$ is a $\mathcal{C}_k[\tau, \trianglelefteq_P]$-formula, then $(\exists^P \trianglelefteq_P. \Phi)(\bar{x})$ is a $\mathcal{P}_k[\tau]$-formula. $\mathcal{P}_k[\tau]$-formulas can be combined as usual in $\mathcal{C}_k$ with Boolean operators and counting quantifiers. Note that $\exists^P$-quantifiers *cannot* be nested. Let $G_1, \dots, G_\ell$ be base graphs, $g \in \mathbb{F}_2$, and $\mathfrak{A} = \mathsf{CFI}(J_{\mathrm{cc}}(G_1, \dots, G_n), g)$. The $\exists^P$-quantifier has the following semantics:

$$(\exists^P \trianglelefteq_P. \Phi)^{\mathfrak{A}} := \left\{ \bar{u} \;\middle|\; \bar{u} \in \Phi^{(\mathfrak{A}, \trianglelefteq_P^{\mathfrak{A}})} \text{ for some } \trianglelefteq_P^{\mathfrak{A}} \in P_{\bar{u}}(\mathfrak{A}) \right\}.$$

▶ **Lemma 17.** *Let $G_1, \dots, G_{k+1}$ be colored base graphs, each with $c > k \geq 3$ color classes, such that $\mathsf{CFI}(G_i, 0) \simeq^k_{\mathcal{C}} \mathsf{CFI}(G_i, 1)$ for every $i \in [k+1]$. Then $\mathsf{CFI}(J_{cc}(G_1, \dots, G_{k+1}), 0)$ and $\mathsf{CFI}(J_{cc}(G_1, \dots, G_{k+1}), 1)$ are $\mathcal{P}_k$-equivalent.*

Proof Sketch. The lemma is proven by a game characterization of $\mathcal{P}_k$. Essentially, because only k of the $k+1$ parts can be pebbled by k pebbles, the twist can always be moved in the pebble-free part, which is not affected by quantifying over pebbled-part individualizations. ◀

Nesting Operators to Define the CFI Query is Necessary. Let $\mathcal{K} := \{G_k \mid k \in \mathbb{N}\}$ be a set of ordered 3-regular base graphs such that G_k has treewidth at least k for every $k \in \mathbb{N}$.

▶ **Lemma 18.** $\mathsf{CFI}(\mathsf{CFI}(G_k, g), 0) \simeq^k_{\mathcal{C}} \mathsf{CFI}(\mathsf{CFI}(G_k, g), 1)$ *for every $k \in \mathbb{N}$ and $g \in \mathbb{F}_2$.*

Proof. The graph G_k is a minor of $\mathsf{CFI}(G_k, g)$ for every $g \in \mathbb{F}_2$ (cf. [11]). Hence, $\mathsf{CFI}(G_k, g)$ has treewidth at least k. The claim follows by Lemma 4. ◀

▶ **Lemma 19.** WSCI^2(IFPC) *defines the CFI query for* $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$.

Proof. IFPC distinguishes 2-orbits of $\mathcal{K}$ and so WSCI(IFPC) canonizes $\mathsf{CFI}(\mathcal{K})$ (Corollary 13) and $\mathsf{CFI}^\omega(\mathcal{K})$ (Lemma 15) and so also distinguishes 2-orbits of $\mathsf{CFI}^\omega(\mathcal{K})$. Thus, WSCI^2(IFPC) canonizes $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$ (Corollary 13) and hence defines the CFI query for $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$. ◀

To show WSCI(IFPC)-undefinability, there are two cases: If a choice is made from an orbit of parts not pebbled by parameters, then CFI graphs of $\mathsf{CFI}(\mathcal{K})$ are distinguished. Otherwise, CFI graphs of $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$ are distinguished only by choices from parameter-pebbled parts, so by (at most) individualizing all pebbled-part vertices, i.e., the graphs are distinguished by $\mathcal{P}_k$. By Lemmas 4 and 17, each case only applies to finitely many $\mathcal{K}$-graphs.

▶ **Lemma 20.** WSCI(IFPC) *does not define the CFI query for* $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$.

Proof Sketch. Suppose, towards a contradiction, that Φ is a WSCI(IFPC)-formula defining the CFI query for $\mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$. Because IFPC is closed under interpretations, we can assume that Φ is a WSC(IFPC)-formula.

Let $\Psi_1(\bar{x}_1), \dots, \Psi_p(\bar{x}_p)$ be all WSC-fixed-point operators in Φ and suppose all $\bar{x}_i$ are element variables (for numeric ones see the full version [30]). Let the number of distinct variables of Φ be k and let $\ell := \ell(k) \geq \max\{k, 3\}$. We consider the subclass $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(\mathcal{K})) \subseteq \mathsf{CFI}(\mathsf{CFI}^\omega(\mathcal{K}))$ and partition $\mathcal{K}$ as follows: Let $\mathcal{K}_{\mathrm{orb}}$ be the set of all $G \in \mathcal{K}$ such that, for every $g \in \mathbb{F}_2$, there are $h \in \mathbb{F}_2$, $j \in [p]$, and a $|\bar{x}_j|$-tuple $\bar{u}$ of $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), h))$ such that

a) all choice-sets during the evaluation of $\Psi_j(\bar{u})$ on $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), h)$ are orbits and
b) some choice-set is $\bar{u}$-unpebbled-part-distinguishing.

Set $\mathcal{K}_{\mathrm{cfi}} := \mathcal{K} \setminus \mathcal{K}_{\mathrm{orb}}$. At least one of $\mathcal{K}_{\mathrm{orb}}$ and $\mathcal{K}_{\mathrm{cfi}}$ is infinite. Assume that $\mathcal{K}_{\mathrm{orb}}$ is infinite. We claim that the CFI query for $\mathsf{CFI}(\mathcal{K}_{\mathrm{orb}})$ is IFPC-definable. There is an IFPC-interpretation that for $G \in \mathcal{K}$ maps $\mathsf{CFI}(G, g)$ and $h \in \mathbb{F}_2$ to $(\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), h), \trianglelefteq)$ such that $\trianglelefteq$ individualizes the vertices of $\ell+1$ many $\mathsf{CFI}(G, 0)$-parts, $\ell+1$ many $\mathsf{CFI}(G, 1)$-parts, and all join vertices. For every $\Psi_j(\bar{x}_j)$, we try all $h \in \mathbb{F}_2$ and all tuples $\bar{u}$ of $\trianglelefteq$-individualized vertices for $\bar{x}_j$. We simulate Ψ_j as long as all choices are made from $\bar{u}$-pebbled parts (which are resolved using $\trianglelefteq$). If that is not the case, we check if the choice-set is $\bar{u}$-unpebbled-part distinguishing. If not, the choice-set is not an orbit and we evaluate to false. Otherwise, the choice-set contains vertices of $\mathsf{CFI}(G, g)$-parts and either of $\mathsf{CFI}(G, 0)$-parts or of $\mathsf{CFI}(G, 1)$-parts. At least one $\mathsf{CFI}(G, 0)$-part and one $\mathsf{CFI}(G, 1)$-part is not $\bar{u}$-pebbled because $|\bar{u}| \leq k < \ell + 1$, which is isomorphic to the $\mathsf{CFI}(G, g)$-parts. So we defined the parity of $\mathsf{CFI}(G, g)$ and IFPC defines the CFI query for $\mathsf{CFI}(\mathcal{K}_{\mathrm{orb}})$ contradicting Lemma 4.

So $\mathcal{K}_{\text{cfi}}$ must be infinite. Then there is an $\ell' > \ell$ such that $G := G_{\ell'} \in \mathcal{K}_{\text{cfi}}$. Hence there is a $g \in \mathbb{F}_2$ such that for all $h \in \mathbb{F}_2$, $j \in [p]$, and all $|\bar{x}_j|$-tuples $\bar{u}$ of $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), h))$

a) some choice-set during the evaluation of $\Psi_j(\bar{u})$ on $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), h)$ is not an orbit or

b) all choice-sets are not $\bar{u}$-unpebbled-part-distinguishing.

We construct a $\mathcal{P}_\ell$-formula equivalent to Φ on $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), 0)$ and $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), 1)$. The general idea is that we quantify over all $\bar{u}$-pebbled-part individualizations. Choices from choice-sets using only vertices of the $\bar{u}$-pebbled parts can be resolved deterministically using the individualization. If this is not always the case, one choice-set will not be an orbit and we evaluate to false. So a $\mathcal{P}_\ell$-formula distinguishes $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), 0)$ and $\mathsf{CFI}(\mathsf{CFI}^{\ell+1}(G, g), 1)$. This finally contradicts Lemma 17: The graphs G and $\mathsf{CFI}(G, g')$ have more than ℓ color classes for every $g' \in \mathbb{F}_2$ and we have $\mathsf{CFI}(\mathsf{CFI}(G, g'), 0) \simeq^k_{\mathcal{C}} \mathsf{CFI}(\mathsf{CFI}(G, g'), 1)$ by Lemma 18. ◀

Proof of Theorem 3. Consider $\mathsf{CFI}^\omega(\mathcal{K})$. The claim follows from Lemmas 15, 19, and 20. ◀

▶ **Corollary 21.** IFPC < WSCI(IFPC) < WSCI^2(IFPC).

It seems natural that WSCI^n(IFPC) < WSCI^{n+1}(IFPC) for every $n \in \mathbb{N}$. Possibly, this hierarchy can be shown by iterating our construction (e.g., using $\mathsf{CFI}((\mathsf{CFI}^\omega)^n(\mathcal{K}))$).

7 Separating IFPC+WSC from IFPC+WSC+I

We define a class of *asymmetric* structures $\mathcal{K}$, i.e., structures without non-trivial automorphisms, for which isomorphism can be reduced to isomorphism of CFI graphs via an interpretation. To do so, we combine CFI graphs and the so-called multipedes.

7.1 Multipedes

We review the multipedes construction [24]. Let $G = (V, W, E, \leq)$ be an ordered bipartite graph, where every vertex in V has degree 3. We obtain the *multipede* $\mathsf{MP}(G)$ as follows. For every vertex $\mathfrak{u} \in W$, there is a vertex pair $F(\mathfrak{u}) = \{u_0, u_1\}$ called a *segment*. We also call $\mathfrak{u} \in W$ a segment. A single vertex u_i is a *foot*. Vertices $\mathfrak{v} \in V$ are *constraint vertices*. For every constraint vertex $\mathfrak{v} \in V$, a degree-3 CFI gadget with three edge-vertex-pairs $\{a^j_0, a^j_1\}$ ($j \in [3]$) is added. Let $N_G(\mathfrak{v}) = \{\mathfrak{u}^1, \mathfrak{u}^2, \mathfrak{u}^3\}$. Then a^j_i is identified with the foot u^j_i for all $j \in [3]$ and $i \in \mathbb{F}_2$. We use the relation-based CFI gadgets, i.e., we do not add further vertices. All base constraint vertices have degree 3 and we obtain a ternary $\{R, \preceq\}$-structure. The coloring $\preceq$ is obtained from $\leq$ such that feet in the same segment have the same color.

The *feet-induced* subgraph by $X \subseteq W$ is $G[[X]] := G[X \cup \{\mathfrak{v} \in V \mid N_G(\mathfrak{v}) \subseteq X\}]$. We extend the notation to the multipede: $\mathsf{MP}(G)[[X]]$ is the induced substructure of all feet whose segment is contained in X. For a tuple $\bar{u}$ of feet of $\mathsf{MP}(G)$, we define $S(\bar{u}) := \{\mathfrak{u} \in W \mid u_i \in F(\mathfrak{u}) \text{ for some } i \leq |\bar{u}|\}$ to be the set of the segments of the u_i.

A bipartite graph $G = (V, W, E, \leq)$ is *odd* if for every $\emptyset \neq X \subseteq W$, there exists a $\mathfrak{v} \in V$ such that $|X \cap N_G(\mathfrak{v})|$ is odd. The graph G is *k-meager*, if for every set $X \subseteq W$ of size $|X| \leq 2k$, it holds that $|\{\mathfrak{v} \in V \mid N_G(\mathfrak{v}) \subseteq X\}| \leq 2|X|$.

▶ **Lemma 22** ([24]). *If G is an odd and ordered bipartite graph, then $\mathsf{MP}(G)$ is asymmetric.*

▶ **Lemma 23** ([24]). *Let G be a k-meager bipartite graph, $\mathfrak{A} = \mathsf{MP}(G)$, and $\bar{u}, \bar{v} \in A^k$. If there is a local isomorphism $\varphi \in \mathsf{Aut}(\mathfrak{A}[[S(\bar{u}\bar{v})]])$ with $\varphi(\bar{u}) = \bar{v}$, then $(\mathfrak{A}, \bar{u}) \simeq^k_{\mathcal{C}} (\mathfrak{A}, \bar{v})$.*

Odd and k-meager graphs exist [24]. A closer inspection shows that for these graphs there are sets of vertices of pairwise large distance. For a bipartite graph $G = (V, W, E)$, a set $X \subseteq W$ is *k-scattered* if all distinct $\mathfrak{u}, \mathfrak{v} \in X$ have distance at least $2k$ in G.

▶ **Lemma 24.** *For every k, there is an odd and k-meager bipartite graph $G = (V, W, E)$ and a k-scattered set $X \subseteq W$ of size $|X| \geq k^2$.*

We will use a k-scattered set X to ensure that in the bijective k-pebble game placing a pebble on one foot of a segment in X creates no restrictions on the other segments in X. For now, fix a bipartite graph $G = (V, W, E, \leq)$. The *attractor* of a set $X \subseteq W$ is

$$\mathrm{attr}(X) := X \cup \bigcup_{u \in V \colon |N_G(u) \setminus X| \leq 1} N_G(u).$$

The set X is *closed* if $X = \mathrm{attr}(X)$. The *closure* $\mathrm{cl}(X)$ of X is the inclusion-wise minimal closed superset of X.

▶ **Lemma 25** ([24]). *Assume that $X \subseteq W$, $|X| \leq k$, G is k-meager, $\mathfrak{A} = \mathsf{MP}(G)$, and $\varphi \in \mathsf{Aut}(\mathfrak{A}[[X]])$. Then there is an extension of φ that is an automorphism of $\mathfrak{A}[[\mathrm{cl}(X)]]$.*

▶ **Lemma 26.** *Let G be $2k$-meager, $X = \{\mathfrak{u}_1, \dots, \mathfrak{u}_\ell\} \subseteq W$ be $6k$-scattered, $Y \subseteq W$ such that $X \cap \mathrm{cl}(Y) = \emptyset$ and $|X| + |Y| < k$, $\mathfrak{A} = \mathsf{MP}(G)$, and $\varphi \in \mathsf{Aut}(\mathfrak{A}[[Y]])$. Then for all $\bar{u}, \bar{u}' \in F(\mathfrak{u}_1) \times \cdots \times F(\mathfrak{u}_\ell)$, there is an extension ψ of φ to $\mathfrak{A}[[X \cup Y]]$ satisfying $\psi(\bar{u}) = \bar{u}'$.*

The previous lemma turns out to be useful in the bijective k-pebble game: If the pebbles are placed on the feet in Y, we can simultaneously for all feet in X place arbitrary pebbles and still maintain a local automorphism. Such sets X will allow us to glue another graph to the multipede at the feet in X: Whatever restrictions on placing pebbles are imposed by the other graph, we still can maintain partial automorphisms in the multipede.

7.2 Gluing Multipedes to CFI Graphs

We now use multipedes to make CFI graphs asymmetric. We alter the CFI graphs in this section. Instead of two edge-vertex-pairs for the same base edge $\{\mathfrak{u}, \mathfrak{v}\}$, we contract the edges between the two vertex pairs and obtain a single edge-vertex-pair with origin $\{\mathfrak{u}, \mathfrak{v}\}$. This preserves all relevant properties of CFI graphs. In this section we write $\mathsf{CFI}(H, f)$ for CFI graphs of this modified construction. A single edge-vertex-pair per base edge removes technical details from the following.

Let $G = (V^G, W^G, E^G, \leq^G)$ be an ordered bipartite graph, $H = (V^H, E^H, \leq^H)$ be an ordered base graph, $f \colon E^H \to \mathbb{F}_2$, and $X \subseteq W^G$ have size $|X| = |E^H|$. We define the *gluing* $\mathsf{MP}(G) \cup_X \mathsf{CFI}(H, f)$ of the multipede $\mathsf{MP}(G) = (A, R^{\mathsf{MP}(G)}, \preceq^{\mathsf{MP}(G)})$ and the CFI graph $\mathsf{CFI}(H, f) = (B, E^{\mathsf{CFI}(H,f)}, \preceq^{\mathsf{CFI}(H,f)})$ at X as follows: We start with the disjoint union of $\mathsf{MP}(G)$ and $\mathsf{CFI}(H, f)$ and identify the i-th edge-vertex-pair of $\mathsf{CFI}(H, f)$ (according to $\leq^H$) with the i-th segment in X (according to $\leq^G$). We turn the edges $E^{\mathsf{CFI}(H,f)}$ into a ternary relation by extending every edge (u, v) to (u, v, v). In that way, we obtain a $\{R, \preceq\}$-structure, where R is the union of $R^{\mathsf{MP}(G)}$ and the triples (u, v, v) defined before and $\preceq$ is the total preorder obtained from combining $\preceq^{\mathsf{MP}(G)}$ and $\preceq^{\mathsf{CFI}(H,f)}$.

▶ **Lemma 27.** *If $\mathsf{MP}(G)$ is asymmetric, then $\mathsf{MP}(G) \cup_X \mathsf{CFI}(H, f)$ is asymmetric.*

Let $\bar{u}$ be a tuple of at most k vertices of $\mathsf{MP}(G) \cup_X \mathsf{CFI}(H, f)$, i.e., $\bar{u}$ contains either gadget vertices of $\mathsf{CFI}(H, f)$ or feet of $\mathsf{MP}(G)$. We call the set of segments $S(\bar{u})$ of all feet in $\bar{u}$ *directly-fixed by* $\bar{u}$ and the segments $\mathrm{cl}(S(\bar{u})) \setminus S(\bar{u})$ *closure-fixed by* $\bar{u}$. A segment $\mathfrak{u} \in X$ is *gadget-fixed by* $\bar{u}$ if the feet of $\mathfrak{u}$ are identified with an edge-vertex-pair with origin $\{\mathfrak{v}, \mathfrak{w}\}$ in $\mathsf{CFI}(H, f)$ such that there is a gadget vertex with origin $\mathfrak{v}$ or $\mathfrak{w}$ in $\bar{u}$. A segment is *fixed by* $\bar{u}$ if it is directly fixed, closure-fixed, or gadget-fixed.

▶ **Lemma 28.** *Let $r \geq k \geq 2$. If H is r-regular, G is $2k$-meager, and X is $6k$-scattered, then at most $r \cdot |\bar{u}|$ segments are fixed. If $\bar{u}$ contains i gadget vertices and ℓ segments in X are directly-fixed, then at most $|\bar{u}| - i - \ell$ segments in X are closure-fixed.*

We now combine winning strategies of Duplicator on multipedes and CFI graphs:

▶ **Lemma 29.** *Let G be $2rk$-meager, H be r-regular and at least $(k+2)$-connected, and X be $6rk$-scattered. Then $\mathsf{MP}(G) \cup_X \mathsf{CFI}(H,0) \simeq^k_{\mathcal{C}} \mathsf{MP}(G) \cup_X \mathsf{CFI}(H,1)$.*

Proof Sketch. Assume $\mathfrak{A} = \mathsf{MP}(G)$, $\mathfrak{B} = \mathsf{CFI}(H,0)$, and $\mathfrak{B}' = \mathsf{CFI}(H,1)$. We show that Duplicator has a winning strategy in the bijective k-pebble game on $\mathfrak{A} \cup_X \mathfrak{B}$ and $\mathfrak{A} \cup_X \mathfrak{B}'$. For a set of segments Y and a tuple $\bar{u}$, we denote by $\bar{u}_Y$ the restriction of $\bar{u}$ to all feet whose segment is contained in Y, by $\bar{u}_{\mathrm{G}}$ the restriction to all gadget vertices, and by $\bar{u}_{\mathrm{F}}$ to all feet. Duplicator maintains the following invariant. At every position $(\mathfrak{A} \cup_X \mathfrak{B}, \bar{u}; \mathfrak{A} \cup_X \mathfrak{B}', \bar{u}')$ in the game, there exist tuples of feet $\bar{v}_{\mathrm{gf}}, \bar{v}_{\mathrm{cf}}$ of $\mathfrak{A} \cup_X \mathfrak{B}$ and $\bar{v}'_{\mathrm{gf}}, \bar{v}'_{\mathrm{cf}}$ of $\mathfrak{A} \cup_X \mathfrak{B}'$ satisfying the following:

1. $\bar{v}_{\mathrm{gf}}$ (respectively $\bar{v}'_{\mathrm{gf}}$) contains for every segment gadget-fixed by $\bar{u}$ exactly one foot and no others.
2. $\bar{v}_{\mathrm{cf}}$ (respectively $\bar{v}'_{\mathrm{cf}}$) contains for every segment in X closure-fixed by $\bar{u}$ exactly one foot and no others.
3. There is a local isomorphism $\varphi \in \mathsf{Aut}(\mathfrak{A}[[S(\bar{u}_{\mathrm{F}}\bar{v}_{\mathrm{gf}}\bar{v}_{\mathrm{cf}})]])$ satisfying $\varphi(\bar{u}_{\mathrm{F}}\bar{v}_{\mathrm{gf}}\bar{v}_{\mathrm{cf}}) = \bar{u}'_{\mathrm{F}}\bar{v}'_{\mathrm{gf}}\bar{v}'_{\mathrm{cf}}$.
4. $(\mathfrak{B}, \bar{u}_X\bar{u}_{\mathrm{G}}\bar{v}_{\mathrm{cf}}) \simeq^k_{\mathcal{C}} (\mathfrak{B}', \bar{u}'_X\bar{u}'_{\mathrm{G}}\bar{v}'_{\mathrm{cf}})$.
5. For every base vertex $\mathfrak{u}$, it holds that $(\mathfrak{B}', \bar{u}_{\mathrm{G}}\bar{v}_{\mathrm{gf}})[V_{\mathfrak{u}}] \cong (\mathfrak{B}, \bar{u}'_{\mathrm{G}}\bar{v}'_{\mathrm{gf}})[V_{\mathfrak{u}}]$, where $V_{\mathfrak{u}}$ is the set of all gadget vertices with origin $\mathfrak{u}$ and all edge vertices with origin $\{\mathfrak{u}, \mathfrak{v}\}$ for some $\mathfrak{v}$.

For Property 3, note that $S(\bar{u}_{\mathrm{F}}\bar{v}_{\mathrm{gf}}\bar{v}_{\mathrm{cf}}) = S(\bar{u}'_{\mathrm{F}}\bar{v}'_{\mathrm{gf}}\bar{v}'_{\mathrm{cf}})$ and $|\bar{u}_{\mathrm{F}}\bar{v}_{\mathrm{gf}}\bar{v}_{\mathrm{cf}}| = |\bar{u}'_{\mathrm{F}}\bar{v}'_{\mathrm{gf}}\bar{v}'_{\mathrm{cf}}| \leq rk$ by Lemma 28 because G is $2rk$-meager. For Property 4, note that $|\bar{u}_X\bar{u}_{\mathrm{G}}\bar{v}_{\mathrm{cf}}| = |\bar{u}'_X\bar{u}'_{\mathrm{G}}\bar{v}'_{\mathrm{cf}}| \leq k$: By Lemma 28, the number of closure-fixed segments in X is at most $|\bar{v}_{\mathrm{cf}}| \leq k - |\bar{u}_{\mathrm{G}}| - |\bar{u}_X|$. Property 5 guarantees that the vertices $\bar{v}_{\mathrm{gf}}$ and $\bar{v}'_{\mathrm{gf}}$ are picked consistently. This is needed because $|\bar{v}_{\mathrm{gf}}|$ exceeds k and thus cannot be included in Property 4.

We play two games. Game I is played with rk pebbles on $(\mathfrak{A}, \bar{u}_{\mathrm{F}}\bar{v}_{\mathrm{gf}}\bar{v}_{\mathrm{cf}}; \mathfrak{A}, \bar{u}'_{\mathrm{F}}\bar{v}'_{\mathrm{gf}}\bar{v}'_{\mathrm{cf}})$. Game II is played with k pebbles on $(\mathfrak{B}, \bar{u}_X\bar{u}_{\mathrm{G}}\bar{v}_{\mathrm{cf}}; \mathfrak{B}', \bar{u}'_X\bar{u}'_{\mathrm{G}}\bar{v}'_{\mathrm{cf}})$. From the winning strategies of Duplicator in both games (Lemmas 4 and 23) we construct a winning strategy on $(\mathfrak{A} \cup_X \mathfrak{B}, \bar{u}; \mathfrak{A} \cup_X \mathfrak{B}', \bar{u}')$. We can do so because in Game I we fixed all gadget-fixed segments and in Game II we fixed all closure-fixed segments in X. When placing a pebble on a gadget vertex, we extend the tuples $\bar{v}_{\mathrm{gf}}$ and $\bar{v}'_{\mathrm{gf}}$ using Lemma 26. When a pebble is placed on a segment not in X, at most one segment in X gets closure fixed and the edge vertices are in the same orbit of the CFI graphs (Lemma 5). ◀

▶ **Theorem 30.** *There is an* FO*-interpretation Θ and, for every $k \in \mathbb{N}$, a pair of ternary $\{R, \preceq\}$-structures $(\mathfrak{A}_k, \mathfrak{B}_k)$ such that $\preceq$ is a total preorder, $\mathfrak{A}_k$ and $\mathfrak{B}_k$ are asymmetric, $\mathfrak{A}_k \simeq^k_{\mathcal{C}} \mathfrak{B}_k$, $\mathfrak{A}_k \not\cong \mathfrak{B}_k$, and $\Theta(\mathfrak{A}_k) \not\cong \Theta(\mathfrak{B}_k)$ are CFI graphs of the same ordered base graph.*

Proof Sketch. We use the gluings constructed before for suitable base graphs and multipedes (Lemma 24). By Lemma 29, the odd and even gluing are $\mathcal{C}_k$-equivalent (but surely not isomorphic) and asymmetric by Lemmas 22 and 27. There is an FO-interpretation Θ removing the multipede: Shorten R-triples (u, v, v) back to edges (u, v) and remove the others. ◀

Proof Sketch of Theorem 1. Consider the $\{R, \preceq\}$-structures $\mathcal{K}$ from Theorem 30. To ensure that the reduct semantics does not add automorphisms, $\preceq$ is encoded into R by attaching paths of different lengths to the vertices. This preserves asymmetry and non-isomorphism. The paths are removed by an FO-interpretation $\Theta_{\mathcal{K}}$. All structures are asymmetric and have a single relation, so IFPC = IFPC+WSC and IFPC does not define isomorphism.

Let Φ_{CFI} be a WSCI(IFPC) = WSC(IFPC)-formula defining the CFI query for ordered base graphs (Corollary 13) and let Θ_{CFI} be the FO-interpretation extracting the CFI graphs from $\mathcal{K}$-structures given by Theorem 30. Then the I(WSC(IFPC))-formula $\mathsf{I}(\Theta_{\text{CFI}} \circ \Theta_{\mathcal{K}}; \Phi_{\text{CFI}})$ defines the isomorphism problem of $\mathcal{K}$-structures. ◀

▶ **Corollary 31.** IFPC+WSC < *PTIME.*

▶ **Corollary 32.** WSC(IFPC) < I(WSC(IFPC)).

Note that the prior corollary refines Corollary 21. We actually expect that

$$\text{WSC(IFPC)} < \text{I(WSC(IFPC))} < \text{WSC(I(WSC(IFPC)))}$$

because it seems unlikely that I(WSC(IFPC)) defines the CFI query of the base graphs of Theorem 3.

▶ **Corollary 33.** IFPC+WSC *is not closed under* IFPC*-interpretations and not even under* 1*-dimensional equivalence-free* FO*-interpretations.*

Using the same structures, we can answer an open question of Dawar and Richerby in [9]:

▶ **Corollary 34.** IFP+SC *is not closed under* 1*-dimensional equivalence-free* FO*-interpretations.*

8 Discussion

We defined the logics IFPC+WSC and IFPC+WSC+I to study the combination of witnessed symmetric choice and interpretations beyond simulating counting. Instead, we provided graph constructions to prove lower bounds. IFPC+WSC+I canonizes CFI graphs if it canonizes the base graphs, but operators have to be nested. We proved that this increase in nesting depth is unavoidable using double CFI graphs obtained by essentially applying the CFI construction twice. Does iterating our construction further show an operator nesting hierarchy in IFPC+WSC+I? We have seen that also in the presence of counting the interpretation operator strictly increases the expressiveness. So indeed both, witnessed symmetric choice and interpretations are needed to possibly capture PTIME. This answers the question to the relation between witnessed symmetric choice and interpretations for IFPC. But it remains open whether IFPC+WSC+I captures PTIME. Here, iterating our CFI construction is of interest again: If one shows an operator nesting hierarchy using this construction, then one in particular will separate IFPC+WSC+I from PTIME because our construction does not change the signature of the structures. Studying this remains for future work.

References

1 Markus Anders and Pascal Schweitzer. Search problems in trees with symmetries: Near optimal traversal strategies for individualization-refinement algorithms. In *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 16:1–16:21. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ICALP.2021.16.

2 Vikraman Arvind and Somenath Biswas. Expressibility of first order logic with a nondeterministic inductive operator. In *STACS 87, 4th Annual Symposium on Theoretical Aspects of Computer Science, Passau, Germany, February 19-21, 1987, Proceedings*, volume 247 of *Lecture Notes in Computer Science*, pages 323–335. Springer, 1987. doi:10.1007/BFb0039616.

3 Andreas Blass and Yuri Gurevich. The logic of choice. *J. Symb. Log.*, 65(3):1264–1310, 2000. doi:10.2307/2586700.

4 Andreas Blass, Yuri Gurevich, and Saharon Shelah. On polynomial time computation over unordered structures. *J. Symb. Log.*, 67(3):1093–1125, 2002. doi:10.2178/jsl/1190150152.

5 Jin-yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12(4):389–410, 1992. doi:10.1007/BF01305232.

6 Ashok K. Chandra and David Harel. Structure and complexity of relational queries. *J. Comput. Syst. Sci.*, 25(1):99–128, 1982. doi:10.1016/0022-0000(82)90012-5.

7 Anuj Dawar, Erich Grädel, and Moritz Lichter. Limitations of the invertible-map equivalences. *J. Log. Comput.*, 2022. doi:10.1093/logcom/exac058.

8 Anuj Dawar and Kashif Khan. Constructing hard examples for graph isomorphism. *J. Graph Algorithms Appl.*, 23(2):293–316, 2019. doi:10.7155/jgaa.00492.

9 Anuj Dawar and David Richerby. A fixed-point logic with symmetric choice. In *Computer Science Logic, 17th International Workshop, CSL 2003, 12th Annual Conference of the EACSL, and 8th Kurt Gödel Colloquium, KGC 2003, Vienna, Austria, August 25-30, 2003, Proceedings*, volume 2803 of *Lecture Notes in Computer Science*, pages 169–182. Springer, 2003. doi:10.1007/978-3-540-45220-1_16.

10 Anuj Dawar and David Richerby. Fixed-point logics with nondeterministic choice. *J. Log. Comput.*, 13(4):503–530, 2003. doi:10.1093/logcom/13.4.503.

11 Anuj Dawar and David Richerby. The power of counting logics on restricted classes of finite structures. In *Computer Science Logic, 21st International Workshop, CSL 2007, 16th Annual Conference of the EACSL, Lausanne, Switzerland, September 11-15, 2007, Proceedings*, volume 4646 of *Lecture Notes in Computer Science*, pages 84–98. Springer, 2007. doi:10.1007/978-3-540-74915-8_10.

12 Anuj Dawar, David Richerby, and Benjamin Rossman. Choiceless Polynomial Time, counting and the Cai-Fürer-Immerman graphs. *Ann. Pure Appl. Logic*, 152(1-3):31–50, 2008. doi:10.1016/j.apal.2007.11.011.

13 Heinz-Dieter Ebbinghaus. Extended logics: the general framework. In *Model-Theoretic Logics*, Perspectives in MathematicalLogic, pages 25–76. Association for Symbolic Logic, 1985.

14 Ronald Fagin. Generalized first-order spectra and polynomial-time recognizable sets. In *Complexity of computation (Proc. SIAM-AMS Sympos. Appl. Math., New York, 1973)*, pages 43–73. SIAM–AMS Proc., Vol. VII, 1974.

15 Françoise Gire and H. Khanh Hoang. An extension of fixpoint logic with a symmetry-based choice construct. *Inf. Comput.*, 144(1):40–65, 1998. doi:10.1006/inco.1998.2712.

16 Erich Grädel, Phokion G. Kolaitis, Leonid Libkin, Maarten Marx, Joel Spencer, Moshe Y. Vardi, Yde Venema, and Scott Weinstein. *Finite Model Theory and Its Applications.* Texts in Theoretical Computer Science. An EATCS Series. Springer, Berlin, Heidelberg, 2007. doi:10.1007/3-540-68804-8.

17 Erich Grädel and Wied Pakusa. Rank logic is dead, long live rank logic! *J. Symb. Log.*, 84(1):54–87, 2019. doi:10.1017/jsl.2018.33.

18 Erich Grädel, Wied Pakusa, Svenja Schalthöfer, and Lukasz Kaiser. Characterising Choiceless Polynomial Time with first-order interpretations. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 677–688. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.68.

19 Martin Grohe. The quest for a logic capturing PTIME. In *Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science, LICS 2008, 24-27 June 2008, Pittsburgh, PA, USA*, pages 267–271. IEEE Computer Society, 2008. doi:10.1109/LICS.2008.11.

20 Martin Grohe. *Descriptive Complexity, Canonization, and Definable Graph Structure Theory.* Cambridge University Press, 2017.

21 Martin Grohe and Daniel Neuen. Canonisation and definability for graphs of bounded rank width. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–13. IEEE Computer Society, 2019. doi:10.1109/LICS.2019.8785682.

22 Yuri Gurevich. Logic and the challenge of computer science. In *Current Trends in Theoretical Computer Science*, pages 1–57. Computer Science Press, 1988.

23 Yuri Gurevich. From invariants to canonization. *Bull. EATCS*, 63, 1997.

24 Yuri Gurevich and Saharon Shelah. On finite rigid structures. *J. Symb. Log.*, 61(2):549–562, 1996. `doi:10.2307/2275675`.

25 Lauri Hella. Logical hierarchies in PTIME. *Information and Computation*, 129(1):1–19, 1996. `doi:10.1006/inco.1996.0070`.

26 Neil Immerman. Expressibility as a complexity measure: results and directions. In *Proceedings of the Second Annual Conference on Structure in Complexity Theory, Cornell University, Ithaca, New York, USA, June 16-19, 1987*. IEEE Computer Society, 1987.

27 Neil Immerman. Languages that capture complexity classes. *SIAM J. Comput.*, 16(4):760–778, 1987. `doi:10.1137/0216051`.

28 Sandra Kiefer, Pascal Schweitzer, and Erkal Selman. Graphs identified by logics with counting. In *Mathematical Foundations of Computer Science 2015 – 40th International Symposium, MFCS 2015, Milan, Italy, August 24-28, 2015, Proceedings, Part I*, volume 9234 of *Lecture Notes in Computer Science*, pages 319–330. Springer, 2015. `doi:10.1007/978-3-662-48057-1_25`.

29 Moritz Lichter. Separating rank logic from polynomial time. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 – July 2, 2021*, pages 1–13. IEEE Computer Society, 2021. `doi:10.1109/LICS52264.2021.9470598`.

30 Moritz Lichter. Witnessed symmetric choice and interpretations in fixed-point logic with counting. *CoRR*, abs/2210.07869, 2022. arXiv preprint. `doi:10.48550/arXiv.2210.07869`.

31 Moritz Lichter and Pascal Schweitzer. Canonization for bounded and dihedral color classes in Choiceless Polynomial Time. In *29th EACSL Annual Conference on Computer Science Logic, CSL 2021, January 25-28, 2021, Ljubljana, Slovenia (Virtual Conference)*, volume 183 of *LIPIcs*, pages 31:1–31:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. `doi:10.4230/LIPIcs.CSL.2021.31`.

32 Moritz Lichter and Pascal Schweitzer. Choiceless Polynomial Time with witnessed symmetric choice. In *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2–5, 2022*, pages 30:1–30:13. ACM, 2022. `doi:10.1145/3531130.3533348`.

33 Rudolf Mathon. A note on the graph isomorphism counting problem. *Inf. Process. Lett.*, 8(3):131–132, 1979. `doi:10.1016/0020-0190(79)90004-8`.

34 Daniel Neuen and Pascal Schweitzer. An exponential lower bound for individualization-refinement algorithms for graph isomorphism. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*, pages 138–150. ACM, 2018. `doi:10.1145/3188745.3188900`.

35 Martin Otto. *Bounded variable logics and counting – A study in finite models*, volume 9 of *Lecture Notes in Logic*. Springer, 1997.

36 Martin Otto. Epsilon-logic is more expressive than first-order logic over finite structures. *J. Symb. Log.*, 65(4):1749–1757, 2000. `doi:10.2307/2695073`.

37 Wied Pakusa, Svenja Schalthöfer, and Erkal Selman. Definability of Cai-Fürer-Immerman problems in Choiceless Polynomial Time. In *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 – September 1, 2016, Marseille, France*, volume 62 of *LIPIcs*, pages 19:1–19:17. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2016. `doi:10.4230/LIPIcs.CSL.2016.19`.

38 David Richerby. *Fixed-point logics with choice*. PhD thesis, University of Cambridge, 2004.

39 Pascal Schweitzer and Constantin Seebach. Resolution with symmetry rule applied to linear equations. In *38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021, March 16-19, 2021, Saarbrücken, Germany (Virtual Conference)*, volume 187 of *LIPIcs*, pages 58:1–58:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. `doi:10.4230/LIPIcs.STACS.2021.58`.

40 Jacobo Torán. On the resolution complexity of graph non-isomorphism. In *Theory and Applications of Satisfiability Testing – SAT 2013 – 16th International Conference, Helsinki, Finland, July 8-12, 2013. Proceedings*, volume 7962 of *Lecture Notes in Computer Science*, pages 52–66. Springer, 2013. doi:10.1007/978-3-642-39071-5_6.

41 Jacobo Torán and Florian Wörz. Number of variables for graph differentiation and the resolution of GI formulas. In *30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference)*, volume 216 of *LIPIcs*, pages 36:1–36:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.CSL.2022.36.

42 Faried Abu Zaid, Erich Grädel, Martin Grohe, and Wied Pakusa. Choiceless Polynomial Time on structures with small abelian colour classes. In *Mathematical Foundations of Computer Science 2014 – 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part I*, volume 8634 of *Lecture Notes in Computer Science*, pages 50–62. Springer, 2014. doi:10.1007/978-3-662-44522-8_5.