Modification Problems Toward Proper (Helly) Circular-Arc Graphs

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Abstract
We present a $9^k \cdot n^{O(1)}$-time algorithm for the proper circular-arc vertex deletion problem, resolving an open problem of van ’t Hof and Villanger [Algorithmica 2013] and Crespelle et al. [Computer Science Review 2023]. Our structural study also implies parameterized algorithms for modification problems toward proper Helly circular-arc graphs.

2012 ACM Subject Classification Theory of computation → Graph algorithms analysis

Keywords and phrases proper (Helly) circular-arc graph, graph modification problem

Digital Object Identifier 10.4230/LIPIcs.MFCS.2023.31


Funding Supported in part by the Hong Kong Research Grants Council (RGC) under grant 15221420 and the National Natural Science Foundation of China (NSFC) under grant 61972330.

1 Introduction

A graph is a circular-arc graph if its vertices can be assigned to arcs on a circle such that there is an edge between two vertices if and only if their corresponding arcs intersect. If none of the arcs properly contains another, then the graph is a proper circular-arc graph. See Figure 1 for two examples of proper circular-arc graphs. Proper circular-arc graphs “form an important subclass of the class of all claw-free graphs,” and their study has been an important step towards finding “a structural characterization of all claw-free graphs” [6]. The structures and recognition of proper circular-arc graphs have been well studied and well understood [19, 8].

Another and earlier motivation for studying (proper) circular-arc graphs is from their relation with (proper) interval graphs, i.e., intersection graphs of intervals on the real line. The intersection graph of a family of sets has a vertex for each set and an edge between two vertices if and only if their the sets they represent have a nonempty intersection. It is easy to see that each (proper) interval graph is a (proper) circular-arc graph, and the connection of these classes has been used in both structural and algorithmic studies of these classes. Indeed, the first linear-time recognition algorithm for proper circular-arc graphs is based on a general observation of both proper circular-arc graphs and proper interval graphs [8]. Neither graph in Figure 1 is a proper interval graph, but removing any vertex from Figure 1(a), or any vertex but $v_5$ from Figure 1(b) leaves a proper interval graph.

Let $G$ be a hereditary (closed under taking induced subgraphs) graph class. Given a graph $G$ and an integer $k$, the $G$ vertex deletion problem asks whether we can remove $k$ vertices from $G$ to make a graph in $G$. It is known that the $G$ vertex deletion problem is NP-hard.
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Figure 1 Two proper circular-arc graphs and their arc models.

when $G$ is nontrivial (i.e., having infinite members and infinite non-members) [15]. These problems have been intensively studied in the framework of parameterized computation. Suppose that the input graph has $n$ vertices and $m$ edges. We say that a graph problem is fixed-parameter tractable (FPT) if there is an algorithm solving it in time $f(k) \cdot n^{O(1)}$, where $f$ is a computable function depending only on $k$ [9]. For example, it is well known that the proper interval vertex deletion problem is FPT [20, 3]. In the algorithm of van ’t Hof and Villanger [20], the kinship between proper circular-arc graphs and proper interval graphs plays a crucial role. They showed that it suffices to destroy all the small forbidden induced subgraphs, and then the graph is already a proper circular-arc graph, on which the proper interval vertex deletion problem can be solved in linear time. They asked whether the proper circular-arc vertex deletion problem is FPT as well, and this open problem was recently raised again by Crespelle et al. [7]. We answer this question affirmatively.

Theorem 1. The proper circular-arc vertex deletion problem can be solved in time $9^k \cdot n^{O(1)}$.

A major difference between the class of proper interval graphs and the class of proper circular-arc graphs is that the later class is not closed under disjoint union. This can be easily observed from their models: while we can always put intervals for two different components side by side, no such accommodation is possible for two sets of arcs if one set of them covers the whole circle. As a matter of fact, if a proper circular-arc graph is not connected, then $G$ is bipartite.

Permutation graphs are the intersection graphs of line segments between two parallel lines, and bipartite permutation graphs are those permutation graphs that are bipartite. Bipartite
permutation graphs are also known as proper interval bigraphs and unit interval bigraphs [14]. It is well known that a co-bipartite graph $H$ is a proper circular-arc graph if and only if $\overline{H}$ is a bipartite permutation graph.

Let $(G, k)$ be an instance to the proper circular-arc vertex deletion problem, and let $V_-$ be a solution. If $G - V_-$ is not connected, then it is a proper interval graph; if $G - V_-$ is not co-connected, then it is the complement of a bipartite permutation graph. We can call the algorithm of Cao [3] and the algorithm of Bożyk et al. [1] to check whether such a set $V_-$ exists, and we are done if it does. In the rest, we may assume that $G - V_-$ is neither a proper interval graph nor the complement of a bipartite graph, hence both connected and co-connected. For this purpose we may assume that $G$ itself is both connected and co-connected; otherwise, there is a unique component $C$ of $\overline{G}$ such that $V(G) \setminus V(C) \subseteq V_-$. Either the instance is trivially FPT, when $n = O(k)$, or it suffices to consider the largest component of $G$ or $\overline{G}$.

The algorithm proceeds as follows. We can destroy all forbidden induced subgraphs of order at most seven by branching. Now $G$ is free of small forbidden induced subgraphs and is both connected and co-connected. Our key observation is that if $G$ is not already a proper circular-arc graph, then $\overline{G}$ must be bipartite. Note that any induced subgraph of a bipartite graph is bipartite, but we have assumed that $G - V_-$ is not the complement of a bipartite graph. Therefore, we are ready to directly return “yes” or “no.”

Since the parameterized algorithm branches on a small set of vertices that intersects every solution, we can easily turn it into an approximation algorithm for the maximum proper circular-arc induced subgraph problem.

▶ Theorem 2. There is a polynomial-time approximation algorithm of approximation ratio 9 for the minimization version of the proper circular-arc vertex deletion problem.

Proper circular-arc graphs also arise naturally when we consider the clique graph (the intersection graph of maximal cliques of a host graph) of a circular-arc graph. The complicated structures of circular-arc graphs are mainly due to the lack of the so-called Helly property: every set of pairwise intersecting arcs has a common intersection. For example, neither model in Figure 1 is Helly: the set \{$v_1, v_2, v_3\} in (a) and the set \{$v_3, v_4, v_5\} in (b) violate the Helly property. A graph is a Helly circular-arc graph if it admits an arc model that is Helly. Since every interval model is Helly, all interval graphs are Helly circular-arc graphs. It is well known that the clique graph of an interval graph, with at most $n$ maximal cliques, is a proper interval graph [13]. The same upper bound holds for the number of maximal cliques in a Helly circular-arc graph, and the clique graph of a Helly circular-arc graph is always a proper circular-arc graph [10]. Let us remark parenthetically that a circular-arc graph may have an exponential number of maximal cliques, e.g., the complement of the union of $p$ disjoint edges, which has $2p$ vertices, each of degree $2p - 2$.

The class of proper Helly circular-arc graphs is sandwiched between proper circular-arc graphs and proper interval graphs. This observation has been crucial for the algorithms for modification problems toward proper interval graphs [3]. A graph is a proper Helly circular-arc graph if it has an arc model that is both proper and Helly. A word of caution is worth on the definition of proper Helly circular-arc graphs. One graph might admit two arc models, one being proper and the other Helly, but no arc model that is both proper and Helly. For example, both models in Figure 1 are proper but neither is Helly, and it is not difficult to make Helly arc models for $S_3$ and $W_4$, but, as the reader may easily verify, neither of them admits an arc model that is both proper and Helly. Therefore, the class of proper Helly circular-arc graphs does not contain all those graphs being both proper circular-arc graphs and Helly circular-arc graphs, but a proper subclass of it. Indeed, a proper circular-arc
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A graph is a proper Helly circular-arc graph if and only if it is \( \{S_3, W_4\} \)-free [16]. Another characterization of proper Helly circular-arc graphs is that they are precisely those graphs whose clique matrices have the circular-ones property for both rows and columns [17].

We then consider modification problems toward proper Helly circular-arc graphs. For this class we also consider the edge deletion and completion problems (a proof of their NP-completeness was provided in the full version). The edge deletion (resp., completion) problem asks whether we can delete (resp., add) at most \( k \) edges to a graph to make it satisfy certain properties. Again, we start by destroying all small forbidden induced subgraphs, up to six vertices. We show that a connected graph free of such induced subgraphs is already a proper Helly circular-arc graph. For the vertex deletion problem, either we remove all but one component, or we remove vertices to get a proper interval graph. The edge deletion problem is even simpler: if the graph is not connected, we cannot make it connected by deleting edges. Thus, depending on whether the graph is connected, either we are already done, or we are solving the proper interval edge deletion problem. This idea can even solve the general deletion problem that allows \( k_1 \) vertex deletions and \( k_2 \) edge deletions. The situation is quite different for the completion problem. We are happy if we can add at most \( k \) edges to make the input graph a proper interval graph. Otherwise, we have to make a connected proper Helly circular-arc graph. After we have dealt with all the small forbidden induced subgraphs, the only nontrivial case is when there is a large component, which contains a long hole \( H \), and several small components. We need to “attach” these small components to vertices on \( H \). Since these operations are local, we can find a solution by dynamic programming. Thus, all three problems are FPT, and they can be done in linear FPT time. Again, the parameterized algorithm for the vertex deletion problem can be easily turned into an approximation algorithm.

Theorem 3. For modification problems toward proper Helly circular-arc graphs, there are
- an \( O(6^k \cdot (m + n)) \)-time algorithm for the vertex deletion problem;
- an \( O(8^k \cdot (m + n)) \)-time algorithm for the edge deletion problem;
- an \( O(14^{k_1 + k_2} \cdot (m + n)) \)-time algorithm for the deletion problem; and
- a \( k^{O(k)} \cdot (m + n) \)-time algorithm for the completion problem.

Moreover, there is an \( O(nm + n^2) \)-time approximation algorithm of approximation ratio 6 for the minimization version of the proper Helly circular-arc vertex deletion problem.

Somewhat surprisingly, modification problems toward circular-arc graphs and its subclasses have not received sufficient attention. We hope our work will inspire more study in this direction. Apart from the two classes in the present paper, the next interesting class is the class of normal Helly circular-arc graphs, a super class of proper Helly circular-arc graphs. They have played crucial roles in solving modification problems to interval graphs [5, 2]. Also related and probably simpler are the modification problems toward unit (Helly) circular-arc graphs. It is well known that a graph is a proper interval graph if and only if it is a unit interval graph. However, there are proper (Helly) circular-arc graphs that are not unit (Helly) circular-arc graphs, e.g., the graph obtained from an even hole of length at least eight by adding edges to connect consecutive even-numbered vertices.

2 Preliminaries

All graphs discussed in this paper are undirected, simple, and finite. The vertex set and edge set of a graph \( G \) is denoted by, respectively, \( V(G) \) and \( E(G) \). Let \( n = |V(G)| \) and \( m = |E(G)| \). A walk in a graph \( G \) is a sequence of vertices and edges in the form of \( v_0, \ldots, v_k \).
Since the edges are determined by the vertices, such a walk can be denoted by $v_0 v_1 \ldots v_\ell$ unambiguously. We say that this walk connects $v_0$ and $v_\ell$, which are the ends of this walk, and refer to it as a $v_0-v_\ell$ walk. The length of a walk is the number of occurrences of edges it contains, and $\ell$ in the previous example. A walk is closed if $\ell \geq 1$ and $v_0 = v_\ell$. A walk is a path if all its vertices are distinct. A closed walk of length $\ell$ is a cycle if it visits precisely $\ell$ vertices; i.e., no repeated vertices except the two ends. The length of a cycle $C$ is denoted as $|C|$. For simplicity, we denote a cycle of length $\ell$ as $v_1 v_2 \ldots v_\ell$ instead of $v_1 v_2 \ldots v_\ell v_1$. The indices are understood to be modulo $\ell$; e.g., $v_0 = v_\ell$ and $v_{-1} = v_{\ell-1}$. A hole is an induced cycle of length at least four. A walk, path, cycle, or hole is odd (resp., even) if its length is odd (resp., even). For $\ell \geq 3$, we use $C_\ell$ to denote an induced cycle on $\ell$ vertices; if we add a new vertex to a $C_\ell$ and make it adjacent to no or all vertices on the cycle, then we end with a $C_\ell^*$ or $W_\ell$, respectively.

The complement graph $\overline{G}$ of a graph $G$ is defined on the same vertex set $V(G)$, where a pair of vertices $u$ and $v$ is adjacent in $\overline{G}$ if and only if $uv \notin E(G)$; e.g., $\overline{C_3}$ is $W_3$. The graph $\overline{C_3}$ is also called a claw. A graph $G$ is connected if every pair of vertices is connected by a path, and co-connected if $G$ is connected.

A circular-arc graph is the intersection graph of a set of arcs on a circle. The set of arcs is called an arc model of this graph. In this paper, all arcs are closed. An arc model is proper if no arc in it properly contains another arc. A graph is a proper circular-arc graph if it has a proper arc model. In case there is a point of the circle avoided by all the arcs in an arc model, we can cut the circle and straighten all the arcs into line segments. Such a graph is an interval graph, i.e., the intersection graph of a set of closed intervals on the real line, and the set of intervals is an interval model of this graph. Proper interval graphs are defined analogously. Clearly, any (proper) interval model can be viewed as a (proper) arc model leaving some point uncovered, and hence all (proper) interval graphs are always (proper) circular-arc graphs.

Let $F$ be a fixed graph. We say that a graph $G$ is $F$-free if $G$ does not contain $F$ as an induced subgraph. For a set $\mathcal{F}$ of graphs, a graph $G$ is $\mathcal{F}$-free if $G$ is $F$-free for every $F \in \mathcal{F}$. If every $F \in \mathcal{F}$ is minimal, i.e., not containing any $F' \in \mathcal{F}$ as a proper induced subgraph, then $\mathcal{F}$ comprises the (minimal) forbidden induced subgraphs of this class. See Figure 2 for some of the forbidden induced subgraphs considered in the present paper. We use $S_3^*$ to denote the graph obtained by adding an isolated vertex to $S_3$.

**Theorem 4** ([19]). A graph is a proper circular-arc graph if and only if it is free of $S_3^*$, $C_\ell^*$ with $\ell \geq 4$, as well as the complements of $S_3$, $F_1$, $F_2$, $F_3$, $F_4$, $C_{2\ell-2}$, and $C_{2\ell-1}$ with $\ell \geq 2$.

Neither the class of circular-arc graphs nor the class of proper circular-arc graphs is closed under taking disjoint unions. Indeed, if a (proper) circular-arc graph $G$ is not a (proper) interval graph, then in any model of $G$, the union of the arcs covers the whole circle. Such a graph is necessarily connected.
Proposition 5 (Folklore). If a proper circular-arc graph $G$ is not connected, then $G$ is a proper interval graph.

Proper circular-arc graphs have three infinite families of forbidden induced subgraphs, namely, \( \{ C^*_\ell | \ell \geq 4 \} \), \( \{ \overline{C}_{2\ell+2} | \ell \geq 2 \} \), and \( \{ \overline{C}_{2\ell-1} | \ell \geq 2 \} \) by Theorem 4. The first of them can be ignored for connected graphs.

Lemma 6. Let $G$ be a connected graph. If $G$ does not contain the complement of $C^*_3$ or the complement of $S_3$, then $G$ is \( \{ C^*_\ell | \ell \geq 5 \} \)-free.

Proof. Suppose for contradiction that there exist an induced cycle $C$ and a vertex $v$ in $G$ with $|C| \geq 5$ and $V(C) \cap N(v) = \emptyset$. Since $G$ is connected, we can find a shortest path from $v$ to $C$. Let the last three vertices on this path be $x$, $y$, and $z$; note that $z$ is on $C$ and $x$ is nonadjacent to any vertex on $C$. We may number the vertices on $C$ such that $C = v_1 v_2 \cdots v_{|C|}$ and $z = v_2$. If $y$ is adjacent to only $v_2$ on $C$, then \( \{ v_1, v_2, v_3, y \} \) induces a claw. If $y$ is also adjacent to both $v_1$ and $v_3$, then \( \{ v_1, v_2, v_3, x, y \} \) induces a claw, and it is similar if $y$ is adjacent to any three consecutive vertices on $C$. Otherwise, $y$ is adjacent to precisely one of $v_1$ and $v_3$. Without loss of generality, assume that $y$ is adjacent to $v_3$ but not $v_1$. Note that $y$ is not adjacent to $v_4$ either, and then \( \{ v_1, v_2, v_3, v_4, x, y \} \) induces a copy of the complement of $S_3$.

An arc model is Helly if every set of pairwise intersecting arcs has a nonempty common intersection. A circular-arc graph is proper Helly if it has an arc model that is both proper and Helly.

Theorem 7 ([17]). A proper circular-arc graph is a proper Helly circular-arc graph if and only if it contains no $W_4$ or $S_3$.

Note that $S^*_3$ contains $S_3$, while all the complements of $F_1, F_2, F_3, F_4$, and \( \{ C_{2\ell}, C_{2\ell-1}^* \} \) contain $W_4$. The following corollaries follow from Theorem 7, together with Theorem 4 and Lemma 6, respectively.

Corollary 8 ([17]). A graph is a proper Helly circular-arc graph if and only if it contains no $\overline{C}^*_3$, $S_3$, $\overline{S}^*_3$, $W_4$, $W_5$, or $\overline{C}^*_6$, or $C^*_\ell$ for $\ell \geq 4$.

Corollary 9. Let $G$ be a connected graph. If $G$ does not contain $\overline{C}^*_3$, $C^*_3$, $S_3$, $\overline{S}^*_3$, $W_4$, $W_5$, or $\overline{C}^*_6$, then $G$ is a proper Helly circular-arc graph.

Recall that proper interval graphs are precisely \( \{ \overline{C}^*_3, S_3, \overline{S}^*_3, C_4, W_4, W_5, \overline{C}^*_6, C^*_\ell \} \)-free graphs [18, 21].

Corollary 10. Let $G$ be a proper Helly circular-arc graph. Then $G$ is a proper interval graph if and only if $G$ does not contain any holes.

The following can be viewed as a constructive version of Corollary 9.\(^1\)

Proposition 11 (⋆). Let $G$ be a connected graph. In $O(m + n)$ time we can either detect an induced subgraph in \( \{ \overline{C}^*_3, C^*_3, S_3, \overline{S}^*_3, W_4, W_5, \overline{C}^*_6 \} \), or build a proper and Helly arc model for $G$.

A graph is a permutation graph if its vertices can be assigned to line segments between two parallel lines such that there is an edge between two vertices if and only if their corresponding segments intersect. The class of permutation graphs has a large number of forbidden induced subgraphs [11]. Fortunately, most of them contain an odd cycle, and thus the structures of forbidden induced subgraphs of bipartite permutation graphs are far simpler.

\(^1\) Proofs of statements marked with ⋆ are given in the full version (attached).
Theorem 12 ([11]). A graph is a bipartite permutation graph if and only if it is free of $F_1, F_2, F_3, C_5,$ and $C_6$ with $\ell \geq 5$.

We correlate proper circular-arc graphs and bipartite permutation graphs.

Theorem 13 (Folklore). The following are equivalent on a graph $G$:

i) $G$ is a proper circular-arc graph and $G$ is bipartite; and

ii) $\overline{G}$ is a bipartite permutation graph.

The following is complement to Proposition 5 in a sense. Note that (proper) circular-arc graphs that are co-bipartite have played crucial roles in understanding these graph classes [19].

Proposition 14 ($\star$). Let $G$ be a proper circular-arc graph. If $\overline{G}$ is not connected, then $\overline{G}$ is a bipartite permutation graph.

Deletions to proper Helly circular-arc graphs

We first study the proper Helly circular-arc vertex deletion problem. We may assume without loss of generality that the input graph cannot be made a proper interval graph by removing $k$ vertices. Therefore, the resulting graph after removing any $k$-solution is connected by Proposition 5. An FPT algorithm is immediate from Corollary 9: after destroying all the copies of $C_3^1, C_4^1, S_3, \overline{S_5}, W_4, W_5,$ and $C_6$ in $G$ by standard branching, we return all vertices except those in a maximum-order component. A similar (and simpler) approach works for the proper Helly circular-arc edge deletion problem. The focus of the following proof is thus on efficient implementations.

Theorem 15. The proper Helly circular-arc vertex deletion problem and the proper Helly circular-arc edge deletion problem can be solved in time $O(6^k \cdot (m + n))$ and $O(10^k \cdot (m + n))$, respectively.

Proof. Let $(G, k)$ be an instance of proper Helly circular-arc vertex deletion. Our algorithm proceeds as follows. We start by calling the algorithm of Cao [3] to check whether there is a set $V_-$ of at most $k$ vertices such that $G \setminus V_-$ is a proper interval graph. If the set is found, then we return “yes.” In the rest, we look for a solution $V_-$ such that $G \setminus V_-$ is not a proper interval graph. By Proposition 5, (note that a proper Helly circular-arc graph is a proper circular-arc graph), $G \setminus V_-$ is connected.

For the general case, the algorithm solves the problem by making recursive calls to itself; we return “no” directly for a recursive call in which $k < 0$. For each component $C$ of $G$, we call the algorithm of Proposition 11. If a subgraph induced by $F \subseteq V(G)$ is found, then the algorithm calls itself $|F|$ times, each with a new instance $(G \setminus v, k - 1)$ for some $v \in F$. Since we need to delete at least one vertex from $F$, the original instance $(G, k)$ is a yes-instance if and only if at least one of the instances $(G \setminus v, k - 1)$ is a yes-instance. Now that $G$ is free of $C_3^1, C_4^1, S_3, \overline{S_5}, W_4, W_5,$ and $C_6$, every component of $G$ is a proper Helly circular-arc subgraph (Corollary 9). We find a component $C$ of $G$ that has the maximum order. We return “yes” if $|V(C)| \geq n - k$, when $V(G) \setminus V(C)$ is a solution, or “no” otherwise. Since each of $C_3^1, C_4^1, S_3, \overline{S_5}, W_4, W_5,$ and $C_6$ has at most 6 vertices, at most 6 recursive calls are made, all with parameter value $k - 1$. By Proposition 11, each recursive call can be made in $O(m + n)$ time. Therefore, the total running time is $O(6^k \cdot (m + n))$.

The algorithm for the edge deletion problem is even simpler. Again, we start by calling the algorithm for proper interval edge deletion problem [3], which takes time $O(4^k \cdot (m + n))$. We proceed only when the answer is “no.” In the recursive calls for the general case, we always
return “no” whenever $k$ is negative or $G$ becomes disconnected; note that a disconnected graph cannot be made connected by edge deletions. We call the algorithm of Proposition 11, and return “yes” if $G$ is already a proper Helly circular-arc graph. Otherwise, an induced subgraph $F$ is found. The algorithm calls itself $|E(F)|$ times, each with a new instance $(G - uv, k - 1)$ for some edge $uv$ in $G[F]$. By Proposition 11, each recursive call can be made in $O(m + n)$ time. Therefore, the total running time is $O(10^k \cdot (m + n))$, where 10 is the number of edges in a $W_5$. ▶

It is straightforward to adapt an approximation algorithm for the proper Helly circular-arc vertex deletion problem from the parameterized algorithm in Theorem 15. From Theorem 15 we can easily derive an FPT algorithm for the combined deletion problem toward proper Helly circular-arc graphs, which allows $k_1$ vertex deletions and $k_2$ edge deletions. We can fill in the gap between the constants in Theorems 15 and 3. Only $S_3$, $W_5$, and $C_6$ have more than eight edges. For an $S_3$, either we delete one edge between two degree-four vertices, or we have to delete both edges incident to a degree-2 vertex. For the other two cases, the ideas are similar. The details are deferred to the full version.

4 Proper Helly circular-arc completion

Compared to the deletion problems, the completion problem toward proper Helly circular-arc graphs is significantly more difficult. For all the deletion problems, we can always assume that the graph is connected, and then by Corollary 9, we are only concerned with small forbidden induced subgraphs. Since adding edges may make a graph connected, we cannot assume connected input graphs for the completion problem.

Every hole in a proper Helly circular-arc graph is a dominating set of the graph, and we can be more specific on the intersection between a hole and the neighborhood of any vertex.

► Proposition 16 (*). Let $H$ be a hole in a proper Helly circular-arc graph. Every vertex in this graph has at least two neighbors on $H$.

It is well known that the maximal cliques of an interval graph can be arranged as a path. Gavril [12] showed that the maximal cliques of a Helly circular-arc graph can be arranged as a circle. This implies that a Helly circular-arc graph has a linear number of maximal cliques.

► Theorem 17 ([12]). A graph $G$ is a Helly circular-arc graph if and only if its maximal cliques can be arranged as a circle so that for every vertex $v$ in $G$, the maximal cliques containing $v$ are consecutive.

We use a clique cycle to denote the circular arrangement of maximal cliques specified in Theorem 17, and a clique path is defined analogously. In a clique path, we call the first and the last cliques end cliques. Note that a clique path can always be viewed as a clique cycle, while if two consecutive cliques of a clique cycle are disjoint, then it can be viewed as a clique path.

Proper interval graphs are precisely claw-free interval graphs, which can be restated as a graph is a proper interval graph if and only if it is claw-free and has a clique path. One may thus expect that a graph is a proper Helly circular-arc graph if and only if it is claw-free and has a clique cycle. As we have mentioned, however, $S_3$ is a Helly circular-arc graph and hence has a clique cycle, but it is not a proper Helly circular-arc graph even though it is claw-free. The following statement can be directly observed from forbidden induced subgraphs of the class of proper Helly circular-arc graphs and of the class of normal Helly circular-arc graphs; see also Lin et al. [17, Theorem 9].

2 An arc model is known to be normal and Helly if no set of three or fewer arcs covers the circle [12, 4].
A graph is a proper Helly circular-arc graph if and only if it is claw-free and it has an arc model in which no set of three or fewer arcs covers the circle.

If a proper Helly circular-arc graph $G$ is not an interval graph, then it has a hole (Corollary 10). The structure of every local part of $G$ is very like a proper interval graph when the hole is long enough (the length at least six). With the removal of two maximal cliques with no edge in between from $G$, the hole is separated into two sub-paths. Since every remaining vertex is adjacent to one of the two sub-paths, the remaining graph has precisely two components.

Let $G$ be a proper Helly circular-arc graph that is not an interval graph. Let $A_1$ and $A_2$ be two maximal cliques of $G$ with no edge between them, and let $B_1$ and $B_2$ be the vertex sets of the two components of $G - (A_1 \cup A_2)$. Let $G_1$ be any proper interval graph on $B_1 \cup A_1 \cup A_2$ in which $A_1$ and $A_2$ are the end cliques, and $N_+ G_i(A_i) \cap B_i = N_+ G_i(A_i) \cap B_1$ for $i = 1, 2$. Replacing $G[B_1 \cup A_1 \cup A_2]$ with $G_1$ gives another proper Helly circular-arc graph.

For the completion problem, we may again assume that the input graph $G$ is free of $C_3^+$, $C_4^+$, $S_3$, $S_5$, $W_4$, $W_5$, and $C_6$. We are done if $G$ is already a proper Helly circular-arc graph. In particular, this is the case when $G$ is a proper interval graph or when $G$ is connected (Corollary 9). Thus, we may assume that $G$ is not connected and it is not a proper interval graph. There must be a hole in $G$ (Corollary 10), and we add either a chord of this hole, or an edge between this hole and every vertex in other components. If there is a hole of length no more than $16k + 16$, then there are only $O(k^2)$ such choices, and we can branch on adding one of them. In the rest, every hole is longer than $16k + 16$ (hence at least half of the vertices on $H$ have no neighbors incident to a $k$-solution). Let $H$ be such a hole, and let $G_0$ be the component of $G$ that contains $H$. Note that $G_0$ is a proper Helly circular-arc graph (Corollary 9). After adding $k$ or fewer edges, if the resulting graph is a proper Helly circular-arc graph, then there must be a hole of length greater than $k$ in the subgraph induced by $V(H)$. Thus, for every vertex $x$ in $V(G) \setminus V(G_0)$, at least two edges must be added between $x$ and $H$ (Proposition 16). We can return “no” if $|V(G_0)| < n - \frac{k}{7}$. Other components have fewer than $k$ vertices while any hole is longer than $16k + 16$, and thus they are already proper interval subgraphs. They are accordingly called small components.

We say that a vertex $x$ is touched by a solution $E_+$ if $x$ is an endpoint of an edge in $E_+$, and a set $X$ of vertices is touched if at least one vertex in $X$ is touched. All vertices in $V(G) \setminus V(G_0)$ are touched, and we are more concerned with touched vertices in $G_0$.

Let $E_+$ be a solution to $G$. If a maximal clique $K$ of $G$ is untouched by $E_+$, then $K$ is a maximal clique of $G + E_+$.

Recall that a clique cycle of a proper Helly circular-arc graph can be found in linear time [4]. We may fix a clique cycle $(K_1, K_2, \ldots, K_L)$ of $G_0$, denoted by $K$, and assume that $H$ and $K$ are numbered such that no neighbor of $v_1$ or $v_{|H|}$ is touched, and $\{v_1, v_{|H|}\} \subseteq K_1$, which is untouched. Note that $G_0$ has at least $|H|$ maximal cliques. Since $H$ is longer than $16k + 16$, few of them are touched by a $k$-solution $E_+$. By Lemma 19 and Proposition 20, these untouched maximal cliques serve as “isolators” of the modifications.

We can guess another untouched maximal clique $K_p$ of $G_0$ that is disjoint from and nonadjacent to $K_1$. By Proposition 20, $K_1$ and $K_p$ are both maximal cliques of a proper Helly circular-arc graph $G + E_+$. Since $K_p$ is disjoint from and nonadjacent to $K_1$, it follows that $G + E_+ - (K_1 \cup K_p)$ is not connected. Then $H$ is broken into two paths in $G - (K_1 \cup K_p)$. Recall that every vertex in $V(G) \setminus V(G_0)$ needs to be connected to a vertex on $H$. When the graph $G + E_+$ is not a proper interval graph, every small component is connected to
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We can then use dynamic programming to calculate the value we need, which can be calculated as follows. Proof. First, for each forbidden induced subgraph, it is possible to add at most k edges to make a graph Helly circular-arc. We then branch on adding edges. After that, the subgraph induced by \( \bigcup_{i=1}^{p} K_i \) is connected. It will remain connected after adding edges. We then branch on adding edges to destroy induced subgraphs in \( G[\bigcup_{i=1}^{p} K_i \cup \bigcup_{j \in S} C_j] \) and holes in the subgraph induced by \( \bigcup_{i=to(a)} K_i \cup \bigcup_{j \in S} C_j \), without adding any edges incident to \( K_{to(a)} \) or \( K_{from(b)} \).

Since \( G_0 \) is a proper Helly circular-arc graph, a vertex is adjacent to at most 4 vertices on H (there is a claw otherwise). A solution is incident to at most 8k vertices, and thus at most 8k vertices on H have touched neighbors. If \( v_b = 1 \) has a touched neighbor, then for some \( i \) with \( 2 \leq i \leq 8k + 2 \), the vertex \( v_{b-1} \) has no touched neighbor. For \( b - a > 8k \), by Lemma 19, we have

\[
\beta(S, a, b) = \min_{1 \leq i \leq 8k+1} \left( \beta(S \setminus S', a, b - i) + \beta(S', b - i, b) \right).
\]

We can then use dynamic programming to calculate \( \beta([1..r], 1, |H|) \) with (1).

We are now ready to summarize the algorithm in Figure 3. The analysis is left to the full version.

## 5 Proper circular-arc vertex deletion

Since we will use properties of both the graph G and its complement, we beg the reader’s attentiveness in reading this section. There are algorithms for the vertex deletion problem toward proper interval graphs and toward bipartite permutation graphs. We are henceforth focused on graphs that are both connected and co-connected. As usual, we can get rid of small forbidden induced subgraphs easily.

**Definition 22.** A graph is reduced if it is both connected and co-connected, and it contains no \( C_3^*, C_5^*, C_4^*, C_6, S_3, \overline{S_3}, F_1, F_2, F_3, \) or \( F_4 \).
1. if there exists an induced subgraph $X$ in $\{ \overline{C_3^s}, C_4^s, S_3, \overline{S_3}, W_4, W_5, \overline{C_5^s} \}$ then
   **branch** on adding missing edges of $X$; "returns "no" if $k$ becomes negative.
2. if $G$ is a proper Helly circular-arc graph then **return** "yes";
3. find a hole $H$ of $G$ and let $G_0$ be the component of $G$ that contains $H$;
4. if $|H| \leq 16k + 16$ then
   **branch** on adding chords of $H$ or edges between $H$ and other components;
5. if $|V(G_0)| < n - \frac{k}{2}$ then **return** "no";
6. if $\beta([1..r], 1, |H|) \leq k$ then **return** "yes";
   else **return** "no."

**Figure 3** The outline of the algorithm for the proper Helly circular-arc completion problem.

Similar to Proposition 11, one can make an algorithm for finding one of the subgraphs listed above when the input graph is not reduced. We omit details since it does not improve our main algorithm. The next lemma is complement and similar to Lemma 6.

**Lemma 23.** A reduced graph is $\{ C_7^s \mid \ell \geq 7 \}$-free.

**Proof.** Let $R$ be a reduced graph. Suppose for contradiction that there exist a hole $H$ of length at least seven and a vertex that is nonadjacent to any vertex on $H$. Since $R$ is connected, we can find a vertex $x$ adjacent to $H$, and another vertex $y$ that is adjacent to $x$ but not to $H$. Let $H = v_1v_2\cdots v_{|H|}$. We argue first that $x$ cannot be adjacent to two consecutive vertices on $H$. Suppose for contradiction that $x$ is adjacent to both $v_1$ and $v_2$. Then $v_1, v_2,$ and $x$ form a triangle. Since $R$ is free of $C_3^s$, it follows that $x$ is adjacent to both $v_4$ and $v_5$. But then $v_1, v_2, v_4, v_5,$ and $x$ induce a $C_4^s$ (note that $v_5$ and $v_1$ are nonadjacent because $\ell \geq 7$).

Assume without loss of generality that $x$ is adjacent to $v_3$. Note that $x$ is adjacent to neither $v_2$ nor $v_4$. If $x$ is adjacent to $v_3$ as well, then $x$ is nonadjacent to $v_6$, and $\{ v_2, \ldots, v_6, x, y \}$ induces an $F_2$. By symmetry, $x$ cannot be adjacent to $v_1$ either. But then $\{ v_1, \ldots, v_5, x, y \}$ induces an $F_1$.

By definition, a reduced graph is $C_3^s$- and $C_5^s$-free.

**Corollary 24.** A reduced graph is $\{ C_{2r+1}^s \mid \ell \geq 1 \}$-free.

By Theorem 4, the definition of reduced graphs, and Lemma 6, a reduced graph is the complement of a proper circular-arc graph if and only if it does not contain any even hole of length at least eight. We will therefore be focused on long even holes. The main structural statement characterizes reduced graphs that contain long even holes.

**Lemma 25.** If a reduced graph contains an even hole of length at least eight, then it is bipartite.

**Proof (sketch).** Let $v_1v_2\cdots v_{\ell}$ be an even hole with $\ell \geq 8$ of a reduced graph $R$, and denote it by $B$. We prove the lemma with a sequence of claims.
1. No vertex on $B$ participates in any triangle.
2. If some odd hole of $R$ intersects $B$, then there exists an odd hole of $R$ whose intersection with $B$ is a nonempty sub-path of $B$.
3. If $V(C) \cap V(B)$ is consecutive for an odd induced cycle $C$, then $|V(C) \cap V(B)| \leq 4$.
4. No odd induced cycle can intersect $B$ (no vertex on $B$ is contained in any odd induced cycle).
Finally, we show that $R$ does not contain any odd cycle at all. Let $C$ be an odd induced cycle that is disjoint from $B$. First assume $C = x_1x_2x_3$. Since $R$ is $C_7^*$-free, every vertex on $B$ is adjacent to at least one vertex on $C$. Assume without loss of generality that $x_1$ has the largest number of neighbors on $B$, and let their indices be $i_1, i_2, \ldots, i_p$, sorted increasingly. Note that all of them have the same parity by the first claim. Since $R$ is $C_6$-free, $i_{j+1} - i_j$ is either two or at least six for all $j = 1, \ldots, p - 1$. Since $p \geq \lceil \frac{\ell}{4} \rceil$, there must be three consecutive ones with differences two; assume without loss of generality, that they are $v_1, v_3,$ and $v_5$. If $\ell = 8$, then $x_1v_5v_6 \cdots v_8v_1$ has length six, and hence $x_1$ must be adjacent to $v_7$ as well; otherwise, $x_1$ has another neighbor on $B$ because $p \geq \lceil \frac{\ell}{4} \rceil \geq 4$. This neighbor forms an $F_3$ with $\{x_1, v_1, v_2, \ldots, v_5\}$. Now that $|C| \geq 5$; let it be $x_1x_2 \cdots x_{|C|}$. We take $v_i \in N(x_1) \cap V(B)$ and $v_j \in N(x_3) \cap V(B)$. The sub-path $v_i v_{i+1} \cdots v_j$ forms an odd cycle with either $x_1x_2x_3$ (when $j - i$ is odd) or $x_3x_4 \cdots x_{|C|}x_1$ (when $j - i$ is even). From this odd cycle we can retrieve an induced odd cycle, which has to intersect both $B$ and $C$ (because both $B$ and $C$ themselves are induced cycles). This contradicts the fourth claim, and concludes the proof of this lemma.

The following is immediate from Theorem 4, Lemma 6, and Lemma 25.

**Corollary 26.** If a reduced graph $R$ is not bipartite, then $R$ is the complement of a proper circular-arc graph.

We are now ready to present the algorithm for the proper circular-arc vertex deletion problem in Figure 4. Let $(G, k)$ be an instance to the problem, and we may assume without loss of generality that $G$ does not contain any small forbidden induced subgraphs on at most seven vertices. If there is a set $V_-$ of $k$ vertices such that $G - V_-$ is a proper interval graph or $\overline{G} - V_-$ is a bipartite permutation graph, then we are done. Hence, we will look for a solution $V_-$ such that $G - V_-$ is both connected and co-connected (Propositions 5 and 14).

For this purpose we may assume that $G$ itself is connected and co-connected: if $G$ is not connected, we can work on the components of $G$ one by one, and it is similar for $\overline{G}$. Thus, $\overline{G}$ is a reduced graph. If $\overline{G}$ is not bipartite, then $G$ is already a proper circular-arc graph (Corollary 26). Otherwise, $\overline{G}$ is bipartite, of which any induced subgraph of it is bipartite. In other words, if there exists a solution $V_-$, then $\overline{G} - V_-$ is a bipartite permutation graph, and this has been handled already.

1. if $(G, k)$ is a yes-instance of proper interval vertex deletion then return “yes”;
2. if $(\overline{G}, k)$ is a yes-instance of bipartite permutation vertex deletion then return “yes”; \quad \text{We’re looking for a solution $V_-$ with both $G - V_-$ and $\overline{G} - V_-$ connected.}
3. branch on deleting vertices of small forbidden induced subgraphs;
4. $C \leftarrow$ maximal vertex sets that are connected and co-connected;
5. if $G[C]$ is co-bipartite for all $C \in C$ then return “no”;
6. $C \leftarrow$ a maximum set from $C$ with $G[C]$ not co-bipartite;
7. if $|V(G) \setminus V(C)| \leq k$ then return “yes”;
   else return “no.”

**Figure 4** The outline of the algorithm for proper circular-arc vertex deletion.

Again, it is quite straightforward to turn this algorithm into an approximation algorithm, and the proofs for Theorems 1 and 2 are left to the full version.
References


