On Kripke, Vietoris and Hausdorff Polynomial Functors

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Abstract

The Vietoris space of compact subsets of a given Hausdorff space yields an endofunctor $\mathscr{V}$ on the category of Hausdorff spaces. Vietoris polynomial endofunctors on that category are built from $\mathscr{V}$, the identity and constant functors by forming products, coproducts and compositions. These functors are known to have terminal coalgebras and we deduce that they also have initial algebras. We present an analogous class of endofunctors on the category of extended metric spaces, using in lieu of $\mathscr{V}$ the Hausdorff functor $\mathcal{H}$. We prove that the ensuing Hausdorff polynomial functors have terminal coalgebras and initial algebras. Whereas the canonical constructions of terminal coalgebras for Vietoris polynomial functors take ω steps, one needs $\omega + \omega$ steps in general for Hausdorff ones. We also give a new proof that the closed set functor on metric spaces has no fixed points.

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1 Introduction

This paper presents results on terminal coalgebras and initial algebras for certain endofunctors on the categories Haus of Hausdorff topological spaces and Met of extended metric spaces. These results are based on the terminal coalgebra construction first presented by Adámek [2] (in dual form) and independently by Barr [8]. Given an endofunctor F, iterate F on the unique morphism $!\colon F1 \to 1$ to obtain the following ω^{op}-chain

$$1 \xleftarrow{!} F1 \xleftarrow{F!} FF1 \xleftarrow{FF!} FFF1 \xleftarrow{FFF!} \cdots \tag{1}$$

Assume that the limit exists, and denote it by V_ω and the limit cone by $\ell_n\colon V_\omega \to F^n 1$ $(n < \omega)$. We obtain a unique morphism $m\colon FV_\omega \to V_\omega$ such that for all $n \in \omega^{\mathsf{op}}$ we have

$$F\ell_n = \big(FV_\omega \xrightarrow{m} V_\omega \xrightarrow{\ell_{n+1}} F^{n+1}1\big). \tag{2}$$

If F preserves the limit V_ω, then m is an isomorphism. Its inverse yields the terminal coalgebra $m^{-1}\colon V_\omega \to FV_\omega$; shortly $\nu F = V_\omega$, and we say that the terminal coalgebra is *obtained in ω steps.*

This technique of *finitary iteration* is the most basic and prominent construction of terminal coalgebras. However finitary iteration requires that the limit in (1) exists and also that it is preserved by the functor. It does *not* apply on Set to the finite power-set functor $\mathscr{P}_\mathsf{f}$. For that functor $FV_\omega \ncong V_\omega$. However, a modification of finitary iteration does apply, as shown by Worrell [23]. One makes a *second infinite iteration*, iterating F on the morphism $m\colon FV_\omega \to V_\omega$ rather than on $!\colon F1 \to 1$, obtaining a chain

$$V_\omega \xleftarrow{m} V_{\omega+1} \xleftarrow{Fm} V_{\omega+2} \xleftarrow{FFm} \cdots \tag{3}$$

Its limit is denoted by $V_{\omega+\omega} = \lim_{n<\omega} V_{\omega+n}$ with the limit cone $\bar{l}_n\colon V_{\omega+\omega} \to V_{\omega+n}$, for $n < \omega$. Worrell's insight was that this second limit, $V_{\omega+\omega}$, is preserved by every finitary functor. We prove that this also works for set functors built from $\mathscr{P}_\mathsf{f}$ using product, coproduct, and composition (which may be non-finitary). These are the *Kripke polynomial functors* mentioned in our title.

We are interested in other settings where terminal coalgebras may be built using either the limit of (1) or the limit of (3). We study fixed points of naturally occurring endofunctors on Hausdorff spaces and metric spaces, endofunctors built from the Vietoris and Hausdorff functors and several other natural constructions.

In the category Top a good analogy of $\mathscr{P}_\mathsf{f}$ is the *Vietoris functor $\mathscr{V}$* assigning to every space X the space of all compact subsets equipped with the Vietoris topology (Section 4). Hofmann et al. [11] define *Vietoris polynomial* functors as those endofunctors on Top built from $\mathscr{V}$, the constant functors, and the identity functor, using product, coproduct, and composition. We study this on the subcategory Haus of Hausdorff spaces and use that $\mathscr{V}\colon \mathsf{Haus} \to \mathsf{Haus}$ preserves limits of ω^{op}-chains, a fact for which we present a new proof. This implies that for *Vietoris polynomial functors* (defined as above but with $\mathscr{V}$ in lieu of $\mathscr{P}_\mathsf{f}$), the terminal coalgebra exists and is the limit of (1). The original proof [11] uses a result by Zenor [24] whose proof is incomplete. The existence of initial algebras follows.

We also present a result for the category Met of metric spaces and nonexpanding maps. The role of the Vietoris functor is played by the *Hausdorff functor $\mathcal{H}$* assigning to every space X the space $\mathcal{H}X$ of all compact subsets with the Hausdorff metric.

Other contributions. In addition to the aforementioned results we show results obtained by either varying the category or the endofunctor. For example, consider again the Hausdorff polynomial functors. Whenever F is such a functor and the constants involved in its construction are complete spaces, νF again turns out to be complete. Analogous results hold for compact spaces, or ultrametric spaces. Finally, we present a proof of the description of $\nu\mathscr{P}_\mathsf{f}$ and V_ω for $\mathscr{P}_\mathsf{f}$ exhibited by Worrell [23] (the latter without a proof).

We simplify a proof of a known negative result: the variation of $\mathcal{H}$ obtained by moving from compact sets to closed sets has no fixed points.

Related work. Our work is more general and hence improves results of Abramsky [1], Hofmann et al. [11], and Worrell [23].

There are numerous results about the existence and construction of terminal coalgebras in the literature. At several places we discuss other possible approaches to our results.

2 Preliminaries

We review a few preliminary points. We assume that readers are familiar with basic notions of category theory as well as algebras and coalgebras for an endofunctor. We denote by Set the category of sets and functions, Top is the category of topological spaces and continuous functions, and Met is the category of *(extended) metric spaces* (so we might have $d(x, y) = \infty$) and non-expanding maps: the functions $f: X \to Y$ where $d(f(x), f(x')) \leq d(x, x')$ holds for every pair $x, x' \in X$. Note that this class of morphisms is smaller than the class of continuous functions between metric spaces.

▶ **Remark 2.1.** Consider an ω^{op}-chain

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \tag{4}$$

1. In Set, the limit L consists of all sequences $(x_n)_{n<\omega}$, $x_n \in X_n$ that are *compatible*: $f_n(x_{n+1}) = x_n$ for every n. The limit projections are the functions $\ell_n: L \to X_n$ defined by $\ell_n((x_i)) = x_n$.
2. In Top, the limit is again carried by the same set L as in Set, and the limit projections ℓ_n are also the same. The topology on L has as a base the sets $\ell_n^{-1}(U)$, for U open in X_n.
3. In Met, the limit is again carried by the same set L, and the same limit projections ℓ_n. The metric on L is defined by $d((x_n), (y_n)) = \sup_{n<\omega} d(x_n, y_n)$.

Smooth Monomorphisms. In addition to terminal coalgebras, we also study initial algebras for the functors of interest in this paper. For this, we call on a general result which allows one to infer the existence of the initial algebra for an endofunctor F from the existence of a terminal coalgebra for F (or in fact of any algebra with monic structure).

For a class $\mathcal{M}$ of monomorphisms we denote by $\mathsf{Sub}_{\mathcal{M}}(A)$ the collection of subobjects of A represented by monomorphisms from $\mathcal{M}$. To say that this is a dcpo means that it is a set which (when ordered by factorization in the usual way) is a poset having directed joins.

▶ **Definition 2.2** [4, Def. 3.1]**.** Let $\mathcal{M}$ be a class of monomorphisms closed under isomorphisms and composition.

1. We say that an object A has *smooth $\mathcal{M}$-subobjects* provided that $\mathsf{Sub}_{\mathcal{M}}(A)$ is a dcpo with bottom $\perp$, where the least element and directed joins are given by colimits of the corresponding diagrams of subobjects.
2. The class $\mathcal{M}$ is *smooth* if every object of $\mathscr{A}$ has smooth $\mathcal{M}$-subobjects.

A category has *smooth monomorphisms* if the class of all monomorphisms is smooth.

▶ **Example 2.3.**

1. The categories Set and Top have smooth monomorphisms, and so does the full subcategory of Hausdorff spaces. This is easy to see.
2. The category Met also has smooth monomorphisms (these are the injective non-expanding maps) [4, Lemma A.1].
   The full subcategory CMS of complete metric spaces does not have smooth monomorphisms. However, strong monomorphisms (isometric embeddings) are smooth in both Met and CMS [4, Lemma A.2].
3. Strong monomorphisms (subspace embeddings) in Top are not smooth [3, Ex. 3.5].

▶ **Theorem 2.4** [4, Cor. 4.4]**.** *Let $\mathcal{M}$ be a smooth class of monomorphisms. If an endofunctor F preserving $\mathcal{M}$ has a terminal coalgebra, then it has an initial algebra.*

Note that loc. cit. states more: given any algebra $m: FA \rightarrowtail A$ where m lies in $\mathcal{M}$, the initial algebra exists and is a subalgebra of (A, m).

3 Kripke Polynomial Functors

We turn to the first collection of functors mentioned in the title of this paper: the Kripke polynomial functors on Set. The name stems from *Kripke structures* used in modal logic. Our definition below is a slight generalization of the (finite) Kripke polynomial functors presented by Jacobs [12, Def. 2.2.1]. (Kripke polynomial functors using the full power-set functor were originally introduced by Rößiger [19].) We admit arbitrary products in lieu of just arbitrary exponents.

▶ **Definition 3.1.** The *Kripke polynomial functors* F are the set functors built from the finite power-set functor, constant functors and the identity functor, by using product, coproduct and composition. In other words, Kripke polynomial functors are built according to the following grammar:

$$F ::= \mathscr{P}_{\mathsf{f}} \mid A \mid \mathsf{Id} \mid \textstyle\prod_{i\in I} F_i \mid \coprod_{i\in I} F_i \mid FF,$$

where A ranges over all sets (and is interpreted as a constant functor) and I is an arbitrary index set.

▶ **Remark 3.2.** The constant functors could be omitted from the grammar since they are obtainable from the rest of the grammar. The constant functor with value 1 is the empty product. For each set A, the constant functor with value A is then a coproduct: $A = \coprod_{a\in A} 1$.

▶ **Example 3.3.** The Kripke polynomial functor $FX = \mathscr{P}_{\mathsf{f}}(A \times X)$ is the type functor of finitely branching labelled transition systems with a set A of actions.

▶ **Remark 3.4.** An endofunctor is *finitary* if it preserves directed colimits. Worrell [23] proved that for every finitary set functor the terminal coalgebra is obtained in $\omega + \omega$ steps. We prove a version of Worrell's result but for Kripke polynomial functors.

There are Kripke polynomial set functors which are not finitary. One example of such a functor is $F(X) = X^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers. There are also finitary set functors which are not Kripke polynomial functors. One example is the functor assigning to a set X the set of *nonempty* finite subsets of X.

Our proof below uses ideas from Worrell's work [23].

▶ **Theorem 3.5.** *Every Kripke polynomial functor F has a terminal coalgebra obtained in $\omega + \omega$ steps: $\nu F = V_{\omega+\omega}$.*

Proof.

1. We first observe that F preserves monomorphisms and intersections of monomorphisms. This is clear for constant functors and for Id, and it is easy to see for $\mathscr{P}_{\mathsf{f}}$. Moreover, these properties are clearly preserved by product, coproduct and composition.
2. Let $(X_n)_{n<\omega}$ be an ω^{op}-chain in Set. Then the canonical morphism $m\colon F(\lim X_n) \to \lim FX_n$ is monic. This is obvious for constant functors and Id. Let us check it for $\mathscr{P}_{\mathsf{f}}$. Denote the limit projections by $\ell_n\colon \lim X_n \to X_n$ and $p_n\colon \lim \mathscr{P}_{\mathsf{f}} X_n \to \mathscr{P}_{\mathsf{f}} X_n$ $(n < \omega)$; the canonical morphism m is unique such that $p_n \cdot m = \mathscr{P}_{\mathsf{f}}\ell_n$. Now given $S \neq T$ in $\mathscr{P}_{\mathsf{f}}(\lim X_n)$, without loss of generality we can pick $x \in T \setminus S$. Using that the ℓ_n are jointly monic, for every $s \in S$ we can choose $n < \omega$ such that $\ell_n(x) \neq \ell_n(s)$. Since S is finite, this choice can be performed independently of $s \in S$. Thus $\ell_n(x) \notin \ell_n[S]$, and hence $\mathscr{P}_{\mathsf{f}}\ell_n(T) \neq \mathscr{P}_{\mathsf{f}}(S)$. Thus, $\mathscr{P}_{\mathsf{f}}\ell_n$ is a jointly monic family. Since $p_n \cdot m = \mathscr{P}_{\mathsf{f}}\ell_n$, we see that m is monic.

3. An induction on Kripke polynomial functors F now shows that $m\colon V_{\omega+1} \to V_\omega$ is monic. We have seen this for the base case functors in item 2. The desired property that m is monic is preserved by products, coproducts and composition. In particular, for a composition FG note that the canonical morphism for FG is the composition

$$FG(\lim X_n) \xrightarrow{Fm} F(\lim GX_n) \xrightarrow{m'} \lim FGX_n,$$

where m is the canonical morphism for G w.r.t. the given ω^{op}-chain and m' the one for F and the ω^{op}-chain $(GX_n)_{n<\omega}$. So this morphism $m' \cdot Fm$ is monic since both m and m' are so and F preserves monomorphisms by item 1.

4. Since F preserves monomorphisms, we see that Fm, FFm etc. are monic. We obtain a decreasing chain of subobjects $V_{\omega+n} \rightarrowtail V_\omega$. Therefore, the limit $V_{\omega+\omega} = \lim_{n<\omega} V_{\omega+n}$ is simply the intersection of these subobjects. From item 1 we know that F preserves this limit. It follows that $\nu F = V_{\omega+\omega}$, as desired. ◀

▶ **Corollary 3.6.** *Every Kripke polynomial functor F on* Set *has an initial algebra.*

This follows from Theorem 3.5, Example 2.3.1, and Theorem 2.4 since F preserves monomorphisms.

▶ **Example 3.7** [23]**.** The functor $\mathscr{P}_{\mathsf{f}}$ has a terminal coalgebra consisting of all finitely branching strongly extensional trees (up to isomorphism of trees). Moreover, the limit V_ω consists of all compactly branching strongly extensional trees. We present a proof of these results in Appendix A (Theorem A.15).

4 Vietoris Polynomial Functors

Hofmann et al. [11] proved that Vietoris polynomial functors on the category Haus of Hausdorff spaces have terminal coalgebras obtained in ω steps. Our proof is slightly different from theirs because we wish to avoid a result stated by Zenor [24] whose proof is incomplete.

Recall that a *base* of a topology is a collection $\mathcal{B}$ of open sets such that every open set is a union of members of $\mathcal{B}$. A *subbase* is a collection of open sets whose finite intersections form a base. For every collection $\mathcal{B}$ of subsets of the space, there is a smallest topology for which $\mathcal{B}$ is a (sub)base, the family of unions of finite intersections from $\mathcal{B}$.

▶ **Definition 4.1.** **1.** Let X be a topological space. We denote by $\mathscr{V}X$ the space of compact subsets of X equipped with the "hit-and-miss" topology. This topology has as a subbase all sets of the following forms:

$$\begin{aligned} U^{\Diamond} &= \{R \in VX : R \cap U \neq \emptyset\} && (R \text{ hits } U),\\ U^{\Box} &= \{R \in VX : R \subseteq U\} && (R \text{ misses } X \setminus U), \end{aligned} \tag{5}$$

where U ranges over the open sets of X. We call $\mathscr{V}X$ the *Vietoris space of* X, also known as the *hyperspace* of X.

2. Recalling that the image of a compact set under a continuous function is also compact, for a continuous function $f\colon X \to Y$ we put $\mathscr{V}f(A) = f[A]$ for every compact subset A of X.

▶ **Remark 4.2.**

1. For a compact Hausdorff space X, Vietoris [22] defined $\mathscr{V}X$ to consist of all *closed* subsets of X. These are the same as the compact subsets in this case. In the coalgebraic literature, $\mathscr{V}X$ has also mostly been studied for spaces X which are compact Hausdorff. However,

the "classic Vietoris space" (using closed subsets) does not yield a functor on Top (see Hofmann et al. [11, Rem. 2.28]). Hofmann et al. [11, Def. 2.27] call the functor $\mathscr{V}$ in Definition 4.1 the *compact Vietoris functor.*
2. Michael [16, Thm. 4.9.8] proved that X is Hausdorff iff so is $\mathscr{V}X$.
3. Vietoris [22] originally proved that for a compact Hausdorff space X (the classic Vietoris space) $\mathscr{V}X$ is compact Hausdorff, too.
4. A *Stone space* is a compact Hausdorff space having a base of clopen sets. If X is a Stone space, so is $\mathscr{V}X$; see [16, Thm. 4.9.9] or [13, Section III.4].

▶ **Proposition 4.3.** *For every continuous function $f\colon X \to Y$ and every open $U \subseteq Y$, $(f^{-1}(U))^{\Diamond} = (\mathscr{V}f)^{-1}(U^{\Diamond})$, and $(f^{-1}(U))^{\Box} = (\mathscr{V}f)^{-1}(U^{\Box})$.*

Proof. Let $R \in \mathscr{V}X$. Observe that

$$R \cap f^{-1}(U) \neq \emptyset \iff f[R] \cap U \neq \emptyset \iff f[R] \in U^{\Diamond} \iff R \in (\mathscr{V}f)^{-1}(U^{\Diamond}).$$

This proves our first assertion for all R. For the second assertion, we have

$$R \subseteq f^{-1}(U) \iff f[R] \subseteq U \iff f[R] \in U^{\Box} \iff R \in (\mathscr{V}f)^{-1}(U^{\Box}).$$ ◀

▶ **Corollary 4.4.** *The mappings $X \mapsto \mathscr{V}X$ and $f \mapsto \mathscr{V}f$ form a functor $\mathscr{V}$ on* Top.

Indeed, Proposition 4.3 shows that for every subbasic open set of $\mathscr{V}Y$ its inverse image under $\mathscr{V}f$ is open in $\mathscr{V}X$. This establishes continuity of $\mathscr{V}f$.

▶ **Notation 4.5.** We denote by Haus, KHaus and Stone the full subcategories of Top given by all Hausdorff spaces, all compact Hausdorff spaces and all Stone spaces, respectively. By Remark 4.2.2–4, $\mathscr{V}$ restricts to these three full subcategories, and we denote the restrictions by $\mathscr{V}$ as well.

▶ **Remark 4.6.**
1. The full subcategories Haus, KHaus and Stone are closed under limits in Top. In particular, the inclusion functors preserve and reflect limits. In fact, KHaus is a full reflective subcategory: the reflection of a space is its Stone-Čech compactification.
2. If an ω^{op}-chain as in (4) consists of surjective continuous maps between compact Hausdorff spaces, then each limit projection $\ell_n\colon \lim_{k<\omega} X_k \to X_n$ is surjective, too. Moreover, Eilenberg and Steenrod [9, Cor. 3.9] prove the surjectivity of projections for all codirected limits of surjections between compact Hausdorff spaces; see also Ribes and Zalesskii [18, Prop. 1.1.10]).
3. If X has a base $\mathcal{B}$ which is closed under finite unions, then the sets $U^{\Diamond}$ and $U^{\Box}$ for $U \in \mathcal{B}$ already form a subbase of $\mathscr{V}X$. Indeed, given a set $\mathcal{S}$ of open subsets of X we have $(\bigcup \mathcal{S})^{\Diamond} = \bigcup\{U^{\Diamond} : U \in \mathcal{S}\}$. Moreover, it is easy to see that

$$(\bigcup \mathcal{S})^{\Box} = \bigcup\{\big(\bigcup \mathcal{F}\big)^{\Box} : \mathcal{F} \subseteq \mathcal{S} \text{ finite}\};$$

"$\supseteq$" is trivial, and for "$\subseteq$" use compactness of $R \in \mathscr{V}X$. Hence, if $\mathcal{S}$ consists of basic open sets from $\mathcal{B}$, then $\bigcup F \in \mathcal{B}$ due to its closure under finite unions. Thus, $(\bigcup \mathcal{S})^{\Box}$ is a union of sets of the form $U^{\Box}$ for $U \in \mathcal{B}$.

▶ **Proposition 4.7.** *The functor $\mathscr{V}\colon \mathsf{Haus} \to \mathsf{Haus}$ preserves limits of ω^{op}-chains.*

Proof. Consider an ω^{op}-chain as in (4). Let $M = \lim \mathscr{V}X_n$, with limit cone $r_n\colon M \to \mathscr{V}X_n$. Let $m\colon \mathscr{V}L \to M$ be the unique continuous map such that $\mathscr{V}\ell_n = r_n \cdot m$ for all $n < \omega$. We shall prove that m is a bijection and then that its inverse is continuous, which proves that m is an isomorphism.

1. Injectivity of m follows from the fact that $\mathscr{V}\ell_n$ $(n < \omega)$ forms a jointly monic family, as we will now prove. Suppose that $A, B \in \mathscr{V}L$ satisfy $\ell_n[A] = \ell_n[B]$ for every $n < \omega$. We prove that $A \subseteq B$; by symmetry $A = B$ follows. Given $a \in A$, we show that every open neighbourhood of a has a nonempty intersection with B. Since B is closed, we then have $a \in B$ (otherwise $L \setminus B$ would be an open neighbourhood of a disjoint from B). It suffices to prove the desired property for the basic open neighbourhoods $\ell_n^{-1}(U)$ of a, for U open in X_n (see Remark 2.1.2). Since $\ell_n[A] = \ell_n[B]$ we have some $b \in B$ which satisfies $\ell_n(a) = \ell_n(b)$. Then $b \in \ell_n^{-1}(U) \cap B$.
2. Surjectivity of m. An element of M is a sequence $(K_n)_{n<\omega}$ of compact (hence closed) subsets $K_n \subseteq X_n$ such that $f_n[K_{n+1}] = K_n$ for every $n < \omega$. We need to find a compact set $K \subseteq L$ such that $\ell_n[K] = K_n$ for every $n < \omega$. With the subspace topology, K_n is itself a compact space. The connecting maps $f_n \colon X_{n+1} \to X_n$ restrict to surjective continuous maps $K_{n+1} \twoheadrightarrow K_n$. Thus, the spaces K_n form an ω^{op}-chain of surjections in KHaus. Let K be the limit with projections $p_n \colon K \twoheadrightarrow K_n$. Then K is a subset of L, and each projection p_n is the restriction of ℓ_n to X_n.
   Let us check that the topology on K is the subspace topology inherited from L. A base of the topology on K is the family of sets $p_n^{-1}(U)$ as U ranges over the open subset of K_n. Each U is of the form $V \cap K_n$ for some open V of X_n, and $p_n^{-1}(U) = \ell_n^{-1}(V) \cap K$. Thus $p_n^{-1}(U)$ is open in the subspace topology, and the converse holds as well.
   The maps p_n are surjective by Remark 4.6.2. Moreover, K is a compact space by Remark 4.6.1. Thus, K is the desired compact set in $\mathscr{V}L$ such that $p_n[K] = K_n$ for all n.
3. Finally, we prove that the inverse $k \colon M \to \mathscr{V}L$, say, of m is continuous. We know that the sets $\ell_n^{-1}(U)$, for U open in X_n, form a base of L. Moreover, this base is closed under finite unions. By Remark 4.6.3 and using Proposition 4.3 we obtain that $\mathscr{V}L$ has a subbase given by the following sets
   $$(\mathscr{V}\ell_n)^{-1}(U^{\Diamond}) = (\ell_n^{-1}(U))^{\Diamond} \quad \text{and} \quad (\mathscr{V}\ell_n)^{-1}(U^{\Box}) = (\ell_n^{-1}(U))^{\Box} \quad \text{for } U \text{ open in } X_n.$$
   It suffices to show that the inverse images of these subbasic open sets of $\mathscr{V}L$ are open in M. For $\mathscr{V}\ell_n^{-1}(U^{\Diamond})$ with U open in X_n we use that $\mathscr{V}\ell_n \cdot k = r_n$ clearly holds to obtain
   $$k^{-1}\big(\mathscr{V}\ell_n^{-1}(U^{\Diamond}) = r_n^{-1}(U^{\Diamond}),$$
   which is a basic open set of M by Remark 2.1.2. For the subbasic open sets $\mathscr{V}\ell_n^{-1}(U^{\Box})$ the proof is similar. ◀

▶ **Corollary 4.8.** *The restrictions of $\mathscr{V}$ to* KHaus *and* Stone *preserve limits of ω^{op}-chains.*

Indeed, use Remark 4.6.1.

▶ **Remark 4.9.** A *codirected limit* is the limit of a diagram whose scheme is of the form P^{op} for a directed poset P. Proposition 4.7 and Corollary 4.8 hold more generally for codirected limits. The argument is the same. This proves a result stated in Zenor [24], but with an incomplete proof.

The following definition is due to Kupke et al. [14] for Stone spaces, whereas Hofmann et al. [11, Def. 2.29] use arbitrary topological spaces, but they later essentially restrict constants to be (compact) Hausdorff, stably compact or spectral spaces.

▶ **Definition 4.10.** The *Vietoris polynomial functors* are the endofunctors on Top built from the Vietoris functor $\mathscr{V}$, the constant functors, and the identity functor, using product, coproduct, and composition. Thus, the Vietoris polynomial functors are built according to the following grammar

$$F ::= \mathscr{V} \mid A \mid \mathsf{Id} \mid \textstyle\prod_{i\in I} F_i \mid \coprod_{i\in I} F_i \mid FF,$$

where A ranges over all topological spaces and I is an arbitrary index set.

▶ **Theorem 4.11.** *Let $F\colon \mathsf{Top} \to \mathsf{Top}$ be a Vietoris polynomial functor, and assume that all constants in F are Hausdorff spaces. Then F has a terminal coalgebra obtained in ω steps, and $\nu F = V_\omega$ is a Hausdorff space.*

Proof. An easy induction on Vietoris polynomial functors F shows that:

1. The functor F has a restriction $F_0\colon \mathsf{Haus} \to \mathsf{Haus}$.
2. The restriction F_0 preserves surjective maps; the most important step being for $\mathscr{V}$ itself, and this uses the fact when $f\colon X \to Y$ is continuous and X and Y are Hausdorff, the inverse images of compact sets are compact.
3. The functor F_0 preserves limits of ω^{op}-chains; the most important step is done in Proposition 4.7.

The terminal coalgebra result for F_0 follows from the fact which we have mentioned in Section 2: νF is the limit of the terminal-coalgebra ω^{op}-chain $F_0^n 1$ ($n < \omega$). Since Haus is closed under limits in Top and $F_0^n 1 = F^n 1$, the functor F has the same terminal coalgebra $\nu F = \lim F^n 1$. ◀

▶ **Corollary 4.12.** *Let $F\colon \mathsf{Top} \to \mathsf{Top}$ be a Vietoris polynomial functor, and assume that all constants in F are Hausdorff spaces. Then F has an initial algebra.*

This follows from Theorem 4.11, Example 2.3.1 and Theorem 2.4, since an easy induction shows that F preserves monomorphisms.

▶ **Corollary 4.13.** *Let $F\colon \mathsf{Top} \to \mathsf{Top}$ be a Vietoris polynomial functor in which all constants are* compact *Hausdorff spaces and only finite coproducts are used. Then the terminal coalgebra νF is a compact Hausdorff space.*

Proof. The functor F restricts to an endofunctor on KHaus. Thus, the terminal-coalgebra ω^{op}-chain $F^n 1$ lies in KHaus. Moreover, KHaus is closed under limits in Top because it is a full reflective subcategory (Remark 4.6.1). Thus, $\nu F = \lim_{n<\omega} F^n 1$ is compact Hausdorff. ◀

▶ **Corollary 4.14.** *Let $F\colon \mathsf{Top} \to \mathsf{Top}$ be a Vietoris polynomial functor in which all constants are Stone spaces and only finite coproducts are used. Then the terminal coalgebra νF is a Stone space.*

The proof is similar.

▶ **Remark 4.15.** Corollary 4.13 essentially appears in work by Hofmann et al. [11, Thm. 3.42] (except for the convergence ordinal). Corollary 4.14 is due to Kupke et al. [14]. Our proof using convergence of the terminal-coalgebra chain is different than the previous ones.

▶ **Example 4.16.** The terminal coalgebra for $\mathscr{V}$ itself was identified by Abramsky [1]. By what we have shown, it is $V_\omega = \lim \mathscr{V}^n 1$. An easy induction on n shows that $\mathscr{V}^n 1$ is $\mathscr{P}_{\mathsf{f}}^n 1$ with the discrete topology; the key point is that each set $\mathscr{P}_{\mathsf{f}}^n 1$ is finite. The topology was described in Remark 2.1.2: it has as a base the sets $\partial_n^{-1}(U)$ as U ranges over the subsets of $\mathscr{P}_{\mathsf{f}}^n 1$. By Corollary 4.14, νF is a Stone space.

In Appendix A, we present for $\mathscr{P}_{\mathsf{f}}$ a concrete description of V_ω as the set of compactly branching strongly extensional trees.

▶ **Remark 4.17.** Note that Theorem 4.11 also holds for Vietoris polynomial functors when we take Haus as our base category. Hofmann et al. [11] consider other full subcategories of Top, and they also study the completeness of the category of coalgebras for Vietoris polynomial functors F (however, they restrict to using finite products and finite coproducts in their definition of Vietoris polynomial functors). For a Vietoris polynomial functor F on Haus, the category of coalgebras is complete [11, Cor. 3.41]. Moreover, every subfunctor of F has a terminal coalgebra [11, Cor. 4.6].

▶ **Remark 4.18.** Hofmann et al. [11, Ex. 2.27(2)] also consider a related construction called the *lower Vietoris space* of X. It is the set of all closed subsets of X with the topology generated by all sets $U^\Diamond$, cf. (5). This again yields a functor on Top: a given continuous function is mapped to $A \mapsto \overline{f[A]}$, where $\overline{f[A]}$ denotes the closure of $f[A]$. Furthermore, one has a corresponding notion of lower Vietoris polynomial functors. They prove that for such functors F on the category of stably compact spaces (defined in [11]), $\mathsf{Coalg}\,F$ is complete [11, Thm. 3.35]. Furthermore, if a lower Vietoris polynomial functor F on Top can be restricted to that category, then it has a terminal coalgebra obtained by finite iteration: $\nu F = V_\omega$ [11, Thm. 3.36]. Similar results hold for the category of spectral spaces and spectral maps.

▶ **Remark 4.19.** Let us mention a very general result which applies in many situations to deliver a terminal coalgebra: Makkai and Paré's Limit Theorem [15, Thm. 5.1.6]. It implies that every accessible endofunctor $F\colon \mathscr{A} \to \mathscr{A}$ on a locally presentable category has an initial algebra and a terminal coalgebra. (Indeed, the theorem implies that the category of F-coalgebras is cocomplete.) This result cannot be used here because Haus is not locally presentable: it does not have a small set of objects that is colimit-dense [3, Prop. 8.2].

▶ **Open Problem 4.20.**
1. Does every Vietoris polynomial functor on Top have a terminal coalgebra?
2. Does every Vietoris polynomial functor on KHaus as in Corollary 4.13 have an initial algebra?

Item 1 above is equivalent to asking whether the result that νF exists for every Vietoris polynomial functor would remain true if we allowed non-Hausdorff constants.

5 Hausdorff Polynomial Functors

Analogously to the Vietoris polynomial functors on Top, we introduce Hausdorff polynomial functors on Met. Closer to the situation of Kripke polynomial functors on Set than to Vietoris polynomial functors on Top, the Hausdorff polynomial functors on Met have terminal coalgebras obtained in $\omega + \omega$ steps.

▶ **Notation 5.1.** The Hausdorff functor $\mathcal{H}\colon \mathsf{Met} \to \mathsf{Met}$ maps a metric space X to the space $\mathcal{H}X$ of all compact subsets of X equipped with the Hausdorff distance[1] given by

$$\bar{d}(S,T) = \max\left(\sup_{x\in S} d(x,T), \sup_{y\in T} d(y,S)\right), \qquad \text{for } S,T \subseteq X \text{ compact},$$

where $d(x,S) = \inf_{y\in S} d(x,y)$. In particular $\bar{d}(\emptyset,T) = \infty$ for nonempty compact sets T. For a non-expanding map $f\colon X \to Y$ we have $\mathcal{H}f\colon S \mapsto f[S]$.

▶ **Remark 5.2.**
1. The functors $\mathscr{V}\colon \mathsf{Top} \to \mathsf{Top}$ and $\mathcal{H}\colon \mathsf{Met} \to \mathsf{Met}$ are closely related: for compact metric spaces X the Vietoris space $\mathscr{V}X$ is precisely the topological space induced by the Hausdorff space $\mathcal{H}X$.

[1] The definition goes back to Pompeiu [17] and was popularized by Hausdorff [10, p. 293].

2. Some authors define $\mathcal{H}X$ to consist of all *nonempty* compact subsets of X. However, Hausdorff [10] did not exclude $\emptyset$, and the above formula works (as already indicated) without such an exclusion.

▶ **Remark 5.3.**

1. For a complete metric space, $\mathcal{H}X$ is complete again (see e.g. Barnsley [7, Thm. 7.1]). Thus $\mathcal{H}$ restricts to a functor on the category CMS of complete metric spaces, which we denote by the same symbol $\mathcal{H}$.
2. Let UMet denote the category of (extended) ultrametric spaces: the full subcategory of Met given by spaces satisfying the following stronger version of the triangle inequality:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

If X is an ultrametric space, then so is $\mathcal{H}X$. To see this, let $S,T,U \in \mathcal{H}X$. Write p for $\max\{\bar{d}(S,T), \bar{d}(T,U)\}$. For each $x \in S$, there is some $y \in T$ such that $d(x,y) \le \bar{d}(S,T)$. For this y, there is some $z \in U$ such that $d(y,z) \le \bar{d}(T,U)$. So

$$d(x,z) \le \max\{d(x,y), d(y,z)\} \le \max\{\bar{d}(S,T), \bar{d}(T,U)\} = p.$$

It follows that $d(x,U) \le p$. This for all $x \in X$ shows that $d(S,U) \le p$. Note that $p = \max\{\bar{d}(U,T), \bar{d}(T,S)\}$. The same argument shows that $\sup_{z\in U} d(z,S) \le p$. So we have $\bar{d}(S,U) \le p$. This proves the ultrametric inequality.

We again denote the restriction of the Hausdorff functor to UMet is denoted by $\mathcal{H}$.

3. For a discrete metric space X (where all distances are 0 or ∞), $\mathcal{H}X$ is the discrete space formed by all finite subsets of X.
4. For an arbitrary metric space X, the finite subsets of X form a dense set in $\mathcal{H}X$. Indeed, given a compact set $S \subseteq X$, for every $\varepsilon > 0$, there exists a finite set $T \subseteq S$ such that S is covered by ε-balls around the points in T. Therefore $d(x,T) \le \varepsilon$ for all $x \in S$, and we have $d(y,S) = 0$ for all $y \in T$. This implies that $\bar{d}(S,T) \le \varepsilon$.

▶ **Example 5.4.** For the Hausdorff functor, a terminal coalgebra is carried by the space of all finitely branching strongly extensional trees equipped with the discrete metric. This follows from the finite power-set functor $\mathscr{P}_{\mathsf{f}}$ having its terminal coalgebra formed by those trees (Example 3.7). Indeed, the terminal-coalgebra chain V_i ($i \in \mathsf{Ord}$) for $\mathcal{H}$ is obtained by equipping the sets in the terminal-coalgebra chain for $\mathscr{P}_{\mathsf{f}}$ with the discrete metric. Furthermore, since limits in Met (or CMS) are set-based, we see that both chains converge in exactly $\omega + \omega$ steps. Therefore $\nu\mathcal{H} = V_{\omega+\omega}$.

It follows that, unlike the Vietoris functor, the Hausdorff functor does not preserve limits of ω^{op}-chains: the terminal-coalgebras chain for $\mathcal{H}(-)$ does not converge before $\omega + \omega$ steps (see Example 5.4.5.4). Thus this functor does not preserve the limit $V_\omega = \lim_{n<\omega} V_n$.

▶ **Definition 5.5.** Let $(X_n)_{n<\omega}$ be an ω^{op}-chain in Met. A cone $r_n\colon M \to X_n$ is *isometric* if for all $x, y \in M$ we have $d(x,y) = \sup_{n\in\mathbb{N}} d(r_n(x), r_n(y))$.

By Remark 2.1.3, limit cones of ω^{op}-chains in Met are isometric.

▶ **Proposition 5.6.** *The Hausdorff functor preserves isometric cones of ω^{op}-chains.*

Proof. Let $(X_n)_{n<\omega}$ be an ω^{op}-chain with connecting maps $f_n\colon X_{n+1} \to X_n$. Given an isometric cone $\ell_n\colon M \to X_n$ $(n < \omega)$, we prove that the cone $\mathcal{H}\ell_n\colon \mathcal{H}M \to \mathcal{H}X_n$ is also isometric:

$$\bar{d}(S,T) = \sup_{n<\omega} \bar{d}(\ell_n(S), \ell_n(T)) \qquad \text{for all compact subset } S,T \subseteq M.$$

We can assume that S and T are nonempty and finite: since finite sets are dense in $\mathcal{H}M$ by Remark 5.3.4, and the maps ℓ_n are (non-expanding whence) continuous, the desired equality then holds for all pairs in $\mathcal{H}M$. The case where S or T is empty is trivial.

Since every ℓ_n is non-expanding, we only need to prove that $\bar{d}(S,T) \leq c$ holds for $c = \sup_{n<\omega} \bar{d}(\ell_n[S], \ell_n[T])$. For this, we show that for every $\varepsilon > 0$, $\bar{d}(S,T) \leq c + \varepsilon$. By the definition of the Hausdorff metric $\bar{d}$, it suffices to prove that for every $x \in S$ we have $d(x,T) \leq c + \varepsilon$. By symmetry, we then also have $d(y,S) \leq c + \varepsilon$ for every $y \in T$.

Given $y \in T$ we have $d(x,y) = \sup_{n<\omega} d(\ell_n(x), \ell_n(y))$. Thus, there is a $k < \omega$ such that

$$d(x,y) \leq d(\ell_k(x), \ell_k(y)) + \varepsilon.$$

Since T is finite, we can choose k such that this inequality holds for all $y \in T$. By definition,

$$\bar{d}(\ell_k(x), \ell_k[T]) = \inf_{y\in T} d(\ell_k(x), \ell_k(y)) \qquad \text{in } X_k.$$

Again using that T is finite, we can pick some $y \in T$ such that $d(\ell_k(x), \ell_k[T]) = d(\ell_k(x), \ell_k(y))$. With this y we conclude that

$$d(x,T) \leq d(x,y) \leq d(\ell_k(x), \ell_k(y)) + \varepsilon = d(\ell_k(x), \ell_k[T]) + \varepsilon \leq \bar{d}(\ell_k[S], \ell_k[T]) + \varepsilon \leq c + \varepsilon. \quad ◀$$

▶ **Remark 5.7.** The Hausdorff functor preserves isometric embeddings and their intersections. Indeed, for every subspace X of a metric space Y, a set $S \subseteq X$ is compact in X iff it is so in Y. Moreover, given $S,T \in \mathcal{H}X$, their distances in $\mathcal{H}X$ and $\mathcal{H}Y$ are the same. Thus, $\mathcal{H}$ preserves isometric embeddings.

Given a collection $X_i \subseteq Y$ $(i \in I)$ of subspaces, a set $S \subseteq \bigcap_{i\in I} X_i$ is compact in X iff it is so in Y (and therefore in every X_i). Thus $\mathcal{H}$ preserves that intersection.

▶ **Definition 5.8.** The *Hausdorff polynomial functors* are the endofunctors on Met built from the Hausdorff functor, the constant functors, and the identity functor, using product, coproduct, and composition. Thus, the Hausdorff polynomial functors are built according to the following grammar (cf. Definition 3.1):

$$F ::= \mathcal{H} \mid A \mid \mathsf{Id} \mid \textstyle\prod_{i\in I} F_i \mid \coprod_{i\in I} F_i \mid FF,$$

where A ranges over all metric spaces and I is an arbitrary index set.

▶ **Theorem 5.9.** *Every Hausdorff polynomial functor $F\colon \mathsf{Met} \to \mathsf{Met}$ has a terminal coalgebra obtained in $\omega + \omega$ steps: $\nu F = V_{\omega+\omega}$.*

Proof. An easy induction over the structure of Hausdorff polynomial functors shows that each such functor F preserves:

1. isometric cones of ω^{op}-chains, and
2. isometric embeddings and their intersections.

The most important step is done in Proposition 5.6 and Remark 5.7.

We conclude that in the terminal-coalgebra chain, the map $m\colon V_{\omega+1} \to V_\omega$ from (2) in the Introduction is an isometric embedding by item 1. By item 2, all of the maps $m, Fm, FFm, \ldots$ in the chain $(V_{\omega+n})_{n<\omega}$ are isometric embeddings. Hence F preserves the intersection of the ensuing subspaces of V_ω viz. the limit $V_{\omega+\omega} = \lim_{n<\omega} V_{\omega+n}$. Consequently, we have $\nu F = V_{\omega+\omega}$. ◀

▶ **Remark 5.10.** Note that if a Hausdorff polynomial functor F uses only contants given by complete metric spaces A, then it has a restriction to an endofunctor on CMS. Indeed, by an easy induction on the structure of F one shows that FX is complete whenever X is complete. Similarly, when F uses constants which are ultrametric spaces, then F has a restriction on UMet.

Since CMS and UMet are closed under limits of ω^{op}-chains in Met, we obtain the following

▶ **Corollary 5.11.** *Every Hausdorff polynomial functor on* CMet *or* UMet *has a terminal coalgebra obtained in* $\omega + \omega$ *steps.*

▶ **Corollary 5.12.** *Every Hausdorff polynomial functor* F *on* Met *or* CMS *has an initial algebra.*

Indeed, since Hausdorff polynomial functors preserve isometric embeddings, this follows from Theorem 5.9, Example 2.3.2, and Theorem 2.4.

▶ **Remark 5.13.** We mentioned another possible approach to terminal coalgebras in Remark 4.19. Let us comment on the situation regarding the results on Met here. The category Met is locally presentable (see e.g. [6, Ex. 2.3]). The Limit Theorem does imply that on Met, the Hausdorff polynomial functors have terminal coalgebras. In more detail, the Hausdorff functor is finitary: this was proved for its restriction to 1-bounded metric spaces [5, Sec. 3], and the proof for $\mathcal{H}$ itself is the same. An easy induction then shows that every Hausdorff polynomial functor is accessible, so that the Limit Theorem can be applied. However, our elementary proof shows that the terminal coalgebra chain converges in $\omega + \omega$ steps. The proof of Makkai and Paré's Limit Theorem does not yield such a bound.

6 Variation: the Closed Subset Functor on Met

We have been concerned with the Hausdorff functor taking a metric space M to the space of its nonempty compact subsets. For two variations, let us consider $\mathscr{P}_{\mathsf{cl}} \colon \mathsf{Met} \to \mathsf{Met}$ taking M to the set of its *closed* subsets, and its subfunctor $\mathscr{P}'_{\mathsf{cl}} \colon \mathsf{Met} \to \mathsf{Met}$ taking M to the set of its nonempty closed subsets. Both $\mathscr{P}_{\mathsf{cl}}M$ and $\mathscr{P}'_{\mathsf{cl}}M$ are given the Hausdorff metric. For a non-expanding map $f \colon X \to Y$, the non-expanding map $\mathscr{P}_{\mathsf{cl}}f \colon \mathscr{P}_{\mathsf{cl}}X \to \mathscr{P}_{\mathsf{cl}}Y$ sends a closed subset S of X to the closure of $f[S]$. This makes $\mathscr{P}_{\mathsf{cl}}$ and $\mathscr{P}'_{\mathsf{cl}}$ functors. Due to the empty set, $\mathscr{P}_{\mathsf{cl}}$ is a closer analog of $\mathcal{H}$. It is natural to ask whether the positive results of Section 5 hold for these functors $\mathscr{P}_{\mathsf{cl}}$ and $\mathscr{P}'_{\mathsf{cl}}$. As proved by van Breugel [20, Prop. 8], the functor $\mathscr{P}_{\mathsf{cl}}$ has no terminal coalgebra. Turning to $\mathscr{P}'_{\mathsf{cl}}$, this functor has an initial algebra given by the empty metric space and a terminal coalgebra carried by a singleton metric space. But $\mathscr{P}'_{\mathsf{cl}}$ has no other fixed points (see van Breugel et al. [21, Cor. 5]), where an object X is a *fixed point* of an endofunctor F if $FX \cong X$. We provide below a different, shorter proof.

▶ **Remark 6.1.**

1. A subset X of a metric space is δ*-discrete* if whenever $x \neq y$ are elements of X, $d(x, y) \geq \delta$. Every subset of a δ-discrete set is δ-discrete, and every such set is closed. Moreover, if C and D are different subsets of a δ-discrete set, then $\bar{d}(C, D) \geq \delta$.
2. A subset S of an ordinal i is *cofinal* if for all $j < i$ there is some $k \in S$ with $j \leq k < i$. If S is not cofinal, then its complement $i \setminus S$ must be so. (But it is possible that both S and $i \setminus S$ are cofinal in i.)

▶ **Theorem 6.2.** *There is no isometric embedding $\mathscr{P}'_{\mathsf{cl}}M \to M$ when $|M| \geq 2$.*

Proof. Suppose towards a contradiction that $\iota\colon \mathscr{P}_{\mathsf{cl}}M \to M$ were an isometric embedding where $|M| \geq 2$. If all distances in M are 0 or ∞, then $\mathscr{P}'_{\mathsf{cl}}M$ is the nonempty power-set of M. In this case, our result follows from the fact that for $|M| \geq 2$, M has more nonempty subsets than elements. Thus we fix distinct points $a, b \in M$ of finite distance, and put $\delta = d(a,b)/2$. Let $A = \{x \in M : d(x,a) \leq \delta\}$, and let $B = M \setminus A$. (In case $d(a,b) = \infty$, we need to adjust this by setting $\delta = \infty$, and B to be the points whose distance to a is finite. But we shall not present the argument in this case.)

We proceed to define an ordinal-indexed sequence of elements $x_i \in M$. We also prove that each set $S_i = \{x_j : j < i\}$ is δ-discrete, and we put

$$X_i = \begin{cases} A & \text{if } \{j < i : x_j \in A\} \text{ is cofinal in } i \\ B & \text{else.} \end{cases}$$

For $i = 0$, put $x_0 = \iota(\{a,b\})$. Given an ordinal $i > 0$, we put

$$x_i = \iota(X_i \cap S_i).$$

Being nonempty (since $i > 0$) and δ-discrete, $X_i \cap S_i$ lies in $\mathscr{P}'_{\mathsf{cl}}M$.

The remainder of our proof consists of showing that for every ordinal i:

$$d(x_j, x_k) \geq \delta \qquad \text{for } 0 \leq j < k \leq i.$$

We proceed by transfinite induction. Assume that our claim holds for every $k < i$ and then prove it for i. The base case $i = 0$ is trivial. For $i > 0$, note first that it follows from the induction hypothesis that S_i is δ-discrete.

Hence, we need only verify that $d(x_j, x_i) \geq \delta$ when $0 \leq j < i$. We argue the case $X_i = A$; when $X_i = B$, the argument is similar, mutatis mutandis. For $j = 0$, recall that $x_0 = \iota(\{a,b\})$ and $x_i = \iota(A \cap S_i)$. Since b has distance at least δ from every element of A, we obtain $\bar{d}(\{a,b\}, A \cap S_i) \geq \delta$. As ι is an isometric embedding, this distance is also $d(x_0, x_i)$. Now let $j > 0$. Since we have $X_i = A$, let k be such that $j \leq k < i$ and $x_k \in A$. Now either $x_j = \iota(A \cap S_j)$ or else $x_j = \iota(B \cap S_j)$.

In the first case, note that $x_k \in A \cap S_i$ since $k < i$, and $x_k \notin S_j$ by the definition of S_j since $k \geq j$. So $A \cap S_j$ and $A \cap S_i$ are different nonempty subsets of the δ-discrete set S_i. Hence, the distance between these sets is at least δ, and therefore we have $d(x_j, x_i) \geq \delta$.

In the second case, $B \cap S_j$ is a nonempty subset of B, and thus again it not equal to $A \cap S_i$. So again we see that $d(x_j, x_i) = \bar{d}(B \cap S_j, A \cap S_i) \geq \delta$.

We now obtain the desired contradiction since (x_i) is an ordinal-indexed sequence of pairwise distinct elements of M. ◀

▶ **Corollary 6.3.**
1. *The functor $\mathscr{P}'_{\mathsf{cl}}\colon \mathsf{Met} \to \mathsf{Met}$ has no fixed points except the empty set and the singletons.*
2. *The functor $\mathscr{P}_{\mathsf{cl}}\colon \mathsf{Met} \to \mathsf{Met}$ admits no isometric embedding $\mathscr{P}_{\mathsf{cl}}M \to M$, whence has no fixed point.*

Proof. The first item is immediate from Theorem 6.2. For the second one, observe that the inclusion map $e\colon \mathscr{P}'_{\mathsf{cl}}M \to \mathscr{P}_{\mathsf{cl}}M$ is an isometric embedding. Assuming that there were an isometric embedding $\iota\colon \mathscr{P}_{\mathsf{cl}}M \to M$, we see that M cannot be empty (since $\mathscr{P}_{\mathsf{cl}}M$ is nonempty) or a singleton (since then $|\mathscr{P}_{\mathsf{cl}}M| = 2$). Hence $|M| \geq 2$. Moreover, we obtain an isometric embedding $\iota \cdot e\colon \mathscr{P}'_{\mathsf{cl}}M \to M$, contradicting Theorem 6.2. ◀

7 Summary

We have investigated versions of the finite power-set functor for the categories Haus and Met. Our main results are that the Vietoris functor $\mathscr{V}$, and indeed all Vietoris polynomial functors, have terminal coalgebras obtained in ω steps of the terminal-coalgebra chain. The same holds for the Hausdorff polynomial functors on Met, but the iteration takes $\omega + \omega$ steps and so the underlying reasons are different.

Our work on the Kripke and Hausdorff polynomial functors highlights a technique which we feel could be of wider interest. To prove that a terminal coalgebra exists in a situation where the limit of the ω^{op}-chain (1) is not preserved by the functor, one could try to find preservation properties which imply that the limit of the ω^{op}-chain $(V_{\omega+n})_n$ was preserved. In Set, we used finitarity and preservation of monomorphisms and intersections, and in Met we used preservation of intersections, isometric embeddings, and isometric cones.

We have also seen that for the functor $\mathscr{P}_{\mathsf{cl}}$ on Met, there is no fixed point and hence no terminal coalgebra. We leave open the question of whether every Vietoris polynomial functor on Top has a terminal coalgebra.

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A Trees and the Limit of the Terminal-Coalgebra Chain for $\mathscr{P}_\mathsf{f}$

We give the description of V_ω for $\mathscr{P}_\mathsf{f}$ due to Worrell [23]. We provide a full exposition to the results which Worrell stated without proof.

A *tree* is a directed graph t with a distinguished node $\mathsf{root}(t)$ from which every other node can be reached by a unique directed path. Every tree in our sense must have a root, so there is no empty tree. All of our trees are *unordered*. We always identify isomorphic trees.

▶ **Definition A.1.**

1. We use the notation t_x for the subtree of t rooted in the node x of t.
2. A tree t is *extensional* if for every node x distinct children y and z of x give different (that is, non-isomorphic) subtrees t_y and t_z.
3. A *graph bisimulation* between two trees t and u is a relation between the nodes of t and the nodes of u with the property that whenever x and y are related: (a) every child of x is related to some child of y, and (b) every child of y is related to some child of x.
4. A *tree bisimulation* between two trees t and u is a graph bisimulation R with the additional properties that that
   a. The nodes $\mathsf{root}(t)$ and $\mathsf{root}(u)$ are related; the roots are not related to other nodes; and
   b. whenever two nodes are related, their parents are also related.
5. Two trees are *tree bisimilar* if there is a tree bisimulation between them.
6. A tree t is *strongly extensional* if every tree bisimulation on it is a subset of the diagonal

$$\Delta_t = \{(x, x) : x \in t\}.$$

   In other words, t is strongly extensional iff distinct children x and y of the same node define subtrees t_x and t_y which are *not* tree bisimilar.

▶ **Remark A.2.**

1. Every composition and every union of tree bisimulations is again a tree bisimulation. In addition, the opposite relation of every tree bisimulation is a tree bisimulation: if R is a tree bisimulation from t to u, then R^{op} is a tree bisimulation from u to t. Consequently, the largest tree bisimulation on every tree is an equivalence relation.
2. A subtree s of a strongly extensional tree t is strongly extensional. Indeed, if R is a tree bisimulation on s, then $R \cup \Delta_t$ is a tree bisimulation on t. Since $R \cup \Delta_t \subseteq \Delta_t$, we have $R \subseteq \Delta_s$.

▶ **Lemma A.3.** *If t and u are strongly extensional and related by a tree bisimulation, then we have $t = u$.*

Proof. Let R be a tree bisimulation between t and u. By Remark A.2, $R^{\mathsf{op}} \cdot R$ is a tree bisimulation on t, whence $R^{\mathsf{op}} \cdot R \subseteq \Delta_t$ by strong extensionality. But every node of t is related to at least one node of u (use induction on the depth of nodes) implying that $R^{\mathsf{op}} \cdot R = \Delta_t$. Similarly, $R \cdot R^{\mathsf{op}} = \Delta_u$. Thus, R (is a function and it) is an isomorphism of trees, and we identify such trees. ◀

▶ **Notation A.4.**

1. Let $\mathcal{T}$ be the class of trees. We define maps $\partial_n \colon \mathcal{T} \to V_n = \mathscr{P}_{\mathsf{f}}^n 1$ as follows: ∂_0 is the unique map to 1, and given the map ∂_n and a tree t, we put

   $$\partial_{n+1}(t) = \{\partial_n(t_x) : x \text{ is a child of the root of } t\}.$$

   On the right we have a subset of $\mathscr{P}_{\mathsf{f}}^n 1$, and this is an element of $\mathscr{P}_{\mathsf{f}}^{n+1} 1$.
2. The trees t and u are *Barr equivalent* provided that $\partial_n t = \partial_n u$ for all n. We write $t \approx u$ in this case.
3. For every tree t, we define maps $\rho^t_n \colon t \to V_n = \mathscr{P}_{\mathsf{f}}^n 1$ in the following way: ρ^t_0 is the unique map $t \to 1$, and for all nodes x of t, $\rho^t_{n+1}(x) = \{\rho^t_n(y) : y \text{ is a child of } x \text{ in } t\}$. This family of maps ρ^t_n is a cone: we have $\rho^t_n = v_{m,n} \cdot \rho^t_m$ for every connecting map $v_{m,n} \colon \mathscr{P}_{\mathsf{f}}^m 1 \to \mathscr{P}_{\mathsf{f}}^n 1$, $m \geq n$. Hence, there is a unique map $\rho^t_\omega \colon t \to V_\omega$ such that $\ell_n \cdot \rho^t_\omega = \rho^t_n$ for all n.

▶ **Remark A.5.** Note that $V_n = \mathscr{P}_{\mathsf{f}}^n 1$ may be described as the set of all extensional trees of height at most n. Indeed, 1 is described as the singleton set consisting of the root-only tree, and every finite set of extensional trees in $V_{n+1} = \mathscr{P}_{\mathsf{f}} V_n$ is represented by the extensional tree obtained by tree-tupling the trees from the given set.

▶ **Remark A.6.**

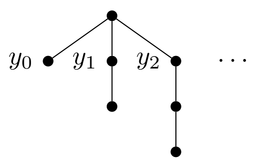
1. If $\rho^t_{n+1}(a) = \rho^t_{n+1}(b)$, then for all children a' of a, there is some child b' of b and $\rho^t_n(a') = \rho^t_n(b')$. This is easy to see from the definition of ρ^t_{n+1}.
2. For all trees t, $\rho^t_i(\mathsf{root}(t)) = \partial_i(t)$. Furthermore, let $b \colon t \to \mathcal{T}$ be given by $b(x) = t_x$. Then $\rho^t_i = \partial_i \cdot b$.

▶ **Definition A.7.** Let $x_0, x_1, \ldots,$ be a sequence of nodes in a tree t, and let y also be a node in t. We write $\lim x_n = y$ to mean that for every n there is some m such that $\rho^t_n(x_p) = \rho^t_n(y)$ whenever $p \geq m$.

A tree t is *compactly branching* if for all nodes x of t, the set of children of x is *sequentially compact*: for every sequence of (y_n) of children of x there is a subsequence (w_n) of (y_n) and some child z of x such that $\lim w_n = z$.

▶ **Example A.8.** The following tree t is not compactly branching:

t:

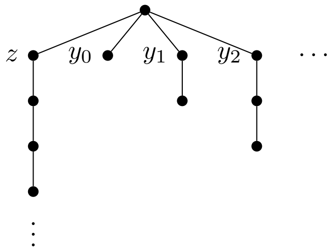


To see this, consider the sequence y_0, y_1, ... Note that for $n \geq m$, $\rho^t_n(y_n) = \partial_i(t_{y_n}) = t_{y_m}$. We claim that for every subsequence (y_{k_n}) of this sequence (y_n) there is no y_p such that $\lim_n y_{k_n} = y_p$. To simplify the notation, we only verify this for the sequence (y_n) itself. It does not converge to any fixed element y_m because for $p > m$,

$$\rho^t_p(y_m) = \partial_p(t_{y_m}) \neq \partial_p(t_{y_p}) = \rho^t_p(y_p).$$

In contrast, the following tree is compactly branching (also observe also that $t \approx t'$):

t':



To check the compactness, consider a sequence of children of the root, say (x_n). If there is an infinite subsequence which is constant, then of course that sequence converges. If not, then there is a subsequence of (x_n), say (w_n), where each w_n is y_k for some $k \geq n$. In this case, $\lim_n(w_n) = z$. This is because for all but finitely many n, $\rho^t_n(z) = \partial_n(t_z) = t_{w_n} = \partial_n(t_{w_n}) = \rho^t_n(w_n)$.

▶ **Lemma A.9.** *If t and u are compactly branching, and if $\rho^t_\omega(\mathsf{root}(t)) = \rho^u_\omega(\mathsf{root}(u))$, then there is a tree bisimulation between t and u which includes $\{(x, y) \in t \times u \colon \rho^t_\omega(x) = \rho^u_\omega(y)\}$.*

Proof. Given compactly branching trees t and u, we define a relation $R \subseteq t \times u$ inductively by

$x \mathrel{R} y$ iff (1) $x = \mathsf{root}(t)$ and $y = \mathsf{root}(u)$, or x and y have R-related parents, and
(2) $\rho^t_\omega(x) = \rho^u_\omega(y)$.

Let us check that R is a tree bisimulation. Suppose that (x, y) are related by R as above, and let x' be a child of x in t. Using Remark A.6.1 we see that for each n, there is some child y'_n of y in u with $\rho^t_n(x') = \rho^u_n(y'_n)$. Consider the sequence y'_0, y'_1, ... Now $\rho^t_n(x') = \rho^u_n(y'_m)$ if $m \geq n$, since ρ^t_n and ρ^u_n form cones: $\rho^t_n(x') = v_{m,n} \cdot \rho^t_m(x') = v_{m,n} \cdot \rho^u_m(y'_m) = \rho^u_n(y'_m)$. By sequential compactness, there is a subsequence z_0, z_1, ..., and also some child z^* of y such that $\lim z_n = z^*$. Being a subsequence, $\rho^t_n(x') = \rho^u_n(z_m)$ whenever $m \geq n$. Let us check that for all n, $\rho^t_n(x') = \rho^u_n(z^*)$. To see this, fix n and let $m \geq n$ be large enough so that for $p \geq m$, $\rho^u_n(z_p) = \rho^u_n(z^*)$. Thus, $\rho^t_n(x') = \rho^u_n(z_m) = \rho^u_n(z^*)$. Thus, $\rho^t_\omega(x') = \rho^u_\omega(z^*)$, which shows $x' \mathrel{R} z^*$, as desired.

The other half of the verification that R is a tree bisimulation is similar. ◀

▶ **Notation A.10.** In this section, V_ω denotes the limit of (1) for the finite power-set functor.

1. We take the elements of V_ω to be compatible sequences (x_n). That is, $x_n \in \mathscr{P}^n_{\mathsf{f}} 1$ and $\mathscr{P}^n_{\mathsf{f}}!(x_{n+1}) = x_n$ for every $n < \omega$. To save on notation, we write x for (x_n). We consider the relation $\rightsquigarrow$ on V_ω defined by

$$x \rightsquigarrow y \quad \text{iff} \quad \text{for all } n,\ y_n \in x_{n+1}. \tag{6}$$

2. Let L^+ be the set of nonempty finite sequences from V_ω. We write such a sequence with the notation $\langle x^1, \ldots, x^n \rangle$. We consider the relation $\Rightarrow$ on L^+ defined by

$$\langle x^1, \ldots, x^n \rangle \Rightarrow \langle y^1, \ldots, y^m \rangle \quad \text{iff} \quad m = n + 1,\ x^1 = y^1,\ \ldots,\ x^n = y^n, \text{ and } x^n \rightsquigarrow y^{n+1}.$$

In other words, $m = n + 1$, $\langle y^1, \ldots, y^{m-1} \rangle = \langle x^1, \ldots, x^n \rangle$, and $x^n \rightsquigarrow y^m$.

3. For each $x \in V_\omega$, let tr_x be the tree whose nodes are the sequences $\langle x, x^2, \ldots, x^n \rangle \in L^+$ whose first entry is x, with root the one-point sequence $\langle x \rangle$, and with graph relation the restriction of $\Rightarrow$. For readers familiar with tree unfoldings of pointed graphs, tr_x is the tree unfolding of the graph $(V_\omega, \rightsquigarrow)$ at the point x.
4. Finally, let

$$T = \{\mathsf{tr}_x : x \in V_\omega\}. \tag{7}$$

Recall the connecting maps $\mathscr{P}_\mathsf{f}^n! \colon \mathscr{P}_\mathsf{f}^{n+1}1 \to \mathscr{P}_\mathsf{f}^n 1$.

▶ **Lemma A.11.** *Let $x \in V_\omega$.*
1. *For all k and all $\langle x, x^2, \ldots, x^n \rangle \in \mathsf{tr}_x$, $\rho_k^{\mathsf{tr}_x}(\langle x, x^2, \ldots, x^n \rangle) = x_k^n$.*
2. *Let R be a tree bisimulation on tr_x. If $\langle x, x^2, \ldots, x^n \rangle \; R \; \langle x, y^2, \ldots, y^n \rangle$, then for all k,*

$$\rho_k^{\mathsf{tr}_x}(\langle x, x^2, \ldots, x^n \rangle) = \rho_k^{\mathsf{tr}_x}(\langle x, y^2, \ldots, y^n \rangle).$$

3. *The tree tr_x is strongly extensional and compactly branching, and $\partial_\omega(\mathsf{tr}_x) = \rho_\omega^{\mathsf{tr}_x}(\langle x \rangle) = x$.*

Proof.
1. By induction on k. For $k = 0$, our result is clear: the codomain of ρ_k is 1. Assume our result for k, fix $x \in L^+$ and $\langle x^1, \ldots, x^n \rangle \in \mathsf{tr}_x$. We first prove that

$$\{y_k : x^n \rightsquigarrow y\} = x_{k+1}^n. \tag{8}$$

Indeed, if $x^n \rightsquigarrow y$, then $y_k \in x_{k+1}^n$. Conversely, if $a \in x_{k+1}^n$, we construct $y \in V_\omega$ such that $x^n \rightsquigarrow y$ with $y_k = a$. Note that

$$x_k^n = \mathscr{P}_\mathsf{f}^k!(x_{k+1}^n) = \mathscr{P}\mathscr{P}_\mathsf{f}^{k-1}!(x_{k+1}^n) = \mathscr{P}_\mathsf{f}^{k-1}![x_{k+1}^n].$$

Since $a \in x_{k+1}^n$, we have $\mathscr{P}_\mathsf{f}^{k-1}!(a) \in x_k^n$. So we let $y_{k-1} = \mathscr{P}_\mathsf{f}^{k-1}!(a)$. We repeat this argument to define y_{k-2}, …, y_1, y_0; the point is that $y_{k-i} \in x_{k-i+1}^n$ for $i = 0, \ldots, k$. Choices are needed when we go the other way from k. Note that

$$\mathscr{P}_\mathsf{f}^{k+1}![x_{k+2}^n] = \mathscr{P}_\mathsf{f}(\mathscr{P}_\mathsf{f}^{k+1}!)(x_{k+2}^n) = \mathscr{P}_\mathsf{f}^{k+2}!(x_{k+2}^n) = x_{k+1}^n.$$

Every set functor preserves surjective functions, and so $\mathscr{P}_\mathsf{f}^{k+1}!$ is surjective. Thus there is some $y_{k+1} \in x_{k+2}^n$ such that $\mathscr{P}_\mathsf{f}^{k+1}!(y_{k+1}) = y_k$. The same argument enables us to find by recursion on i a sequence $y_{k+i+1} \in x_{k+i+2}^n$ such that $\mathscr{P}_\mathsf{f}^{k+i+1}!(y_{k+i+1}) = y_{k+i}$. This defines y such that $x^n \rightsquigarrow y$ according to (6) with $y_k = a$.
The induction step is now easy:

$$\begin{aligned}
\rho_{k+1}^{\mathsf{tr}_x}(\langle x, x^2, \ldots, x^n \rangle) &= \{\rho_k^{\mathsf{tr}_x}(\langle x, x^2, \ldots, x^n, y \rangle) : x^n \rightsquigarrow y\} \\
&= \{y_k : x^n \rightsquigarrow y\} && \text{by induction hypothesis} \\
&= x_{k+1}^n && \text{by (8).}
\end{aligned}$$

2. This again is an induction on k, and the steps are similar to what we have just seen. We also note that tuples in tr_x related by a tree bisimulation must have the same length.
3. Note first that by item 1 with $n = 1$, we have $\rho_k^{\mathsf{tr}_x}(\langle x \rangle) = x_k$ for all k. This implies that $\rho_\omega^{\mathsf{tr}_x}(\langle x \rangle) = x$. For the strong extensionality, let R be a tree bisimulation on tr_x. Suppose that $\langle x, x^2, \ldots, x^n \rangle$ and $\langle x, y^2, \ldots, y^n \rangle$ are related by R. Using items 1 and 2, we see that for all k, we have $x_k^n = y_k^n$. Thus $x^n = y^n$. In addition, since R is a tree bisimulation,

the parents of the two nodes under consideration are also related by R. So the same argument shows that $x^{n-1} = y^{n-1}$. Continuing in this way shows that $x^{n-2} = y^{n-2}$, ..., $x^2 = y^2$. Hence $\langle x, x^2, \ldots, x^n \rangle = \langle x, y^2, \ldots, y^n \rangle$.
Finally, we verify that tr_x is compactly branching. To simplify the notation a little, we shall show this for children of the root $\langle x \rangle$. So suppose we have an infinite sequence $\langle x, y^1 \rangle$, $\langle x, y^2 \rangle$, ... Recall that each set $\mathscr{P}_\mathsf{f}^n 1$ is finite. By successively thinning the sequence y^1, y^2, ..., we may assume that for all $n \in \omega$ and all $p, q \geq n$, $y^n_p = y^n_q$. Let $z \in V_\omega$ be the 'diagonal' sequence $z_n = y^n_n$. Since every $\langle x, y^n \rangle$ is a child of the root $\langle x \rangle$ (in symbols: $\langle x \rangle \Rightarrow \langle x, y^n \rangle$), we have $x \rightsquigarrow y^n$. This implies that for all n, we have $z_n = y^n_n \in x_{n+1}$, whence $x \rightsquigarrow z$. Thus, $\langle x, z \rangle$ is a child of the root of tr_x. Recall from item 1 that $\rho_n^{\mathsf{tr}_x}(\langle x, z \rangle) = z_n$. So we obtain the desired conclusion: $\lim \langle x, y^n \rangle = \langle x, z \rangle$. ◀

▶ **Lemma A.12.** *For every tree t there is a Barr-equivalent tree $t^* \in T$ such that t^* is strongly extensional and compactly branching.*

Proof. Given any tree t, we have $x = \partial_\omega(t) \in V_\omega$. For all n, $x_n = \partial_n(t)$. The tree $t^* = \mathsf{tr}_x$ in Lemma A.11.3 is strongly extensional and compactly branching. Recall that the root of t^* is $\langle x \rangle$. By Lemma A.11.1, we have that for all $n < \omega$,

$$\partial_n(t^*) = \rho_n^{t^*}(\mathsf{root}(\mathsf{tr}_x)) = \rho_n^{t^*}(\langle x \rangle) = x_n = \partial_n(t).$$ ◀

▶ **Lemma A.13.** *The set T defined in (7) is the set of all compactly branching, strongly extensional trees.*

Proof. By Lemma A.11.3 we know that every tree in T is strongly extensional and compactly branching. For the reverse inclusion, let t be compactly branching and strongly extensional. Let t^* be as in Lemma A.12 for t. By Lemmas A.3 and A.9, $t = t^*$. Thus $t \in T$. ◀

▶ **Definition A.14.** Let D be the set of finitely branching strongly extensional trees. Let $\delta\colon D \to \mathscr{P}_\mathsf{f} D$ take a strongly extensional tree t to the (finite) set of its subtrees t_x.

In this definition, we use Remark A.2.2: a subtree of a strongly extensional tree is strongly extensional.

▶ **Theorem A.15** [23]. *For the finite power-set functor $\mathscr{P}_\mathsf{f}$ the following hold:*
1. *the maps $\partial_n\colon T \to \mathscr{P}_\mathsf{f}^n 1$ given by $\partial_n(\mathsf{tr}_x) = x_n$ form a limit of (2); thus, $V_\omega \cong T$,*
2. *the coalgebra (D, δ) is terminal.*

Proof.
1. The map $\varphi\colon V_\omega \to T$ given by $\varphi(x) = \mathsf{tr}_x$ is obviously surjective. Suppose that $\mathsf{tr}_x = \mathsf{tr}_y$. The roots of these trees are $\langle x \rangle$ and $\langle y \rangle$. For all n, we have that
$$x_n = \rho_n^{\mathsf{tr}_x}(\langle x \rangle) = \rho_n^{\mathsf{tr}_y}(\langle y \rangle) = y_n.$$
Thus $\partial_\omega(\langle x \rangle) = \partial_\omega(\langle y \rangle)$. By Lemmas A.3 and A.9, $x = y$. So φ is injective. The formula for ∂_n comes from Lemma A.11.1.
2. We use Theorem 3.5. The map $m\colon V_{\omega+1} \to V_\omega$ in (2) assigns to a finite set of trees in V_ω their tree-tupling. Its image is the set of all strongly extensional, compactly branching trees which are finitely branching at the root. An easy induction on n shows that $V_{\omega+n}$ is the set of all compactly branching, strongly extensional trees t with the property that the topmost n levels of t are finitely branching. With this description, $V_{\omega+n} \subseteq D$, and the limit $V_{\omega+\omega}$ is simply the intersection $D = \bigcap_n V_{\omega+n}$. This shows that the carrier set of $\nu\mathscr{P}_\mathsf{f}$ is D. For the structure map δ, note that $m\colon \mathscr{P}_\mathsf{f} V_\omega \to V_\omega$ in (2) is tree-tupling, as are $\mathscr{P}_\mathsf{f} m$, $\mathscr{P}_\mathsf{f}\mathscr{P}_\mathsf{f} m$, etc. It follows that in the intersection, D, the coalgebra structure is the inverse of tree-tupling. ◀

This concludes our work showing that for the finite power-set functor $\mathscr{P}_\mathsf{f}$, V_ω is the set T of strongly extensional, compactly branching trees, and the terminal coalgebra $\nu\mathscr{P}_\mathsf{f}$ is the set D of finitely branching, strongly extensional trees.