Sampling from the Random Cluster Model on Random Regular Graphs at All Temperatures via Glauber Dynamics*

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Abstract

We consider the performance of Glauber dynamics for the random cluster model with real parameter $q > 1$ and temperature $\beta > 0$. Recent work by Helmuth, Jenssen and Perkins detailed the ordered/disordered transition of the model on random Δ-regular graphs for all sufficiently large q and obtained an efficient sampling algorithm for all temperatures β using cluster expansion methods. Despite this major progress, the performance of natural Markov chains, including Glauber dynamics, is not yet well understood on the random regular graph, partly because of the non-local nature of the model (especially at low temperatures) and partly because of severe bottleneck phenomena that emerge in a window around the ordered/disordered transition.

Nevertheless, it is widely conjectured that the bottleneck phenomena that impede mixing from worst-case starting configurations can be avoided by initialising the chain more judiciously. Our main result establishes this conjecture for all sufficiently large q (with respect to Δ). Specifically, we consider the mixing time of Glauber dynamics initialised from the two extreme configurations, the all-in and all-out, and obtain a pair of fast mixing bounds which cover *all* temperatures β, including in particular the bottleneck window. Our result is inspired by the recent approach of Gheissari and Sinclair for the Ising model who obtained a similar-flavoured mixing-time bound on the random regular graph for sufficiently low temperatures. To cover all temperatures in the RC model, we refine appropriately the structural results of Helmuth, Jenssen and Perkins about the ordered/disordered transition and show spatial mixing properties "within the phase", which are then related to the evolution of the chain.

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1 Introduction

For real numbers $q, \beta > 0$ and a graph $G = (V, E)$, the *random cluster model* on G with parameters q and β is a probability distribution on the set $\Omega = \Omega_G$ of all assignments $\mathcal{F} : E \to \{0, 1\}$; we typically refer to assignments in Ω as configurations. For a configuration $\mathcal{F}$, we say that edges mapped to 1 are *in-edges*, and edges mapped to 0 are *out-edges*. We

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use $\mathrm{In}(\mathcal{F})$ to denote the set of edges e with $\mathcal{F}(e) = 1$, $\mathrm{Out}(\mathcal{F})$ to denote the set of edges e with $\mathcal{F}(e) = 0$, $|\mathcal{F}|$ for the cardinality of $\mathrm{In}(\mathcal{F})$ and $c(\mathcal{F})$ for the number of connected components in the graph $(V, \mathrm{In}(\mathcal{F}))$. Then, the weight of $\mathcal{F}$ in the RC model is given by $w_G(\mathcal{F}) = q^{c(\mathcal{F})}(e^\beta - 1)^{|\mathcal{F}|}$.

For integer values of q, the RC model is closely connected to the (ferromagnetic) Ising/-Potts models; $q = 2$ is the Ising model and $q \geq 3$ is the Potts model whose configurations are all possible assignments of q colours to the vertices of the graph where an assignment σ has weight proportional to $\mathrm{e}^{\beta m(\sigma)}$ with $m(\sigma)$ being the number of monochromatic edges under σ. The RC model is an alternative edge representation of the models (for integer q) that has also been studied extensively in its own right due to its intricate behaviour (see, e.g., [19]).

We will be primarily interested in sampling from the so-called *Gibbs distribution* on Ω induced by these weights, denoted by $\pi_G(\cdot)$, where for a configuration $\mathcal{F}$, $\pi_G(\mathcal{F}) = w_G(\mathcal{F})/Z_G$ where the normalising factor $Z_G = \sum_{\mathcal{F}' \in \Omega_G} w_G(\mathcal{F}')$ is the aggregate sum of weight of all configurations (known as the partition function). We focus on the *Glauber dynamics* which is a classical Markov chain for sampling from Gibbs distributions which is a particularly useful tool for developing approximate sampling algorithms. We will refer to Glauber dynamics for the RC model as the RC dynamics. Roughly, the RC dynamics is a Markov chain $(X_t)_{t \geq 0}$ initialised at some configuration X_0 which evolves by iteratively updating at each step $t \geq 1$ a randomly chosen edge based on whether its endpoints belong to the same component in the graph $(V, \mathrm{In}(X_t))$. The mixing time of the chain is the number of steps to get within total variation distance $\leq 1/4$ from π_G, see Section 2 for details.

Our goal is to obtain a fast algorithm for the RC model using Glauber dynamics on the random regular graph. There are two key obstacles that arise, especially at low temperatures (large β): (i) Glauber dynamics for the RC model has a highly non-local behaviour, and (ii) there are severe bottleneck phenomena and worst case graphs which prohibit a general fast-convergence result, and more generally an efficient algorithm. The random regular graph is a particularly interesting testbed in this front since it exhibits all the relevant phase transition phenomena and has also been used as the main gadget in hardness reductions [15].

To overview the phenomena that are most relevant for us, the following picture was detailed in a remarkable development by Jenssen, Helmuth, and Perkins [22]: for $\Delta \geq 5$ and all sufficiently large q, they established the ordered/disordered transition occurring at some β_c satisfying $\beta_c = (1 + o_q(1))\frac{2 \log q}{\Delta}$ (see also [15] for integer $q \geq 3$).[1] Roughly, for $\beta < \beta_c$ a typical configuration of the model is disordered, whereas for $\beta > \beta_c$ it is ordered: disordered configurations resemble the all-out configuration (in that all components are of size $O(\log n)$) whereas ordered configurations resemble the all-in configuration (where there is a giant component with $\Omega(n)$ vertices). The two types of configurations coexist at $\beta = \beta_c$, i.e., each appears with some probability bounded away from zero. The methods in [22] are based on cluster expansion techniques which also yielded an efficient sampling algorithm at all temperatures $\beta > 0$. This is a surprising algorithmic result given that the coexistence causes multimodality in π_G and severe bottleneck phenomena for Markov chains in a window around β_c; it was shown for instance in [22] that the RC dynamics (and the related non-local Swendsen-Wang dynamics) have exponential mixing time, essentially because of the number of steps needed for the chain to move from ordered to disordered (and vice versa).

These results pose a rather bleak landscape for the RC dynamics; yet, on random regular graphs it is widely conjectured that the multimodality and the associated bottlenecks can be circumvented by initialising the chain more judiciously, in particular at either the all-out

[1] Recent results of Bencs, Borbényi, and Csikvári [1] yield the exact formula $\beta_c = \log \frac{q-2}{(q-1)^{1-2/\Delta}-1}$ for all $q > 2$ and $\Delta \geq 3$, which was previously only known for integer q [15].

or the all-in configurations (depending on whether $\beta \leq \beta_c$). However the tools available for analysing Markov chains are typically insensitive to the initial configuration, and even more so when working at a critical range of the parameters.

Our main result establishes this conjecture for all $\Delta \geq 5$ and q sufficiently large (conditions which we inherit from [22]). For an integer n such that Δn is even, let $\mathcal{G}_{n,\Delta}$ denote the set of all Δ-regular graphs with n vertices.[2] Throughout, we use $O(1)$ to denote a constant depending on q, β, Δ but independent of n.

▶ **Theorem 1.** *Let $\Delta \geq 5$ be an integer. There exists $C = C(\Delta) > 0$ such that, for all sufficiently large q, the following holds for any $\beta > 0$, w.h.p. over $G \sim \mathcal{G}_{n,\Delta}$.*

1. *For $\beta < \beta_c$, the mixing time of the RC dynamics starting from all-out is $O(n \log n)$.*
2. *For $\beta > \beta_c$, the mixing time of the RC dynamics starting from all-in is $O(n^C)$. For integer q, the mixing time is in fact $O(n \log n)$.*

Note that Theorem 1 implies an $O(n \log n)$ sampling algorithm from the Potts model for all $\beta \neq \beta_c$ (and all sufficiently large q). Intuitively, and as we will see later in more detail, Theorem 1 asserts that the RC dynamics starting from all-in mixes quickly within the set of ordered configurations for $\beta > \beta_c$, and similarly it mixes well within the disordered set of configurations starting from all out when $\beta < \beta_c$. In fact the same is true for $\beta = \beta_c$ and hence the RC dynamics can be used to sample even at criticality, see Remark 12 for details.

Finally, let us note that the RC dynamics can be used analogously to the theorem above to produce a sample within total variation distance ε of π_G for any $\varepsilon \geq \mathrm{e}^{-\Theta(n)}$, by running it for a number of steps which is $\log(1/\varepsilon)$ times the corresponding mixing time bound.[3] The lower bound on the error comes from the total variation distance between π_G and the conditional "ordered" and "disordered" configurations, see Lemma 3.

1.1 Further related work

Our approach to proving Theorem 1 is inspired from a recent paper by Gheissari and Sinclair [16] who established similar flavoured results for the Ising model ($q = 2$) on the random regular graph for large β. To obtain our results for all β, we adapt suitably their notion of "spatial mixing within the phase", see Section 2.2 for details.

Among the results in [16], it was established that Glauber dynamics on the random regular graph, initialised appropriately, mixes in $O(n \log n)$ time when β is sufficiently large.[4] More recently, Gheissari and Sinclair [17] obtained mixing-time bounds for the RC dynamics on the lattice $\mathbb{Z}^d$ under appropriate boundary conditions. They also analyse the mixing time starting from a mixture of the all-in/all-out initialisation. Note that the phase transition on grid lattices is qualitatively different than that of the random regular graph; there, instead of a window/interval of temperatures, the three points β_u, β_u^* and β_c all coincide into a single phase transition point. See also [6, 23] for related algorithmic results on $\mathbb{Z}^d$ using cluster expansion methods.

For the random regular graph, Blanca and Gheissari [4] showed for all integer $\Delta \geq 3$ and real $q \geq 1$ that the mixing time is $O(n \log n)$ provided that $\beta < \beta_u(q, \Delta)$ where β_u is the uniqueness threshold on the tree. A sampling algorithm (not based on MCMC) for

[2] We write $G \sim \mathcal{G}_{n,\Delta}$ to denote a graph in $\mathcal{G}_{n,\Delta}$ chosen uniformly at random, and we say that a property holds w.h.p. for $G \sim \mathcal{G}_{n,\Delta}$ as a shorthand for "with probability $1 - o_n(1)$ over a graph $G \in \mathcal{G}_{n,\Delta}$ chosen uniformly at random.

[3] The standard submultiplicative argument to bootstrap the total-variation distance goes through using the monotonicity of the RC model (to account for the constraint on the initial configuration), see also [25].

[4] Note that, for $q = 2$, an $O(n^{10})$ upper bound for the RC dynamics on any graph G was previously known at all temperatures β by Guo and Jerrum [20] (see also [13]).

$\beta < \beta_c(q,\Delta)$ and $q, \Delta \geq 3$ was designed by Efthymiou [12] (see also [3]), albeit achieving weaker approximation guarantees. Coja-Oghlan et al. [10] showed that, for all integer $q, \Delta \geq 3$ and $\beta \in (\beta_u, \beta'_u)$ the mixing time is $\mathrm{e}^{\Omega(n)}$ where $\beta'_u = \log(1 + \frac{q}{\Delta-2}) > \beta_u$ is (conjectured to be) another uniqueness threshold on the tree (see [21, 24]). More generally, for integer $q \geq 3$, the hardness results/techniques of [18, 15] yield that for any $\beta > \beta_c$, there are graphs G where the mixing time of the RC dynamics is $\exp(n^{\Omega(1)})$ and the problem of appoximately sampling on graphs of max-degree Δ becomes #BIS-hard; on the other hand, for $\beta \leq (1 - o_{q,\Delta}(1))\beta_c$ it has been shown in [11, 7] that the cluster-expansion technique of [22] yields a sampling algorithm on any max-degree Δ graph.

As a final note, another model of interest where analogous mixing results for Glauber dynamics (initialised appropriately) should be obtainable is for sampling independent sets on random bipartite regular graphs. However, in contrast to the RC/Potts models, the phase transition there is analogous to that of the Ising model, and hence, establishing the relevant spatial mixing properties close to the criticality threshold is likely to require different techniques, see, e.g., [9] for more discussion.

1.2 Independent results of Blanca and Gheissari

In an independent and simultaneous work, Blanca and Gheissari [5] obtain related (but incomparable) results. For $\Delta \geq 3, q \geq 1$ and arbitrarily small $\tau > 0$, they show for sufficiently large β a mixing time bound of $O(n^{1+\tau})$ for the RC dynamics on the random regular graph starting from an arbitrary configuration (and obtain an analogous result for the grid and the Swendsen-Wang dynamics). Our result instead applies to all β for the random regular graph (even the critical window) by taking into consideration the initial configuration; the two papers have different approaches to obtain the main ingredients.

2 Proof of Theorem 1

We start with the formal description of the RC dynamics. Given a graph $G = (V, E)$ and an initial configuration $X_0 : E \to \{0, 1\}$, the RC dynamics on G is a Markov chain $(X_t)_{t\geq 0}$ on the set of configurations Ω_G. Let $p := 1 - \mathrm{e}^{-\beta}$ and $\hat{p} := \frac{p}{(1-p)q+p}$ (note that for $q > 1$ it holds that $\hat{p} \in (p/q, p)$). For $t \geq 0$, to obtain X_{t+1} from X_t:

1. Choose u.a.r. an edge $e \in E$. If e is a cut-edge in the graph $(V, \mathrm{In}(X_t) \cup \{e\})$, set $X_{t+1}(e) = 1$ with probability $\hat{p}$ (and $X_{t+1}(e) = 0$ otherwise). Else, set $X_{t+1}(e) = 1$ with probability p, and $X_{t+1}(e) = 0$ otherwise.
2. Set $X_{t+1}(f) = X_t(f)$ for all $f \in E \backslash \{e\}$.

It is a standard fact that the distribution of X_t converges to the RC distribution π_G. Let $T_{\mathrm{mix}}(G; X_0) = \min_{t\geq 0}\{t \mid \mathrm{dist}_{\mathrm{TV}}(X_t, \pi_G) \leq 1/4\}$ be the number of steps needed to get within total-variation distance $\leq 1/4$ from π_G starting from X_0, and $T_{\mathrm{mix}}(G) = \max_{X_0} T_{\mathrm{mix}}(G; X_0)$ be the mixing time from the worst starting state.

2.1 The ordered and disordered phases on random regular graphs

We review in more detail the ordered/disordered transition, following [22].

▶ **Definition 2.** *For $\Delta \geq 3$, let $\eta = \eta(\Delta) \in (0, 1/2)$ be a small constant (see Definition 17). For $G \in \mathcal{G}_{n,\Delta}$, the* ordered phase *is the set of configurations $\Omega^{\mathrm{ord}} := \{\mathcal{F} \in \Omega : |\mathrm{In}(\mathcal{F})| \geq (1-\eta)|E|\}$, whereas the* disordered phase *is the set $\Omega^{\mathrm{dis}} := \{\mathcal{F} \in \Omega : |\mathrm{In}(\mathcal{F})| \leq \eta|E|\}$. For $q, \beta > 0$, let $\pi_G^{\mathrm{ord}}, \pi_G^{\mathrm{dis}}$ be the conditional distributions of π_G on $\Omega^{\mathrm{ord}}, \Omega^{\mathrm{dis}}$, respectively.*

We will use the following result of Helmuth, Jenssen and Perkins [22, Lemma 9].

▶ **Lemma 3** ([22, Theorem 1]). *Let $\Delta \geq 5$ be an integer. Then, for all sufficiently large q, there exists $\beta_c > 0$ satisfying $\beta_c = (1 + o_q(1))\frac{2\log q}{\Delta}$ such that the following holds for any $\beta > 0$ w.h.p. for $G \sim \mathcal{G}_{n,\Delta}$.*

$$\text{if } \beta < \beta_c, \text{ then } \left\|\pi_G - \pi_G^{\text{dis}}\right\|_{\text{TV}} = \mathrm{e}^{-\Omega(n)}; \qquad \text{if } \beta > \beta_c, \text{ then } \left\|\pi_G - \pi_G^{\text{ord}}\right\|_{\text{TV}} = \mathrm{e}^{-\Omega(n)}. \quad (1)$$

Moreover, there exists $\zeta = \zeta(\Delta) > 0$ with $\zeta < \eta$ such that

$$\begin{aligned} &\text{for } \beta \leq \beta_c, \quad \pi_G^{\text{dis}}\Big(|\operatorname{In}(\mathcal{F})| \geq \zeta|E|\Big) = \mathrm{e}^{-\Omega(n)}, \quad \text{and} \\ &\text{for } \beta \geq \beta_c, \quad \pi_G^{\text{ord}}\Big(|\operatorname{In}(\mathcal{F})| \leq (1-\zeta)|E|\Big) = \mathrm{e}^{-\Omega(n)}. \end{aligned} \quad (2)$$

Proof. The claims about the total variation distance are shown in [22, Theorem 1, Items (2), (3), (8)]. Equation (2) shows a bit of slack in the definitions of Ω^{dis} and Ω^{ord} that will be useful later; it follows essentially from the same theorem, we defer the details to Lemma 25 of the full version [14]. ◀

2.2 Main ingredient: Weak spatial mixing within a phase

Let $G = (V, E)$ be a graph. For $v \in V$ and $r \geq 0$, let $B_r(v)$ denote the set of all vertices in V whose distance from v is at most r. Let $\pi = \pi_G$ be the RC distribution on G and let $\pi_{\mathcal{B}_r^+(v)}$ be the conditional distribution of π where all edges in $E\backslash E(B_r(v))$ are "in". We define analogously $\pi_{\mathcal{B}_r^-(v)}$ by conditioning the edges in $E\backslash E(B_r(v))$ to be "out".

▶ **Definition 4.** *Let G be a graph with m edges. Let $q, \beta > 0$ be reals and $r \geq 1$ be an integer. We say that the graph G has* WSM within the ordered phase at radius r *if for every $v \in V(G)$ and every edge e incident to v, $\|\pi_{\mathcal{B}_r^+(v)}(e \mapsto \cdot) - \pi_G^{\text{ord}}(e \mapsto \cdot)\|_{\text{TV}} \leq \frac{1}{100m}$. Analogously, we say that G has* WSM within the disordered phase at radius r *if for every $v \in V(G)$ and every edge e incident to v, $\|\pi_{\mathcal{B}_r^-(v)}(e \mapsto \cdot) - \pi_G^{\text{dis}}(e \mapsto \cdot)\|_{\text{TV}} \leq \frac{1}{100m}$.*

The bulk of our arguments consists of showing the following two theorems.

▶ **Theorem 5.** *Let $\Delta \geq 5$ be an integer. There exists $M = M(\Delta) > 0$ such that for all q sufficiently large, the following holds for any $\beta \geq \beta_c$. W.h.p. over $G \sim \mathcal{G}_{n,\Delta}$, G has WSM within the ordered phase at a radius r which satisfies $r \leq \frac{M}{\beta}\log n$.*

The upper bound on the radius r in terms of $1/\beta$ ensures that we can remove the dependence on β of the mixing time in Theorem 1 (caused by a loose bound on the mixing time on the tree, see Lemma 9 below). For the disordered phase, we have

▶ **Theorem 6.** *For all integer $\Delta \geq 5$, for all q sufficiently large and any $\beta \leq \beta_c$, w.h.p. over $G \sim \mathcal{G}_{n,\Delta}$, G has WSM within the disordered phase at a radius r which satisfies $r \leq \frac{1}{3}\log_{\Delta-1} n$.*

2.3 Second ingredient: Local mixing on tree-like neighbourhoods

We first define a local version of RC dynamics where we perform only updates in a small ball around a vertex. Here, we need to consider the extreme boundary conditions that all vertices outside of the ball belong in distinct components ("free boundary") and where they belong to the same component ("wired boundary"); we will refer to these two chains as the free and wired RC dynamics, respectively. For the random regular graph, these "local-mixing" considerations are strongly connnected to the Δ-regular tree.

Formally, given a graph $G = (V, E)$ and a subset $U \subseteq V$, let $G[U]$ be the induced subgraph of G on U. The tree-excess of a connected graph G is given by $|E| - |V| + 1$. For a vertex v in G and integer $r \geq 0$, let $B_r(v)$ denote the set of vertices at distance at most r from v and $S_r(v)$ those at distance exactly r from v. For $K > 0$, a max-degree Δ graph G is locally K-treelike if for every $v \in V$ and $r \leq \frac{1}{3}\log_{\Delta-1}|V|$, the graph $G[B_r(v)]$ has tree excess $\leq K$.

▶ **Lemma 7** (see, e.g., [16, Lemma 5.8])**.** *For any integer $\Delta \geq 3$, there is $K > 0$ such that w.h.p. $G \sim G_{n,\Delta}$ is locally K-treelike.*

For a graph G, a vertex ρ in G and an integer $r \geq 1$, the *free RC dynamics* on $B_r(\rho)$ is the RC dynamics where all edges outside of $B_r(\rho)$ are conditioned to be out and only edges of G with both endpoints in $B_r(\rho)$ are updated.

▶ **Lemma 8** ([4, Lemma 6.5])**.** *Let $\Delta \geq 3$ be an integer, and $q, K > 1$, $\beta > 0$ be reals. There exists $C > 0$ such that the following holds for any Δ-regular graph G and integer $r \geq 1$.*

Suppose that $\rho \in V$ is such that $G[B_r(\rho)]$ is K-treelike. Then, with $n = |B_r(\rho)|$, the mixing time of the free RC dynamics on $B_r(\rho)$ is $\leq Cn\log n$.

To define the wired RC dynamics, for a graph G, a vertex ρ in G and an integer $r \geq 1$, let H be the graph obtained by removing all vertices and edges outside of $B_r(\rho)$, and adding a new vertex v_∞ connected to all vertices in $S_r(\rho)$. The *wired RC dynamics* on $B_r(\rho)$ is the RC dynamics on H where the edges adjacent to v_∞ are conditioned to be in and only edges of G with both endpoints in $B_r(\rho)$ are updated. Denote by $\hat{\pi}_{B_r(\rho)}$ the stationary distribution of the wired RC dynamics. Note that when the graph outside of $B_r(\rho)$ is connected, $\hat{\pi}_{B_r(\rho)}$ induces the same distribution as $\pi_{\mathcal{B}^+_r(\rho)}$.[5]

▶ **Lemma 9.** *Let $\Delta \geq 3$ be an integer, and $q, K > 1$, $\beta > 0$ be reals. There exists $\hat{C} > 0$ such that the following holds for every Δ-regular graph $G = (V, E)$ and any integer $r \geq 1$.*

Suppose that $\rho \in V$ is such that $G[B_r(\rho)]$ is K-treelike. Then, with $n = |B_r(\rho)|$, the mixing time of the wired RC dynamics on $B_r(\rho)$ is $\leq \hat{C}n^3(q^4\mathrm{e}^\beta)^{\Delta r}$.

▶ Remark 10. For integer $q > 1$, the mixing time bound in Lemma 9 can be improved to $O(n\log n)$ using results of [2], see Appendix A.2. in the full version [14] for details.

2.4 Proof of Theorem 1

In this section, we will prove the $\beta > \beta_c$ part of Theorem 1, given below as Theorem 11 for convenience. The proof of the $\beta < \beta_c$ part of Theorem 1 is in Appendix A of the full version [14].

▶ **Theorem 11.** *Let $\Delta \geq 5$ be an integer. Then, for all sufficiently large q, there exists $C = C(q, \Delta)$ such that the following holds w.h.p. for $G \sim \mathcal{G}_{n,\Delta}$. For $\beta > \beta_c$, the mixing time of the RC dynamics starting from all-in is $O(n^C)$. For integer q, the mixing time is in fact $O(n\log n)$.*

Proof of Theorem 11 (Theorem 1(b)). The argument resembles that of [16], a bit of care is required to combine the pieces. Consider $G = (V, E) \sim \mathcal{G}_{n,\Delta}$ with $n = |V|$ and $m = |E|$. Let q be sufficiently large so that both Lemma 3 and Theorem 5 apply; assume also that Lemma 7 applies so that G is locally K-treelike.

[5] More precisely, the weight of a configuration $\mathcal{F} : E(B_r(\rho)) \to \{0, 1\}$ in $\hat{\pi}_{B_r(\rho)}$ is proportional to $q^{\hat{c}(\mathcal{F})}(\mathrm{e}^\beta - 1)^{|\mathcal{F}|}$ where $\hat{c}(\mathcal{F})$ denotes the number of components in the graph $(B_r(\rho), \mathrm{In}(F))$ that do not include any of the vertices in $S_r(\rho)$ (since all of these belong to the same component in the wired dynamics and hence contribute just a single extra factor of q).

Consider arbitrary $\beta > \beta_c$ and set $\beta_0 := \log(q^{1.9/\Delta} + 1)$. Since $\beta_c = (1 + o_q(1))\frac{2\log q}{\Delta}$, we have that $\beta \geq \beta_0$ for all sufficiently large q. By Lemma 3, a graph $G = (V, E) \sim \mathcal{G}_{n,\Delta}$ satisfies w.h.p. $\left\|\pi_G - \pi_G^{\mathrm{ord}}\right\|_{\mathrm{TV}} = \mathrm{e}^{-\Omega(n)}$ and $\pi_G^{\mathrm{ord}}\big(|\operatorname{In}(\mathcal{F})| \leq (1-\zeta)|E|\big) = \mathrm{e}^{-\Omega(n)}$. Moreover, by Theorem 5, G has WSM within the ordered phase at radius r for some $r \leq \frac{M}{\beta}\log n$, where $M = M(\Delta) > 0$ is a constant independent of β. Note that by taking q large, we can ensure that β_0 and hence β are at least M so that $r \leq \frac{1}{3}\log_{\Delta-1} n$. Theorem 11 will follow by showing that the mixing time is bounded by $T = O(n^{2+\log W})$, where $W = \Delta^{2M/\beta_0}\mathrm{e}^{M\Delta(\Delta+1)}$ is independent of q, β; for integer q we will show the stronger upper bound $\tilde{T} = O(n\log n)$.

Consider the RC dynamics $(X_t)_{t\geq 0}$ with X_0 being the all-in configuration on the edges. Consider also the "ordered" RC dynamics $\hat{X}_t$ with $\hat{X}_0 \sim \pi_G^{\mathrm{ord}}$ where we reject moves that lead to configurations outside of Ω^{ord}; note that $\hat{X}_t \sim \pi_G^{\mathrm{ord}}$ for all $t \geq 0$. For $t \geq 0$, let $\mathcal{E}_t$ be the event that $\operatorname{In}(\hat{X}_t) \geq (1-\zeta)|E|$ and let $\mathcal{E}_{<t} := \bigcap_{t'=0,\ldots,t-1}\mathcal{E}_{t'}$. From Lemma 3 we have that $\pi_G^{\mathrm{ord}}(\mathcal{E}_t) \geq 1 - \mathrm{e}^{-\Omega(n)}$ and hence by a union bound $\pi_G^{\mathrm{ord}}(\mathcal{E}_{<t}) \geq 1 - t\mathrm{e}^{-\Omega(n)}$ as well.

We couple the evolution of X_t and $\hat{X}_t$ using the monotone coupling, i.e., at every step of the two chains choose the same edge e_t to update and use the same uniform number $U_t \in [0,1]$ to decide whether to include e_t in each of $X_{t+1}, \hat{X}_{t+1}$. Using the monotonicity of the model for $q \geq 1$ (and in particular that $p > \hat{p}$), under the monotone coupling, for all $t \geq 0$ such that $\mathcal{E}_{<t}$ holds (and hence no reject move has happened in $\hat{X}_t$ so far), we have that $\hat{X}_t \leq X_t$ (i.e., $\operatorname{In}(\hat{X}_t) \subseteq \operatorname{In}(X_t)$). To complete the proof, it therefore suffices to show that

$$\Pr(X_T \neq \hat{X}_T) \leq 1/4. \tag{3}$$

Consider an arbitrary time $t \geq 0$. By a union bound, we have that

$$\Pr\left(X_t \neq \hat{X}_t\right) \leq \sum_e \Pr\left(X_t(e) \neq \hat{X}_t(e)\right) \leq m\Pr\left(\overline{\mathcal{E}_{<t}}\right) + \sum_e \Pr\left(X_t(e) \neq \hat{X}_t(e) \mid \mathcal{E}_{<t}\right). \tag{4}$$

Fix an arbitrary edge e incident to some vertex v, and let (X_t^v) be the wired RC dynamics on $G[B_r(v)]$. We couple the evolution of (X_t^v) with that of (X_t) and $(\hat{X}_t)$ using the monotone coupling analogously to above, where in X_t^v we ignore updates of edges outside the ball $G[B_r(v)]$). We have $X_t^v \geq X_t$ for all $t \geq 0$, and hence, conditioned on $\mathcal{E}_{<t}$, we have that $X_t^v \geq X_t \geq \hat{X}_t$. It follows that

$$\begin{aligned}\Pr\left(X_t(e) \neq \hat{X}_t(e) \mid \mathcal{E}_{<t}\right) &= \Pr(X_t(e) = 1 \mid \mathcal{E}_{<t}) - \Pr(\hat{X}_t(e) = 1 \mid \mathcal{E}_{<t}) \\ &\leq |\Pr(X_t^v(e) = 1 \mid \mathcal{E}_{<t}) - \Pr(\hat{X}_t(e) = 1 \mid \mathcal{E}_{<t})|.\end{aligned}$$

For any two events A, B, we have $|\Pr(A) - \Pr(A \mid B)| \leq 2\Pr(\overline{B})$, so using this for $B = \mathcal{E}_{<t}$ and A the events $\{X_t^v(e) = 1\}, \{\hat{X}_t(e) = 1\}$, the triangle inequality gives

$$\Pr\left(X_t(e) \neq \hat{X}_t(e) \mid \mathcal{E}_{<t}\right) \leq 4\Pr\left(\overline{\mathcal{E}_{<t}}\right) + |\Pr(X_t^v(e) = 1) - \Pr(\hat{X}_t(e) = 1)|$$

Note that $\Pr(\hat{X}_t(e) = 1) = \pi_G^{\mathrm{ord}}(e \mapsto 1)$, so another application of triangle inequality gives

$$\begin{aligned}\Pr\left(X_t(e) \neq \hat{X}_t(e) \mid \mathcal{E}_{<t}\right) \leq 4\Pr\left(\overline{\mathcal{E}_{<t}}\right) &+ \left|\Pr(X_t^v(e) = 1) - \pi_{\mathcal{B}_r^+(v)}(e \mapsto 1)\right| \\ &+ \left|\pi_{\mathcal{B}_r^+(v)}(e \mapsto 1) - \pi_G^{\mathrm{ord}}(e \mapsto 1)\right|.\end{aligned} \tag{5}$$

Since G has WSM within the ordered phase at radius r, we have that

$$\left|\pi_{\mathcal{B}_r^+(v)}(e \mapsto 1) - \pi_G^{\mathrm{ord}}(e \mapsto 1)\right| \leq 1/(100m). \tag{6}$$

Moreover, let T_v be the mixing time of the wired RC dynamics on $G[B_r(v)]$ and let $N_v = |E(B_r(v))| \le \Delta^{r+1}$. Since $r \le \frac{1}{3}\log_{\Delta-1} n$, $G[B_r(v)]$ is K-treelike, so from Lemma 9, with $\hat{C} = O(1)$ denoting the constant there (and absorbing a couple of factors of Δ into it),

$$T_v \le \hat{C}(N_v)^3(q^4\mathrm{e}^{\beta})^{\Delta r} \le \hat{C}N_v\Delta^{2r}q^{4\Delta r}\mathrm{e}^{\beta\Delta r} = \hat{C}N_v\Big(\Delta^{2M/\beta}q^{4\Delta M/\beta}\mathrm{e}^{M\Delta}\Big)^{\log n} \le \hat{C}N_vW^{\log n},$$

where in the last inequality we used that $\beta > \beta_0$, $\beta_0 > \frac{1}{\Delta}\log q$ and $W = \Delta^{2M/\beta_0}\mathrm{e}^{M\Delta(\Delta+1)}$. For $T = \Theta(n^{2+\log W})$, we have $T \ge 40T_v\frac{m}{N_v}\log m$, so by Chernoff bounds, with probability $1-\exp(-n^{\Omega(1)})$, we have at least $10T_v\log m$ updates inside $B_r(v)$ among $t = 1,\dots,T$. For integer $k \ge 1$ the distance from stationarity after kT_v steps is at most $(1/4)^k$, we obtain

$$\big|\Pr(X_T^v(e)=1) - \pi_{\mathcal{B}_r^+(v)}(e\mapsto 1)\big| \le \exp(-n^{\Omega(1)}) + \mathrm{e}^{-4\log m} \le 1/m^3. \tag{7}$$

Plugging (6) and (7) into (5) for $t = T$, and then back into (4), we obtain using $\Pr(\overline{\mathcal{E}_{<T}}) \le T\mathrm{e}^{-\Omega(n)}$ that $\Pr(X_T \ne \hat{X}_T) \le 5mT\mathrm{e}^{-\Omega(n)} + m/m^3 + 1/100 \le 1/4$, as needed.

For integer q, to get the improved mixing time bound $O(n\log n)$ in Theorem 11 the reasoning is similar. The main difference is that for integer q, we have that for any vertex v the mixing time T_v is bounded by $T_v = O(N_v\log N_v)$ (cf. Remark 10), and therefore the above argument yields a mixing time upper bound of $O(n(\log n)^2)$. With a bit more care, for $\tilde{T} = \Theta(n\log n)$, we show in Appendix A.3 in the full version [14] using a log-Sobolev inequality that

$$\big|\Pr(X_{\tilde{T}}^v(e)=1) - \pi_{\mathcal{B}_r^+(v)}(e\mapsto 1)\big| \le 1/m^3, \tag{8}$$

which analogously to above yields $\Pr(X_{\tilde{T}} \ne \hat{X}_{\tilde{T}}) \le 1/4$, and hence the desired mixing time bound of $O(n\log n)$ for integer q. ◀

▶ Remark 12. All the ingredients to show the coupling of the RC dynamics starting from all-in with π_G^{ord} (i.e., (3)) work even at criticality, i.e., for $\beta = \beta_c$; a similar observation applies at $\beta = \beta_c$ for π_G^{dis} when starting the RC dynamics from all-out. The difference at criticality is that π_G is a mixture of π_G^{ord} and π_G^{dis}, i.e., to obtain a sample for π_G, one should output a sample for π_G^{ord} with some probability Q and otherwise a sample from π_G^{dis}. The value of Q can be computed in time $\tilde{O}(n^2)$ by approximating the corresponding partition functions, by using, e.g., the algorithms in [22, 8] (or even the RC dynamics itself). See also [22, Theorems 2 & 3] for precise results characterising the distribution of Q; it is shown for example that Q converges to $1/(q+1)$ as q grows large.

3 Proof outline of the WSM within the ordered phase

3.1 Locally tree-like expanders

Analogously to [22], we work a bit more generally with Δ-regular expanders, which are also tree-like. The *expansion profile* of an n-vertex graph $G = (V,E)$ for $\varepsilon > 0$ is given by

$$\phi_G(\varepsilon) := \min_{S\subseteq V;\ 0<|S|\le\varepsilon n} \frac{|E(S,V\backslash S)|}{\Delta|S|}.$$

Then the classes $G_{\Delta,\delta}$ and $\mathcal{G}_{\Delta,\delta,K}$ are as follows.

▶ **Definition 13.** *Let $\Delta \ge 5$ be an integer, and $\delta \in (0,1/2), K > 0$ be reals. $G_{\Delta,\delta}$ is the class of Δ-regular graphs such that $\phi_G(1/2) \ge 1/10$ and $\phi_G(\delta) \ge 5/9$. $\mathcal{G}_{\Delta,\delta,K}$ is the class of all locally K-treelike graphs $G \in \mathcal{G}_{\Delta,\delta}$.*

We use the following lemma.

▶ **Lemma 14** ([22, Proposition 37]). *Fix $\Delta \geq 5$. There is a constant $\delta \in (0, 1/2)$ such that w.h.p. a uniformly random Δ-regular graph belongs to $\mathcal{G}_{\Delta,\delta}$.*

Lemma 14 and Lemma 7 show that there is also a positive integer K such that, w.h.p, $G \in \mathcal{G}_{\Delta,\delta,K}$. Next we state an important property of expanders from [26].

▶ **Lemma 15** ([26, Lemma 2.3]). *Let $G = (V, E)$ be a regular graph and consider $E' \subseteq E$ with $|E'| \leq \theta|E|$ for some $\theta \in (0, \phi_G(1/2))$. Then $(V, E \backslash E')$ has a component of size at least $\left(1 - \frac{\theta}{2\phi_G(1/2)}\right)|V|$.*

We use Lemma 15 to establish the existence of a giant component.

▶ **Definition 16.** *The size of a component of a graph is the number of vertices in the component. A* giant component *in an n-vertex graph is a component whose size is greater than $n/2$. Given a graph $G = (V, E)$ and a subset $F \subseteq E$, $G[F]$ denotes the graph (V, F).*

▶ **Definition 17.** *Fix $\Delta \geq 5$. Fix $\delta \in (0, 1/2)$ satisfying Lemma 14. Let $\eta = \min(\delta/5, 1/100)$.*

▶ **Corollary 18.** *Fix integers $\Delta \geq 5$ and $K \geq 0$ and a real number $\delta \in (0, 1/2)$. Let G be a graph in $\mathcal{G}_{\Delta,\delta,K}$ and let $\mathcal{F}$ be a configuration in Ω^{ord} or a partial configuration with $|\operatorname{In}(\mathcal{F})| \geq (1 - \eta)|E|$. Then there is a giant component in $G[\operatorname{In}(\mathcal{F})]$ whose size is at least $(1 - \delta)|V|$.*

Proof. Apply Lemma 15 with $E' = \operatorname{Out}(\mathcal{F})$ and $\theta = \eta = \min(\delta/5, 1/100)$. Note $|\operatorname{Out}(\mathcal{F})| \leq \eta|E|$ and $\phi_G(1/2) \geq 1/10$. Thus the lemma say that $G[\operatorname{In}(\mathcal{F})]$ has a component of size at least $\left(1 - \frac{\delta/5}{2 \cdot 1/10}\right)|V| = (1 - \delta)|V| > |V|/2$. ◀

3.2 Sketch of proof of Theorem 5

Let $\Delta \geq 5$ be an integer. Consider any sufficiently large q and any $\beta \geq \beta_c$. For sufficiently large n, choose a "radius" $r \approx \frac{1}{\beta} \log n$ and let $G = (V, E) \sim \mathcal{G}_{n,\Delta}$. Fix a vertex $v \in V$ and an edge e incident to v. We wish to show, with sufficiently high probability, that $\|\pi_{\mathcal{B}_r^+(v)}(e \mapsto \cdot) - \pi_G^{\text{ord}}(e \mapsto \cdot)\|_{\text{TV}} \leq 1/(100|E|)$.

Our goal is essentially to construct a coupling of $\mathcal{F}^+ \sim \pi_{\mathcal{B}_r^+(v)}$ and $\mathcal{F}^{\text{ord}} \sim \pi^{\text{ord}}$, such that $\Pr(\mathcal{F}^+(e) \neq \mathcal{F}^{\text{ord}}(e))$ is sufficiently small. In order to construct the coupling, we take advantage of the fact that $G[B_r(v)]$ is locally tree-like. In fact, we identify a suitable subgraph of $G[B_r(v)]$ without cycles and restrict the coupling to this subgraph.

Consider a breadth-first search from v in $G[B_r(v)]$. Let T_0 be the rooted tree consisting of all forward edges in this breadth-first search. All other edges in $B_r(v)$ are called "excess edges". W.h.p., since $G \sim \mathcal{G}_{n,\Delta}$, there are at most K excess edges in $B_r(v)$ for some absolute constant $K > 0$. In particular, since G is locally tree-like, we can identify integers r_1 and r_2 satisfying $r \geq r_1 > r_2 \geq 0$ such that $E(B_{r_1}(v)) \setminus E(B_{r_2}(v))$ contains no excess edges and $r_1 - r_2 \geq r/(2K) = \Omega(r)$. The fact that $r_1 - r_2 = \Omega(r)$ ensures that $B_{r_1}(v) \setminus B_{r_2}(v)$ is a sufficiently large subgraph of G, and the coupling focuses on this subgraph.

In order to describe the coupling process we need a small amount of notation. A *partial configuration* $\mathcal{F}$ is a map from the edges of G to the set $\{0, 1, *\}$. In-edges and out-edges (those that are mapped to 1 or to 0) are "revealed" and edges that are mapped to $*$ are "unrevealed". A refinement of a partial configuration is obtained by revealing more edges. We use $\mathcal{F} \subseteq \mathcal{F}'$ to denote the fact that $\mathcal{F}'$ refines $\mathcal{F}$.

In the coupling, we generate a sequence of edge subsets $F_0 \subseteq F_1 \subseteq \cdots \subseteq E$ such that, after iteration i, the edges in F_i are revealed. We also construct two sequences of partial configurations $\mathcal{F}_0^+ \subseteq \mathcal{F}_1^+ \subseteq \cdots \subseteq \mathcal{F}^+$ and $\mathcal{F}_0^{\text{ord}} \subseteq \mathcal{F}_1^{\text{ord}} \subseteq \cdots \subseteq \mathcal{F}^{\text{ord}}$, maintaining the invariant that the revealed edges in $\mathcal{F}_i^{\text{ord}}$ and $\mathcal{F}_i^+$ are exactly the edges in F_i. The coupling will have the crucial property that $\mathcal{F}^{\text{ord}} \sim \pi^{\text{ord}}$ and $\mathcal{F}^+ \sim \pi_{\mathcal{B}_{r_1}^+(v)}$

- The process starts with iteration $i = 0$. The initial set F_0 of revealed edges is all edges except those in $E(B_{r_1}(v))$. In $\mathcal{F}_0^{\text{ord}}$ these revealed edges are sampled from the projection $\pi_{F_0}^{\text{ord}}$ of π^{ord} onto F_0. It is likely that the configuration $\mathcal{F}_0^{\text{ord}}$ has at least $(1-\eta)|E|$ in-edges. If not, then the coupling terminates (unsuccessfully), generating $\mathcal{F}^{\text{ord}}$ and $\mathcal{F}^+$ from the right distributions. We will show that the probability of this unsuccessful termination is low. On the other hand, if $\mathcal{F}_0^{\text{ord}}$ has a least $(1-\eta)|E|$ in-edges, then we are off to a good start. All configurations refining $\mathcal{F}_0^{\text{ord}}$ are in Ω^{ord}, so the projection of π and π^{ord} onto subsequent edges that get revealed are the same (making it easier to continue the coupling). At this point $\mathcal{F}_0^+$ is taken to be the configuration with revealed edges F_0 where all revealed edges are in-edges.
- After iteration $i = 0$, iterations continue with $i = 1, 2, \ldots$ until an edge is revealed whose distance from v is at most r_2 or until the in-edges in $\mathcal{F}_i^{\text{ord}}$ induce a giant component, and this giant component contains all vertices on the boundary of F_i. We will show that it is very unlikely that an edge at distance at most r_2 from v is reached. So it is likely the giant component in $\mathcal{F}_i^{\text{ord}}$ contains all vertices on the boundary of F_i. This is a good situation because the conditional distribution of π, conditioned on refining $\mathcal{F}_i^{\text{ord}}$ and the conditional distribution of $\mathcal{F}^+$, conditioned on refining $\mathcal{F}_i^+$ induce the same distribution on edges incident to v, which enables us to show that $\Pr(\mathcal{F}^+(e) \neq \mathcal{F}^{\text{ord}}(e))$ is sufficiently small.
- The process at iteration $i+1$ is as follows. W_i is taken to be the set of all vertices on the boundary of F_i whose components (induced by the in-edges in $\mathcal{F}_i^{\text{ord}}$) are all small. By "boundary" we mean that vertices in W_i are adjacent to revealed edges, and to unrevealed edges. If W_i is empty, then the coupling finishes. Otherwise, a vertex $w_i \in W_i$ is chosen to be as far from v as possible. The edges in the subtree of T_0 below the parent of w_i are revealed in F_{i+1}.

The main remaining ingredient in the proof is showing that the unsuccessful terminations of the coupling are unlikely. To do this, we use the polymer framework of [22]. (Ordered) polymers are defined using an inductive definition. For a set of edges $A \subseteq E$, let $\mathcal{B}_0(A) = A$, and inductively for $j = 0, 1, 2, \ldots$ define $\mathcal{B}_{j+1}(A)$ to be the set of all edges such that they are either in $\mathcal{B}_j(A)$ or edges that are incident to a vertex that has at least $5\Delta/9$ incident edges in $\mathcal{B}_j(A)$. Let $\mathcal{B}_\infty(A) = \bigcup_{j\in\mathbb{N}} \mathcal{B}_j(A)$. An ordered polymer of a configuration $\mathcal{F}$ is a connected component of $B_\infty(\text{Out}(\mathcal{F}))$. The bulk of the work is to prove the following lemma, which is repeated in the appendix of the full version [14] (with more detail) as Lemma 44.

▶ **Lemma 19.** *Fix $\Delta \geq 5$ and $K, M > 0$. Suppose that $\beta \geq 3M$. Suppose that n is sufficiently large so that $r := \frac{M}{\beta} \log_{\Delta-1} n > K$ and $|B_r(v)| \leq 9\Delta n/200$. Define r_1 as above. Let $\mathcal{F}^{\text{ord}}$ and $\mathcal{F}^+$ be generated by the process. Then at least one of the following conditions holds.*

1. *$\mathcal{F}^{\text{ord}}$ and $\mathcal{F}^+$ agree on the edges that are incident to v.*
2. *$|\text{In}(\mathcal{F}^{\text{ord}}) \setminus E(B_{r_1}(v))| < (1-\eta)|E|$.*
3. *$\mathcal{F}^{\text{ord}}$ contains a polymer of size at least $\frac{r}{400\Delta(1+K)} - 1$.*

To complete the proof of Theorem 5, we show that items 2 and 3 are unlikely. The proof that item 2 is unlikely, Lemma 45 in the full version [14], follows from the slack specified in Equation (2) of Lemma 3. The proof that item 3 is unlikely, Lemma 29 in the full

version [14], follows from an analysis on the size of polymers by adapting appropriately the cluster expansion techniques of [22] (a bit of extra work is needed there to capture the $1/\beta$ dependence in the size of the polymer, see [14, Lemma 29] for details).

The proof of WSM for the disordered phase (Theorem 6) follows a similar strategy, the details are given in Appendix D of the full version [14].

References

1 Ferenc Bencs, Márton Borbényi, and Péter Csikvári. Random cluster model on regular graphs. *Communications in Mathematical Physics*, pages 1–46, 2022.

2 Antonio Blanca, Zongchen Chen, Daniel Štefankovič, and Eric Vigoda. The Swendsen-Wang dynamics on trees. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques* (APPROX/RANDOM 2021), volume 207, pages 43:1–43:15, 2021.

3 Antonio Blanca, Andreas Galanis, Leslie Ann Goldberg, Daniel Štefankovič, Eric Vigoda, and Kuan Yang. Sampling in uniqueness from the Potts and random-cluster models on random regular graphs. *SIAM Journal on Discrete Mathematics*, 34(1):742–793, 2020.

4 Antonio Blanca and Reza Gheissari. Random-cluster dynamics on random regular graphs in tree uniqueness. *Communications in Mathematical Physics*, 386(2):1243–1287, 2021.

5 Antonio Blanca and Reza Gheissari. On the tractability of sampling from the Potts model at low temperatures via Swendsen-Wang dynamics. *CoRR*, abs/2304.03182, 2023.

6 Christian Borgs, Jennifer Chayes, Tyler Helmuth, Will Perkins, and Prasad Tetali. Efficient sampling and counting algorithms for the Potts model on $\mathbb{Z}^d$ at all temperatures. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing,* (STOC '20) , pages 738–751, 2020.

7 Charlie Carlson, Ewan Davies, Nicolas Fraiman, Alexandra Kolla, Aditya Potukuchi, and Corrine Yap. Algorithms for the ferromagnetic Potts model on expanders. In *63rd Annual Symposium on Foundations of Computer Science* (FOCS 2022), pages 344–355, 2022.

8 Zongchen Chen, Andreas Galanis, Leslie A. Goldberg, Will Perkins, James Stewart, and Eric Vigoda. Fast algorithms at low temperatures via markov chains†. *Random Structures & Algorithms*, 58(2):294–321, 2021.

9 Zongchen Chen, Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. Sampling colorings and independent sets of random regular bipartite graphs in the non-uniqueness region. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms,* (SODA '22), pages 2198–2207, 2022.

10 Amin Coja-Oghlan, Andreas Galanis, Leslie Ann Goldberg, Jean Bernoulli Ravelomanana, Daniel Štefankovič, and Eric Vigoda. Metastability of the Potts ferromagnet on random regular graphs. *Communications in Mathematical Physics*, pages 1–41, 2023.

11 Matthew Coulson, Ewan Davies, Alexandra Kolla, Viresh Patel, and Guus Regts. Statistical physics approaches to unique games. In *35th Computational Complexity Conference,* (CCC 2020), volume 169 of *LIPIcs*, pages 13:1–13:27, 2020.

12 Charilaos Efthymiou. On sampling symmetric Gibbs distributions on sparse random graphs and hypergraphs. In *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*, pages 57:1–57:16, 2022.

13 Weiming Feng, Heng Guo, and Jiaheng Wang. Sampling from the ferromagnetic Ising model with external fields. *CoRR*, abs/2205.01985, 2022. `arXiv:2205.01985`.

14 Andreas Galanis, Leslie Ann Goldberg, and Paulina Smolarova. Sampling from the random cluster model on random regular graphs at all temperatures via Glauber dynamics, 2023. `arXiv:2305.13239`.

15 Andreas Galanis, Daniel Štefankovic, Eric Vigoda, and Linji Yang. Ferromagnetic Potts model: Refined #bis-hardness and related results. *SIAM Journal on Computing*, 45(6):2004–2065, 2016.

16 Reza Gheissari and Alistair Sinclair. Low-temperature Ising dynamics with random initializations. In *54th Annual ACM SIGACT Symposium on Theory of Computing,* (STOC '22), pages 1445–1458, 2022.

17 Reza Gheissari and Alistair Sinclair. Spatial mixing and the random-cluster dynamics on lattices. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms,* (SODA '23) , pages 4606–4621, 2023.

18 Leslie Ann Goldberg and Mark Jerrum. Approximating the partition function of the ferromagnetic Potts model. *Journal of the ACM*, 59(5):1–31, 2012.

19 Geoffrey Grimmett. *The random-cluster model*, volume 333. Springer, 2006.

20 Heng Guo and Mark Jerrum. Random cluster dynamics for the Ising model is rapidly mixing. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms,* (SODA 2017), pages 1818–1827, 2017.

21 Olle Häggström. The random-cluster model on a homogeneous tree. *Probability Theory and Related Fields*, 104:231–253, 1996.

22 Tyler Helmuth, Matthew Jenssen, and Will Perkins. Finite-size scaling, phase coexistence, and algorithms for the random cluster model on random graphs. *Annales de l'Institut Henri Poincare (B) Probabilites et statistiques*, 59(2):817–848, 2023.

23 Tyler Helmuth, Will Perkins, and Guus Regts. Algorithmic Pirogov-Sinai theory. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, ,* (STOC '19) , pages 1009–1020, 2019.

24 Johan Jonasson. The random cluster model on a general graph and a phase transition characterization of nonamenability. *Stochastic Processes and their Applications*, 79(2):335–354, 1999.

25 Yuval Peres and Peter Winkler. Can extra updates delay mixing? *Communications in Mathematical Physics*, 323:1007–1016, 2013.

26 Luca Trevisan. Lecture notes on graph partitioning, expanders and spectral methods, 2016. URL: `https://lucatrevisan.github.io/books/expanders-2016.pdf`.