

## STRUCTURAL ASPECTS OF TILINGS

ALEXIS BALLIER, BRUNO DURAND, AND EMMANUEL JEANDEL

Laboratoire d'informatique fondamentale de Marseille (LIF) Aix-Marseille Université, CNRS  
39 rue Joliot-Curie, 13 453 Marseille Cedex 13, France

*E-mail address:* alexis.ballier@lif.univ-mrs.fr

*E-mail address:* bruno.durand@lif.univ-mrs.fr

*E-mail address:* emmanuel.jeandel@lif.univ-mrs.fr

---

**ABSTRACT.** In this paper, we study the structure of the set of tilings produced by any given tile-set. For better understanding this structure, we address the set of finite patterns that each tiling contains.

This set of patterns can be analyzed in two different contexts: the first one is combinatorial and the other topological. These two approaches have independent merits and, once combined, provide somehow surprising results.

The particular case where the set of produced tilings is countable is deeply investigated while we prove that the uncountable case may have a completely different structure.

We introduce a pattern preorder and also make use of Cantor-Bendixson rank. Our first main result is that a tile-set that produces only periodic tilings produces only a finite number of them. Our second main result exhibits a tiling with exactly one vector of periodicity in the countable case.

### 1. Introduction

Tilings are basic models for geometric phenomena of computation: local constraints they formalize have been of broad interest in the community since they capture geometric aspects of computation [15, 1, 9, 13, 6]. This phenomenon was discovered in the sixties when tiling problems happened to be crucial in logic: more specifically, interest shown in tilings drastically increased when Berger proved the undecidability of the so-called domino problem [1] (see also [8] and the well known book [2] for logical aspects). Later, tilings were basic tools for complexity theory (see the nice review of Peter van Emde Boas [16] and some of Leonid Levin's paper such as [12]).

Because of growing interest for this very simple model, several research tracks were aimed directly on tilings: some people tried to generate the most complex tilings with the most simple constraints (see [15, 9, 13, 6]), while others were most interested in structural aspects (see [14, 5]).

In this paper we are interested in structural properties of tilings. We choose to focus on finite patterns tilings contain and thus introduce a natural preorder on tilings: a tiling

---

*1998 ACM Subject Classification:* G.2.m.

*Key words and phrases:* tiling, domino, patterns, tiling preorder, tiling structure.



is extracted from another one if all finite patterns that appear in the first one also appear in the latter. We develop this combinatorial notion in Section 2.1. This approach can be expressed in terms of topology (subshifts of finite type) and we shall explain the relations between both these approaches in Section 2.2.

It is important to stress that both these combinatorial and topological approaches have independent merits. Among the results we present, different approaches are indeed used for proofs. More specifically, our first main result (Theorem 3.8) states that if a tile-set produces only periodic tilings then it produces only finitely many of them; despite its apparent simplicity, we did not find any proof of Theorem 3.8 in the literature. Our other main result (Theorem 3.11) which states that in the countable case a tiling with exactly one vector of periodicity exists is proved with a strong help of topology.

Our paper is organized as follows: Section 2 is devoted to definitions (combinatorics, topology) and basic structural remarks. In Section 3 we prove the existence of minimal and maximal elements in tilings enforced by a tile-set. Then we present an analysis in terms of Cantor-Bendixson derivative which provides powerful tools. We study the particular case where tilings are countable and present our main results. We conclude by some open problems.

## 2. Definitions

### 2.1. Tilings

We present notations and definitions for tilings since several models are used in literature: Wang tiles, geometric frames of rational coordinates, local constraints... All these models are equivalent for our purposes since we consider very generic properties of them (see [3] for more details and proofs). We focus our study on tilings of the plane although our results hold in higher dimensions.

In our definition of tilings, we first associate a state to each cell of the plane. Then we impose a local constraint on them. More formally,  $Q$  is a finite set, called the *set of states*. A *configuration*  $c$  consists of cells of the plane with states, thus  $c$  is an element of  $Q^{\mathbb{Z}^2}$ . We denote by  $c_{i,j}$  or  $c(i, j)$  the state of  $c$  at the cell  $(i, j)$ .

A tiling is a configuration which satisfies a given finite set of finite constraints everywhere. More specifically we express these constraints as a set of allowed patterns: a configuration is a tiling if around any of its cells we can see one of the allowed patterns:

**Definition 2.1** (patterns). A *pattern*  $P$  is a finite restriction of a configuration i.e., an element of  $Q^V$  for some finite domain  $V$  of  $\mathbb{Z}^2$ . A pattern *appears* in a configuration  $c$  (resp. in some other pattern  $P'$ ) if it can be found somewhere in  $c$  (resp. in  $P'$ ); i.e., if there exists a vector  $t \in \mathbb{Z}^2$  such that  $c(x+t) = P(x)$  on the domain of  $P$  (resp. if  $P'(x+t)$  is defined for  $x \in V$  and  $P'(x+t) = P(x)$ ).

By language extension we say that a pattern is *absent* or *omitted* in a configuration if it does not appear in it.

**Definition 2.2** (tile-sets and tilings). A *tile-set* is a tuple  $\tau = (Q, \mathcal{P}_\tau)$  where  $\mathcal{P}_\tau$  is a finite set of patterns on  $Q$ . All the elements of  $\mathcal{P}_\tau$  are supposed to be defined on the same domain denoted by  $V$  ( $\mathcal{P}_\tau \subseteq Q^V$ ).

A *tiling* by  $\tau$  is a configuration  $c$  equal to one of the patterns on all cells:

$$\forall x \in \mathbb{Z}^2, c|_{V+x} \in \mathcal{P}_\tau$$

We denote by  $\mathcal{T}_\tau$  the set of tilings by  $\tau$ .

Notice that in the definition of one tile-set we can allow patterns of different definition domains provided that there are a finite number of them.

An example of a tile-set defined by its allowed patterns is given in Fig. 1. The produced tilings are given in Fig. 2; the meaning of the edges in the graph will be explained later; tilings are represented modulo shift. In  $A_i$  and  $B_i$ ,  $i$  is an integer that represents the size of the white stripe.

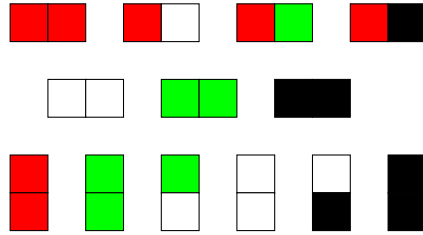
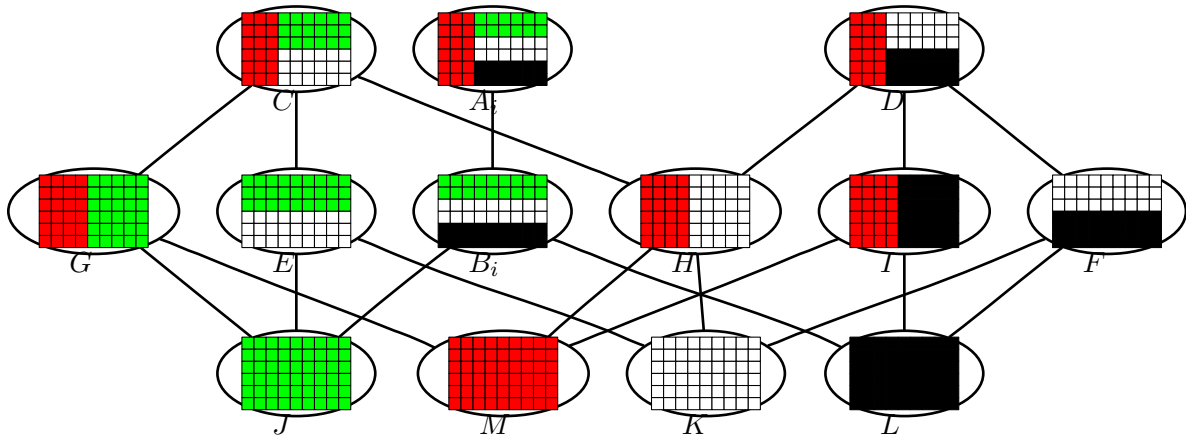


Figure 1: Allowed patterns



An edge represents a relation  $Q < P$  if  $P$  is above  $Q$ . Transitivity edges are not depicted. As an example  $K < E$  and  $K < C$ .

Figure 2: Hasse diagram of the order  $<$  with the tile-set defined in Fig. 1

Throughout the following, it will be more convenient for us to define tile-sets by the set of their *forbidden patterns*: a tile-set is then given by a finite set  $\mathcal{F}_\tau$  of forbidden patterns ( $\mathcal{F}_\tau = Q^V \setminus \mathcal{P}_\tau$ ); a configuration is a tiling if no forbidden pattern appears.

Let us now introduce the following natural preorder, which will play a central role in our paper:

**Definition 2.3** (Preorder). Let  $x, y$  be two tilings, we say that  $x \preceq y$  if any pattern that appears in  $x$  also appears in  $y$ .

We say that two tilings  $x, y$  are equivalent if  $x \preceq y$  and  $y \preceq x$ . We denote this relation by  $x \approx y$ . In this case,  $x$  and  $y$  contain the same patterns. The equivalence class of  $x$  is denoted by  $\langle x \rangle$ . We write  $x \prec y$  if  $x \preceq y$  and  $x \not\approx y$ .

Some structural properties of tilings can be seen with the help of this preorder. The Hasse diagram in Fig. 2 correspond to the relation  $\prec$ .

We choose to distinguish two types of tilings: A tiling  $x$  is of *type a* if any pattern that appears in  $x$  appears infinitely many times;  $x$  is of *type b* if there exists a pattern that appears only once in  $x$ . Note that any tiling is either of *type a* or of *type b*: suppose that there is a pattern that appears only a finite number of times in  $x$ ; then the pattern which is the union of those patterns appears only once.

If  $x$  is of *type b*, then the only tilings equivalent to  $x$  are its shifted: there is a unique way in  $\langle x \rangle$  to grow around the unique pattern.

## 2.2. Topology

In the domain of symbolic dynamics, topology provides both interesting results and is also a nice condensed way to express some combinatorial proofs [10, 7]. The benefit of topology is a little more surprising for tilings since they are essentially static objects. Nevertheless, we can get nice results with topology as will be seen in the sequel.

We see the space of configurations  $Q^{\mathbb{Z}^2}$  as a metric space in the following way: the distance between two configurations  $c$  and  $c'$  is  $2^{-i}$  where  $i$  is the minimal offset (for e.g. the euclidean norm) of a point where  $c$  and  $c'$  differ:

$$d(c, c') = 2^{-\min\{|i|, c(i) \neq c'(i)\}}$$

We could also endow  $Q$  with the discrete topology and then  $Q^{\mathbb{Z}^2}$  with the product topology, thus obtaining the same topology as the one induced by  $d$ .

In this topology, a basis of open sets is given through the patterns: for each pattern  $P$ , the set  $\mathcal{O}_P$  of all configurations  $c$  which contains  $P$  in their center (i.e., such that  $c$  is equal to  $P$  on its domain) is an open set, usually called a cylinder. Furthermore cylinders such defined are also closed (their complements are finite unions of  $\mathcal{O}_{P'}$  where  $P'$  are patterns of same domain different from  $P$ ). Thus  $\mathcal{O}_P$ 's are clopen.

**Proposition 2.4.**  *$Q^{\mathbb{Z}^2}$  is a compact perfect metric space (a Cantor space).*

We say that a set of configurations  $S$  is shift-invariant if any shifted version of any of its configurations is also in  $S$ ; i.e., if for every  $c \in S$ , and every  $v \in \mathbb{Z}^2$  the configuration  $c'$  defined by  $c'(x) = c(x + v)$  is also in  $S$ . We denote such a shift by  $\sigma_v$ .

**Remark 2.5.** Our definition of pattern preorder 2.3 can be reformulated in a topological way :  $x \preceq y$  if and only if there exists shifts  $(\sigma_i)_{i \in \mathbb{N}}$  such that  $\sigma_i(y) \xrightarrow{i \rightarrow \infty} x$ . We say that  $x$  can be extracted from  $y$ .

For a given configuration  $x$ , we define the topological closure of shifted forms of  $x$ :  $\Gamma(x) = \{\sigma_v(x), v \in \mathbb{Z}^2\}$  where  $\sigma_{i,j}$  represents a shift of vector  $v$ .

We see that  $x \preceq y$  if and only if  $\Gamma(x) \subseteq \Gamma(y)$ . Remark that  $x$  is minimal for  $\prec$  if and only if  $\langle x \rangle$  is closed.

As sets of tilings can be defined by a finite number of forbidden patterns, they correspond to *subshifts* of finite type<sup>1</sup>. In the sequel, we sometimes use arbitrary subshifts; they correspond to a set of configurations with a potentially infinite set of forbidden patterns.

### 3. Main results

#### 3.1. Basic structure

Let us first present a few structural results. First, the existence of minimal classes for  $\prec$  is well known.

**Theorem 3.1** (minimal elements). *Every set of tilings contains a minimal class for  $\prec$ .*

In the context of tilings, those that belong to minimal classes are often called *quasiperiodic*, while in language theory they are called *uniformly recurrent* or *almost periodic*. Those quasiperiodic configurations admit a nice characterization: any pattern that appears in one of them can be found in any sufficiently large pattern (placed anywhere in the configuration).

For a combinatorial proof of this theorem see [5]. Alternatively, here is a scheme of a topological proof: consider a minimal subshift of  $\mathcal{T}_\tau$  (such a subshift exists, see e.g. [14]) then every tiling in this set is in a minimal class.

An intensively studied class of tilings is the set of self-similar tilings. These tilings indeed are minimal elements (quasiperiodic) but one can find other kinds of minimal tilings (e.g. the nice approach of Kari and Culik in [4]).

The existence of maximal classes of tilings is not trivial and we have to prove it:

**Theorem 3.2** (maximal elements). *Every set of tilings contains a maximal class for  $\prec$ .*

*Proof.* Let us prove that any increasing chain has a least upper bound. The theorem is then obtained by Zorn's lemma.

Consider  $T_i$  an increasing chain of tiling classes. Consider the set  $P$  of all patterns that this chain contains. As the set of all patterns is countable,  $P$  is countable too,  $P = \{p_i\}_{i \in \mathbb{N}}$ .

Now consider two tilings  $T_k$  and  $T_l$ , any pattern that appears in  $T_k$  or  $T_l$  appears in  $T_{\max(k,l)}$ . Thus we can construct a sequence of patterns  $(p'_i)_{i \in \mathbb{N}}$  such that  $p'_i$  contains all  $p_j$ ,  $j \leq i$  and  $p'_{i-1}$ . Note that  $p'_i$  is correctly tiled by the considered tile-set.

The sequence of patterns  $p'_i$  grows in size. By shift invariance, we can center each  $p'_i$  by superimposing an instance of  $p'_{i-1}$  found in  $p'_i$  over  $p'_{i-1}$ . We can conclude that this sequence has a limit and this limit is a tiling that contains all  $p_i$ , hence is an upper bound for the chain  $T_i$ . ■

Note that this proof also works when the *set of states*  $Q$  and/or the *set of forbidden patterns*  $\mathcal{F}_\tau$  are countably infinite (neither compactness nor finiteness is assumed). However it is easy to construct examples where  $Q$  is infinite and there does not exist a minimal tiling.

Note that we actually prove that every chain has not only a upper bound, but also a least upper bound. Such a result does not hold for lower bound: We can easily build chains with lower bounds but no greatest lower bound.

---

<sup>1</sup>*Subshifts* are closed shift-invariant subsets of  $Q^{\mathbb{Z}^2}$

### 3.2. Cantor-Bendixson

In this section we use the topological derivative and define Cantor-Bendixson rank; then we discuss properties of sets of tilings from this viewpoint. Most of the results presented in this section are direct translations of well known results in topology [11].

A configuration  $c$  is said to be *isolated* in a set of configurations  $S$  if there exists a pattern  $P$  (of domain  $V$ ) such that  $c$  is the only configuration in  $S$  that contains the pattern  $P$  in its center ( $\forall x \in V, c(x) = P(x)$ ). We say that  $P$  *isolates*  $c$ . This corresponds to the topological notion: a point is isolated if there exists an open set that contains only this point. As an example, in Fig. 3, the tilings  $A_i$  are isolated, the pattern isolating an  $A_i$  is the boundary between red, white, black and green parts of it.

The topological derivative of a set  $S$  is formed by its elements that are not isolated. We denote it by  $S'$ .

If  $S$  is a set of tilings, or more generally a subshift, we get some more properties. If  $P$  isolates a configuration in  $S$  then a shifted form of  $P$  isolates a shifted form of this configuration. Any configuration of  $S$  that contains  $P$  is isolated.

As a consequence, if  $S = \mathcal{T}_\tau$ , then  $S' = \mathcal{T}_{\tau'}$  where  $\tau'$  forbids the set  $\mathcal{F}_\tau \cup \{P \mid P \text{ isolates some configuration in } \mathcal{T}_\tau\}$ .

Note that  $S'$  is not always a set of tilings, but remains a subshift. Let us examine the example shown in Fig. 3.  $S'$  is  $S$  minus the classes  $A_i$ . However any set of tilings (subshift of finite type) that contains  $C, B_i$  and  $D$  also contains  $A_i$ . Hence  $S'$  is not of finite type in this example.

We define inductively  $S^{(\lambda)}$  for any ordinal  $\lambda$  :

- $S^{(0)} = S$
- $S^{(\alpha+1)} = (S^{(\alpha)})'$
- $S^{(\lambda)} = \bigcap_{\alpha < \lambda} S^{(\alpha)}$  when  $\lambda$  is a limit ordinal.

Notice that there exists a countable ordinal  $\lambda$  such that  $S^{(\lambda+1)} = S^{(\lambda)}$ . Indeed, at each step of the induction, the set of forbidden patterns increases, and there is at most countably many patterns. We call the least such ordinal the *Cantor-Bendixson rank* of  $S$  [11].

An element  $c$  is of *rank*  $\lambda$  in  $S$  if  $\lambda$  is the least ordinal such that  $c \notin S^{(\lambda)}$ . If no such  $\lambda$  exists,  $c$  is of infinite rank. For instance all strictly quasiperiodic configurations (quasiperiodic configurations that are not periodic) are of infinite rank. We write  $\rho(x)$  the rank of  $x$ .

An example of what Cantor-Bendixson ranks look like is shown in Fig. 3, the first row contains the tilings of rank 1, the second row the ones of rank 2 etc.

Ranked tilings have many interesting properties. First of all, as any  $\mathcal{T}_\tau^{(\lambda)}$  is shift-invariant, a tiling has the same rank as its shifted forms.

Note that at each step of the inductive definition, the set of isolated points is at most countable (there are less isolated points than patterns). As a consequence, if all tilings are ranked,  $\mathcal{T}_\tau$  is countable, as a countable union (the Cantor-Bendixson rank is countable) of countable sets.

The converse is also true:

**Theorem 3.3.**  $\mathcal{T}_\tau$  is countable if and only if all tilings are ranked.

*Proof.* Let  $\lambda$  be the Cantor-Bendixson rank of  $\mathcal{T}_\tau$ .  $\mathcal{T}_\tau^{(\lambda)} = \mathcal{T}_\tau^{(\lambda+1)}$  is a perfect set (no points are isolated). As a consequence,  $\mathcal{T}_\tau^{(\lambda)}$  must be either empty or uncountable (classical

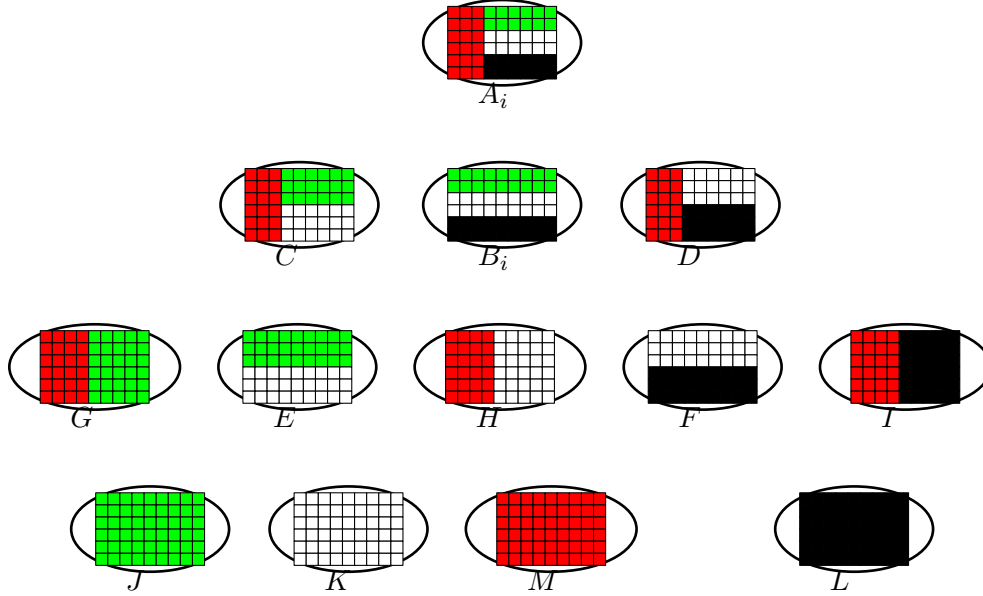


Figure 3: Cantor-Bendixson ranks

application of Baire’s Theorem :  $\mathcal{T}_\tau^{(\lambda)}$  is compact thus has the Baire property and a non empty perfect set with the Baire property cannot be countable).

As  $\mathcal{T}_\tau$  is countable,  $\mathcal{T}_\tau^{(\lambda)} = \emptyset$ . ■

**Remark 3.4.** Strictly quasiperiodic tilings only appear when the number of possible tilings is uncountable [5]. As a consequence, if all tilings are ranked, strictly quasiperiodic tilings do not appear, thus all minimal tilings are periodic. In this case we therefore may expect all tilings to be somehow simple. We’ll study this case later in this paper.

As the topology of  $Q^{\mathbb{Z}^2}$  has a basis of clopens  $\mathcal{O}_P$ ,  $Q^{\mathbb{Z}^2}$  is a 0-dimensional space, thus any subset of  $Q^{\mathbb{Z}^2}$  is also 0-dimensional. As any (non empty) perfect 0-dimensional compact metric space is isomorphic to the Cantor Space we obtain:

**Theorem 3.5** (Cardinality of tiling spaces). *A set of tilings is either finite, countable or has the cardinality of continuum.*

Note that the proof of this result does not make use of the continuum hypothesis.

We now present the connection between our preorder  $\prec$  and the Cantor-Bendixson rank.

**Proposition 3.6.** *Let  $x$  and  $y$  be two ranked tilings such that  $x \prec y$ . Then  $\rho(x) > \rho(y)$ .*

*Proof.* By definition of  $\prec$ , any pattern that appears in  $x$  also appears in  $y$ . As a consequence, if  $P$  isolates  $x$  in  $S^{(\lambda)}$ , then  $x$  is the only tiling of  $S^{(\lambda)}$  that contains  $P$  hence  $y$  cannot be in  $S^{(\lambda)}$ . ■

Thus tilings of Cantor-Bendixson rank 1 (minimal rank) are maximal tilings for  $\prec$ . Conversely if all tilings are ranked, tilings of maximal rank exist and are minimal tilings. These tilings are periodic, see remark 3.4.

Another consequence is that if all tilings are ranked, no infinite increasing chain for  $\prec$  exists because such chain would induce an infinite decreasing chain of ordinals:

**Theorem 3.7.** *If  $\mathcal{T}_\tau$  is countable, there is no infinite increasing chain for  $\prec$ .*

### 3.3. The countable case

In the context of Cantor-Bendixson ranks, the case of countable tilings was revealed as an important particular case. Let us study this case in more details.

If the number of tilings is finite, the situation is easy: any tiling is periodic. Our aim is to prove that in the countable case, there exists a tiling  $c$  which has exactly one vector of periodicity (such a tiling is sometimes called weakly periodic in the literature).

We split the proof in three steps :

- There exists a tiling which is not minimal;
- There exists a tiling  $c$  which is at level 1, that is such that all tilings less than  $c$  are minimal;
- Such a tiling has exactly one vector of periodicity.

The first step is a result of independent interest. To prove the last two steps we use Cantor-Bendixson ranks.

Recall that in our case any minimal tiling is periodic (no strictly quasiperiodic tiling appears in a countable setting [5]). The first step of the proof may thus be reformulated:

**Theorem 3.8.** *If all tilings produced by a tile-set are periodic, then there are only finitely many of them.*

It is important to note that a compactness argument is not sufficient to prove this theorem, there is no particular reason for a converging sequence of periodic tilings with strictly increasing period to converge towards a non periodic tiling: there indeed exist such sequences with a periodic limit.

*Proof.* We are in debt to an anonymous referee who simplified our original proof.

Suppose that a tile-set produces infinitely many tilings, but only periodic ones.

As the set of tilings is infinite and compact, one of them is obtained as a limit of the others: There exists a tiling  $X$  and a sequence  $X_i$  of distinct tilings such that  $X_i \rightarrow X$ .

Now by assumption  $X$  is periodic of period  $p$  for some  $p$ . We may suppose that no  $X_i$  has  $p$  as a period. Denote by  $M$  the pattern which is repeated periodically.

$X_i \rightarrow X$  means that  $X_i$  contains in its center a square of size  $q(i) \times q(i)$  of copies of  $M$ , where  $q$  is a growing function.

For each  $i$ , consider the largest square of  $X_i$  consisting only of copies of  $M$ . Such a largest square exists, as it is bounded by a period of  $X_i$ . Let  $k$  be the size of this square. Now, the boundary of this square contains a  $p \times p$  pattern which is not  $M$  (otherwise this is not the largest square).

By shifting  $X_i$  so that this pattern is at the center, we obtain a tiling  $Y_i$  which contains a  $p \times p$  pattern at the origin which is not  $M$  adjacent to a  $k/2 \times k/2$  square consisting of copies of  $M$  in one of the four quarter planes.

By taking a suitable limit of these  $Y_i$ , we will obtain a tiling which contains a  $p \times p$  pattern which is not  $M$  in its center adjacent to a quarter plane of copies of  $M$ .

Such a tiling cannot be periodic. ■



This proof does not assume that the set of forbidden patterns  $\mathcal{F}_\tau$  is finite, therefore it is still valid for any shift-invariant closed subset (subshift) of  $Q^{\mathbb{Z}^2}$ .

Now we prove stronger results about the Cantor-Bendixson rank of  $\mathcal{T}_\tau$ . Let  $\alpha$  be the Cantor-Bendixson rank of  $\mathcal{T}_\tau$ . Since  $(\mathcal{T}_\tau)^{(\alpha)} = \emptyset$ ,  $\alpha$  cannot be a limit ordinal: Suppose that it is indeed a limit ordinal, therefore  $\bigcap_{\beta < \alpha} (\mathcal{T}_\tau)^{(\beta)} = \emptyset$  is an empty intersection of closed sets in  $Q^{\mathbb{Z}^2}$  therefore by compactness there exists  $\gamma < \alpha$  such that  $\bigcap_{\beta < \gamma} (\mathcal{T}_\tau)^{(\beta)} = \emptyset$  and therefore  $\mathcal{T}_\tau$  can not have rank  $\alpha$ . Hence  $\alpha$  is a successor ordinal,  $\alpha = \beta + 1$ .

However, we can refine this result :

**Lemma 3.9.** *The rank of  $\mathcal{T}_\tau$  cannot be the successor of a limit ordinal.*

*Proof.* Suppose that  $\beta = \cup_{i < \omega} \beta_i$ . Since  $(\mathcal{T}_\tau)^{(\beta+1)} = \emptyset$ ,  $(\mathcal{T}_\tau)^{(\beta)}$  is finite (otherwise it would have a non-isolated point by compactness), it contains only periodic tilings.

Let  $p$  be the least common multiple of the periods of the tilings in  $(\mathcal{T}_\tau)^{(\beta)}$ . Let  $M$  be the set of patterns of size  $2p \times 2p$  that do not admit  $p$  as a period. Let  $x_i$  be an element that is isolated in  $(\mathcal{T}_\tau)^{(\beta_i)}$ .

As there is only a finite number of  $p$ -periodic tilings, we may suppose w.l.o.g. that no  $x_i$  admit  $p$  as a period.

For any  $i$ , there exists a pattern of  $M$  that appears in  $x_i$ . Let  $x'_i$  be the tiling with this pattern at its center. By compactness, one can extract a limit  $x'$  of the sequence  $(x'_i)_{i \in \mathbb{N}}$ ,  $x'$  is by construction in  $\bigcap_i (\mathcal{T}_\tau)^{(\beta_i)} = \mathcal{T}_\tau^{(\beta)}$ . However,  $x'$  does not contain a  $p$ -periodic pattern at its center, that is a contradiction. ■

We write  $\alpha = \lambda + 2$  the rank of  $\mathcal{T}_\tau$ .

We already proved that there exists a non minimal tiling but this is not sufficient to conclude that there exists a tiling at level 1<sup>2</sup>. However, we achieve this as a corollary of the previous lemma:  $(\mathcal{T}_\tau)^{(\lambda)}$  is infinite (otherwise  $(\mathcal{T}_\tau)^{(\lambda+1)}$  would be empty) and contains a non periodic tiling by theorem 3.8. This non periodic tiling  $c$  is not minimal (otherwise it would be strictly quasiperiodic and then  $\mathcal{T}_\tau$  would not be countable). Now  $c$  is at level 1 : any tiling less than  $c$  is in  $(\mathcal{T}_\tau)^{(\lambda+1)}$  therefore periodic (hence minimal).

If a tiling  $x$  is of *type a* and is ranked, then it has a vector of periodicity: consider the pattern  $P$  that isolates it in the last topological derivative of  $\mathcal{T}_\tau$  that it belongs to. Since  $x$  is of *type a*, this pattern appears twice in it, therefore there exists a shift  $\sigma$  such that  $\sigma(x)$  contains  $P$  at its center.  $x = \sigma(x)$  because  $P$  isolates  $x$ .

As any tiling of *type a* has a vector of periodicity, it remains to prove that  $c$  is of *type a*:

**Lemma 3.10.**  *$c$  is of type a.*

*Proof.* Suppose the converse : there exists a pattern  $P$  that appears only once in  $c$ . Considering the union of this pattern  $P$  and a pattern that isolates  $c$ , we may assume that  $P$  isolates  $c$ .  $c$  has only a finite number of tilings smaller than itself: they lie in  $\mathcal{T}_\tau^{(\lambda+1)}$  which is finite, and are all periodic, say of period  $p$ . As  $P$  isolates  $c$ , none of these tilings contain  $P$ .

Consider the patterns of size  $2p \times 2p$  of  $T$  that are not  $p$ -periodic. If those patterns can appear arbitrary far from  $P$  then one can extract a tiling from  $c$  (thus smaller than  $c$ ) that is not  $p$ -periodic and does not contain  $P$ ; this is not possible.

<sup>2</sup>We actually can prove that the level 1 exists: There is no infinite decreasing chain whose lower bound is a periodic configuration

Therefore there is a pattern in  $c$  that contains  $P$  (thus appears only once) and any other part of  $c$  is  $p$ -periodic (one can gather all non  $p$ -periodic parts of  $c$  around  $P$ ), as depicted in Fig. 4(a).

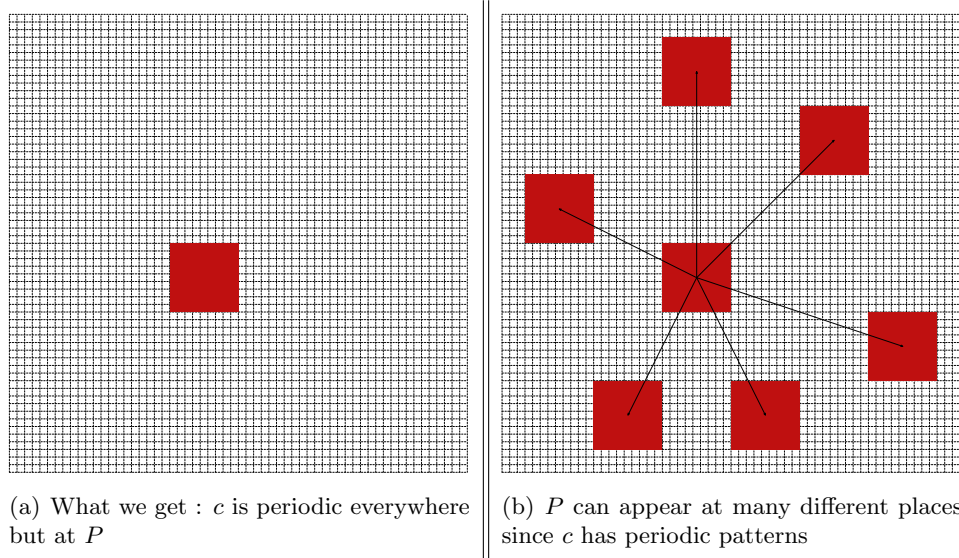


Figure 4: What can happen if  $c$  is of *type b*?

This non periodic part could also be inserted at infinitely many different positions in  $c$  since the tiling rules are of bounded radius, as depicted in Fig. 4(b). Hence the number of tilings is not countable. ■

$c$  is of *type a*,  $c$  is not periodic,  $c$  has a vector of periodicity, therefore our theorem 3.11 holds :

**Theorem 3.11.** *If  $\tau$  is a tile-set that produces a countable number of tilings then it produces a tiling with exactly one vector of periodicity.*

#### 4. Open problems

We are interested in proving more precise results for the order  $\prec$  for a countable set of tilings: we wonder whether the order  $\prec$  has at most finitely many levels, as it is the case in Fig. 2. We know how to construct a tile-set so that the maximal level is any arbitrary integer see e.g. Fig.5 for level 3.

We also intend to prove a similar result for uncountable sets of tilings; the problem is that we are tempted to think that if the set of tilings is uncountable, then a quasiperiodic tiling must appear. However, this is not true: imagine a tile-set that admits a vertical line of white or black cells with red on the left and green on the right. The uncountable part is due to the vertical line that itself contains a quasiperiodic of dimension 1 but not of dimension 2. This tile-set produces tilings that looks like  $H$  in Fig. 2, except that the vertical line can have two different colors without any constraint.

A generalization of lemma 3.9 would be to prove that the Cantor-Bendixson rank of a countable set of tilings cannot be infinite; we know how to construct sets of tilings that have

an arbitrary large but finite Cantor-Bendixson rank, but we do not know how to obtain a set of tilings of rank greater than  $\omega$ .

## References

- [1] R. Berger. The undecidability of the domino problem. *Memoirs American Mathematical Society*, 66:1966, 1966.
- [2] Egon Börger, Erich Grädel, and Yuri Gurevich. *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer, 1997.
- [3] Julien Cervelle and Bruno Durand. Tilings: recursivity and regularity. *Theor. Comput. Sci.*, 310(1-3):469–477, 2004.
- [4] Karel Culik and Jarkko Kari. On aperiodic sets of Wang tiles. *Lecture Notes in Computer Science*, 1337:153–??, 1997.
- [5] Bruno Durand. Tilings and quasiperiodicity. *Theor. Comput. Sci.*, 221(1-2):61–75, 1999.
- [6] Bruno Durand, Leonid A. Levin, and Alexander Shen. Complex tilings. In *STOC*, pages 732–739, 2001.
- [7] Walter Helbig Gottschalk and Gustav Arnold Hedlund. *Topological Dynamics*. American Mathematical Society, Providence, Rhode Island, 1955.
- [8] Y. Gurevich and I. Koriakov. A remark on Berger’s paper on the domino problem. *Siberian Journal of Mathematics*, 13:459–463, 1972. (in Russian).
- [9] William P. Hanf. Nonrecursive tilings of the plane. i. *J. Symb. Log.*, 39(2):283–285, 1974.
- [10] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical Systems Theory*, 3:320–375, 1969.
- [11] Kazimierz Kuratowski. *Topology, Vol. I, 3rd edition*. NY: Academic Press, 1966.
- [12] Leonid A. Levin. Average case complete problems. *SIAM J. Comput.*, 15(1):285–286, 1986.
- [13] Dale Myers. Nonrecursive tilings of the plane. ii. *J. Symb. Log.*, 39(2):286–294, 1974.
- [14] C. Radin and M. Wolff. Space tilings and local isomorphism, 1992.
- [15] R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12:177–209, 1971.
- [16] P. van Emde Boas. Dominoes are forever. Research report 83-04, University of Amsterdam. Department of Mathematics., 1983.

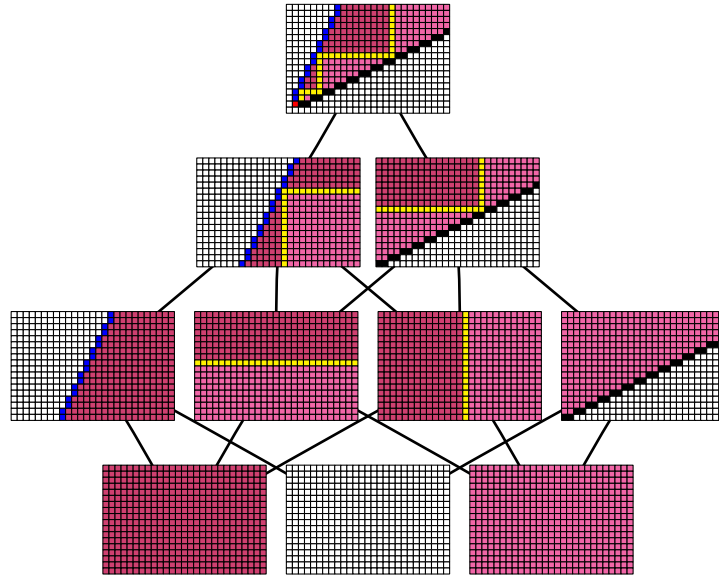


Figure 5: An example of a tile-set that produces countably many tilings and a tiling at level 3