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ON THE INDUCED MATCHING PROBLEM

IYAD A. KANJ [1], MICHAEL J. PELSMAJER [2], MARCUS SCHAEFER [1], AND GE XIA [3]

[1] DePaul University, Chicago, IL 60604, USA.
E-mail address: {ikanj,mschaefer}@cs.depaul.edu

[2] Illinois Institute of Technology, Chicago, IL 60616, USA.
E-mail address: pelsmajer@iit.edu

[3] Lafayette College, Easton PA 18042, USA.
E-mail address: gexia@cs.lafayette.edu

Abstract. We study extremal questions on induced matchings in several natural graph classes. We argue that these questions should be asked for twinless graphs, that is graphs not containing two vertices with the same neighborhood. We show that planar twinless graphs always contain an induced matching of size at least $n/40$ while there are planar twinless graphs that do not contain an induced matching of size $(n+10)/27$. We derive similar results for outerplanar graphs and graphs of bounded genus. These extremal results can be applied to the area of parameterized computation. For example, we show that the induced matching problem on planar graphs has a kernel of size at most $40k$ that is computable in linear time; this significantly improves the results of Moser and Sikdar (2007). We also show that we can decide in time $O(91^k + n)$ whether a planar graph contains an induced matching of size at least k.

Introduction

A matching in a graph is an *induced matching* if it occurs as an induced subgraph of the graph; we let $mim(G)$ denote the size of a maximum induced matching in G. Determining whether a graph has an induced matching of size at least k is NP-complete for general graphs and remains so even if restricted to bipartite graphs of maximum degree 4, planar bipartite graphs, 3-regular planar graphs (see [4] for a detailed history). Furthermore, approximating a maximum induced matching is difficult: the problem is APX-hard, even for $4r$-regular graphs [4, 14].

In terms of the parameterized complexity of the induced matching problem on general graphs, it is known that the problem is $W[1]$-*hard* [9]. Hence, according to the parameterized

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complexity hypothesis, it is unlikely that the problem is *fixed-parameter tractable*, that is, solvable in time $f(k)n^c$ for some constant c independent of k.

There are several classes of graphs for which the problem turns out to be polynomial time solvable, for example chordal graphs and outerplanar graphs (see [4] for a survey and [8] for the result on outerplanar graphs).

Very recently, Moser and Sidkar [8] considered the parameterized complexity of PLANAR-IM: finding an induced matching of size at least k in a planar graph. They showed that PLANAR-IM has a *linear problem kernel*, but left the constant in the kernel size undetermined. Their result automatically implies that the problem is fixed-parameter tractable.

In the current paper we take a combinatorial approach to the problem establishing lower and upper bounds on the size of induced matchings in certain graph classes. In particular, an application of our results to PLANAR-IM gives a significantly smaller problem kernel than the one given in [8]. We also apply the results to obtain a practical parameterized algorithm for PLANAR-IM that can be extended to graphs of bounded genus and could be used as a heuristic for general graphs.

Let us consider the induced matching problem from the point of view of extremal graph theory: How large can a graph be without containing an induced matching of size at least k? Of course, dense graphs such as K_n and $K_{n,n}$ pose an immediate obstacle to this question being meaningful, but they can easily be eliminated by restricting the maximum or the average degree of the graph. Indeed, for strong edge colorings the maximum degree restriction is popular: a *strong edge coloring* with k colors is a partition of the edge set into at most k induced matchings [12]. A greedy algorithm shows that graphs of maximum degree Δ have a strong edge chromatic number of at most $2\Delta(\Delta - 1) + 1$, and, of course, Δ is an immediate lower bound. If we are only interested in a large induced matching though, perhaps we need not restrict the maximum degree. On the other hand, bounding only the average degree of a graph allows pathological examples such as $K_{1,n}$, which has average degree less than 2 but only a single-edge induced matching. This example illustrates another obstacle to a large induced matching: *twins*. Two vertices u and v are said to be twins if $N(u) = N(v)$. Obviously, at most one of u and v can be an endpoint of an edge in an induced matching and if one of them can, either can. Thus, from the extremal point of view (and since they can be easily recognized and eliminated) we should study the induced matching problem on graphs without twins. Twinlessness does not allow us to drop the bounded average degree requirement however, as shown by removing a perfect matching from $K_{n,n}$, which yields a twinless graph with a maximum induced matching of size 2.

We begin by studying twinless graphs of bounded average degree. Those graphs might still not have large induced matchings since they could contain very dense subgraphs (Remark 1.3 elaborates on this point). One way of dealing with this problem is to extend the average degree requirement to all subgraphs. In Section 1 we see that a slightly weaker condition is sufficient, namely a bound on the chromatic number of the graph. We can show that a graph of average degree at most d and chromatic number at most k contains an induced matching of size $\Theta(n^{1/(d+1)})$.

While we cannot expect to substantially improve the dependency on the average degree of this result in general (see Remark 1.2), we do investigate the case of planar graphs and graphs of bounded genus, for which we can show the existence of induced matchings of linear size. Indeed, a planar twinless graph always contains an induced matching of size $n/40$. We also know that this bound cannot be improved beyond $(n + 10)/27$ (Remark 2.10). Planar graphs and graphs of bounded genus are discussed in Section 2.

We next investigate the case of outerplanar graphs: an outerplanar graph of minimum degree 2 always contains an induced matching of size $n/7$ (even without assuming twinlessness), and this result is tight (Section 3). Our bounds fit in with a long series of combinatorial results on finding sharp bounds on the size of induced structures in subclasses of planar graphs (see for example [5, 11, 1, 10]).

We also use our combinatorial results to obtain fixed-parameter algorithms for the induced matching problem. For example, we show that PLANAR-IM can be solved in time $O(91^k + n)$ by a very practical algorithm, while—on the more theoretical side—there is an algorithm deciding it in time $O(2^{159\sqrt{k}} + n)$ using the Lipton-Tarjan [7] separator theorem. Both results easily extend to graphs of bounded genus.

For graph-theoretic terminology we refer the reader to West [13]. For background on parameterized complexity, we recommend Downey and Fellows [3].

1. Induced matchings in graphs of bounded average degree

We can show that twinless graphs of bounded average degree and bounded chromatic number contain large induced matchings. At the core of the proof is a combinatorial result due to Füredi and Tuza [6, Theorem 9.13] on systems of strong representatives. For lack of space, we omit the details.

Theorem 1.1. *A twinless graph G with $\chi(G) \leq k$ and average degree at most d must contain an induced matching of size at least*

$$\left(\frac{d}{2}\left(\frac{n-1}{2k(d+1)}\right)^{1/(d+1)} - (d+1)\right)/(k-1)$$

which is $\Theta(n^{1/(d+1)})$ where $n = |V(G)|$.

Remark 1.2. Consider the following bipartite graph: take a set A of ℓ vertices, and for every $d/2$ element subset of A create a new vertex and connect it to the vertices of the subset.

This graph has $n = \ell + \binom{\ell}{d/2}$ vertices, and its largest induced matching has size $\ell/(d/2)$. Moreover, its average degree is $2 \cdot \frac{d}{2}\binom{\ell}{d/2}/\left(\ell + \binom{\ell}{d/2}\right) \leq d$. For d fixed, $\ell/(d/2)$ is of order $n^{2/d}$, which shows that the bound of the theorem (while not being tight) has the right form.

Remark 1.3. The preceding example can be extended to show that bounding the chromatic number is necessary: take the graph as constructed in the previous remark and add all edges between the ℓ vertices of A. Assuming $d \geq 4$, this gives a graph of average degree at most $d + 2$. However, the largest induced matching in this graph has size 1.

2. Planar graphs and graphs of bounded genus

2.1. Matchings and induced matchings

To find large induced matchings in graphs we often proceed in two steps: we first find a large matching in the graph and then turn it into an induced matching. To make this approach work, we need assumptions on the graph: to obtain a large matching, we assume an upper bound on $\alpha(G)$, the size of the largest independent set in G. To turn the matching into an induced matching, we assume that the graph is twinless and all minors of G have a large independent set.

Lemma 2.1. *A graph G with $\alpha(G) \leq \alpha n$ contains a matching of size at least $(1-\alpha)n/2$, where $n = n(G)$.*

Proof. Let $M \subseteq E$ be a maximal matching in G on vertex set $V(M)$. Then $I = V - V(M)$ is an independent set. By assumption, $|I| \leq \alpha n$. Adding $|V(M)|$ to either side gives us $n \leq \alpha n + |V(M)|$, and, therefore, $|V(M)| \geq (1-\alpha)n$. ∎

Lemma 2.2. *Assume that any minor $H \preceq G$ of a graph G fulfills $\alpha(H) \geq \alpha\, n(H)$. Then any matching M in G contains an induced matching in G of size at least $\alpha|M|$.*

Proof. Remove all vertices not in $V(M)$ and contract the edges of M (removing duplicate edges). The resulting graph is a minor of G, and, by assumption, has an independent set of size $\alpha|M|$. The edges in M which were contracted to the vertices in the independent set, form an induced matching in G. ∎

By this lemma a matching of size k in a planar graph contains an induced matching of size $k/4$. In [2] the authors show that a 3-connected planar graph contains a matching of size at least $(n+4)/3$, which allows us to draw the following conclusion.

Corollary 2.3. *A 3-connected planar graph contains an induced matching of size $(n+4)/12$.*

To apply the two lemmas to planar graphs and graphs of bounded genus we need some generalizations of Euler's theorem to hypergraphs. We say a *hypergraph $\mathcal{H}$ is embeddable in a surface* if the bipartite incidence graph obtained from $\mathcal{H}$ by replacing each of its edges by a vertex adjacent to all the vertices in the edge is embeddable in that surface.

Lemma 2.4. *A hypergraph of genus at most g on n vertices has at most $2n + 4g - 4$ edges containing at least three vertices, unless $n = 1$ and $g = 0$.*

If $\mathcal{H}$ is a hypergraph of genus g such that all edges have size 2, we can take the associated bigraph G of genus g and contract away all the the vertices that correspond to edges of $\mathcal{H}$. This produces a graph of genus g with $|V(\mathcal{H})|$ vertices and $|E(\mathcal{H})|$ edges, to which we may apply the following consequence of Euler's Theorem.

Lemma 2.5 (Euler)**.** *A graph of genus g on n vertices contains at most $3n + 6g - 6$ edges if $n \geq 2$.*

By splitting edges of a hypergraph into those of size at least three, those of size two, and those that contain a single vertex, we can derive the following.

Lemma 2.6. *A hypergraph of genus at most g on n vertices has at most $6n + 10g - 9$ edges if $n \geq 2$.*

We are now ready to give a lower bound on the size of induced matchings in twinless graphs of bounded genus. This includes the planar case, but in the next section we will give an improved bound for that case. We need a result due to Heawood [13] that states that a graph of genus at most g can be colored using at most $(7 + \sqrt{1+48g})/2$ colors. The statement remains true for the plane case, $g = 0$, by virtue of the Four-Color Theorem.

Theorem 2.7. *A twinless graph of genus at most g contains an induced matching of size at least $(n - 10g)/(49 + 7\sqrt{1+48g})$, where n is the number of vertices of the graph.*

Proof. Let G be a twinless graph of genus at most g, and assume temporarily that G does not contain any isolated vertex. Let $M \subseteq E$ be a maximal matching in G on vertex set $V(M)$. Then $I = V - V(M)$ is an independent set. Let $\mathcal{H}$ be the hypergraph with vertex set $V(M)$ and edges $N(v)$, $v \in I$. Then $\mathcal{H}$ is a hypergraph of genus at most g (as its bipartite incidence graph is a subgraph of G), and by Lemma 2.6, has at most $6|V(M)|+10g-9$ edges (note that we can assume $|V(M)| \geq 2$ since otherwise G consists of a single vertex, in which case there is nothing to prove). As G contains no twins, each edge of $\mathcal{H}$ uniquely corresponds to a vertex in I, so $|I| \leq 6|V(M)| + 10g - 9$ and, therefore, $|V(M)| \geq (|V| - 10g + 9)/7$. The original graph might have contained at most one isolated vertex (since it is twinless), so $|V(M)| \geq (n - 10g)/7$ and G has a matching of size at least $(n - 10g)/14$.

By Heawood's theorem and the Four-Color Theorem, a graph of genus at most g can be colored using at most $(7+\sqrt{1 + 48g})/2$ colors. Hence, G and any of its minors always contain independent sets on a $2/(7 + \sqrt{1 + 48g})$ fraction of their vertices. Then by Lemma 2.2, G has an induced matching of size at least $2(n - 10g)/[14(7 + \sqrt{1 + 48g})] = (n - 10g)/(49 + 7\sqrt{1 + 48g})$. ∎

A simple consequence of Theorem 2.7 not involving the concept of twinlessness is the following:

Corollary 2.8. *A planar graph of minimum degree at least* 3 *contains an induced matching of size at least* $(n + 8)/20$*, where* n *is the number of vertices of the graph.*

Proof. Since the graph has minimum degree at least 3 it cannot contain degree 1 and 2 vertices. Then by Lemma 2.4, the hypergraph constructed in the proof of Theorem 2.7 (for $g = 0$) contains at most $2|V(M)| - 4$ edges. However, it is now possible that more than one vertex in the independent set results in the same edge of the hypergraph. However, there can be at most two vertices sharing the same neighborhood, since a planar graph does not contain a $K_{3,3}$. Therefore, the size of the independent set is at most $4|V(M)| - 8$, and thus the graph contains a matching of size at least $(n+8)/5$. Using Lemma 2.2, it can be turned into an induced matching of size at least $(n + 8)/20$. ∎

Theorem 2.7 implies that a planar twinless graph always contains an induced matching of size $n/56$. This lower bound can still be improved as shown in the following theorem.

Theorem 2.9. *A twinless planar graph contains an induced matching of size at least* $n/40$*, where* n *is the number of vertices of the graph.*

Remark 2.10. We do not have a matching upper bound to complement Theorem 2.9, but we can get close. We can construct a graph whose largest induced matching has size $(n + 10)/27$.

3. Induced matchings in outerplanar graphs

The main result of this section is that a nontrivial connected outerplanar graph with minimum degree 2 has an induced matching of size $\lceil \frac{n}{7} \rceil$. This result is sharp, as will be seen later. We first consider a special case, which will arise later in the proof of the main result. We refer the reader to [13] for the terminology on the block decomposition of a graph.

Lemma 3.1. *Suppose that* G *is a connected graph for which the block-cutpoint tree is a path and all blocks are triangles or cut-edges; or, equivalently,* G *is the union of a path of* *length $\ell \geq 1$ and at most ℓ triangles, with each edge of the path in at most one triangle, and exactly one edge of each triangle in the path. If $2 \leq n(G) \leq 5$ then $mim(G) \geq \lceil \frac{n(G)+1}{6} \rceil$ and if $n(G) \geq 6$ then $mim(G) \geq \lceil \frac{n(G)+3}{6} \rceil$.*

Corollary 3.2. *Let G be a 2-connected outerplanar graph with exactly one non-leaf face, such that every leaf face is a 3-face. Then for any vertex v, $mim(G - v) \geq \lceil \frac{n(G)}{6} \rceil$.*

To prove the main result of this section, we use induction after separating the graph into components (by removing vertices that form a certain cut in the graph). To apply the inductive statement, each of these components must have minimum degree 2. This, however, may not be true after the removal of the cut-set from the graph. We next define an operation, called the *patching operation*, that patches each of these components so that its minimum degree is 2.

Definition 3.3. Let H be an outerplanar graph with $n(H) \geq 4$ and with at most two degree 1 vertices. We define an operation that can be applied to H, called the *patching operation*, to obtain a graph H' as follows.

(a) If there is no degree 1 vertex in H let $H' = H$.
(b) If there is exactly one degree 1 vertex u in H, let u' be its neighbor. If $\deg_H(u') \geq 3$, let $H' = H - u$. Otherwise ($\deg_H(u') = 2$), let v be the other neighbor of u'. Let v' be a vertex after v on the boundary walk in $H - \{u, u'\}$. Let $H' = (H - u) + u'v'$.
(c) If there are exactly two degree 1 vertices u and v in H, let u' be the neighbor of u and v' be the neighbor of v. Remove u from H and add the edge $u'v$. Let H' be the resulting graph.

Proposition 3.4. *Let H be an outerplanar graph with $n(H) \geq 4$ and with at most two degree 1 vertices. Moreover, when H has exactly two degree 1 vertices u and v, then adding a path from u to v leaves H outerplanar. Let H' be the graph resulting from the application of the patching operation to H. Then H' is an outerplanar graph such that: (1) the minimum degree of H' is 2, (2) $n(H') \geq n(H) - 1$, and (3) $mim(H) \geq mim(H')$.*

Theorem 3.5. *A nontrivial connected outerplanar graph G of minimum degree 2 has an induced matching of size $\lceil \frac{n}{7} \rceil$.*

Proof. Clearly the statement is true if $3 \leq n \leq 7$. Therefore, we may assume in the remainder of the proof that $n \geq 8$, and that, inductively, the statement is true for any graph with fewer than n vertices.

Let u be a cut-point in G which is in at most one non-leaf block. Let $B_1, \cdots, B_\ell$ be all the leaf blocks containing u, let $B_0 = G - \bigcup_{i=1}^{\ell}[V(B_i) - u]$, and let $n_i = n(B_i)$, for $i = 0, \cdots, \ell$. If G has no cut-points, let u be any vertex in G, and let $B_0 = G$.

Let B_i, where $i \in \{1, \cdots, \ell\}$ be a block such that $n_i \geq 7$. Let B_i' be the block obtained from B_i by deleting the chord of each 3-face of B_i. Suppose that B_i' is not a cycle. Clearly, any leaf face in B_i' must be of length at least 4.

Suppose that B_i' has a leaf face of length at least 6, with boundary $F = (u_1, \ldots, u_r, u_1)$ such that u_1u_r is a chord and $u_1 \neq u$. Let $H = G - \{u_1, u_2, u_3, u_4, u_5\}$, and note that none of the vertices in H is a cut-point in G. Therefore, H is an outerplanar graph with at most two degree 1 vertices. Apply the patching operation to H to obtain a graph H'. Then H' is a connected outerplanar graph with minimum degree two. Inductively, we have $mim(H') \geq \lceil \frac{n(H')}{7} \rceil$. Since $n(H') \geq n(H) - 1$ and $mim(H) \geq mim(H')$ by Proposition 3.4, we have $mim(H) \geq \lceil \frac{n(H)-1}{7} \rceil = \lceil \frac{n(G)-6}{7} \rceil$. A maximum induced matching in H plus edge u_2u_3 is an induced matching in G, because any edge of $E(B_i) - E(B'_i)$ incident to u_2 or u_3 has at its other endpoint u_1, u_4, or u_5, by the construction of B'_i and F. We conclude that $mim(G) \geq \lceil \frac{n(G)-6}{7} \rceil + 1 = \lceil \frac{n(G)+1}{7} \rceil$, which suffices.

If B'_i contains a leaf face $F = (u_1, \cdots, u_r, u_1)$ with $r = 4$ or $r = 5$, and such that $u_1 \neq u$ and $u_r \neq u$, then similar to the above, we let $H = G - \{u_1, \cdots, u_r\}$. Again note that none of the vertices in H is a cut-point in G. Using the same analysis as in the above paragraph, we obtain $mim(G) \geq \lceil \frac{n(G)+1}{7} \rceil$.

Assuming that $n_i \geq 7$ and that B'_i is not a cycle, it follows now that every leaf face in B'_i has length 4 or 5 and is incident to the cut-point u in G. Therefore, B'_i has exactly two leaf faces that contain u, and each of length 4 or 5. Let $F = (u_1, \cdots, u_r, u_1)$ and $F' = (u'_1, \cdots, u'_s, u'_1)$ where $r, s \in \{4, 5\}$, $u = u_1 = u'_1$, and u_1u_r and $u'_1u'_s$ are chords. Note that it is possible that $u_r = u'_s$. Let H be the graph obtained from B_i by removing the vertices in $F \cup F'$; then H is a path so it has at most two vertices of degree 1. If $n(H) \geq 1$ then the edges u_2u_3 and $u'_2u'_3$ give an induced matching in B_i of size 2. Since $n_i \leq 10$, B_i has a matching M_i of size at least $\lceil \frac{n_i+4}{7} \rceil$. If $n(H)$ is 2 or 3, then H has a maximum induced matching of size at least 1, which together with edges u_2u_3 and $u'_2u'_3$ give an induced matching in B_i of size 3. Since in this case $n_i \leq 12$, we conclude that B_i has an induced matching M_i of at least $\lceil \frac{n_i}{6} \rceil$. Moreover, no edge of M_i is incident on the cut-point u of G. Now if $n(H) \geq 4$, we apply the patching operation to H to obtain an outerplanar graph of minimum degree two. Inductively, $mim(H') \geq \lceil \frac{n(H')}{7} \rceil$, and hence $mim(H) \geq \lceil \frac{n(H)-1}{7} \rceil$. Now any induced matching in H plus edges u_2u_3 and $u'_2u'_3$ gives an induced matching M_i in B_i such that none of the edges in M_i is incident on u. It follows that $mim(G) \geq 2 + mim(H) \geq 2 + \lceil \frac{n(H)-1}{7} \rceil \geq 2 + \lceil \frac{n_i-9-1}{7} \rceil \geq \lceil \frac{n_i+4}{7} \rceil$. Therefore, in this case B_i contains an induced matching M_i, none of its edges is incident on u, of size at least $\lceil \frac{n_i+4}{7} \rceil$.

Now, for any $i \in \{1, \cdots, \ell\}$ we have the following:

If $n_i \leq 6$, then clearly B_i contains an induced matching M_i, none of its edges incident on u, of size at least $\lceil \frac{n_i}{6} \rceil$. Simply let M_i be any edge in B_i that is not incident on u.

If $n_i \geq 7$ and B'_i is a cycle, then B_i satisfies the conditions of Corollary 3.2, and B_i has an induced matching M_i of size at least $\lceil \frac{n_i}{6} \rceil$, none of its edges is incident on u (by choosing $v = u$ in Corollary 3.2).

If $n_i \geq 7$, and B'_i is not a cycle, then from the above discussion, B_i has an induced matching of size at least $min\{\lceil \frac{n_i+4}{7} \rceil, \lceil \frac{n_i}{6} \rceil\}$.

Let $M = \bigcup_{i=1}^{\ell} M_i$. Let $H = B_0 - u$ and note that H has at most two degree 1 vertices. If $n(H) \leq 3$, then clearly $mim(H) \geq \lceil \frac{n_0}{6} \rceil$. If $n(H) \geq 4$, apply the patching operation to H to obtain an outerplanar graph H' of minimum degree 2. Now by applying the inductive statement to H', we get $mim(B_0) \geq \lceil \frac{n_0-2}{7} \rceil$. Let M_0 be a maximum induced matching in $B_0 - u$, and note that since none of the induced matching edges in $M \cup M_0$ is incident on u, $M \cup M_0$ is an induced matching in G.

If G has no cut-points, then G is 2-connected and we let $B_1 = G$. In this case we have $mim(G) \geq min\{\lceil \frac{n(G)+4}{7} \rceil, \lceil \frac{n(G)}{6} \rceil\} \geq \lceil \frac{n(G)}{7} \rceil$.

Now we can assume that $\ell \geq 1$. Note that in this case we have $n_0 + n_1 + \cdots + n_\ell = n + \ell$.

If at least one block B_i has $|M_i| \geq \lceil \frac{n_i+4}{7} \rceil$, then by using $\lceil \frac{n_i}{7} \rceil$ as a lower bound on the size of the matching in each block B_j where $j \in \{1, \cdots, \ell\}$ and $j \neq i$, we get:

$$|M \cup M_0| \geq \sum_{j=1, j\neq i}^{\ell} \lceil \frac{n_j}{7} \rceil + \lceil \frac{n_i+4}{7} \rceil + \lceil \frac{n_0-2}{7} \rceil \geq \lceil \frac{n+2+\ell}{7} \rceil \geq \lceil \frac{n}{7} \rceil.$$

Otherwise, we can use $\lceil \frac{n_i}{6} \rceil$ as a lower bound on the size of each block B_i where $i \in \{1, \cdots, \ell\}$. If $\ell \geq 2$, we have:

$$|M \cup M_0| \geq \sum_{i=1}^{\ell} \lceil \frac{n_i}{6} \rceil + \lceil \frac{n_0-2}{7} \rceil \geq \sum_{i=1}^{\ell} \lceil \frac{n_i}{7} \rceil + \lceil \frac{n_0-2}{7} \rceil \geq \lceil \frac{n+\ell-2}{7} \rceil \geq \lceil \frac{n}{7} \rceil.$$

If $\ell = 1$ and $n_1 \leq 5$, by picking M to be any edge that is not incident on u in block B_1, we get:

$$|M \cup M_0| \geq 1 + \lceil \frac{n_0-2}{7} \rceil = \lceil \frac{n_0+5}{7} \rceil \geq \lceil \frac{n}{7} \rceil.$$

If $\ell = 1$ and $n_1 \geq 6$, we have:

$$\begin{aligned} |M \cup M_0| &\geq \lceil \frac{n_1}{6} \rceil + \lceil \frac{n_0-2}{7} \rceil \geq \lceil \frac{7n_1+6n_0-12}{42} \rceil = \lceil \frac{6(n_1+n_0)+n_1-12}{42} \rceil \\ &\geq \lceil \frac{6n+6+n_1-12}{42} \rceil \geq \lceil \frac{n}{7} \rceil. \end{aligned}$$

This completes the induction and the proof. ■

Figure 1 shows an example of a graph in which the size of the maximum induced matching is exactly $\lceil n/7 \rceil$. A graph in this family consists of a cycle of length 2ℓ ($\ell \geq 3$) with ℓ gadgets attached as indicated in the figure. The total number of vertices in this graph is 7ℓ, and it is easy to verify that the maximum induced matching has size exactly ℓ.

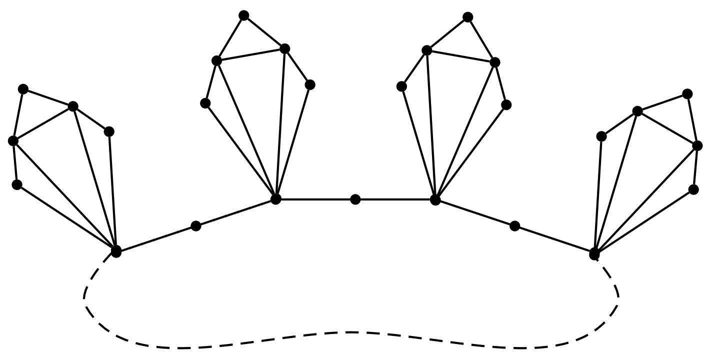

Figure 1: An illustration of a family of outerplanar graphs for which the lower bound on the size of an induced matching is tight.

4. Applications to parameterized computation

In this section we apply our previous results to obtain parameterized algorithms for IM on graphs of bounded genus. Let (G, k) be an instance of IM where G has n vertices and genus g for some integer constant $g \geq 0$.

4.1. A problem kernel

We first show how to kernelize the instance (G, k) when G is planar (i.e., for the case $g = 0$). We then extend the results to graphs with genus g for any integer constant $g > 0$.

Theorem 2.9 shows that any twinless planar graph on n vertices has an induced matching of at least $n/40$ edges. Observing that if u is a vertex in G that has a twin then $mim(G) = mim(G - u)$, by repeatedly removing every vertex in G with a twin, we end up with a twinless graph G' such that G has an induced matching of size k if and only if G' does. If $k \le n(G')/40$ then the instance (G', k) of IM can be accepted; otherwise, the instance (G', k) is a kernel of (G, k) with $n(G') \le 40k$, and we can work on (G', k).

Therefore, our task amounts to reducing the graph G to the twinless graph G'. We describe next how this can be done in linear time.

Assume that G is given by its adjacency list and that the vertices in G are labeled by the integers $1, \ldots, n$. We can further assume that the neighbors of every vertex appear in the adjacency list in increasing order. If this is not the case, we create the desired adjacency list by enumerating the vertices in increasing order, and inserting each vertex in the neighborhood list of each of its adjacent vertices. This can be easily done in $O(n)$ time.

For every vertex v of degree d, we associate a d-digit number $x_v = v_1 \cdots v_d$, where $v_1, \ldots, v_d$ are the neighbors of v in the order they appear in the adjacency list of v (i.e., in increasing order). We perform a radix sort on the numbers associated with the vertices of G using only the first three or less (leftmost) digits of these numbers. Since each digit is a number in the range $1 \ldots n$, and there are at most $O(n)$ numbers (twice the number of edges in the planar graph), radix sort takes $O(n)$ time. Let π be this sorted list. Observe that two vertices u and v are twins if and only if $x_u = x_v$. Moreover, since the graph is planar, and hence does not contain the complete bipartite graph $K_{r,r}$ for any integer $r \ge 3$, any twin vertices of degree at least 3 must have their numbers adjacent in π (otherwise there would be at least 3 vertices with the same neighborhood). Therefore, we can recognize the twins in G as follows. Process the numbers in π in order: Let x_u and x_v be two adjacent numbers in π, and assume that x_u appears before x_v. We check whether u and v are twins by comparing the corresponding digits of x_u and x_v. If u and v are twins, we mark u. When we have finished this process, we remove all marked vertices from the graph. We let G' be the resulting graph. Since for each number x_u in π we spend time proportional to the number of digits in x_u and that of the number appearing next to x_u in π, the running time is proportional to the sum of the degrees of the vertices in G, which is $O(n)$.

Theorem 4.1. *Let (G, k) be an instance of* IM *where G is a planar graph on n vertices. Then in $O(n)$ time we can compute an instance (G', k') where (G', k') is a kernel of (G, k), $k' \le k$, and such that either $n(G') \ge 40k'$ and we can accept the instance (G, k), or $n(G') < 40k'$.*

The above theorem gives a kernel of size $40k$ for PLANAR-IM, and is a significant improvement on the results in [8] where a kernel of size $O(k)$ was derived without the constant in the asymptotic notation being specified. The above results give a concrete value for the bound on the kernel size. Moreover, this value is moderately small and the analysis techniques are much simpler when compared to the technique of decomposing a planar graph into regions used in [8].

The same technique can be used to eliminate twin vertices from a graph with genus g. Using Euler's formula on $K_{r,r}$ with the fact that faces in an embedded bipartite graph have length at least 4, it can be easily shown that:

Proposition 4.2. *A graph with genus g does not contain the complete bipartite graph $K_{r,r}$ for any $r > 2 + 2\sqrt{g}$.*

Using Theorem 2.7 and Proposition 4.2, Theorem 4.1 can now be generalized to graphs with bounded genus.

Theorem 4.3. *Let (G, k) be an instance of* IM *where G is a graph on n vertices with genus g. Then in $O(gn)$ time we can compute an instance (G', k') where (G', k') is a kernel of (G, k), $k' \leq k$, and such that either $n(G') \geq (49 + 7\sqrt{1 + 48g})k' + 10g$ and we can accept the instance (G, k), or $n(G') < (49 + 7\sqrt{1 + 48g})k' + 10g$.*

4.2. Parameterized algorithms for IM on graphs with bounded genus

We again begin with the planar case. Assume that we have an instance (G, k) of Planar-IM. By Theorem 4.1, we can assume that after an $O(n)$ preprocessing time, the number of vertices n in G satisfies $n \leq 40k$. We will show how to design a parameterized algorithm for the Planar-IM problem. Our algorithm is a bounded-search-tree algorithm that uses the Lipton-Tarjan separator theorem [7]. Our results answer an open question posed by [8] of whether a bounded-search-tree algorithm exists for Planar-IM. We also show at the end of this section how these results can be extended to bounded genus graphs.

Theorem 4.4 ([7]). *Given a planar graph $G = (V, E)$ on n vertices, there is a linear time algorithm that partitions V into vertex-sets A, B, S such that:*

1. *$|A|, |B| \leq 2n/3$;*
2. *$|S| \leq \sqrt{8n}$; and*
3. *S separates A and B, i.e. there is no edge between a vertex in A and and a vertex in B.*

Given an instance (G, k) of Planar-IM, where $G = (V, E)$ and $|V| = n$, we partition V into vertex-sets A, B, S according to the Lipton-Tarjan theorem. Let G_A, G_B, and G_S be the subgraphs of G induced by the vertices in A, B, and S, respectively. The idea is simple: separate the graph by enumerating a possible status for the vertices in S, and then use a divide-and-conquer approach. However, special care needs to be taken when enumerating the vertices in S as this enumeration is not straightforward. We outline the general approach below

Each vertex u in S is either an endpoint of an edge in the induced matching or not. Therefore, we assign each vertex u two possible statuses: status 0 if u is an endpoint of an edge in the induced matching and 1 if it is not. Suppose that we have assigned a status to every vertex u in S. If the assigned status to u is 0, we simply remove u (and its incident edges) from G. If the assigned status to u is 1 and there is an edge uu' where $u' \in S$ and the status assigned to u' is 1, then uu' has to be an edge in the induced matching if our enumeration is correct. Therefore, we can add uu' to the matching and remove all the neighbors of u and u' from G. If the assigned status to u is 1, and no vertex $u' \in S$ exists such that the assigned status to u' is 1, then we further assign u two statuses: status 1_A if u is matched to a vertex in G_A in the induced matching, and status 1_B if u is matched to a vertex in G_B. In the former case, we add u to G_A and remove all its neighbors in G_B, and in the latter case, we add u to B and remove all its neighbors in G_A.

After assigning each vertex in S a status from $\{0, 1_A, 1_B\}$, and updating the graph according to the above description, G_A and G_B are separated, and we can recurse on them to compute an induced matching M_A of G_A and M_B of G_B. We then return $M_A \cup M_B$ plus all the edges uu' where $u, u' \in S$, and the assigned status to u and u' is 1. Note that since our enumeration might be incorrect, the returned set of edges may not correspond to an induced matching. Therefore, we will need to verify that the returned set corresponds to an induced matching before returning it.

If there is an induced matching of at least k edges in G, then it is not difficult to see that at least one enumeration will return such an induced matching. Otherwise, no enumeration can find an induced matching of at least k edges, and we reject the instance.

Finally, note that in the recursive calls, some of the vertices in G_A and G_B may have already been assigned the status 1, and we need to respect the assigned statuses in any possible future enumeration of those vertices in G_A and G_B.

A standard analysis shows that the running time of the algorithm is $O(2^{25\sqrt{n}})$. Noting that $n \leq 40k$, we have the following theorem:

Theorem 4.5. *In time $O(2^{159\sqrt{k}} + n)$, it can be determined whether a planar graph on n vertices has an induced matching of at least k edges.*

The above results can be extended to bounded genus graphs.

Theorem 4.6. *Let G be a graph on n vertices with genus g. In time $O(2^{O(\sqrt{gk})} + n)$, it can be determined whether G has an induced matching of at least k edges.*

Due to the large constant in the exponent of the running time of the above algorithms, it is clear that these algorithms are far from being practical. We shall present in the next section more practical parameterized algorithms for IM on bounded genus graphs.

5. Practical algorithms for IM on graphs of bounded genus

We start with the planar case. Let (G, k) be an instance of PLANAR-IM where G has n vertices. By Theorem 4.1, we can assume that after an $O(n)$ preprocessing time, the number of vertices n in G satisfies $n \leq 40k$.

Let M be a maximal matching in G and let $I = V(G) - V(M)$. If $V(M)$ contains more than $8k$ vertices, then by contracting each edge of M in $G_M = G(V(M))$ then applying the Four-Color Theorem to G_M, we conclude that G_M, and hence G, has an induced matching of at least k edges, and we can accept the instance (G, k). Assume that $V(M) < 8k$.

The algorithm will look for a set of exactly k edges that form an induced matching. These edges will have at most $2k$ endpoints in $V(M)$. Therefore, we start by enumerating every subset $S \subseteq V(M)$ of size at most $2k$. There are at most $\sum_{i=0}^{2k} \binom{8k}{i}$ such subsets. Let S be such a subset. We work under the assumption that every vertex in S is an endpoint of an edge in the induced matching until we either find the desired induced matching, or this assumption turns out to be false. In the latter case we enumerate the next subset S.

If two vertices u and v in S are adjacent, then uv must be an edge in the induced matching; therefore, in this case we include uv, remove every neighbor of u and v from G, and reduce k by 1. After we have included (in the induced matching) every edge both of whose endpoints are in S, every remaining vertex in S must be matched with a vertex in I. Observe that if there is a vertex $w \in I$ that is adjacent to at least two vertices in S, then none of the edges joining w to S is in the induced matching. Hence, w could not be an endpoint to an edge in the matching, and w can be removed from I. After removing every such vertex w from I, each remaining vertex in I is adjacent to at most one vertex in

S. Now if our original choice of the set S was correct, then by choosing a neighbor in I for every vertex in S, we should obtain an induced matching in G of size k. If such a choice is not possible (for example, a vertex in S does not have a neighbor in I), or the total number of edges in the induced matching at the end of this process is less than k, then our choice of S was incorrect, and we enumerate the next subset S of $V(M)$ of size at most $2k$. After we have enumerated all subsets of $V(M)$ of size at most $2k$, either we have found an induced matching of at least k edges, or no such matching exists. Noting that there are at most $\sum_{i=0}^{2k}\binom{8k}{i} \leq (2k+1)\binom{8k}{2k}$ such subsets, and that the number of vertices in G is $O(k)$, we have the following theorem:

Theorem 5.1. Planar-IM *can be solved in* $O(\binom{8k}{2k}k^2 + n) = O(91^k + n)$ *time.*

This algorithm is more practical for small values of the parameter k than the one described previously. We can generalize the result to bounded genus graphs:

Theorem 5.2. *The* IM *problem on graphs with n vertices and genus g can be solved in* $O(\binom{(7+\sqrt{1+48g})k}{2k}k^2 + n)$ *time.*

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