

DECIDABILITY OF THE INTERVAL TEMPORAL LOGIC $AB\bar{B}$ OVER THE NATURAL NUMBERS

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ABSTRACT. In this paper, we focus our attention on the interval temporal logic of the Allen's relations "meets", "begins", and "begun by" ($AB\bar{B}$ for short), interpreted over natural numbers. We first introduce the logic and we show that it is expressive enough to model distinctive interval properties, such as accomplishment conditions, to capture basic modalities of point-based temporal logic, such as the until operator, and to encode relevant metric constraints. Then, we prove that the satisfiability problem for $AB\bar{B}$ over natural numbers is decidable by providing a small model theorem based on an original contraction method. Finally, we prove the EXPSPACE-completeness of the problem.

1. Introduction

Interval temporal logics are modal logics that allow one to represent and to reason about time intervals. It is well known that, on a linear ordering, one among thirteen different binary relations may hold between any pair of intervals, namely, "ends", "during", "begins", "overlaps", "meets", "before", together with their inverses, and the relation "equals" (the so-called Allen's relations [1])¹. Allen's relations give rise to respective unary modal operators, thus defining the modal logic of time intervals HS introduced by Halpern and Shoham

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¹We do not consider here the case of ternary relations. Amongst the multitude of *ternary* relations among intervals there is one of particular importance, which corresponds to the binary operation of concatenation of meeting intervals. The logic of such a ternary interval relation has been investigated by Venema in [20]. A systematic analysis of its fragments has been recently given by Hodkinson et al. [13].



in [12]. Some of these modal operators are actually definable in terms of others; in particular, if singleton intervals are included in the structure, it suffices to choose as basic the modalities corresponding to the relations “begins” B and “ends” E , and their transposes \bar{B} , \bar{E} . HS turns out to be highly undecidable under very weak assumptions on the class of interval structures over which its formulas are interpreted [12]. In particular, undecidability holds for any class of interval structures over linear orderings that contains at least one linear ordering with an infinite ascending or descending chain, thus including the natural time flows \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} . Undecidability of HS over finite structures directly follows from results in [15]. In [14], Lodaya sharpens the undecidability of HS showing that the two modalities B, E suffice for undecidability over dense linear orderings (in fact, the result applies to the class of all linear orderings [11]). Even though HS is very natural and the meaning of its operators is quite intuitive, for a long time such sweeping undecidability results have discouraged the search for practical applications and further investigations in the field. A renewed interest in interval temporal logics has been recently stimulated by the identification of some decidable fragments of HS, whose decidability does not depend on simplifying semantic assumptions such as locality and homogeneity [11]. This is the case with the fragments $B\bar{B}$, $E\bar{E}$ (logics of the “begins/begun by” and “ends/ended by” relations) [11], A , $A\bar{A}$ (logics of temporal neighborhood, whose modalities capture the “meets/met by” relations [10]), and D , $D\bar{D}$ (logics of the subinterval/superinterval relations) [3, 16].

In this paper, we focus our attention on the product logic $AB\bar{B}$, obtained from the join of $B\bar{B}$ and A (the case of $\bar{A}E\bar{E}$ is fully symmetric), interpreted over the linear order \mathbb{N} of the natural numbers (or a finite prefix of it). The decidability of $B\bar{B}$ can be proved by translating it into the point-based propositional temporal logic of linear time with temporal modalities F (sometime in the future) and P (sometime in the past), which has the finite (pseudo-)model property and is decidable, e.g., [9]. In general, such a reduction to point-based temporal logics does not work: formulas of interval temporal logics are evaluated over pairs of points and translate into binary relations. For instance, this is the case with A . Unlike the case of $B\bar{B}$, when dealing with A one cannot abstract away from the left endpoint of intervals, as contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint. The decidability of $A\bar{A}$, and thus that of its fragment A , over various classes of linear orderings has been proved by Bresolin et al. by reducing its satisfiability problem to that of the two-variable fragment of first-order logic over the same classes of structures [4], whose decidability has been proved by Otto in [18]. Optimal tableau methods for A with respect to various classes of interval structures can be found in [6, 7]. A decidable metric extension of A over the natural numbers has been proposed in [8]. A number of undecidable extensions of A , and $A\bar{A}$, have been given in [2, 5].

$AB\bar{B}$ retains the simplicity of its constituents $B\bar{B}$ and A , but it improves a lot on their expressive power (as we shall show, such an increase in expressiveness is achieved at the cost of an increase in complexity). First, it allows one to express assertions that may be true at certain intervals, but at no subinterval of them, such as the conditions of accomplishment. Moreover, it makes it possible to easily encode the until operator of point-based temporal logic (this is possible neither with $B\bar{B}$ nor with A). Finally, meaningful metric constraints about the length of intervals can be expressed in $AB\bar{B}$, that is, one can constrain an interval to be at least (resp., at most, exactly) k points long. We prove the decidability of $AB\bar{B}$ interpreted over \mathbb{N} by providing a small model theorem based on an original contraction method. To prove it, we take advantage of a natural (equivalent) interpretation of $AB\bar{B}$ formulas over grid-like structures based on a bijection between the set of intervals over \mathbb{N}

and (a suitable subset of) the set of points of the $\mathbb{N} \times \mathbb{N}$ grid. In addition, we prove that the satisfiability problem for $AB\bar{B}$ is EXPSpace-complete (that for A is NEXPTIME-complete). In the proof of hardness, we use a reduction from the exponential-corridor tiling problem.

The paper is organized as follows. In Section 2 we introduce $AB\bar{B}$. In Section 3, we prove the decidability of its satisfiability problem. We first describe the application of the contraction method to finite models and then we generalize it to infinite ones. In Section 4 we deal with computational complexity issues. Conclusions provide an assessment of the work and outline future research directions. Missing proofs can be found in [17].

2. The interval temporal logic $AB\bar{B}$

In this section, we briefly introduce syntax and semantics of the logic $AB\bar{B}$, which features three modal operators $\langle A \rangle$, $\langle B \rangle$, and $\langle \bar{B} \rangle$ corresponding to the three Allen's relations A ("meets"), B ("begins"), and \bar{B} ("begun by"), respectively. We show that $AB\bar{B}$ is expressive enough to capture the notion of accomplishment, to define the standard until operator of point-based temporal logics, and to encode metric conditions. Then, we introduce the basic notions of atom, type, and dependency. We conclude the section by providing an alternative interpretation of $AB\bar{B}$ over labeled grid-like structures.

2.1. Syntax and semantics

Given a set \mathcal{Prop} of propositional variables, formulas of $AB\bar{B}$ are built up from \mathcal{Prop} using the boolean connectives \neg and \vee and the unary modal operators $\langle A \rangle$, $\langle B \rangle$, $\langle \bar{B} \rangle$. As usual, we shall take advantage of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[A]\varphi = \neg\langle A \rangle\neg\varphi$, $[B]\varphi = \neg\langle B \rangle\neg\varphi$, $\top = p \vee \neg p$, and $\perp = p \wedge \neg p$, with $p \in \mathcal{Prop}$. Hereafter, we denote by $|\varphi|$ the size of φ .

We interpret formulas of $AB\bar{B}$ in interval temporal structures over natural numbers endowed with the relations "meets", "begins", and "begun by". Precisely, we identify any given ordinal $N \leq \omega$ with the prefix of length N of the linear order of the natural numbers and we accordingly define \mathbb{I}_N as the set of all non-singleton closed intervals $[x, y]$, with $x, y \in \mathbb{N}$ and $x < y$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_N$, the Allen's relations "meets" A , "begins" B , and "begun by" \bar{B} are defined as follows (note that \bar{B} is the inverse relation of B):

- "meets" relation: $[x, y] A [x', y']$ iff $y = x'$;
- "begins" relation: $[x, y] B [x', y']$ iff $x = x'$ and $y' < y$;
- "begun by" relation: $[x, y] \bar{B} [x', y']$ iff $x = x'$ and $y < y'$.

Given an *interval structure* $\mathcal{S} = (\mathbb{I}_N, A, B, \bar{B}, \sigma)$, where $\sigma : \mathbb{I}_N \rightarrow \mathcal{P}(\mathcal{Prop})$ is a labeling function that maps intervals in \mathbb{I}_N to sets of propositional variables, and an initial interval I , we define the semantics of an $AB\bar{B}$ formula as follows:

- $\mathcal{S}, I \models a$ iff $a \in \sigma(I)$, for any $a \in \mathcal{Prop}$;
- $\mathcal{S}, I \models \neg\varphi$ iff $\mathcal{S}, I \not\models \varphi$;
- $\mathcal{S}, I \models \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \models \varphi_1$ or $\mathcal{S}, I \models \varphi_2$;
- for every relation $R \in \{A, B, \bar{B}\}$, $\mathcal{S}, I \models \langle R \rangle\varphi$ iff there is an interval $J \in \mathbb{I}_N$ such that $I R J$ and $\mathcal{S}, J \models \varphi$.

Given an interval structure \mathcal{S} and a formula φ , we say that \mathcal{S} *satisfies* φ if there is an interval I in \mathcal{S} such that $\mathcal{S}, I \models \varphi$. We say that φ is *satisfiable* if there exists an interval structure that satisfies it. We define the *satisfiability problem* for $AB\bar{B}$ as the problem of establishing whether a given $AB\bar{B}$ -formula φ is satisfiable.

We conclude the section with some examples that account for $AB\bar{B}$ expressive power. The first one shows how to encode in $AB\bar{B}$ conditions of accomplishment (think of formula φ as the assertion: “Mr. Jones flew from Venice to Nancy”): $\langle A \rangle (\varphi \wedge [B](\neg\varphi \wedge [A]\neg\varphi) \wedge [\bar{B}]\neg\varphi)$. Formulas of point-based temporal logics of the form $\psi \mathbf{U} \varphi$, using the standard until operator, can be encoded in $AB\bar{B}$ (where atomic intervals are two-point intervals) as follows: $\langle A \rangle ([B]\perp \wedge \varphi) \vee \langle A \rangle (\langle A \rangle ([B]\perp \wedge \varphi) \wedge [B](\langle A \rangle ([B]\perp \wedge \psi)))$. Finally, metric conditions like: “ φ holds over a right neighbor interval of length greater than k (resp., less than k , equal to k)” can be captured by the following $AB\bar{B}$ formula: $\langle A \rangle (\varphi \wedge \langle B \rangle^k \top)$ (resp., $\langle A \rangle (\varphi \wedge [B]^{k-1} \perp)$, $\langle A \rangle (\varphi \wedge [B]^k \perp \wedge \langle B \rangle^{k-1} \top)$)².

2.2. Atoms, types, and dependencies

Let $\mathcal{S} = (\mathbb{I}_{\mathbb{N}}, A, B, \bar{B}, \sigma)$ be an interval structure and φ be a formula of $AB\bar{B}$. In the sequel, we shall compare intervals in \mathcal{S} with respect to the set of subformulas of φ they satisfy. To do that, we introduce the key notions of φ -atom, φ -type, φ -cluster, and φ -shading.

First of all, we define the *closure* $\mathcal{Cl}(\varphi)$ of φ as the set of all subformulas of φ and of their negations (we identify $\neg\neg\alpha$ with α , $\neg\langle A \rangle\alpha$ with $[A]\neg\alpha$, etc.). For technical reasons, we also introduce the *extended closure* $\mathcal{Cl}^+(\varphi)$, which is defined as the set of all formulas in $\mathcal{Cl}(\varphi)$ plus all formulas of the forms $\langle R \rangle\alpha$ and $\neg\langle R \rangle\alpha$, with $R \in \{A, B, \bar{B}\}$ and $\alpha \in \mathcal{Cl}(\varphi)$.

A φ -*atom* is any non-empty set $F \subseteq \mathcal{Cl}^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{Cl}^+(\varphi)$, we have $\alpha \in F$ iff $\neg\alpha \notin F$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{Cl}^+(\varphi)$, we have $\gamma \in F$ iff $\alpha \in F$ or $\beta \in F$ (intuitively, a φ -atom is a maximal *locally consistent* set of formulas chosen from $\mathcal{Cl}^+(\varphi)$). Note that the cardinalities of both sets $\mathcal{Cl}(\varphi)$ and $\mathcal{Cl}^+(\varphi)$ are *linear* in the number $|\varphi|$ of subformulas of φ , while the number of φ -atoms is *at most exponential* in $|\varphi|$ (precisely, we have $|\mathcal{Cl}(\varphi)| = 2|\varphi|$, $|\mathcal{Cl}^+(\varphi)| = 14|\varphi|$, and there are at most $2^{7|\varphi|}$ distinct atoms).

We also associate with each interval $I \in \mathcal{S}$ the set of all formulas $\alpha \in \mathcal{Cl}^+(\varphi)$ such that $\mathcal{S}, I \models \alpha$. Such a set is called φ -*type* of I and it is denoted by $\mathcal{T}_{\text{type}_{\mathcal{S}}}(I)$. We have that every φ -type is a φ -atom, but not vice versa. Hereafter, we shall omit the argument φ , thus calling a φ -atom (resp., a φ -type) simply an atom (resp., a type).

Given an atom F , we denote by $\mathcal{Obs}(F)$ the set of all *observables* of F , namely, the formulas $\alpha \in \mathcal{Cl}(\varphi)$ such that $\alpha \in F$. Similarly, given an atom F and a relation $R \in \{A, B, \bar{B}\}$, we denote by $\mathcal{Req}_R(F)$ the set of all *R-requests* of F , namely, the formulas $\alpha \in \mathcal{Cl}(\varphi)$ such that $\langle R \rangle\alpha \in F$. Taking advantage of the above sets, we can define the following two relations between atoms F and G :

$$F \xrightarrow{A} G \quad \text{iff} \quad \mathcal{Req}_A(F) = \mathcal{Obs}(G) \cup \mathcal{Req}_B(G) \cup \mathcal{Req}_{\bar{B}}(G);$$

$$F \xrightarrow{B} G \quad \text{iff} \quad \begin{cases} \mathcal{Obs}(F) \cup \mathcal{Req}_{\bar{B}}(F) \subseteq \mathcal{Req}_{\bar{B}}(G) \subseteq \mathcal{Obs}(F) \cup \mathcal{Req}_{\bar{B}}(F) \cup \mathcal{Req}_B(F), \\ \mathcal{Obs}(G) \cup \mathcal{Req}_B(G) \subseteq \mathcal{Req}_B(F) \subseteq \mathcal{Obs}(G) \cup \mathcal{Req}_B(G) \cup \mathcal{Req}_{\bar{B}}(G). \end{cases}$$

²It is not difficult to show that $AB\bar{B}$ subsumes the metric extension of A given in [8]. A simple game-theoretic argument shows that the former is in fact strictly more expressive than the latter.

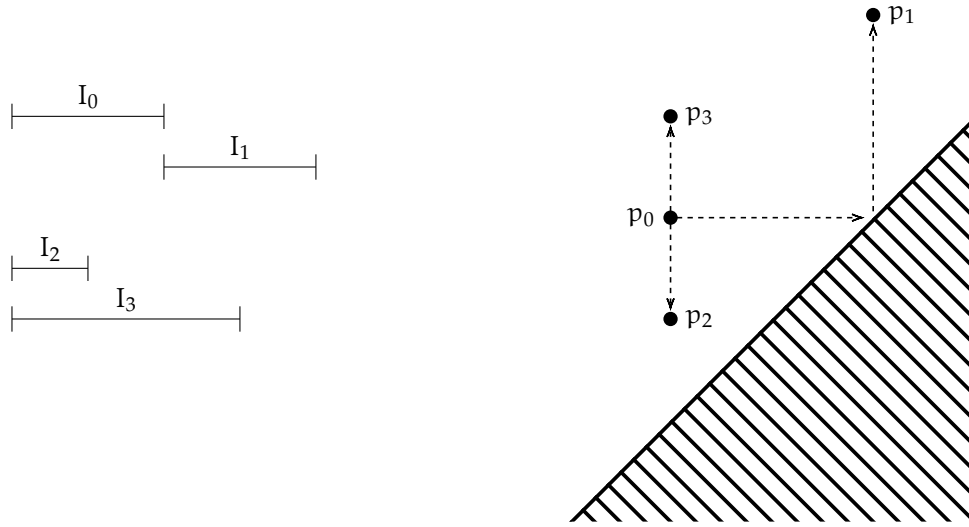


Figure 1: Correspondence between intervals and points of a discrete grid.

Note that the relation \xrightarrow{B} is transitive, while \xrightarrow{A} is not. Moreover, both \xrightarrow{A} and \xrightarrow{B} satisfy a *view-to-type dependency*, namely, for every pair of intervals I, J in \mathcal{S} , we have that

$$\begin{aligned} I \ A \ J & \text{ implies } \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{A} \mathcal{T}ype_{\mathcal{S}}(J) \\ I \ B \ J & \text{ implies } \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{B} \mathcal{T}ype_{\mathcal{S}}(J). \end{aligned}$$

Relations \xrightarrow{A} and \xrightarrow{B} will come into play in the definition of consistency conditions (see Definition 2.1).

2.3. Compass structures

The logic $AB\bar{B}$ can be equivalently interpreted over grid-like structures (the so-called compass structures [20]) by exploiting the existence of a natural bijection between the intervals $I = [x, y]$ and the points $p = (x, y)$ of an $N \times N$ grid such that $x < y$. As an example, Figure 1 depicts four intervals I_0, \dots, I_3 such that $I_0 \ A \ I_1$, $I_0 \ B \ I_2$, and $I_0 \ \bar{B} \ I_3$, together with the corresponding points p_0, \dots, p_3 of a discrete grid (note that the three Allen’s relations A, B, \bar{B} between intervals are mapped to corresponding spatial relations between points; for the sake of readability, we name the latter ones as the former ones).

Definition 2.1. Given an $AB\bar{B}$ formula φ , a (consistent and fulfilling) *compass* (φ -)structure of length $N \leq \omega$ is a pair $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, where \mathbb{P}_N is the set of points $p = (x, y)$, with $0 \leq x < y < N$, and \mathcal{L} is function that maps any point $p \in \mathbb{P}_N$ to a (φ -)atom $\mathcal{L}(p)$ in such a way that

- for every pair of points $p, q \in \mathbb{P}_N$ and every relation $R \in \{A, B\}$, if $p \ R \ q$ holds, then $\mathcal{L}(p) \xrightarrow{R} \mathcal{L}(q)$ follows (**consistency**);
- for every point $p \in \mathbb{P}_N$, every relation $R \in \{A, B, \bar{B}\}$, and every formula $\alpha \in \mathcal{R}eq_R(\mathcal{L}(p))$, there is a point $q \in \mathbb{P}_N$ such that $p \ R \ q$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(q))$ (**fulfillment**).

We say that a compass (φ -)structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ features a formula α if there is a point $p \in \mathbb{P}_N$ such that $\alpha \in \mathcal{L}(p)$. The following proposition implies that the satisfiability problem

for $AB\bar{B}$ is reducible to the problem of deciding, for any given formula φ , whether there exists a φ -compass structure that features φ .

Proposition 2.2. *An $AB\bar{B}$ -formula φ is satisfied by some interval structure if and only if it is featured by some (φ) -compass structure.*

3. Deciding the satisfiability problem for $AB\bar{B}$

In this section, we prove that the satisfiability problem for $AB\bar{B}$ is decidable by providing a “small-model theorem” for the satisfiable formulas of the logic. For the sake of simplicity, we first show that the satisfiability problem for $AB\bar{B}$ interpreted over *finite* interval structures is decidable and then we generalize such a result to all (finite or infinite) interval structures.

As a preliminary step, we introduce the key notion of shading. Let $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ be a compass structure of length $N \leq \omega$ and let $0 \leq \mathbf{y} < N$. The *shading of the row \mathbf{y}* of \mathcal{G} is the set $Shading_{\mathcal{G}}(\mathbf{y}) = \{\mathcal{L}(x, \mathbf{y}) : 0 \leq x < \mathbf{y}\}$, namely, the set of the atoms of all points in \mathbb{P}_N whose vertical coordinate has value \mathbf{y} (basically, we interpret different atoms as different colors). Clearly, for every pair of atoms F and F' in $Shading_{\mathcal{G}}(\mathbf{y})$, we have $Req_A(F) = Req_A(F')$.

3.1. A small-model theorem for finite structures

Let φ be an $AB\bar{B}$ formula. Let us assume that φ is featured by a *finite* compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, with $N < \omega$. In fact, without loss of generality, we can assume that φ belongs to the atom associated with a point $\mathbf{p} = (0, \mathbf{y})$ of \mathcal{G} , with $0 < \mathbf{y} < N$. We prove that we can restrict our attention to compass structures $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, where N is bounded by a double exponential in $|\varphi|$. We start with the following lemma that proves a simple, but crucial, property of the relations \xrightarrow{A} and \xrightarrow{B} (the proof can be found in [17]).

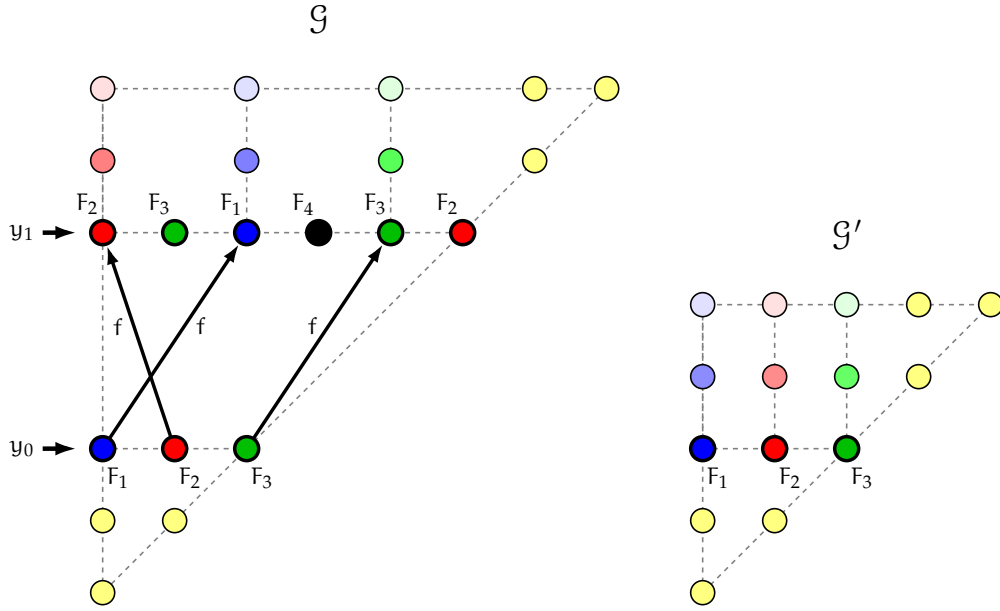
Lemma 3.1. *If $F \xrightarrow{A} H$ and $G \xrightarrow{B} H$ hold for some atoms F, G, H , then $F \xrightarrow{A} G$ holds as well.*

The next lemma shows that, under suitable conditions, a given compass structure \mathcal{G} may be reduced in length, preserving the existence of atoms featuring φ .

Lemma 3.2. *Let \mathcal{G} be a compass structure featuring φ . If there exist two rows $0 < \mathbf{y}_0 < \mathbf{y}_1 < N$ in \mathcal{G} such that $Shading_{\mathcal{G}}(\mathbf{y}_0) \subseteq Shading_{\mathcal{G}}(\mathbf{y}_1)$, then there exists a compass structure \mathcal{G}' of length $N' < N$ that features φ .*

Proof. Suppose that $0 < \mathbf{y}_0 < \mathbf{y}_1 < N$ are two rows of \mathcal{G} such that $Shading_{\mathcal{G}}(\mathbf{y}_0) \subseteq Shading_{\mathcal{G}}(\mathbf{y}_1)$. Then, there is a function $f : \{0, \dots, \mathbf{y}_0 - 1\} \rightarrow \{0, \dots, \mathbf{y}_1 - 1\}$ such that, for every $0 \leq x < \mathbf{y}_0$, $\mathcal{L}(x, \mathbf{y}_0) = \mathcal{L}(f(x), \mathbf{y}_1)$. Let $k = \mathbf{y}_1 - \mathbf{y}_0$, $N' = N - k$ ($< N$), and $\mathbb{P}_{N'}$ be the portion of the grid that consists of all points $\mathbf{p} = (x, \mathbf{y})$, with $0 \leq x < \mathbf{y} < N'$. We extend f to a function that maps points in $\mathbb{P}_{N'}$ to points in \mathbb{P}_N as follows:

- if $\mathbf{p} = (x, \mathbf{y})$, with $0 \leq x < \mathbf{y} < \mathbf{y}_0$, then we simply let $f(\mathbf{p}) = \mathbf{p}$;
- if $\mathbf{p} = (x, \mathbf{y})$, with $0 \leq x < \mathbf{y}_0 \leq \mathbf{y}$, then we let $f(\mathbf{p}) = (f(x), \mathbf{y} + k)$;
- if $\mathbf{p} = (x, \mathbf{y})$, with $\mathbf{y}_0 \leq x < \mathbf{y}$, then we let $f(\mathbf{p}) = (x + k, \mathbf{y} + k)$.


 Figure 2: Contraction \mathcal{G}' of a compass structure \mathcal{G} .

We denote by \mathcal{L}' the labeling of $\mathbb{P}_{N'}$ such that, for every point $p \in \mathbb{P}_{N'}$, $\mathcal{L}'(p) = \mathcal{L}(f(p))$ and we denote by \mathcal{G}' the resulting structure $(\mathbb{P}_{N'}, \mathcal{L}')$ (see Figure 2). We have to prove that \mathcal{G}' is a *consistent* and *fulfilling* compass structure that features φ (see Definition 2.1). First, we show that \mathcal{G}' satisfies the consistency conditions for the relations B and A ; then we show that \mathcal{G}' satisfies the fulfillment conditions for the \bar{B} -, B -, and A -requests; finally, we show that \mathcal{G}' features φ .

CONSISTENCY WITH RELATION B . Consider two points $p = (x, y)$ and $p' = (x', y')$ in \mathcal{G}' such that $p B p'$, i.e., $0 \leq x = x' < y' < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$ by distinguishing among the following three cases (note that exactly one of such cases holds):

- (1) $y < y_0$ and $y' < y_0$,
- (2) $y \geq y_0$ and $y' \geq y_0$,
- (3) $y \geq y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since \mathcal{G} is a (consistent) compass structure, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}'(p')$. If $y \geq y_0$ and $y' \geq y_0$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geq y_0$. Similarly, we have either $f(p') = (f(x'), y' + k) = (f(x), y' + k)$ or $f(p') = (x' + k, y' + k) = (x + k, y' + k)$. This implies $f(p) B f(p')$ and thus, since \mathcal{G} is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.

If $y \geq y_0$ and $y' < y_0$, then, since $x < y' < y_0$, we have by construction $f(p) = (f(x), y + k)$ and $f(p') = p'$. Moreover, if we consider the point $p'' = (x, y_0)$ in \mathcal{G}' , we easily see that (i) $f(p'') = (f(x), y_1)$, (ii) $f(p) B f(p'')$ (whence $\mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p''))$), (iii) $\mathcal{L}(f(p'')) = \mathcal{L}(p'')$, and (iv) $p'' B p'$ (whence $\mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p')$). It thus follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p'')) = \mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by exploiting the transitivity of the relation \xrightarrow{B} , we obtain $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$.

CONSISTENCY WITH RELATION A. Consider two points $p = (x, y)$ and $p' = (x', y')$ such that $p A p'$, i.e., $0 \leq x < y = x' < y' < N'$. We define $p'' = (y, y + 1)$ in such a way that $p A p''$ and $p' B p''$ and we distinguish between the following two cases:

- (1) $y \geq y_0$,
- (2) $y < y_0$.

If $y \geq y_0$, then, by construction, we have $f(p) A f(p'')$. Since \mathcal{G} is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{A} \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

If $y < y_0$, then, by construction, we have $\mathcal{L}(p'') = \mathcal{L}(f(p''))$. Again, since \mathcal{G} is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) = \mathcal{L}(p) \xrightarrow{A} \mathcal{L}(p'') = \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

In both cases we have $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p'')$. Now, we recall that $p' B p''$ and that, by previous arguments, \mathcal{G}' is consistent with the relation B. We thus have $\mathcal{L}'(p') \xrightarrow{B} \mathcal{L}'(p'')$. Finally, by applying Lemma 3.1, we obtain $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p')$.

FULFILLMENT OF B-REQUESTS. Consider a point $p = (x, y)$ in \mathcal{G}' and some B-request $\alpha \in \text{Req}_B(\mathcal{L}'(p))$ associated with it. Since, by construction, $\alpha \in \text{Req}_B(\mathcal{L}(f(p)))$ and \mathcal{G} is a (fulfilling) compass structure, we know that \mathcal{G} contains a point $q' = (x', y')$ such that $f(p) B q'$ and $\alpha \in \text{Obs}(\mathcal{L}(q'))$. We prove that \mathcal{G}' contains a point p' such that $p B p'$ and $\alpha \in \text{Obs}(\mathcal{L}'(p'))$ by distinguishing among the following three cases (note that exactly one of such cases holds):

- (1) $y < y_0$
- (2) $y' \geq y_1$,
- (3) $y \geq y_0$ and $y' < y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p) B q' = p'$ and $\alpha \in \text{Obs}(\mathcal{L}'(p'))$ ($= \text{Obs}(\mathcal{L}(f(p'))) = \text{Obs}(\mathcal{L}(q'))$).

If $y' \geq y_1$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geq y_0$. We define $p' = (x, y' - k)$ in such a way that $p B p'$. Moreover, we observe that either $f(p') = (f(x), y')$ or $f(p') = (x + k, y')$, depending on whether $x < y_0$ or $x \geq y_0$, and in both cases $f(p') = q'$ follows. This shows that $\alpha \in \text{Obs}(\mathcal{L}'(p'))$ ($= \text{Obs}(\mathcal{L}(f(p'))) = \text{Obs}(\mathcal{L}(q'))$).

If $y \geq y_0$ and $y' < y_1$, then we define $\bar{p} = (x, y_0)$ and $\bar{q} = (x', y_1)$ and we observe that $f(p) B \bar{q}$, $\bar{q} B q'$, and $f(\bar{p}) = \bar{q}$. From $f(p) B \bar{q}$ and $\bar{q} B q'$, it follows that $\alpha \in \text{Req}_B(\mathcal{L}(\bar{q}))$ and hence $\alpha \in \text{Req}_B(\mathcal{L}(\bar{p}))$. Since \mathcal{G} is a (fulfilling) compass structure, we know that there is a point p' such that $\bar{p} B p'$ and $\alpha \in \text{Obs}(\mathcal{L}(\bar{p}'))$. Moreover, since $\bar{p} B p'$, we have $f(p') = p'$, from which we obtain $p B p'$ and $\alpha \in \text{Obs}(\mathcal{L}(p'))$.

FULFILLMENT OF \bar{B} -REQUESTS. The proof that \mathcal{G}' fulfills all \bar{B} -requests of its atoms is symmetric with respect to the previous one.

FULFILLMENT OF A-REQUESTS. Consider a point $p = (x, y)$ in \mathcal{G}' and some A-request $\alpha \in \text{Req}_A(\mathcal{L}'(p))$ associated with p in \mathcal{G}' . Since, by previous arguments, \mathcal{G}' fulfills all \bar{B} -requests of its atoms, it is sufficient to prove that either $\alpha \in \text{Obs}(\mathcal{L}'(p'))$ or $\alpha \in \text{Req}_{\bar{B}}(\mathcal{L}'(p'))$, where $p' = (y, y + 1)$. This can be easily proved by distinguishing among the three cases $y < y_0 - 1$, $y = y_0 - 1$, and $y \geq y_0$.

FEATURED FORMULAS. Recall that, by previous assumptions, \mathcal{G} contains a point $p = (0, y)$, with $0 < y < N$, such that $\varphi \in \mathcal{L}(p)$. If $y \leq y_0$, then, by construction, we have

$\varphi \in \mathcal{L}'(\mathfrak{p})$ ($= \mathcal{L}(f(\mathfrak{p})) = \mathcal{L}(\mathfrak{p})$). Otherwise, if $\mathfrak{y} > \mathfrak{y}_0$, we define $\mathfrak{q} = (0, \mathfrak{y}_0)$ and we observe that $\mathfrak{q} \bar{B} \mathfrak{p}$. Since \mathcal{G} is a (consistent) compass structure and $\langle \bar{B} \rangle \varphi \in \mathcal{Cl}^+(\varphi)$, we have that $\varphi \in \mathcal{Req}_{\bar{B}}(\mathcal{L}(\mathfrak{q}))$. Moreover, by construction, we have $\mathcal{L}'(\mathfrak{q}) = \mathcal{L}(f(\mathfrak{q}))$ and hence $\varphi \in \mathcal{Req}_{\bar{B}}(\mathcal{L}'(\mathfrak{q}))$. Finally, since \mathcal{G}' is a (fulfilling) compass structure, we know that there is a point \mathfrak{p}' in \mathcal{G}' such that $f(\mathfrak{q}) \bar{B} \mathfrak{p}'$ and $\varphi \in \mathcal{Obs}(\mathcal{L}'(\mathfrak{p}'))$. \square

On the grounds of the above result, we can provide a suitable upper bound for the length of a minimal finite interval structure that satisfies φ , if there exists any. This yields a straightforward, but inefficient, 2EXPSpace algorithm that decides whether a given $AB\bar{B}$ -formula φ is satisfiable over finite interval structures.

Theorem 3.3. *An $AB\bar{B}$ -formula φ is satisfied by some finite interval structure iff it is featured by some compass structure of length $N \leq 2^{2^{7|\varphi|}}$ (i.e., double exponential in $|\varphi|$).*

Proof. One direction is trivial. We prove the other one (“only if” part). Suppose that φ is satisfied by a finite interval structure \mathcal{S} . By Proposition 2.2, there is a compass structure \mathcal{G} that features φ and has finite length $N < \omega$. Without loss of generality, we can assume that N is minimal among all finite compass structures that feature φ . We recall from Section 2.2 that \mathcal{G} contains at most $2^{7|\varphi|}$ distinct atoms. This implies that there exist at most $2^{2^{7|\varphi|}}$ different shadings of the form $Shading_{\mathcal{G}}(\mathfrak{y})$, with $0 \leq \mathfrak{y} < N$. Finally, by applying Lemma 3.2, we obtain $N \leq 2^{2^{7|\varphi|}}$ (otherwise, there would exist two rows $0 < \mathfrak{y}_0 < \mathfrak{y}_1 < N$ such that $Shading_{\mathcal{G}}(\mathfrak{y}_0) = Shading_{\mathcal{G}}(\mathfrak{y}_1)$, which is against the hypothesis of minimality of N). \square

3.2. A small-model theorem for infinite structures

In general, compass structures that feature φ may be infinite. Here, we prove that, without loss of generality, we can restrict our attention to sufficiently “regular” infinite compass structures, which can be represented in double exponential space with respect to $|\varphi|$. To do that, we introduce the notion of periodic compass structure.

Definition 3.4. An infinite compass structure $\mathcal{G} = (\mathbb{P}_{\omega}, \mathcal{L})$ is *periodic*, with *threshold* $\tilde{\mathfrak{y}}_0$, *period* $\tilde{\mathfrak{y}}$, and *binding* $\tilde{\mathfrak{g}} : \{0, \dots, \tilde{\mathfrak{y}}_0 + \tilde{\mathfrak{y}} - 1\} \rightarrow \{0, \dots, \tilde{\mathfrak{y}}_0 - 1\}$, if the following conditions are satisfied:

- for every $\tilde{\mathfrak{y}}_0 + \tilde{\mathfrak{y}} \leq \mathfrak{x} < \mathfrak{y}$, we have $\mathcal{L}(\mathfrak{x}, \mathfrak{y}) = \mathcal{L}(\mathfrak{x} - \tilde{\mathfrak{y}}, \mathfrak{y} - \tilde{\mathfrak{y}})$,
- for every $0 \leq \mathfrak{x} < \tilde{\mathfrak{y}}_0 + \tilde{\mathfrak{y}} \leq \mathfrak{y}$, we have $\mathcal{L}(\mathfrak{x}, \mathfrak{y}) = \mathcal{L}(\tilde{\mathfrak{g}}(\mathfrak{x}), \mathfrak{y} - \tilde{\mathfrak{y}})$.

Figure 3 gives an example of a periodic compass structure (the arrows represent some relationships between points induced by the binding function $\tilde{\mathfrak{g}}$). Note that any periodic compass structure $\mathcal{G} = (\mathbb{P}_{\omega}, \mathcal{L})$ can be finitely represented by specifying (i) its threshold $\tilde{\mathfrak{y}}_0$, (ii) its period $\tilde{\mathfrak{y}}$, (iii) its binding $\tilde{\mathfrak{g}}$, and (iv) the labeling \mathcal{L} restricted to the portion $\mathbb{P}_{\tilde{\mathfrak{y}}_0 + \tilde{\mathfrak{y}} - 1}^{\tilde{\mathfrak{y}}_0}$ of the domain.

The following theorem leads immediately to a 2EXPSpace algorithm that decides whether a given $AB\bar{B}$ -formula φ is satisfiable over infinite interval structures (the proof is provided in [17]).

Theorem 3.5. *An $AB\bar{B}$ -formula φ is satisfied by an infinite interval structure iff it is featured by a periodic compass structure with threshold $\tilde{\mathfrak{y}}_0 < 2^{2^{7|\varphi|}}$ and period $\tilde{\mathfrak{y}} < 2|\varphi| \cdot 2^{2^{7|\varphi|}}$.*

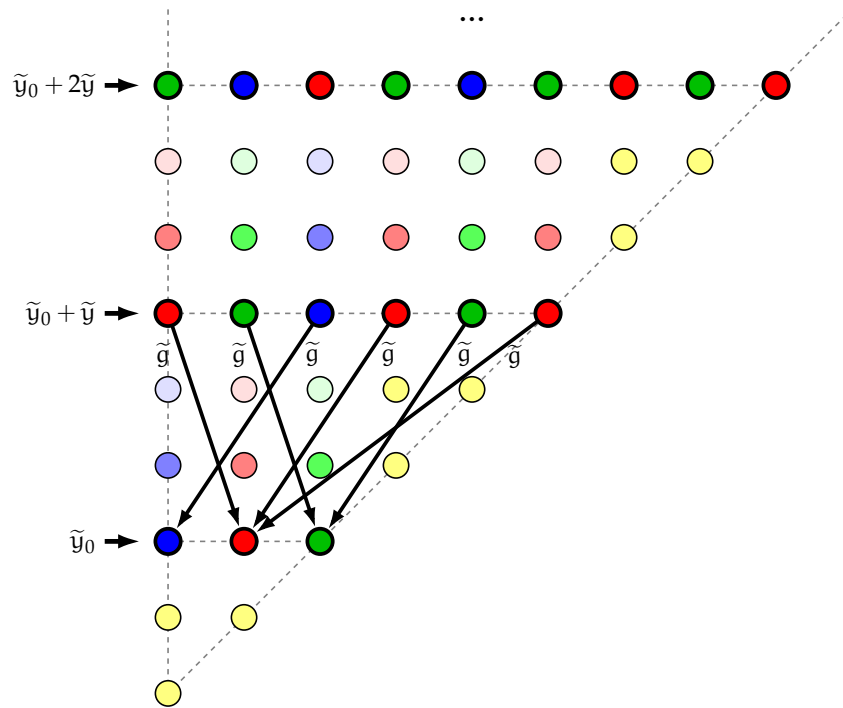


Figure 3: A periodic compass structure with threshold \tilde{y}_0 , period \tilde{y} , and binding \tilde{g} .

4. Tight complexity bounds to the satisfiability problem for $AB\bar{B}$

In this section, we show that the satisfiability problem for $AB\bar{B}$ interpreted over (either finite or infinite) interval temporal structures is EXPSPACE-complete.

The EXPSPACE-hardness of the satisfiability problem for $AB\bar{B}$ follows from a reduction from the *exponential-corridor tiling problem*, which is known to be EXPSPACE-complete [19]. Formally, an instance of the exponential-corridor tiling problem is a tuple $\mathcal{T} = (\mathbb{T}, t_{\perp}, t_{\top}, H, V, n)$ consisting of a finite set \mathbb{T} of tiles, a bottom tile $t_{\perp} \in \mathbb{T}$, a top tile $t_{\top} \in \mathbb{T}$, two binary relations H, V over \mathbb{T} (specifying the horizontal and vertical constraints), and a positive natural number n (represented in unary notation). The problem consists in deciding whether there exists a tiling $f : \mathbb{N} \times \{0, \dots, 2^n - 1\} \rightarrow \mathbb{T}$ of the infinite discrete corridor of height 2^n , that associates the tile t_{\perp} (resp., t_{\top}) with the bottom (resp., top) row of the corridor and that respects the horizontal and vertical constraints H and V , namely,

- i) for every $x \in \mathbb{N}$, we have $f(x, 0) = t_{\perp}$,
- ii) for every $x \in \mathbb{N}$, we have $f(x, 2^n - 1) = t_{\top}$,
- iii) for every $x \in \mathbb{N}$ and every $0 \leq y < 2^n$, we have $f(x, y) H f(x + 1, y)$,
- iv) for every $x \in \mathbb{N}$ and every $0 \leq y < 2^n - 1$, we have $f(x, y) V f(x, y + 1)$.

The proof of the following lemma, which reduces the exponential-corridor tiling problem to the satisfiability problem for $AB\bar{B}$, can be found in [17]. Intuitively, such a reduction exploits (i) the correspondence between the points $p = (x, y)$ inside the infinite corridor $\mathbb{N} \times \{0, \dots, 2^n - 1\}$ and the intervals of the form $I_p = [y + 2^n x, y + 2^n x + 1]$, (ii) $|\mathbb{T}|$ propositional variables which represent the tiling function f , (iii) n additional propositional variables which represent (the binary expansion of) the y -coordinate of each row of the corridor, and

(iv) the modal operators $\langle A \rangle$ and $\langle B \rangle$ by means of which one can enforce the local constraints over the tiling function f (as a matter of fact, this shows that the satisfiability problem for the AB fragment is already hard for EXPSPACE).

Lemma 4.1. *There is a polynomial-time reduction from the exponential-corridor tiling problem to the satisfiability problem for $AB\bar{B}$.*

As for the EXPSPACE-completeness, we claim that the existence of a compass structure \mathcal{G} that features a given formula φ can be decided by verifying suitable local (and stronger) consistency conditions over all pairs of contiguous rows. In fact, in order to check that these local conditions hold between two contiguous rows \mathbf{y} and $\mathbf{y} + 1$, it is sufficient to store into memory a bounded amount of information, namely, (i) a counter \mathbf{y} that ranges over $\{1, \dots, 2^{2^{7|\varphi|}} + |\varphi| \cdot 2^{2^{7|\varphi|}}\}$, (ii) the two guessed shadings S and S' associated with the rows \mathbf{y} and $\mathbf{y} + 1$, and (iii) a function $g : S \rightarrow S'$ that captures the horizontal alignment relation between points with an associated atom from S and points with an associated atom from S' . This shows that the satisfiability problem for $AB\bar{B}$ can be decided in exponential space, as claimed by the following lemma. Further details about the decision procedure, including soundness and completeness proofs, can be found in [17].

Lemma 4.2. *There is an EXPSPACE non-deterministic procedure that decides whether a given formula of $AB\bar{B}$ is satisfiable or not.*

Summing up, we obtain the following tight complexity result.

Theorem 4.3. *The satisfiability problem for $AB\bar{B}$ interpreted over (prefixes of) natural numbers is EXPSPACE-complete.*

5. Conclusions

In this paper, we proved that the satisfiability problem for $AB\bar{B}$ interpreted over (prefixes of) the natural numbers is EXPSPACE-complete. We restricted our attention to these domains because it is a common commitment in computer science. Moreover, this gave us the possibility of expressing meaningful metric constraints in a fairly natural way. Nevertheless, we believe it possible to extend our results to the class of all linear orderings as well as to relevant subclasses of it. Another restriction that can be relaxed is the one about singleton intervals: all results in the paper can be easily generalized to include singleton intervals in the underlying structure $\mathbb{I}_{\mathbb{N}}$. The most exciting challenge is to establish whether the modality \bar{A} can be added to $AB\bar{B}$ preserving decidability (and complexity). It is easy to show that there is not a straightforward way to lift the proof for $AB\bar{B}$ to $AB\bar{B}\bar{A}$ (notice that $\langle A \rangle$, $\langle B \rangle$, and $\langle \bar{B} \rangle$ are all future modalities, while $\langle \bar{A} \rangle$ is a past one).

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