Uniqueness of Normal Forms is Decidable for Shallow Term Rewrite Systems*

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Abstract

Uniqueness of normal forms ($\mathrm{UN}^=$) is an important property of term rewrite systems. $\mathrm{UN}^=$ is decidable for ground (i.e., variable-free) systems and undecidable in general. Recently it was shown to be decidable for linear, shallow systems. We generalize this previous result and show that this property is decidable for shallow rewrite systems, in contrast to confluence, reachability and other properties, which are all undecidable for flat systems. Our result is also optimal in some sense, since we prove that the $\mathrm{UN}^=$ property is undecidable for two superclasses of flat systems: left-flat, left-linear systems in which right-hand sides are of depth at most two and right-flat, right-linear systems in which left-hand sides are of depth at most two.

Keywords and phrases term rewrite systems, uniqueness of normal forms, decidability, shallow rewrite systems, flat rewrite systems

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1 Introduction

Term rewrite systems (TRSs), finite sets of rules, are useful in many computer science fields including theorem proving, rule-based programming, and symbolic computation. An important property of TRSs is confluence (also known as the Church-Rosser property), which implies uniqueness of normal forms ($\mathrm{UN}^=$). Normal forms are expressions to which no rule is applicable. A TRS has the $\mathrm{UN}^=$ property if there are *not* distinct normal forms n, m such that $n \overset{*}{\leftrightarrow}_R m$, where $\overset{*}{\leftrightarrow}_R$ is the symmetric closure of the rewrite relation induced by the TRS R.

Uniqueness of normal forms is an interesting property in itself and well-studied [9]. Confluence can be a requirement too strong for some applications such as lazy programming. Additionally, in the proof-by-consistency approach for inductive theorem proving, consistency is often ensured by requiring the $\mathrm{UN}^=$ property.

We study the decidability of uniqueness of normal forms. Uniqueness of normal forms is decidable for ground systems [12], but is undecidable in general [12]. Since the property is undecidable in general, we would like to know for which classes of rewrite systems, beyond ground systems, we can decide $\mathrm{UN}^=$. In [13, 14] a polynomial time algorithm for this property was given for linear, shallow rewrite systems. A rewrite system is *linear* if variables occur at

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most once in each side of any rule. It is *shallow* if variables occur only at depth zero or depth one in each side of any rule. It is *flat* if both the left- and right-hand sides of all the rules have height zero or one. An example of a linear flat (in fact, ground) system that has $\mathrm{UN}^=$ but not confluence is $\{f(c) \to 1,\ c \to g(c)\}$. More sophisticated examples can be constructed using a sequential 'or' function in which the second argument gives rise to a nonterminating computation.

In this paper, we consider the class of shallow systems, i.e., we drop the linearity restriction of [13], and a subset of this class, the flat systems. For flat systems many properties are known to be undecidable including confluence, reachability, joinability, and existence of normal forms [7, 11, 3]. On the other hand, the word problem is known to be decidable for shallow systems [1]. This paper shows that the uniqueness of normal forms problem is decidable for the class of shallow term rewrite systems, which is a significant generalization of [13] and also somewhat surprising since so many properties are undecidable for this class of systems. We also prove the undecidability of $\mathrm{UN}^=$ for two subclasses of linear systems: left-hand sides are linear, flat and right-hand sides are of depth at most two and conversely right-flat, right-linear, and depth two left-hand sides, which shows that our result is optimal as far as depth restrictions are involved and close to optimal as far as linearity and depth restrictions are concerned (the problem is undecidable for the linear, depth-two class [11]).

The structure of our decidability proof is as follows: in [13, 14] it was shown that $\mathrm{UN}^=$ for shallow systems can be reduced to $\mathrm{UN}^=$ for flat systems, (ii) checking $\mathrm{UN}^=$ for flat systems can be reduced to searching for equational proofs between terms drawn from a finite set of terms, and (iii) existence of equational proofs between terms in part (ii) is done thanks to the decidability of the word problem by Comon et al. [1].

Our strategy for part (ii) above, assuming a flat TRS, R, is to show that a sufficiently small witness to non-$UN^=$ for R exists if, and only if, any witness at all exists. To see this, say $\langle M, N\rangle$ is a minimal witness to non-$UN^=$ (in that the sum of the sizes of M and N is minimal). We show that we can replace certain subterms of M and N that are not equivalent to constants with variables, obtaining a witness $\langle M', N'\rangle$. If the heights of M' and N' are both strictly less than the maximum of $\{1, C\}$, where C is the number of constants in our rewrite system, then $\langle M', N'\rangle$ is sufficiently small. Otherwise, M' or N' must have a big subterm (i.e. a subterm whose height is greater than, or equal to, the number of constants), and this subterm is equivalent to a constant. However, in this case (when there is a constant that is equivalent to a big subterm of a component of a minimal witness), we can show that there is a small witness to non-$UN^=$. So, in all cases, we end up with a small witness.

Comparison with related work. Viewed at a very high level, the proof of decidability shows some similarity with other decidability proofs of properties of rewrite systems such as [2]. The basic insight seems to be that, just as in algebra the terms that equal 0 are crucial in a sense, so in rewriting are the terms that reduce to (or are equivalent to) constants. Of course, this observation is about as helpful in proofs of decidability as a compass is to someone lost in a maze. The details in both scenarios are vital and there are many twists and turns. The proof of undecidability shows some similarity with proofs in [12, 4].

1.1 Definitions

Terms. A *signature* is a set $\mathcal{F}$ along with a function *arity*: $\mathcal{F} \to \mathbb{N}$. Members of $\mathcal{F}$ are called *function symbols*, and *arity*(f) is called the *arity* of the function symbol f. Function symbols of arity zero are called *constants*. Let X be a countable set disjoint from $\mathcal{F}$ that we shall call the set of *variables*. The set $\mathcal{T}(\mathcal{F}, X)$ of *$\mathcal{F}$-terms over X* is defined to be the smallest set that contains X and has the property that $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, X)$ whenever

$f \in \mathcal{F}$, $n = arity(f)$, and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, X)$. The set of function symbols with arity n is denoted by $\mathcal{F}_n$; in particular, the set of constants is denoted by $\mathcal{F}_0$. We use $root(t)$ to refer to the outermost function symbol of t.

The *size*, $|t|$, of a term t is the number of occurrences of constants, variables and function symbols in t. So, $|t| = 1$ if t is a constant or a variable, and $|t| = 1 + \Sigma_{i=1}^n |t_i|$ if $t = f(t_1, \ldots, t_n)$ for $n > 0$. The *height* of a term t is 0 if t is a constant or a variable, and $1 + max\{height(t_1), \ldots, height(t_n)\}$ if $t = f(t_1, \ldots, t_n)$. If a term t has height zero or one, then it is called *flat*. A *position* of a term t is a sequence of natural numbers that is used to identify the locations of subterms of t. The subterm of $t = f(t_0, \ldots, t_{n-1})$ at position p, denoted $t|_p$, is defined recursively: $t|_\lambda = t$, $t|_k = t_k$, for $0 \leq k \leq n-1$, and $t|_{k.p} = (t|_k)|_p$. If $t = f(t_0, \ldots, t_{n-1})$, then we call $t_0, \ldots, t_{n-1}$ the *depth*-1 subterms of t. If all variables appearing in t are either t itself or depth-1 subterms of t, then we say that t is *shallow*. The notation $g[a]$ focuses on (any) one occurrence of subterm a of term g, and $s\{u \mapsto v\}$ denotes the term obtained from term s by replacing all occurrences of the subterm u in s by term v.

A *substitution* is a mapping $\sigma : X \to \mathcal{T}(\mathcal{F}, X)$ that is the identity on all but finitely many elements of X. Substitutions are generally extended to a homomorphism on $\mathcal{T}(\mathcal{F}, X)$ in the following way: if $t = f(t_1, \ldots, t_k)$, then (abusing notation) $\sigma(t) = f(\sigma(t_1), \ldots, \sigma(t_k))$. Oftentimes, the application of a substitution to a term is written in postfix notation. A *unifier* of two terms s and t is a substitution σ (if it exists) such that $s\sigma = t\sigma$. We assume familiarity with the concept of *most general unifier* [9], which is unique up to variable renaming and denoted by *mgu*.

Term Rewrite Systems. A *rewrite rule* is a pair of terms, (l, r), usually written $l \to r$. For the rule $l \to r$, the *left-hand side* is $l \notin X$, and the *right-hand side* is r. Notice that l cannot be a variable. A rule, $l \to r$, can be applied to a term, t, if there exists a substitution, σ, such that $l\sigma = t'$, where t' is a subterm of t; in this case, t is rewritten by replacing the subterm $t' = l\sigma$ with $r\sigma$. The process of replacing the subterm $l\sigma$ with $r\sigma$ is called a *rewrite*. A *root rewrite* is a rewrite where $t' = t$. A rule $l \to r$ is *flat* (resp. shallow) if both l and r are flat (resp. shallow). The rule $l \to r$ is *collapsing* if r is a variable. A *term rewrite system* (or *TRS*) is a pair, $(\mathcal{T}, R)$, where R is a finite set of rules and $\mathcal{T}$ is the set of terms over some signature. A TRS, R, is *flat* (resp. shallow) if all of the rules in R are flat (resp. shallow). If we think of $\to$ as a relation, then $\xrightarrow{+}$ and $\xrightarrow{*}$ denote its transitive closure, and reflexive and transitive closure, respectively. Also, $\leftrightarrow$, $\xleftrightarrow{+}$, and $\xleftrightarrow{*}$ denote the symmetric closure, symmetric and transitive closure, and symmetric, transitive, and reflexive closure, respectively. We put an 'r' over arrows to denote a root rewrite, i.e., $\xleftrightarrow{r}$.

A *derivation* is a sequence of terms, $t_1, \ldots, t_n$, such that $t_i \to t_{i+1}$ for $i = 1, \ldots, n-1$; this sequence is often denoted by $t_1 \to t_2 \to \ldots \to t_n$. A *proof* is a sequence, $t_1, \ldots, t_n$, such that $t_i \leftrightarrow t_{i+1}$ for $i = 1, \ldots, n-1$; this sequence is generally denoted by $t_1 \leftrightarrow t_2 \leftrightarrow \ldots \leftrightarrow t_n$. If R is a rewrite system, then a proof is *over* R if it can be constructed using rules in R. If π is a proof, we say that $\pi \in s \xleftrightarrow{*} t$ if π is of the form $s \leftrightarrow \ldots \leftrightarrow t$ (it is possible for the proof sequence to consist of a single term, in which case $s = t$ and the proof is simply a sequence with a single element, s). We say that $\pi \in s \xleftrightarrow{+} t$ if $\pi \in s \xleftrightarrow{*} t$ and the proof sequence contains at least two terms. We write $s \xleftrightarrow{*} t$ (resp. $s \xleftrightarrow{+} t$) to denote that there is a proof, π, with $\pi \in s \xleftrightarrow{*} t$ (resp. $\pi \in s \xleftrightarrow{+} t$).

A *normal form* is a term, $t \in \mathcal{T}(\mathcal{F}, X)$, such that no subterm of t can be rewritten. A term that is not a normal form, i.e., one with a subterm that *can* be rewritten, is called *reducible*. We denote the set of all normal forms for R by NF_R, or simply NF. A rewrite system R is $UN^=$ if it is *not* the case that R has two distinct normal forms, M and N, such that $M \xleftrightarrow{*} N$. If such a pair exists, then we say that the pair, $\langle M, N \rangle$, is a *witness* to non-$UN^=$. The *size of a witness*, denoted $|\langle M, N\rangle|$, is $|M| + |N|$. A *minimal witness* is a witness with minimal size. Finally, we define $SubMinWit_R$ to be set of all terms M' such that $\langle M, N\rangle$ is a minimal witness, and M' is a subterm of M.

2 Preliminary Results

We begin with a few simple results, whose proofs are omitted to save space, on when rules apply. They are used throughout the paper to show that normal forms are preserved under certain transformations. Before we begin, notice that it is relatively simpler to preserve normal forms when the relevant TRS is linear. For instance, imagine any *flat and linear* TRS such that $f(g(a), h(b))$ is a normal form. Since $g(a)$ is evidently a normal form, $f(g(a), g(a))$ would also be a normal form, when the TRS is linear. If the TRS is not linear, then there could be a rule of the form $f(x, x) \to t$, making $f(g(a), g(a))$ reducible. The results below handle such complications presented by non-linear rules.

▶ **Definition 1.** Let R be a rewrite system, and let $l \to r = \rho \in R$ be a rule. The *pattern of* ρ, denoted $Patt(\rho)$, is a set of equations $\{i = j \mid \; l|_i = l|_j, l|_i, l|_j \in X\}$.

▶ **Definition 2.** Let $t \in \mathcal{T}(\mathcal{F}, X)$ be a term with $root(l) = root(t)$. If $A = \{i_1, i_2, \dots, i_k\}$ is the set of positions that appear in equations in $Patt(\rho)$, then the *pattern of* t *with respect to* ρ, denoted $Patt_\rho(t)$, is the set $\{i_a = i_b \mid t|_{i_a} = t|_{i_b}, i_a, i_b \in A\}$.

Note that $Patt_\rho(t)$ is undefined if $root(l) \neq root(t)$.

▶ **Lemma 3.** *Let R be a flat TRS. Let $t \in \mathcal{T}(\mathcal{F}, X)$ be a term, and let $l \to r = \rho \in R$ be a rule. Then ρ can be applied to t at λ if, and only if, (i) $l|_i = t|_i$ whenever $l|_i$ is a constant, and (ii) $Patt_\rho(t)$ is defined and $Patt(\rho) \subseteq Patt_\rho(t)$.*

Consider the term $f(a, x, x, g(b))$. Let's assume that it is a normal form. We want to know if altering depth-1 subterms can make the term reducible. Clearly, replacing x with a constant could *potentially* make the term reducible, depending on the rules in the rule set. But what about replacing any of the depth-1 subterms with a normal form containing a fresh variable? Notice that such a replacement could not make condition (i) of the above lemma true if it had been false. But what if condition (i) is true and condition (ii) is false? Could replacing a depth-1 subterm, or even several depth-1 subterms, with terms containing fresh variables make condition (ii) true? This question is answered by the following proposition.

▶ **Proposition 4.** *Let R be a flat TRS, and let $M = f(s_1, \dots, s_m)$ be a normal form for R. Let $S = \{t_{i_1}, \dots, t_{i_n}\}$ be a set of normal forms, where $n \le m$ and each term contains at least one fresh variable (relative to M). Further, say that $t_{i_j} \neq t_{i_k}$ whenever $s_{i_j} \neq s_{i_k}$ for all $i_j, i_k \in \{i_1, \dots, i_n\}$. If M' is what one obtains from M by replacing each s_{i_j} with t_{i_j}, then $M' \in NF_R$.*

▶ **Lemma 5.** *If R is any TRS such that $f(t_1, \dots, t_m) \in SubMinWit_R$, then $t_i \overset{*}{\leftrightarrow}_R t_j$ is impossible for $t_i \neq t_j$. This is equivalent to saying that there is no term s that is equivalent to both t_i and t_j via R.*

2.1 Normal Forms Equivalent to Constants

Let E be a finite set of equations. Following the authors of [1], we extend E to $\widehat{E}$ by closing under the following inference rules:

1. $\dfrac{g = d,\; l = r}{d\sigma = r\sigma}$ if $l, g \notin X$ and $\sigma = mgu(l, g)$

2. $\dfrac{x = d,\, y = r}{d = r\{y \mapsto x\}}$ if $y \in X$ and $x \in \mathcal{F}_0 \cup X$

3. $\dfrac{g[a] = d,\, a = b}{g[b] = d}$ if $a, b \in \mathcal{F}_0$

Notice that if E is flat, then $\widehat{E}$ is flat, as well.

We can think of a rewrite system as a set of equations: if $s \to t$ is a rule in R, then $s \leftrightarrow t$ is its corresponding equation. We write E_R for the set of equations obtained in this way from a rewrite system R. Clearly, if s and t are terms in $\mathcal{T}(\mathcal{F}, X)$, then they are R-equivalent if and only if they are E_R equivalent. Also, from [1] we know that terms are E_R equivalent if, and only if, they are $\widehat{E_R}$-equivalent. In [1], the authors show that, if R is a shallow TRS and $s, t \in \mathcal{T}(\mathcal{F}, X)$, then there is a procedure that produces, for any proof, $\pi \in s \overset{*}{\leftrightarrow}_R t$, over R, a new proof, which is denoted by $\pi_{1rr} \in s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$, over $\widehat{E_R}$, such that there is at most one root rewrite step in π_{1rr}.

Consider the following example: $R = \{f(x,x) \to c, f(x,x) \to g(a,x), g(a,x) \to g(a,x), a \to h(b), b \to h(c)\}$. It is easy to check that $\widehat{E_R} = E_R \cup \{c \leftrightarrow g(a,x)\}$. We use $\widehat{E_R}$ to search for a minimal witness to non-$UN^=$ for R; in particular, we will use the fact that for every proof $s \overset{*}{\leftrightarrow}_R t$, there is a proof $s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$ with at most one root rewrite.

Clearly, c is an R-normal form, so if we are looking for a minimal witness to non-$UN^=$ for R, $\langle c, ?\rangle$ might be a good first guess. We know that $c \leftrightarrow_{\widehat{E_R}} f(x,x)$, so maybe $\langle c, f(u,v)\rangle$ is a minimal witness, for some normal forms u and v. This is not possible. First, notice that $f(x,x)$ appears on the LHS of a rule, so $f(t,t)$ cannot be a normal form, for arbitrary term t. Second, notice that if $f(t,t)$ is equivalent to another normal form, then we can assume it is of the form $f(u,v)$, because we have already "used up" our only root rewrite by using $c \leftrightarrow_{\widehat{E_R}} f(x,x)$. So, maybe we can plug some term, t, into x, and then rewrite one instance of it to a normal form u, and another instance of it to a normal form v, obtaining a minimal witness of the form $\langle c, f(u,v)\rangle$? This cannot be the case, because if $\langle c, f(u,v)\rangle$ is a minimal witness, then (by Lemma 5 and the fact that $u \overset{*}{\leftrightarrow} v$) $\langle u, v\rangle$ would violate the minimality of $\langle c, f(u,v)\rangle$. So, we should consider $c \leftrightarrow_{\widehat{E_R}} g(a,x)$ as *the* (one and only) rewrite step in our proof. We know that a is not a normal form, and must, therefore, be rewritten to one - $h(h(c))$. But what about x? Should we plug anything into it? Say we were to plug t into x, and then rewrite t to some normal form, u. This would be unnecessary, because non-linearity is not an issue here, and so we can leave x as it is. So, $\langle c, g(h(h(c)), x)\rangle$ is a minimal witness, and the relevant proof looks like: $c \leftrightarrow_{\widehat{E_R}} g(a,x) \leftrightarrow_{\widehat{E_R}} g(h(b),x) \leftrightarrow_{\widehat{E_R}} g(h(h(c)),x)$.

Now, here is the interesting part. Notice that we have *four* R-normal forms equivalent to constants, but only *three* constants in R, i.e, $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$, $h(c) \overset{*}{\leftrightarrow}_{\widehat{E_R}} b$, $h(h(c)) \overset{*}{\leftrightarrow}_{\widehat{E_R}} a$, and $g(h(h(c)),x) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$. From the Pigeonhole Principle, we can conclude that there must be some constant in R that is equivalent to two distinct normal forms (of course, we already knew this, but in general this technique will be useful). We generalize the lessons learned from this example in the following results.

▶ **Lemma 6.** *Let R be a flat TRS. Let $\langle M_0, M_1\rangle$ be a minimal witness to non-$UN^=$ for R, and say $M = f(t_1, \ldots, t_m)$ is a subterm of M_0. Let c be a constant, and let $c \overset{r}{\leftrightarrow}_{\widehat{E_R}} f(s_1, \ldots, s_m)$ $\overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t_1, \ldots, t_m) = M$ be a proof with a single root rewrite. If s_i is not a constant, then $height(t_i) = 0$.*

Proof. Let S_{const} be the set of positive integers, i, such that $s_i \in \mathcal{F}_0$. If none of the s_i's is a variable, then there is nothing to show; so, assume at least one of the s_i's is a variable. Now, let

$$s'_j = \begin{cases} s_j & \text{if } j \in S_{const} \\ x_{s_j} & \text{otherwise} \end{cases} \quad \text{and} \quad t'_j = \begin{cases} t_j & \text{if } j \in S_{const} \\ x_{s_j} & \text{otherwise} \end{cases}$$

where x_{s_j} is a fresh variable not appearing in M_0 or M_1, and $x_{s_i} = x_{s_j}$ if and only if $s_i = s_j$. We show that (i) $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m)$, (ii) $f(t'_1, \ldots, t'_m) \in NF_R$, and (iii) for $i \notin S_{const}$, $height(t_i) = 0$.

Part (i). If $j \notin S_{const}$, then $s'_j = t'_j = x_{s_j}$. So, say $j \in S_{const}$. In this case, $s'_j = s_j \overset{*}{\leftrightarrow}_{\widehat{E_R}} t_j = t'_j$. So, $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m)$. *Part (ii).* Let $j, j' \notin S_{const}$, and say $t_j \neq t_{j'}$. In order to apply Proposition 4, we need to show that $t'_j \neq t'_{j'}$. From Lemma 5, we know that $s_j \neq s_{j'}$, and hence $t'_j = x_{s_j} \neq x_{s_{j'}} = t'_{j'}$. Therefore, we can apply Proposition 4 to obtain that $f(t'_1, \ldots, t'_m) \in NF_R$. *Part (iii).* Notice that, by (i) and $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$, we have $f(t_1, \ldots, t_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m) = N$. Also, since N contains at least one fresh variable not appearing in M_0 or M_1, we know that $M \neq N$ and $C[N] \neq M_0$ or M_1, where $C[]$ is a context and $M_0 = C[M]$. Hence $\langle C[N], M_1 \rangle$ is a witness to non-$UN^=$, with $|C[N]| \leq |M_0|$. But $\langle M_0, M_1 \rangle$ is a minimal witness, so $|C[N]| = |C[M]|$ and $|N| = |M|$. Since $|t'_i| = 1$ for all $i \notin S_{const}$, it must be the case that $|t_i| = 1$. Thus, we have that $height(t_i) = height(t'_i) = 0$ for all $i \notin S_{const}$. ∎

▶ **Corollary 7.** *Under the same assumptions as Lemma 6 plus the assumption that at least one of the s_i's is a constant, there is a j such that $s_j \in \mathcal{F}_0$ and $height(t_j) = height(f(t_1, \ldots, t_m)) - 1$ with $1 \leq j \leq m$.*

Proof. Since $height(t_i) = 0$ whenever $s_i \notin \mathcal{F}_0$, we know that $height(t_i) \leq height(t_j)$ whenever $s_i \notin \mathcal{F}_0$ and $s_j \in \mathcal{F}_0$. So, amongst the direct subterms of $f(t_1, \ldots, t_m)$ with maximal height, there must be one, t_j, such that $s_j \in \mathcal{F}_0$. ∎

▶ **Proposition 8.** *Let R be a flat TRS, and let $c \in \mathcal{F}_0$. Let $\langle M, N \rangle$ be a minimal witness, and let N' be a subterm of N such that $height(N') = k$. Further, let $\pi \in c \overset{*}{\leftrightarrow} N'$ be a proof over R. Then we can find either (i) $1 + k$ distinct normal forms equivalent to constants, the normal forms having heights $0, 1, \ldots, k$, or (ii) a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that N_0 and N_1 are flat.*

Proof. We proceed by induction on $height(N')$. For the base case we assume that $height(N') = 0$. If the proof is trivial, i.e., if $c = N'$, then we have $1 = 1 + height(N')$ normal form (with height zero) equivalent to a constant. So, assume that π has at least one step.

We know that there is a proof, $\pi_{1rr} \in c \overset{+}{\leftrightarrow}_{\widehat{E_R}} N'$, such that there is only one root rewrite step in π_{1rr}. Since the first step in π_{1rr} is necessarily a root rewrite, π_{1rr} must have the form $c \overset{r}{\leftrightarrow} w\sigma = N'$, where the rule applied is $c \to w$ or $w \to c$, and $height(w) = 0$ (notice that if $c \overset{r}{\leftrightarrow} u \overset{*}{\leftrightarrow} N'$ for some term u with $height(u) > 0$, then we would need a second root rewrite to get back to N'). If $w \in X$, then $x \leftrightarrow c \leftrightarrow y$, where x, y are distinct variables. Therefore, $\langle x, y \rangle$ is a witness to non-$UN^=$ with x and y flat. If $w \in \mathcal{F}_0$, then we have found $1 = 1 + height(N')$ normal form (with height zero) equivalent to a constant.

For the inductive step, assume that $height(N') > 0$, and that the proposition holds for any height strictly less than $height(N')$. Now, π_{1rr} has the form

$$c \overset{r}{\leftrightarrow}_{\widehat{E_R}} f(t_1, \ldots, t_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(u_1, \ldots, u_m) = N'$$

and $t_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} u_i$ for $1 \leq i \leq m$. We have two cases: (i) there is an i such that $t_i \in \mathcal{F}_0$, and (ii) there is no such i. For (i), by Corollary 7, there exists an i such that t_i is a constant and

$height(u_i) = k - 1$. So, we can apply the inductive hypothesis to conclude that we have either (i) $1 + (1 + (height(N') - 1)) = 1 + height(N')$ distinct normal forms, with heights $0, 1, \ldots, height(N')$, equivalent to constants (the first $height(N') - 1$ normal forms come from the inductive hypothesis, and the final normal form is N' itself, which is equivalent to c), or (ii) a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that N_0 and N_1 are flat.

In case (ii), if $c \leftrightarrow_{\widehat{E_R}} f(s_1, \ldots, s_m)$ is the rule used for $c \leftrightarrow_{\widehat{E_R}} f(t_1, \ldots, t_m)$, then s_i is a variable for $1 \leq i \leq m$. We need to show that $f(s_1, \ldots, s_m) \in NF_R$. From Lemma 5, we know that $t_i \neq t_j$ whenever $u_i \neq u_j$ for $1 \leq i, j \leq m$. Since $t_i \neq t_j$ implies that $s_i \neq s_j$, we see that $s_i \neq s_j$ whenever $u_i \neq u_j$. We can assume that the variables $s_1, \ldots, s_m$ are fresh relative to $f(u_1, \ldots, u_m)$, and so we can replace u_i with s_i in $f(u_1, \ldots, u_m)$, obtaining $f(s_1, \ldots, s_m) \in NF_R$ by Proposition 4. Since $f(s_1, \ldots, s_m)$ is a normal form, we can replace the variables appearing in $f(s_1, \ldots, s_m)$ with fresh variables to produce a new normal form, $f(s'_1, \ldots, s'_m)$, such that $f(s_1, \ldots, s_m) \leftrightarrow_{\widehat{E_R}} c \leftrightarrow_{\widehat{E_R}} f(s'_1, \ldots, s'_m)$. So, $\langle f(s_1, \ldots, s_m), f(s'_1, \ldots, s'_m) \rangle$ is our witness with $f(s_1, \ldots, s_m)$ and $f(s'_1, \ldots, s'_m)$ flat. ∎

▶ **Corollary 9.** *Let R be a flat TRS, and let $c \in \mathcal{F}_0$. Let $\langle M, N \rangle$ be a minimal witness, and let N' be a subterm of N, with $height(N') \geq |\mathcal{F}_0|$. Further, let $\pi \in c \overset{*}{\leftrightarrow}_R N'$ be a proof over R. Then we can find either (i) a witness, $\langle M_0, M_1 \rangle$, to non-$UN^=$, such that M_0 and M_1 are flat, or (ii) a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that $height(N_0), height(N_1) \leq |\mathcal{F}_0|$.*

Proof. By Proposition 8, we know that we can find either (a) a witness, $\langle M_0, M_1 \rangle$, to non-$UN^=$, such that M_0 and M_1 are flat, or (b) $1 + height(N')$ distinct normal forms equivalent to constants. If (a) is the case, then we are done. So assume that (b) is true. Since there are $1 + height(N') > |\mathcal{F}_0|$ normal forms equivalent to, at most, $|\mathcal{F}_0|$ constants, we know, by the Pigeonhole Principle, that a single constant is equivalent to two distinct normal forms. From the above observation, we know that the normal forms have heights 0, 1, 2, …, $height(N')$. The smallest (height-wise) $1 + |\mathcal{F}_0|$ normal forms each have height no more than $|\mathcal{F}_0|$. So, we know that we can find a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that $height(N_0), height(N_1) \leq |\mathcal{F}_0|$. ∎

▶ **Proposition 10.** *Let R be a flat TRS. Then, either (i) there does not exist a constant $c \in \mathcal{F}_0$ and normal form $N \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N$ and $height(N) \geq |\mathcal{F}_0|$, or (ii) there exists a witness, $\langle N_0, N_1 \rangle$ to non-$UN^=$ for R such that $height(N_0), height(N_1) \leq k = max\{1, |\mathcal{F}_0|\}$. Further, there is an effective procedure to decide whether (i) or (ii) is the case.*

Proof. Consider all ground[1] normal forms over the signature of the rewrite system, i.e., consisting of constants and function symbols appearing in the finitely many rules of R, with height less than, or equal to, k; we use $NF_{\leq k}$ to denote this set. Notice that if there is a constant, $c \in \mathcal{F}_0$, and an element of $SubMinWit_R$, N, with $height(N) \geq |\mathcal{F}_0|$, such that $c \overset{*}{\leftrightarrow} N$, then by Corollary 9 there is a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$ for R with $height(N_0), height(N_1) \leq k$. By a result in [1], the word problem is decidable for flat systems. So, we can construct the set of all pairs, (s, t), such that $s, t \in NF_{\leq k}$ and $s \overset{*}{\leftrightarrow}_R t$. If we do not find a witness to non-$UN^=$ in $NF_{\leq k}$, then we know that there is no $c \in \mathcal{F}_0$ and $N \in SubMinWit_R$ such that $height(N) \geq |\mathcal{F}_0|$ and $c \overset{*}{\leftrightarrow}_R N$. Otherwise, we have found the witness $\langle N_0, N_1 \rangle$ with $height(N_0), height(N_1) \leq k$. ∎

[1] As in [13, 14], for nonlinear rewrite systems also we can expand the signature of the rewrite system with 3α new constants, where α is the maximum arity of a function symbol in the rules, and focus on ground normal forms.

2.2 Shrinking Witnesses

Say $\langle f(a, g(b, f(c, x))), h(y, y, h(a, b, c))\rangle$ is a witness to non-$UN^=$ for some TRS. Can we replace big subterms of a component of the witness, without changing the fact that it is a witness, i.e., if we replace $g(b, f(c, x))$ with a variable, z, will $\langle f(a, z), h(y, y, h(a, b, c))\rangle$ still be a witness? We show that we can replace depth-1 subterms that are *not* equivalent to a constant with a variable. This shrinks the size of the witness; in particular, only depth-1 subterms of such a shrunk witness that are equivalent to a constant can have height greater than, or equal to, the number of constants in the TRS. So, a shrunk minimal witness either has small components, or there is a large subterm of a component of a minimal witness that is equivalent to a constant. If the latter is the case, then we know, by Corollary 9, that there is a small witness.

▶ **Definition 11.** Let R be a rewrite system. For each term (up to renaming of variables), t, we can add a new variable $x_{\overline{t}}$ to X without altering the relation $\overset{*}{\leftrightarrow}$, where $x_{\overline{s}} = x_{\overline{t}}$ if, and only if, $s \overset{*}{\leftrightarrow}_R t$. Let $t = f(t_1, \ldots, t_n)$ be a term in $\mathcal{T}(\mathcal{F}, X)$. Then we define

$$\phi(t) = \begin{cases} x_{\overline{t}} & \text{if } t \text{ is not equivalent to a constant} \\ t & \text{otherwise} \end{cases}$$

Let $u = f(u_1, \ldots, u_m)$ for $m > 0$ and $v \in X$. We define the function α that maps terms to terms as follows: $\alpha(u) = f(\phi(u_1), \ldots, \phi(u_m))$ and $\alpha(v) = v$.

Notice that $\alpha(c) = c$ for $c \in \mathcal{F}_0$, since α only affects depth-1 subterms.

▶ **Lemma 12.** *Let R be a flat TRS, and let $u \leftrightarrow_R v$ be a proof over R, where $u \leftrightarrow_R v$ is not a root rewrite. Then, there is a proof $\alpha(u) \overset{*}{\leftrightarrow}_R \alpha(v)$.*

Proof. Say $u = f(u_1, \ldots, u_m)$ and $v = f(v_1, \ldots, v_m)$ (notice that if $u \leftrightarrow_R v$ is not a root rewrite, then neither u nor v can have height zero). Since the rewrite is not a root rewrite, we know that there are u_i and v_i such that $u_i \leftrightarrow_R v_i$, and $u_j = v_j$ for all $j \neq i$. If u_i, v_i are equivalent to a constant, then $\phi(u_i) = u_i$ and $\phi(v_i) = v_i$, and hence $\alpha(u) \leftrightarrow_R \alpha(v)$. If u_i, v_i are not equivalent to a constant, then $\phi(u_i) = x_{\overline{u_i}} = x_{\overline{v_i}} = \phi(v_i)$, and hence $\alpha(u) = \alpha(v)$. ∎

▶ **Lemma 13.** *Let R be a flat TRS, and let $u \leftrightarrow_R v$ be a proof over R, where $u \leftrightarrow_R v$ is a root rewrite. If the rewrite has the form $u = w\sigma \to x\sigma = v$ (i.e. it uses a collapsing rule $w \to x$), then $\alpha(u) \leftrightarrow_R \phi(v)$; otherwise $\alpha(u) \leftrightarrow_R \alpha(v)$.*

Proof. In case of a collapsing rule, any instantiations of x appearing as depth-1 subterms of u are equal to v, and so they are replaced by $\phi(v)$ in $\alpha(u)$. Since constants in w are never replaced, $\alpha(u) \leftrightarrow_R \phi(v)$. Otherwise, if s is a depth-1 subterm of u or v that is an instantiation of a shared variable, then every depth-1 instance of s is replaced by $\phi(s)$ in $\alpha(u)$ and $\alpha(v)$. So, $\alpha(u) \leftrightarrow_R \alpha(v)$. ∎

▶ **Proposition 14.** *Let R be a flat TRS. Let s and t be terms not equivalent to a constant and $\pi \in s \overset{*}{\leftrightarrow} t$ be a proof over R. Then, either there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$.*

Proof. We know that there is a proof, π_{1rr}, over $\widehat{E_R}$ with at most one root rewrite. If π_{1rr} has zero steps, then $\alpha(s) = \alpha(t)$, and so $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$. Assume that π_{1rr} has at least one step, and say that it has the form $s = s_0 \leftrightarrow_{\widehat{E_R}} \ldots \leftrightarrow_{\widehat{E_R}} s_k = t$ for some $k \geq 1$. We consider three cases: (i) π_{1rr} has no root rewrite; (ii) the only root rewrite in π_{1rr} uses a collapsing rule; and (iii) the only root rewrite in π_{1rr} does not use a collapsing rule.

In cases (i) and (iii), we know, by lemmas 12 and 13, that there is a proof $\alpha(s_i) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(s_{i+1})$ for $0 \leq i \leq k-1$. Therefore, there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$.

In case (ii), let $w\sigma = s_j \leftrightarrow_{\widehat{E_R}} s_{j+1} = x\sigma$ be the instance of the collapsing rule, $w \to x$, for some $0 \leq j \leq k-1$. For $i < j$, we know that there is a proof $\alpha(s_i) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(s_{i+1})$. By Lemma 13, we know that $\alpha(s_j) \leftrightarrow_{\widehat{E_R}} \phi(s_{j+1})$, and so there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \phi(s_{j+1})$. Since the terms in π_{1rr} cannot be equivalent to a constant (since s, t are not equivalent to a constant), we know that $\phi(s_{j+1}) = x_{\overline{s_{j+1}}}$, and so the proof is complete ∎

▶ **Remark 15.** *As mentioned above, for any term v not equivalent to a constant, $\phi(v)$ can be chosen so that it does not appear as a subterm of any finite number of terms. Therefore, $\phi(s_{j+1})$ can be chosen so that it does not appear as a subterm of $s_0, s_1, \ldots, s_k$.*

▶ **Proposition 16.** *Let R be a flat TRS, and let $\langle M, N\rangle$ be a minimal witness to non-$UN^=$ for R, with M, N not equivalent to a constant. Then either $\langle \alpha(M), y\rangle$ or $\langle \alpha(M), \alpha(N)\rangle$ is a witness for some variable, y.*

Proof. We know from Proposition 14 that either there is a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or there is a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(N)$. So, we need to show that (i) $\alpha(M)$, $\alpha(N)$, and y are normal forms, and that (ii) $\alpha(M) \neq y$ (whenever $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$) and $\alpha(M) \neq \alpha(N)$.

For (i), we need to show that if s and t are depth-1 subterms of M (or N) that are not equivalent to constants, then $\phi(s) \neq \phi(t)$ whenever $s \neq t$. So, say that $s \neq t$. If $s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$, then $\langle s, t\rangle$ would violate the minimality of $\langle M, N\rangle$, since $|s| + |t| < |M| \leq |M| + |N|$. So, we know that s and t are not equivalent, and hence $\phi(s) \neq \phi(t)$. We know by Proposition 4 that $\alpha(M)$ and $\alpha(N)$ are normal forms, because the variables replacing subterms of M and N can be chosen so that they are fresh. Since variables are always normal forms, we know that $\alpha(M)$, $\alpha(N)$, and y are normal forms.

For (ii), if M is not a variable, then $\alpha(M)$ is not a variable, and hence $\alpha(M) \neq y$. If M is a variable, then, by Remark 15, we can choose y so that it does not appear as a subterm of M. So, $\alpha(M) = M \neq y$.

To see that $\alpha(M) \neq \alpha(N)$, we need to consider two cases. If $root(M) \neq root(N)$, then clearly $\alpha(M) \neq \alpha(N)$, since α does not affect the outermost function symbol. If $root(M) = root(N)$, then it must be the case that $M|_i \neq N|_i$ for some integer, i. In order for $\alpha(M) = \alpha(N)$ to be true, $M|_i$ and $N|_i$ must be replaced by the same variable. But this only happens when $M|_i$ and $N|_i$ are equivalent, and if $M|_i$ and $N|_i$ were equivalent, then (setting $M' = M|_i$ and $N' = N|_i$) $\langle M', N'\rangle$ would be a witness with $|M'| < |M|$ and $|N'| < |N|$. This would violate the minimality of $\langle M, N\rangle$, so $M|_i$ and $N|_i$ cannot be equivalent, and hence $M|_i$ and $N|_i$ must be replaced by distinct variables. Therefore, $\alpha(M) \neq \alpha(N)$. ∎

3 Decidability for Flat and Shallow Rewrite Systems

▶ **Lemma 17.** *Let R be a flat TRS, and say that there is no constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$. Let $\langle M, N\rangle$ be a minimal witness to non-$UN^=$ for R. Then $height(\alpha(M)), height(\alpha(N)) \leq k = max\{1, |\mathcal{F}_0|\}$.*

Proof. We know that (i) all depth-1 subterms of $\alpha(M)$ and $\alpha(N)$ that are not equivalent to a constant are necessarily variables, and (ii) there is no constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$. Hence, the depth-1 subterms of $\alpha(M)$ and $\alpha(N)$ are either (i) variables or (ii) elements of $SubMinWit_R$ with height strictly less than $|\mathcal{F}_0|$. This means that the heights of $\alpha(M)$ and $\alpha(N)$ are at most $max\{1, |\mathcal{F}_0|\}$. ∎

▶ **Theorem 1.** *Let R be a flat TRS. If there is a witness to non-$UN^=$ for R, then there exists a witness, $\langle N_0, N_1 \rangle$, with $height(N_0), height(N_1) \leq k = max\{1, |\mathcal{F}_0|\}$. Hence $UN^=$ is decidable for R.*

Proof. By Proposition 10, we know that there is either (i) *no* constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$, or (ii) a witness, $\langle N_0, N_1 \rangle$ to non-$UN^=$ for R such that $height(N_0), height(N_1) \leq k$. Further, there is an effective procedure to decide if (i) or (ii) is the case.

If (ii) is the case, then we have our witness. So, assume that (i) is the case, and let $\langle M, N \rangle$ be a minimal witness to non-$UN^=$ for R. If M and N are equivalent to a constant, c, and $height(M)$, $height(N) < |\mathcal{F}_0|$, then we are done. So, we assume (without loss of generality) that M, N are not equivalent to a constant, and thus we can apply Proposition 14. Hence there is either a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(N)$. By Lemma 17, we know that $height(\alpha(M)), height(\alpha(N)) \leq k$. Hence, by Proposition 16, either $\langle \alpha(M), y \rangle$ or $\langle \alpha(M), \alpha(N) \rangle$ is a witness to non-$UN^=$ with $height(\alpha(M)), height(\alpha(N)), |y| \leq k$.

So, if there is a witness to non-$UN^=$ for R, then there is a witness, $\langle N_0, N_1 \rangle$, with $height(N_0)$, $height(N_1) \leq k$. The following algorithm, on input R, determines if R is $UN^=$: Enumerate all ground normal forms over the signature of the rewrite system, i.e., consisting of constants and function symbols appearing in the finitely many rules of R, with height less than, or equal to, k; say they are $N_0, \ldots, N_n$. In [1], the authors show that the word problem is decidable for shallow TRS. So, for $0 \leq i < j \leq n$, check if $N_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} N_j$. If $N_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} N_j$ for some $0 \leq i < j \leq n$, then R is not $UN^=$; otherwise, R is $UN^=$. ∎

Now that we have shown that $UN^=$ is decidable for flat rewrite systems, we extend this result to shallow rewrite systems. We do this by *flattening* a shallow rewrite system, i.e., transforming a shallow rewrite system into a flat one in a way that preserves $UN^=$.

▶ **Theorem 2.** *Let R be a shallow TRS. Then $UN^=$ is decidable for R.*

4 Undecidability of $UN^=$ for some Rewrite Systems

We show that $UN^=$ is undecidable for certain rewrite systems by showing that a decision procedure for $UN^=$ for these rewrite systems could be used to construct a decision procedure for the Post Correspondence Problem (PCP) [8]. As PCP is undecidable, so must $UN^=$ be for these rewrite systems. Note that, in this section, we sometimes use concatenation to denote the application of a unary function, i.e., $f(g(h(c)))$ could, for convenience, be denoted by $fgh(c)$. We consider rewrite systems with rules that have flat right-hand sides, and left-hand sides with height at most two. For each PCP instance, P, with tiles $\tau_1, \ldots, \tau_k$ (each tile is basically a pair of strings) and tile alphabet Γ, we construct a TRS, R_P, with flat right-hand sides, and left-hand sides with height at most two. Let T be the set of tiles, and say Γ_{bot} is the set of words appearing on the bottom of a tile, and Γ_{top} is the set of words appearing on the top of a tile in T. We construct the TRS as follows:

1. For each tile τ_i, $f(i(x), u(y), v(z)) \to f(x, y, z)$, where $u \in \Gamma_{top}$ is on the top of tile τ_i, and $v \in \Gamma_{bot}$ is on the bottom of τ_i.
2. For constants s and α, $f(s, s, s) \to \alpha$.
3. For each $a \in \Gamma$ and each tile τ_i, $f(i(x), a(y), a(y)) \to \beta$, where β is a constant.
4. $s \to s$, $f(x, y, z) \to f(x, y, z)$, $a(x) \to a(x)$, and $i(x) \to i(x)$ for every $a \in \Gamma$ and tile τ_i.

The rules from item (1) are used to construct the bulk of the proof. The rule from (3) allows you to reach the normal form β once a PCP instance has been constructed. Notice that α and β are the only two normal forms for R_P. The rules from item (4) do not play a non-trivial role in any proof—they exist simply to eliminate the possibility of there being more than two normal forms. Notice that $u, v \in \Gamma^+$ appear on the left-hand side of a rule. This means that the left-hand side can have height strictly greater than two. However, putting u, v on the left-hand side of a rule is just a convenience, as such a rule can be simulated by rules with flat right-hand sides, and left-hand sides with height at most two. For instance, let $u = \gamma_m \dots \gamma_1$ and $v = \delta_n \dots \delta_1$, for $\gamma_i, \delta_j \in \Gamma$ and $n \geq m$. In this case, the rule $f(i(x), u(y), v(z)) \to f(x, y, z)$ can be simulated by:

$$\begin{array}{rcl}
f(i(x), y, \delta_n(z)) & \to & f^{(n-1)}(x, y, z) \\
& \vdots & \\
f^{(m+1)}(x, y, \delta_{m+1}(z)) & \to & f^{(m)}(x, y, z) \\
f^{(m)}(x, \gamma_m(y), \delta_m(z)) & \to & f^{(m-1)}(x, y, z) \\
& \vdots & \\
f^{(2)}(x, \gamma_2(y), \delta_2(z)) & \to & f^{(1)}(x, y, z) \\
f^{(1)}(x, \gamma_1(y), \delta_1(z)) & \to & f(x, y, z)
\end{array}$$

We given an outline of the proof of correctness and omit details for lack of space.

▶ **Lemma 18.** *A minimal proof over R_P cannot contain a backward application of rule type 1 at the root position immediately followed by a forward application at the root position of rule type 1.*

▶ **Lemma 19.** *Let $\alpha \overset{*}{\leftrightarrow} \beta$ be a proof over R_P with minimal length. Then the proof must have the form $\alpha \leftrightarrow f(s, s, s) \overset{+}{\leftrightarrow} f(i(t'), a(t), a(t)) \leftrightarrow \beta$.*

▶ **Corollary 20.** *Let P be a PCP instance. If R_P is not* UN$^=$*, then there is a solution to the PCP instance.*

It is straight-forward to show that if there is a solution to P, then R_P violates UN$^=$. So, if P is an instance of PCP, then there is a solution to P if and only if R_P violates UN$^=$. Since PCP is undecidable, we have the following theorem.

▶ **Theorem 3.** *$UN^=$ is undecidable for TRS with rules that have flat, linear right-hand sides and left-hand sides with height at most two.*

A slight modification of the rules can produce another result. Consider the following rule set:

1. For each tile τ_i, $f(i(x), u(y), v(z)) \leftarrow f(x, y, z)$, where $u \in \Gamma_{top}$ is on the top of tile τ_i, and $v \in \Gamma_{bot}$ is on the bottom of τ_i.
2. For constants s and α, $f(s, s, s) \to \alpha$.
3. For each $a \in \Gamma$ and each tile τ_i, $f(i(x), a(y), a(y)) \leftarrow \beta$.
4. $\beta \to \gamma$, where γ is a constant.
5. $s \to s$, $f(x, y, z) \to f(x, y, z)$, $a(x) \to a(x)$, and $i(x) \to i(x)$ for every $a \in \Gamma$ and tile τ_i.

Notice that now α and γ are the only two normal forms. Let $\alpha \overset{*}{\leftrightarrow} \gamma$ be a proof over R_P with minimal length. Then the proof must have the form $\alpha \leftrightarrow f(s, s, s) \overset{+}{\leftrightarrow} f(i(t'), a(t), a(t)) \leftrightarrow \beta \leftrightarrow \gamma$. So, we have the following corollary.

▶ **Corollary 21.** *$UN^=$ is undecidable for TRS with rules that have linear, flat left-hand sides, and right-hand sides with height at most two.*

5 Conclusion

The $UN^=$ property of TRSs is shown to be decidable for the shallow class and undecidable for the class of TRSs in which one side of the rule is allowed to be at most depth-two and the other side is flat and linear. Among the fundamental properties of TRSs only the word problem and the $UN^=$ property are now known to be decidable for the shallow class. An important direction for future research is to give a complete classification of the basic properties for all 15 classes obtained by combinations of linearity and depth restrictions on variables in each side of TRSs (see also [11] in this regard).

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