Non-Commutative Infinitary Peano Arithmetic

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Abstract

Does there exist any sequent calculus such that it is a subclassical logic and it becomes classical logic when the exchange rules are added? The first contribution of this paper is answering this question for infinitary Peano arithmetic. This paper defines infinitary Peano arithmetic with non-commutative sequents, called non-commutative infinitary Peano arithmetic, so that the system becomes equivalent to Peano arithmetic with the omega-rule if the the exchange rule is added to this system. This system is unique among other non-commutative systems, since all the logical connectives have standard meaning and specifically the commutativity for conjunction and disjunction is derivable. This paper shows that the provability in non-commutative infinitary Peano arithmetic is equivalent to Heyting arithmetic with the recursive omega rule and the law of excluded middle for Sigma-0-1 formulas. Thus, non-commutative infinitary Peano arithmetic is shown to be a subclassical logic. The cut elimination theorem in this system is also proved. The second contribution of this paper is introducing infinitary Peano arithmetic having antecedent-grouping and no right exchange rules. The first contribution of this paper is achieved through this system. This system is obtained from the positive fragment of infinitary Peano arithmetic without the exchange rules by extending it from a positive fragment to a full system, preserving its 1-backtracking game semantics. This paper shows that this system is equivalent to both non-commutative infinitary Peano arithmetic, and Heyting arithmetic with the recursive omega rule and the Sigma-0-1 excluded middle.

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1 Introduction

Substructural logics, which are logical systems without some of the contraction rule, the weakening rule, and the exchange rule, have been actively studied in both mathematical logic and computer science. For example, linear logic, which is a logical system without the contraction rule or the weakening rule is successful [9].

Does there exist any sequent calculus such that it is a subclassical logic and it becomes classical logic when the exchange rules are added? The first contribution of this paper is answering this question for infinitary Peano arithmetic. This paper defines infinitary Peano arithmetic with non-commutative sequents, called non-commutative infinitary Peano arithmetic, so that the system becomes equivalent to Peano arithmetic with the omega-rules if the exchange rules are added to this system. This paper shows that the provability in non-commutative infinitary Peano arithmetic is equivalent to Heyting arithmetic with

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the recursive omega rules and the law of excluded middle for Σ^0_1 formulas. Thus, non-commutative infinitary Peano arithmetic is shown to be a subclassical logic.

Arithmetic without the exchange rule has not been studied yet. For infinitary arithmetic without the exchange rules, only its positive fragment was investigated in [4, 6, 2]. For a full system without the exchange rules, only the classical sequent calculus without the exchange rules is studied [13].

The second contribution of this paper is introducing infinitary Peano arithmetic having antecedent-grouping and no right exchange rules. The first contribution is achieved through this system. This system is obtained from the positive fragment of infinitary Peano arithmetic without the exchange rules by extending it from a positive fragment to a full system, preserving its 1-backtracking game semantics. This paper shows that this system is equivalent to both non-commutative infinitary Peano arithmetic, and Heyting arithmetic with the recursive omega rules and the Σ^0_1 excluded middle.

This paper will define non-commutative infinitary Peano arithmetic NCIPA as well as the arithmetic IPA^- having antecedent-grouping and no right exchange rules, and prove (1) NCIPA becomes equivalent to Peano arithmetic IPA with the ω-rules when the exchange rules are added to the system, (2) NCIPA is equivalent to Heyting arithmetic with the recursive ω-rules, called IHA, and the law EM_1 of excluded middle for Σ^0_1 formulas, (3) the cut elimination theorem in NCIPA, (4) the cut elimination theorem in IPA^-, and (5) translations between NCIPA and IPA^-.

IPA^- was inspired by 1-backtracking game semantics [5, 8]. [4] proved correspondence between its positive fragment and 1-backtracking game, by which a winning strategy corresponds to a proof. [2] also defined a sound and complete semantics for the fragment using interactive realizers.

IPA^- in this paper is a full system obtained from the positive fragment by adding implication. IPA^- is described by using a sequent $\Gamma \vdash \Delta$ with antecedent-grouping where formulas in the antecedent Γ are grouped and structural rules can be used only inside a group. We can also use the weakening rule and the contraction rule in the succedent Δ, but cannot use the exchange rule.

EM_1 is the principle $\forall x_1 \ldots x_n(A \vee \neg A)$ for a Σ^0_1 formula A. This principle gives logical systems between intuitionistic logic and classical logic, which have been studied actively, in particular, for hidden algorithms in their proofs [1, 3, 11] and for their relation with continuation-passing style programs [7].

We design NCIPA from IPA^- so that it is based on ordinary sequents without antecedent-grouping, and the grouping information is represented by the length of a sequence of formulas. The translations between NCIPA and IPA^- will be defined so that they map the length of a sequence of formulas and the grouping information into each other. The equivalence between NCIPA and $\mathrm{IHA} + \mathrm{EM}_1$ is proved from the translations between NCIPA and IPA^-, and the equivalence between IPA^- and $\mathrm{IHA} + \mathrm{EM}_1$.

The implication from provability in $\mathrm{IHA} + \mathrm{EM}_1$ to provability in IPA^- is proved by using the cut elimination theorem in IPA^-.

The implication from provability in IPA^- to provability in $\mathrm{IHA} + \mathrm{EM}_1$ is proved by using flag formulas. A flag formula is a Π^0_1 formula and is defined for each formula in the succedent when its proof is given. Given a proof of $\Gamma \vdash A_1, \ldots, A_n$, if the flag formula F_i of A_i is true, then every succedent in the proof is of length more than or equal to i. Flag formulas enable us to find the minimum length of the succedents in a proof even if the proof is infinite. The key idea is case analysis by a flag formula, which we can use since EM_1 proves $F_i \vee \neg F_i$.

The cut elimination theorem in NCIPA is proved by the translations between NCIPA and IPA^- and the cut elimination theorem in IPA^-.

A potential application of the equivalence results is program extraction with the halting problem oracle. Since NCIPA and IPA^- are equivalent to $\mathrm{IHA} + \mathrm{EM}_1$ and EM_1 corresponds to the halting problem oracle, we can extract a program with the halting problem oracle from a proof in NCIPA or IPA^-. This program can be interpreted as a learning algorithm, using 1-backtracking and learning in the limit [7].

Section 2 defines infinitary arithmetic IPA and IHA. Section 3 presents IPA^-, and its cut elimination theorem is proved in Section 4. Section 5 proves the implication from $IHA+EM_1$ to IPA^-, and Section 6 proves the other implication from IPA^- to $IHA + EM_1$. Section 7 defines NCIPA, and shows the equivalence between NCIPA with the exchange rules and IPA. Section 8 gives the translations between NCIPA and IPA^-, and shows the equivalence between NCIPA and $IHA + EM_1$. The cut elimination theorem in NCIPA is proved in Section 9.

2 Infinitary Arithmetic

We define the system IPA. It is Peano arithmetic based on infinitary logic where the inference rules $(\forall R)$ and $(\exists L)$ are replaced by the ω-rules with countably many assumptions, and it does not have induction rules. The induction principles are derivable.

▶ Definition 2.1 (language). The language of IPA is a first-order language generated from the following symbols: We have variables $x, y, z, \ldots$. Constants are numerals $0, 1, 2, \ldots$, denoted by $n, m, i, j, \ldots$.

Function symbols are denoted by $f, g, \ldots$. We assume that the set of function symbols is recursive, and the set of functions represented by function symbols is the same as the set of primitive recursive functions.

Terms are denoted by $s, t, \ldots$.

Predicate symbols are denoted by $P, Q, \ldots$. We assume that the set of predicate symbols is recursive, and the set of predicates represented by predicate symbols is the same as the set of primitive recursive predicates. We have 0-ary predicate symbols $\top$ and $\bot$, which mean the truth and the falsity respectively.

Formulas are defined by $A, B, C, \ldots ::= P(t_1, \ldots, t_n) | A \wedge B | A \vee B | A \to B | \forall x A | \exists x A$, where P is a predicate symbol of arity n. We will write $\neg A$ for $A \to \bot$.

A sentence is a closed formula. A sequence $A_1, \ldots, A_n$ $(n \geq 0)$ of sentences is denoted by $\Gamma, \Delta, \Pi, \Sigma, \ldots$. $|\Gamma|$ denotes its length. A^n denotes $A, \ldots, A$ (n times). $A[t/x]$ is the formula obtained from A by replacing x by t.

Sequents are of the form $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ $(n, m \geq 0)$ where A_i, B_i sentences. We respect order of sentences in a sequence and a sequent.

IPA is based on infinitary logic where assumptions can be countably many. A proof in this system is defined as a well-founded recursive tree by inference rules.

We have the following inference rules given in Figure 1. In the rules $(Ax\ R)$ and $(Ax\ L)$, true and false refer to the truth value in the standard model. The rules $(\forall R)$ and $(\exists L)$ denote an inference of its conclusion from some recursive function f such that $f(m)$ is the code of a proof of the m-th assumption, for example, $\Gamma \vdash \Delta, A[m/x]$ for $(\forall R)$.

We give an accurate definition of a proof inductively as follows: $\lceil \cdot \rceil$ is a standard coding function and $\lceil e \rceil$ is a code of a syntactical object e. (1) For an inference rule except $(\forall R)$ or $(\exists L)$, $(\lceil L \rceil, \lceil S \rceil, P_1, \ldots, P_n)$ is a proof of the sequent S if its name is L, its instance is the inference of S from $S_1 \ldots S_n$, and P_i is a proof of S_i for $1 \leq i \leq n$. (2) For an inference rule among $(\forall R)$ and $(\exists L)$, $(\lceil L \rceil, \lceil S \rceil, f)$ is a proof of the sequent S if its name is L and its instance is the inference of S from $S_1[n/x]$ (for all n) and f is a recursive function such that $f(n)$ is the code of a proof of $S_1[n/x]$.

We will write $\Gamma \vdash_{\mathrm{IPA}} \Delta$ to denote that the sequent $\Gamma \vdash \Delta$ is provable in IPA. We will also use $\vdash_T$ for some other systems T we will introduce later.

We define the system IHA. It is Heyting arithmetic based on infinitary logic where the inference rules $(\forall R)$ and $(\exists L)$ are replaced by the recursive ω-rules.

The language is the same as that of IPA except that its sequents are intuitionistic sequents $A_1, \ldots, A_n \vdash B$ or $A_1, \ldots, A_n \vdash$.

$$\frac{}{\Gamma, A \vdash \Delta}\ (Ax\ L)\ (A \quad \text{a false atomic formula})$$

$$\frac{}{\Gamma \vdash \Delta, A}\ (Ax\ R)\ (A \quad \text{a true atomic formula})$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge R) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L1) \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L2)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee R1) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee R2) \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L)$$

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to R) \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Sigma}{\Gamma, A \to B \vdash \Delta, \Sigma}\ (\to L)$$

$$\frac{\Gamma \vdash \Delta, A[m/x] \quad (\text{for all } m)}{\Gamma \vdash \Delta, \forall x A}\ (\forall R) \qquad \frac{\Gamma, A[m/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ (\forall L) \qquad \frac{\Gamma \vdash \Delta, A[m/x]}{\Gamma \vdash \Delta, \exists x A}\ (\exists R)$$

$$\frac{\Gamma, A[m/x] \vdash \Delta \quad (\text{for all } m)}{\Gamma, \exists x A \vdash \Delta}\ (\exists L) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ (weak\ R) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (weak\ L)$$

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (cont\ R) \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (cont\ L)$$

$$\frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ (exch\ R) \qquad \frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ (exch\ L)$$

Figure 1 Inference Rules of IPA

The inference rules are the same as those of IPA except that their sequents are restricted to intuitionistic sequents.

The law EM_1 of excluded middle for Σ^0_1 formulas is defined as the axiom schema $\forall x_1 \ldots x_n (A \vee \neg A)$ for a Σ^0_1 formula A. This is a weaker version of the law of excluded middle. The system $\mathrm{IHA} + \mathrm{EM}_1$ strictly includes IHA and is strictly included in IPA.

Note that the identity rule $\Gamma, A \vdash A$ is provable. It is shown by induction on A.

3 The system IPA$^-$

We define the logical system IPA$^-$ of Peano arithmetic having the recursive ω-rules, antecedent-grouping, and no right exchange rules.

The language of IPA$^-$ is the same as that of IPA except that its sequents are different. A sequent in IPA$^-$ is of the form $\Gamma \vdash A_1, \ldots, A_n$ where Γ is a sequence of sentences and n symbols of the symbol $-$. An example of the sequent is $A_1, -, A_2, A_3, -, A_4, -, A_5, A_6 \vdash B_1, B_2, B_3$.

In the sequent $\Gamma_0, -, \Gamma_1, -, \Gamma_2, \ldots, -, \Gamma_n \vdash A_1, \ldots, A_n$ where Γ_i is a sequence of sentences and does not contain the symbol $-$, the group Γ_0 means an initial group, and the group Γ_i corresponds to A_i.

$\Gamma, \Delta, \ldots$ denote a sequence of both sentences and symbols $-$. We will write $-^n$ for $-, \ldots, -$ (n times).

We have the following inference rules given in Figure 2

A proof in this system is defined as a well-founded recursive tree in a similar way to IPA. A proof of a formula A means a proof of the sequent $- \vdash A$.

Intuitive meaning of provable sequents is given by using the familiar interpretation of

$$\frac{}{\Gamma, A \vdash \Delta}\ (Ax\ L)\quad (A \quad \text{a false atomic formula})$$

$$\frac{}{\Gamma \vdash \Delta, A}\ (Ax\ R)\quad (A \quad \text{a true atomic formula})$$

$$\frac{\Gamma, - \vdash \Delta, A \wedge B, A \quad \Gamma, - \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge R)$$

$$\frac{\Gamma_1, A \wedge B, \Gamma_2, A \vdash \Delta}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta}\ (\wedge L1) \qquad \frac{\Gamma_1, A \wedge B, \Gamma_2, B \vdash \Delta}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta}\ (\wedge L2)$$

$$\frac{\Gamma, - \vdash \Delta, A \vee B, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee R1) \qquad \frac{\Gamma, - \vdash \Delta, A \vee B, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee R2)$$

$$\frac{\Gamma_1, A \vee B, \Gamma_2, A \vdash \Delta \quad \Gamma_1, A \vee B, \Gamma_2, B \vdash \Delta}{\Gamma_1, A \vee B, \Gamma_2 \vdash \Delta}\ (\vee L)$$

$$\frac{\Gamma, A \vdash \Delta, A \to B}{\Gamma \vdash \Delta, A \to B}\ (\to R1) \qquad \frac{\Gamma, - \vdash \Delta, A \to B, B}{\Gamma \vdash \Delta, A \to B}\ (\to R2)$$

$$\frac{\Gamma_1, A \to B, \Gamma_2, - \vdash \Delta, A \quad \Gamma_1, A \to B, \Gamma_2, B \vdash \Delta}{\Gamma_1, A \to B, \Gamma_2 \vdash \Delta}\ (\to L)$$

$$\frac{\Gamma, - \vdash \Delta, \forall x A, A[m/x] \quad (\text{for all } m)}{\Gamma \vdash \Delta, \forall x A}\ (\forall R) \qquad \frac{\Gamma_1, \forall x A, \Gamma_2, A[m/x] \vdash \Delta}{\Gamma_1, \forall x A, \Gamma_2 \vdash \Delta}\ (\forall L)$$

$$\frac{\Gamma, - \vdash \Delta, \exists x A, A[m/x]}{\Gamma \vdash \Delta, \exists x A}\ (\exists R) \qquad \frac{\Gamma_1, \exists x A, \Gamma_2, A[m/x] \vdash \Delta \quad (\text{for all } m)}{\Gamma_1, \exists x A, \Gamma_2 \vdash \Delta}\ (\exists L)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, - \vdash \Delta, A}\ (weak\ R) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (weak\ L)$$

Figure 2 Inference Rules of IPA$^-$

a sequent in the sequent calculus LK in the standard model of numbers as follows: If $\Gamma_0, -, \Gamma_1, \ldots, -, \Gamma_n \vdash A_1, \ldots, A_n$ is provable, then (1) $\Gamma_0 \vdash$ is true, or (2) $\Gamma_0, \Gamma_1, \ldots, \Gamma_i \vdash A_i$ is true for some i. Each inference rule is sound by this interpretation. Theorem 6.1 will provide more information.

If $\Gamma_1, -, \Pi, -, \Gamma_2 \vdash \Delta$ is provable, then $\Gamma_1, -, \Pi', -, \Gamma_2 \vdash \Delta$ is provable where Π does not contain $-$ symbols and Π' is obtained from Π by exchange, weakening, and contraction. This will be shown in Proposition 3.5. On the other hand, we cannot use right exchange, nor left exchange over formulas in different groups.

We explain this system with some examples. In the examples, we assume the identity lemma $\Gamma_1, A, \Gamma_2 \vdash \Delta, A$, which will be shown in Lemma 3.4 after the examples.

▶ Example 3.1. The first example is given in Figure 3, which shows the conjunction of IPA$^-$ is commutative.

▶ Example 3.2. The next example shows how this system respects the order of formulas. We have three provable sequents

$$-, A, -, B \vdash A, \bot,$$
$$-, A, -, B \vdash \bot, A,$$
$$-, A, -, B \vdash \bot, B.$$

$$\dfrac{\dfrac{}{-, A\wedge B, -, B \vdash B\wedge A, B}(Id)}{-, A\wedge B, - \vdash B\wedge A, B}(\wedge L2) \quad \dfrac{\dfrac{}{-, A\wedge B, -, A \vdash B\wedge A, A}(Id)}{-, A\wedge B, - \vdash B\wedge A, A}(\wedge L1)$$
$$\dfrac{\qquad\qquad\qquad\qquad}{-, A\wedge B \vdash B\wedge A}(\wedge R)$$

Figure 3 Example 3.1

On the other hand the sequent

$$-, A, -, B \vdash B, \bot$$

is not provable. The first sequent is provable since the initial and the first groups give the assumption A, which proves the first formula A. The second sequent is provable since the initial, the first, and the second groups give the assumptions A, B, which prove the second formula A. The third sequent is provable similarly to the second sequent, since the initial, the first, and the second groups give the assumptions A, B, which prove the second formula B. Formally the first sequent is proved by

$$\dfrac{\dfrac{\dfrac{}{-, A \vdash A}(Id)}{-, A, - \vdash A, \bot}(weak\ R)}{-, A, -, B \vdash A, \bot}(weak\ L)$$

and the second and the third sequents are proved by (Id).

On the other hand, the fourth sequent is not provable, since we have neither of the following cases: (1) the initial group is empty, which proves the contradiction, nor (2) the initial and the first groups give the assumption A, which proves the first formula B, nor (3) the initial, the first, and the second groups give the assumptions A, B, which prove the second formula $\bot$.

▶ Example 3.3. Suppose P be a predicate symbol. Let $A(x) = \exists y P(x, y)$. The following is a proof of an instance $\forall x(A(x) \vee \neg A(x))$ of EM_1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \pi_m\\ -, -, P(n,m) \vdash A(n)\vee\neg A(n), \neg A(n) \quad (\text{for all } m)\end{matrix}}{-, -, A(n) \vdash A(n)\vee\neg A(n), \neg A(n)}}{-, - \vdash A(n)\vee\neg A(n), \neg A(n)}}{- \vdash A(n)\vee\neg A(n) \qquad (\text{for all } n)}}{- \vdash \forall x(A(x)\vee\neg A(x))}$$

If $P(n, m)$ is false, the proof π_m is the axiom $(Ax\ L)$. If $P(n, m)$ is true, the proof π_m is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-, -, - \vdash A(n)\vee\neg A(n), A(n), P(n,m)}(Ax\ R)}{-, - \vdash A(n)\vee\neg A(n), A(n)}}{- \vdash A(n)\vee\neg A(n)}}{-, - \vdash A(n)\vee\neg A(n), \neg A(n)}(weak\ R)}{-, -, P(n,m) \vdash A(n)\vee\neg A(n), \neg A(n)}(weak\ L)$$

Remark that $A \vee \neg A$ is not provable for a Π^0_2 formula because this system does not have exchange rules.

We will explain game theoretic semantics in a general way first. Backtracking is a feature we may add to any formal game G between two players, E (Eloise) and A (Abelard), defining

$$\dfrac{\dfrac{}{\Gamma_1, B \to C, \Gamma_2, B, - \vdash \Delta, B \to C, B}\ (Id) \quad \dfrac{\dfrac{}{\Gamma_1, B \to C, \Gamma_2, B, C, - \vdash \Delta, B \to C, C}\ (Id)}{\Gamma_1, B \to C, \Gamma_2, B, C \vdash \Delta, B \to C}\ (\to R2)}{\dfrac{\Gamma_1, B \to C, \Gamma_2, B \vdash \Delta, B \to C}{\Gamma_1, B \to C, \Gamma_2 \vdash \Delta, B \to C}\ (\to R1)}\ (\to L)$$

Figure 4 Proof of Lemma 3.4

a new game bck(G). Informally, for a while a play on bck(G) runs like a play in G. However, in addition to the moves of G, Player E (Eloise) can make a new kind of move, called backtracking. We imagine that each new position of the play p is added to some stack. Using backtracking, E can move back to some previous position p of the play, provided p is recorded in the stack, erase all positions of the stack coming after p in the stack, and eventually perform an ordinary move from p (this new move is added to the stack).

In some cases, E has no recursive winning strategies over G, but some recursive winning strategies if we allow her to backtrack (i.e. E has some recursive winning strategy over bck(G)). "Backtracking" allows E to win more games using recursive winning strategies. The intuitive reason is that, in bck(G), E is not forced to provide a winning move at once, but she can find this winning move after several attempts, by trial-and-error.

Backtracking defines a new method for unwinding proofs. Assume that A is any implication-free arithmetical formula, and call TA the Tarski game for A. Then E has a recursive winning strategy over TA if and only if A is intuitionistically provable. In contrast, E has recursive winning strategy over bck(TA) if and only if A has a classical proof using only EM_1.

Assume the players play A_i at i-the stage. Then the stack of the plays is represented by $A_1, A_2, \ldots, A_n$. This stack can be simulated by the sequent $A_1, A_2, \ldots, A_n$, if the sequent calculus does not have the exchange rules and it respects the order of formulas. The weakening rules give backtracking, since it changes $A_1, A_2, \ldots, A_n, B$ to $A_1, A_2, \ldots, A_n$. Following this idea, for the positive fragment of IPA^-, [4] showed that it has 1-backtracking game semantics, and a proof in the system corresponds to a winning strategy in the game. Kobayashi [10] and the authors of this paper are preparing to show that IPA^- has a nice game theoretic semantics with 1-backtracking.

We will present several properties of IPA^-.

▶ Lemma 3.4. The following is derivable.

$$\dfrac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta, A}\ (Id)$$

This lemma is shown by induction on A in a standard way. We explain only the implication case $A = B \to C$, which is proved in Figure 4.

$\sharp_-\Gamma$ denotes the number of the symbol $-$ in Γ. $(\Gamma)_0$ is defined to be Γ if Γ does not contain $-$. $(\Gamma, -, \Pi)_0$ is defined to be Γ if Γ does not contain $-$.

We will show some structural rules are admissible in this system. In the antecedent, we can use weakening by (*weak L*2), and contraction by (*cont L*). In the same group in the antecedent, we can also use exchange by (*exch L*).

▶ Proposition 3.5. (1) The following are derivable.

$$\dfrac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ (Ax\ L2) \quad (A \text{ a false atomic formula})$$

$$\dfrac{}{\Gamma \vdash \Delta_1, A, \Delta_2}\ (Ax\ R2) \quad (A \text{ a true atomic formula})$$

(2) The following are admissible.

$$\frac{\Gamma_1, \top, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\ (\top E) \qquad \frac{\Gamma_1, -, \Gamma_2, -, \Gamma_3 \vdash \Delta_1, A, A, \Delta_2}{\Gamma_1, -, \Gamma_2, \Gamma_3 \vdash \Delta_1, A, \Delta_2}\ (cont\ R)\ (\sharp_-\Gamma_3 = |\Delta_2|, \sharp_-\Gamma_2 = 0)$$

$$\frac{\Gamma_1, -, \Gamma_2 \vdash \Delta_1, \bot, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\bot E)\ (\sharp_-\Gamma_2 = |\Delta_2|) \qquad \frac{\Gamma_1, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ (weak\ L2)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{\Gamma_1, -, \Gamma_2 \vdash \Delta_1, A, \Delta_2}\ (weak\ R2)\ (\sharp_-\Gamma_2 = |\Delta_2|, (\Gamma_2)_0 = \phi)$$

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Sigma, \Delta}\ (weak\ R3)\ (\sharp_-\Pi = |\Sigma|)$$

$$\frac{\Gamma_1, -, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, -, \Gamma_2 \vdash \Delta}\ (move) \qquad \frac{\Gamma_1, A, \Gamma_2, A, \Gamma_3 \vdash \Delta}{\Gamma_1, A, \Gamma_2, \Gamma_3 \vdash \Delta}\ (cont\ L) \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (exch\ L)$$

4 Cut Elimination in IPA$^-$

We will show the cut elimination theorem for IPA$^-$.

▶ Definition 4.1. We define the cut rule:

$$\frac{\Gamma_1, - \vdash \Gamma_2, A \quad \Delta_1, A, \Sigma_1 \vdash \Delta_2, \Sigma_2}{\Gamma_1, \Delta_1, \Sigma_1 \vdash \Gamma_2, \Delta_2, \Sigma_2}\ (cut)$$

where $\sharp_-\Sigma_1 = |\Sigma_2|$.

We have the cut elimination theorem in IPA$^-$.

▶ Theorem 4.2 (Cut Elimination). If $\Gamma \vdash \Delta$ is provable in IPA$^-$ with the cut rule, then it is provable in IPA$^-$.

In order to prove this theorem, we use the following rule:

$$\frac{\Gamma_1, - \vdash \Gamma_2, A \quad \Gamma_1, A, \Delta_1 \vdash \Gamma_2, \Delta_2}{\Gamma_1, \Delta_1 \vdash \Gamma_2, \Delta_2}\ (cut2)$$

In the next lemma we will prove the rule $(cut2)$ can be eliminated.

▶ Lemma 4.3. (1) If we have a proof

$$\begin{array}{c} \vdots\ \pi_1 \\ \Gamma_1, -, \Pi_1 \vdash \Gamma_2, A, \Pi_2 \end{array}$$

in IPA$^-$ where $\sharp_-\Gamma_1 = |\Gamma_2|$, and the proof π_2

$$\frac{\begin{array}{c}\vdots \\ \Gamma_1, A, \Delta_1^i \vdash \Gamma_2, \Delta_2^i \quad (i \in I)\end{array}}{\Gamma_1, A, \Delta_1 \vdash \Gamma_2, \Delta_2}\ (Rule)$$

in IPA$^-$ where $(Rule)$ is a logical rule which introduces the formula A, and we have proofs

$$\begin{array}{c} \vdots\ \pi_3^i \\ \Gamma_1, \Delta_1^i \vdash \Gamma_2, \Delta_2^i \end{array}$$

for $i \in I$ in IPA$^-$, then $\Gamma_1, \Delta_1, \Pi_1 \vdash \Gamma_2, \Delta_2, \Pi_2$ is provable in IPA$^-$.

(2) If we have a proof

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma_1, - \vdash \Gamma_2, A\end{array} \quad \begin{array}{c}\vdots\ \pi_2\\ \Gamma_1, A, \Delta_1 \vdash \Gamma_2, \Delta_2\end{array}}{\Gamma_1, \Delta_1 \vdash \Gamma_2, \Delta_2}\ (cut2)$$

in IPA^- with the rule ($cut2$) and the subproofs π_1 and π_2 do not contain the rule ($cut2$), then the conclusion $\Gamma_1, \Delta_1 \vdash \Gamma_2, \Delta_2$ is provable in IPA^-.

$|\pi|$ denotes the height of the proof π. We can prove (1) and (2) simultaneously by induction on $(A, |\pi_1| + |\pi_2|)$. For each step, we will first show (1) and use (1) to show (2).

5 From IHA + EM$_1$ to IPA$^-$

This section proves the implication from $\mathrm{IHA}+\mathrm{EM}_1$ to IPA^-. We will use the cut elimination theorem for the proof.

▶ Proposition 5.1. (1) If $\Gamma \vdash \Delta$ is provable in IHA, then $-^{|\Pi|+|\Delta|}, \Gamma \vdash \Pi, \Delta$ is provable in IPA^- for any Π.

(2) If $\Gamma \vdash A$ is provable in IHA, then $-, \Gamma \vdash A$ is provable in IPA^-.

The proof idea is simulating each inference rule of IHA by inference rules of IPA^-. One difference is that a logical rule in IPA^- has a redundant principal formula. For example, the right conjunction rule in IPA^- is

$$\dfrac{\Gamma, - \vdash \Delta, A \wedge B, A \quad \Gamma, - \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge R)$$

and on the other hand the right conjunction rule in IHA is

$$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge R)$$

This difference is covered by putting $A \wedge B$ by ($weak\ R2$) in Proposition 3.5. The other difference is the existence of $-$, which is handled by moving $-$ by ($move$) in Proposition 3.5.

▶ Theorem 5.2. If $\Gamma, \mathrm{EM}_1 \vdash A$ is provable in IHA, then $-, \Gamma \vdash A$ is provable in IPA^-.

Proof. By Proposition 5.1 (2), $-, \Gamma, \mathrm{EM}_1 \vdash A$ is provable in IPA^-.

We can show $- \vdash \mathrm{EM}_1$ in IPA^- in a similar way to Example 3.3. By (cut) we have $-, \Gamma \vdash A$ in IPA^- with (cut). By Theorem 4.2, we have $-, \Gamma \vdash A$ in IPA^-. ◻

6 From IPA$^-$ to IHA + EM$_1$

This section proves the direction from IPA^- to $\mathrm{IHA} + \mathrm{EM}_1$.

In order to discuss proofs in infinitary logic, we will have to formalize proofs in codes and discuss some recursive functions from proofs to proofs. However, for space limitation, we will not describe those codes in details.

Since it is well known that the cut elimination theorem holds for IHA [12], we will use the following cut rule in IHA:

$$\dfrac{\Gamma \vdash A \quad \Pi, A \vdash \Sigma}{\Gamma, \Pi \vdash \Sigma}\ (cut)$$

In our proof, we will use the idea of flag formulas. A flag formula F_i is a Π^0_1 formula and is assigned to each formula A_i in a sequent $\Gamma \vdash A_1, \ldots, A_n$ when its proof is given. Given a proof of $\Gamma \vdash A_1, \ldots, A_n$, if F_i is true, then the proof does not include any sequent with its succedent of length less than i.

In order to prove the theorem, we need the minimum length of the succedents in the sequents in a given proof. Even if a proof is given, we cannot effectively find the minimum length since a proof may be infinite. The idea is using a set of Π^0_1 formulas to describe the minimum length. When $F_1, \ldots, F_m$ are true and $F_{m+1}, \ldots, F_n$ are false, we know the minimum length is m. The point is that we can effectively assign these Π^0_1 formulas even for an infinite proof.

For example, for the proof

$$\dfrac{\dfrac{\dfrac{}{-, A \vdash A}\ (Ax\ R)}{-, A, - \vdash A, \bot}\ (weak\ R)}{-, A, -, B \vdash A, \bot}\ (weak\ L)$$

where A and B are true atomic formulas. Let F_1 and F_2 be the flag formulas for A and $\bot$ respectively. We will define F_1 to be true and F_2 to be false, which means the minimum length is 1.

We will explain how to define flag formulas by example. The first example is a proof ending with the rule $(\wedge R)$:

$$\dfrac{\begin{matrix}\vdots\ \pi_1 \\ \Gamma, - \vdash \Delta, A \wedge B, A\end{matrix} \quad \begin{matrix}\vdots\ \pi_2 \\ \Gamma, - \vdash \Delta, A \wedge B, B\end{matrix}}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge R)$$

Let the minimum length be m and the minimum length for π_i be m_i for $i = 1, 2$. Let the flag formulas for the proof be $F_1, \ldots, F_n$, and the flag formulas for π_i be $F^i_1, \ldots, F^i_n$ for $i = 1, 2$. We can calculate m by $m = \min(m_1, m_2)$, but we do it by using flag formulas instead. We define $F_j = F^1_j \wedge F^2_j$ for $j = 1, \ldots, n$.

The second example is a proof ending with the rule $(\forall R)$:

$$\dfrac{\begin{matrix}\vdots\ \pi_k \\ \Gamma, - \vdash \Delta, \forall x A, A[k/x] \quad (\text{for all } k)\end{matrix}}{\Gamma \vdash \Delta, \forall x A}\ (\forall R)$$

Let the minimum length be m and the minimum length for π_k be m_k. Let the flag formulas for the proof be $F_1, \ldots, F_n$, and the flag formulas for π_k be $F^k_1, \ldots, F^k_n$. We cannot effectively calculate $m = \min_k(m_k)$, but we can do it by flag formulas. We define $F_j = \forall x F^x_j$ for $j = 1, \ldots, n$, which informally means the infinite conjunction $F^1_j \wedge F^2_j \wedge \ldots$. Note that F_j is also a Π^0_1 formula when F^k_j $(k = 0, 1, 2, \ldots)$ are Π^0_1.

▶ Theorem 6.1. There exists a recursive function such that if IPA^- proves the sequent $\Gamma_0, -, \Gamma_1, -, \Gamma_2, \ldots, -, \Gamma_n \vdash A_1, A_2, \ldots, A_n$ where $n \geq 1$ and Γ_i does not contain any $-$ symbol, then the function computes the codes of Π^0_1 formulas $F_1, \ldots, F_n$ and the codes of proofs of the following in $\mathrm{IHA} + \mathrm{EM}_1$ from the code of the proof of the sequent $\Gamma_0, -, \Gamma_1, -, \Gamma_2, \ldots, -, \Gamma_n \vdash A_1, A_2, \ldots, A_n$:

(1) $\neg F_1, \Gamma_0 \vdash$,
(2) $F_i, \neg F_{i+1}, \Gamma_0, \ldots, \Gamma_i \vdash A_i$ $(1 \leq i < n)$,
(3) $F_n, \Gamma_0, \ldots, \Gamma_n \vdash A_n$.

▶ Theorem 6.2. If IPA^- proves $\Gamma_0, -, \Gamma_1 \vdash A$, then $\mathrm{IHA} + \mathrm{EM}_1$ proves $\Gamma_0, \Gamma_1 \vdash A$.

Proof. By Theorem 6.1 with $n = 1$, there exists the Π^0_1 formula F_1 such that $\mathrm{IHA} + \mathrm{EM}_1$ proves (1) $\neg F_1, \Gamma_0 \vdash$, and (2) $F_1, \Gamma_0, \Gamma_1 \vdash A$. By weakening and $(\vee L)$, we get $\neg F_1 \vee F_1, \Gamma_0, \Gamma_1 \vdash A$. We have $\mathrm{EM}_1 \vdash \neg F_1 \vee F_1$. Hence, by the cut rule, we have the claim. ◻

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ (Ax\ L)\ (A \text{ a false atomic formula})$$

$$\frac{}{\Gamma \vdash \Delta_1, A, \Delta_2}\ (Ax\ R)\ (A \text{ a true atomic formula})$$

$$\frac{\Gamma, \top \vdash \Delta, A \wedge B, A \quad \Gamma, \top \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B}\ (\wedge R)$$

$$\frac{\Gamma_1, A \wedge B, \Gamma_2, A \vdash \Delta, D, D}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta, D}\ (\wedge L1) \qquad \frac{\Gamma_1, A \wedge B, \Gamma_2, B \vdash \Delta, D, D}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta, D}\ (\wedge L2)$$

$$\frac{\Gamma, \top \vdash \Delta, A \vee B, A}{\Gamma \vdash \Delta, A \vee B}\ (\vee R1) \qquad \frac{\Gamma, \top \vdash \Delta, A \vee B, B}{\Gamma \vdash \Delta, A \vee B}\ (\vee R2)$$

$$\frac{\Gamma_1, A \vee B, \Gamma_2, A \vdash \Delta, D, D \quad \Gamma_1, A \vee B, \Gamma_2, B \vdash \Delta, D, D}{\Gamma_1, A \vee B, \Gamma_2 \vdash \Delta, D}\ (\vee L)$$

$$\frac{\Gamma, A \vdash \Delta, A \to B, A \to B}{\Gamma \vdash \Delta, A \to B}\ (\to R1) \qquad \frac{\Gamma, \top \vdash \Delta, A \to B, B}{\Gamma \vdash \Delta, A \to B}\ (\to R2)$$

$$\frac{\Gamma_1, A \to B, \Gamma_2, \top \vdash \Delta, D, A \quad \Gamma_1, A \to B, \Gamma_2, B \vdash \Delta, D, D}{\Gamma_1, A \to B, \Gamma_2 \vdash \Delta, D}\ (\to L)$$

$$\frac{\Gamma, \top \vdash \Delta, \forall x A, A[m/x] \quad (\forall m)}{\Gamma \vdash \Delta, \forall x A}\ (\forall R) \qquad \frac{\Gamma_1, \forall x A, \Gamma_2, A[m/x] \vdash \Delta, D, D}{\Gamma_1, \forall x A, \Gamma_2 \vdash \Delta, D}\ (\forall L)$$

$$\frac{\Gamma, \top \vdash \Delta, \exists x A, A[m/x]}{\Gamma \vdash \Delta, \exists x A}\ (\exists R) \qquad \frac{\Gamma_1, \exists x A, \Gamma_2, A[m/x] \vdash \Delta, D, D \quad (\forall m)}{\Gamma_1, \exists x A, \Gamma_2 \vdash \Delta, D}\ (\exists L)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta, B}\ (sweak) \qquad \frac{\top, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (\top E) \qquad \frac{\Gamma \vdash \bot, \Delta}{\Gamma \vdash \Delta}\ (\bot E)$$

Figure 5 Inference Rules of NCIPA

7 Non-Commutative Infinitary Peano Arithmetic

We define non-commutative infinitary Peano arithmetic NCIPA. In the next section we will prove that NCIPA is a subsystem of IPA essentially equivalent to IPA^-.

The language is defined to be the same as that of IPA. The inference rules are given by Figure 5. The rule $(sweak)$ means symmetric weakening. A proof in this system is defined as a well-founded recursive tree in a similar way to IPA.

Intuitive meaning of provable sequents is given by using the familiar interpretation of a sequent in the sequent calculus LK in the standard model of numbers as follows: If $\Pi, A_1, \ldots, A_n \vdash B_1, \ldots, B_n$ is provable, then (1) $\Pi \vdash$ is true, or (2) $\Pi, A_1, \ldots, A_i \vdash B_i$ is true for some i. If $A_1, \ldots, A_n \vdash C_1, \ldots, C_m, B_1, \ldots, B_n$ is provable, then (1) $\vdash C_i$ is true for some i, or (2) $A_1, \ldots, A_i \vdash B_i$ is true for some i.

This system is obtained from IPA^- by coding grouping information by the length of a sequence of formulas. We explain it by example.

▶ Example 7.1. The sequent

$$A_1, -, A_2, A_3, -, A_4, -, A_5, A_6 \vdash B_1, B_2, B_3$$

$$\cfrac{\cfrac{}{A \wedge B, \top, B \vdash B \wedge A, B, B}\,(Id)}{A \wedge B, \top \vdash B \wedge A, B}\,(\wedge L2) \quad \cfrac{\cfrac{}{A \wedge B, \top, A \vdash B \wedge A, A, A}\,(Id)}{A \wedge B, \top \vdash B \wedge A, A}\,(\wedge L1)$$
$$\overline{\qquad A \wedge B \vdash B \wedge A \qquad}\,(\wedge R)$$

Figure 6 Example 7.2

in IPA$^-$ is coded by the sequent

$$A_1, \top, A_2, A_3, \top, A_4, \top, A_5, A_6 \vdash B_1, B_1, B_1, B_2, B_2, B_3, B_3, B_3$$

in NCIPA. The atomic formula $\top$ is used for separating groups. The group $\top, A_5, A_6$ corresponds to B_3, B_3, B_3. The group $\top, A_4$ corresponds to B_2, B_2. The group $\top, A_2, A_3$ corresponds to B_1, B_1, B_1. We can decode this information by counting formulas from the right to the left on both sides. This decoding may not be unique, but it is unique up to the provability in IPA$^-$. This translation is formally defined in Definition 8.4.

We explain this system by the same examples as those in Section 3. In the examples, we assume the identity lemma $\Gamma_1, A, \Gamma_2 \vdash \Delta, A$, which will be shown as Lemma 7.4 after the examples.

▶ Example 7.2. The first example in Figure 6 shows the conjunction of NCIPA is commutative.

▶ Example 7.3. The next example shows how this system respects the order of formulas. We have three provable sequents

$$A, B \vdash A, \bot,$$
$$A, B \vdash \bot, A,$$
$$A, B \vdash \bot, B.$$

On the other hand the sequent

$$A, B \vdash B, \bot$$

is not provable. The first sequent is provable since $A \vdash A$ is true. The second sequent is provable since $A, B \vdash A$ is true. The third sequent is provable since $A, B \vdash B$ is true. Formally the first sequent is proved by

$$\cfrac{\cfrac{}{A \vdash A}\,(Id)}{A, B \vdash A, \bot}\,(sweak)$$

and the second and the third sequents are proved by (Id). On the other hand, the fourth sequent is not provable, since $A \vdash B$ is not true and $A, B \vdash \bot$ is not true.

▶ Lemma 7.4. The following is derivable.

$$\cfrac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta, A}\,(Id)$$

This lemma is shown by induction on A in a similar way to Lemma 3.4.

Remark. (1) $(\bot E)$ is necessary for making a binary left logical rule for the empty succedent admissible. It is used in the proof of Proposition 8.6. For example, the following is admissible.

$$\cfrac{\Gamma_1, A \vee B, \Gamma_2, A \vdash \quad \Gamma_1, A \vee B, \Gamma_2, B \vdash}{\Gamma_1, A \vee B, \Gamma_2 \vdash}\,(\vee L)$$

(2) $(\top E)$ is necessary since $\vdash \top \vee \bot, \bot$ would not be provable otherwise, though it is indeed provable by

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\top \vdash \top \vee \bot, \top}\ (Ax\ R)}{\vdash \top \vee \bot}\ (\vee R1)}{\top \vdash \top \vee \bot, \bot}\ (sweak)}{\vdash \top \vee \bot, \bot}\ (\top E)$$

▶ Proposition 7.5. The following are admissible.

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, B, \Delta_2}\ (sweak2) \quad (|\Gamma_2| = |\Delta_2|)$$

$$\frac{\Gamma_1, A, A, \Gamma_2 \vdash \Delta_1, B, B, \Delta_2}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, B, \Delta_2}\ (scont) \quad (|\Gamma_2| = |\Delta_2|)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ (weak\ L) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ (\bot I) \qquad \frac{\Gamma_1, \top, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ (replace\ L)$$

$$\frac{\Gamma \vdash \Delta_1, A, A, \Delta_2}{\Gamma \vdash \Delta_1, A, \Delta_2}\ (cont\ R) \qquad \frac{\Gamma_1, \top, A, \Gamma_2 \vdash \Delta_1, B, B, \Delta_2}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, B, \Delta_2}\ (\top E2) \quad (|\Gamma_2| = |\Delta_2|)$$

We define the system NCIPA+EX as the system NCIPA with the following inference rules $(exch\ L)$ and $(exch\ R)$.

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (exch\ R) \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (exch\ L)$$

When the exchange rules are added to NCIPA, the coding information is lost and it becomes equivalent to IPA.

▶ Theorem 7.6. $\Gamma \vdash \Delta$ is provable in NCIPA+EX if and only if $\Gamma \vdash \Delta$ is provable in IPA.

The system NCIPA is a subclassical logic.

▶ Theorem 7.7. $\Gamma \vdash A$ is provable in NCIPA if and only if $\Gamma, \mathrm{EM}_1 \vdash A$ is provable in IHA.

We will complete the proof of this theorem in Section 8.

8 Translations between NCIPA and IPA$^-$

This section gives translations between NCIPA and IPA$^-$ in both directions and proves that they preserve provability. By using these translations, we will prove the equivalence theorem between NCIPA and IHA + EM$_1$.

First, we give a translation from NCIPA to IPA$^-$. To translate $\Gamma \vdash \Delta$, we insert the same number of $-$ symbols as $|\Delta|$ into Γ by adding a single $-$ symbol in front of each formula from the right. For example, the sequent $A_1, A_2, A_3, A_4 \vdash B_1, B_2$ in NCIPA is translated into the sequent $A_1, A_2, -, A_3, -, A_4 \vdash B_1, B_2$ in IPA$^-$.

▶ Definition 8.1 (Translation from NCIPA to IPA$^-$). We translate a sequent $\Gamma \vdash \Delta$ in NCIPA into the sequent $\Gamma^{-|\Delta|} \vdash \Delta$ in IPA$^-$, where $(\Gamma_0, A_1, A_2, \ldots, A_n)^{-n}$ is defined as $\Gamma_0, -, A_1, -, A_2, \ldots, -, A_n$ and $(A_1, A_2, \ldots, A_m)^{-n}$ $(m < n)$ is defined as $-^{n-m}, -, A_1, -, A_2, \ldots, -, A_m$.

▶ Example 8.2. The NCIPA-proof of $A \wedge B \vdash B \wedge A$ in Example 7.2 is translated into the IPA$^-$-proof given in Figure 7.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{-, A \wedge B, -, \top, -, B \vdash B \wedge A, B, B}(Id)}{-, A \wedge B, -, \top, B \vdash B \wedge A, B}(cont\ R)}{-, A \wedge B, -, \top \vdash B \wedge A, B}(\wedge L2)}{-, A \wedge B, - \vdash B \wedge A, B}(\top E) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{-, A \wedge B, -, \top, -, A \vdash B \wedge A, A, A}(Id)}{-, A \wedge B, -, \top, A \vdash B \wedge A, A}(cont\ R)}{-, A \wedge B, -, \top \vdash B \wedge A, A}(\wedge L1)}{-, A \wedge B, - \vdash B \wedge A, A}(\top E)$$
$$\dfrac{\cdots \quad \cdots}{-, A \wedge B \vdash B \wedge A}(\wedge R)$$

Figure 7 Example for NCIPA to IPA$^-$

$$\dfrac{\dfrac{\dfrac{}{\top, A \wedge B, \top, B \vdash B \wedge A, B \wedge A, B, B}(Id)}{\top, A \wedge B, \top \vdash B \wedge A, B \wedge A, B}(\wedge L2) \quad \dfrac{\dfrac{}{\top, A \wedge B, \top, A \vdash B \wedge A, B \wedge A, A, A}(Id)}{\top, A \wedge B, \top \vdash B \wedge A, B \wedge A, A}(\wedge L1)}{\top, A \wedge B \vdash B \wedge A, B \wedge A}(\wedge R)$$

Figure 8 Example for IPA$^-$ to NCIPA

▶ Proposition 8.3. $\Gamma \vdash_{\mathrm{NCIPA}} \Delta$ implies $\Gamma^{-|\Delta|} \vdash_{\mathrm{IPA}^-} \Delta$.

This is proved by induction on the proof.

Next, we define a translation from IPA$^-$ to NCIPA. To translate $\Gamma \vdash \Delta$, we replace $-$ by $\top$ in Γ, and the succedent is produced from Δ by multiplying the i-th formula by $n_i + 1$ when the i-th group in Γ has n_i formulas. An example is given in Example 7.1. A^n denotes $A, \ldots, A$ (n times).

▶ Definition 8.4 (Translation from IPA$^-$ to NCIPA). We translate a sequent $\Gamma \vdash \Delta$ in IPA$^-$ into the sequent $\Gamma^\top \vdash \Delta^\Gamma$ in NCIPA, where $\Gamma^\top$ is defined as $\Gamma_0, \top, \Gamma_1, \top, \Gamma_2, \ldots, \top, \Gamma_n$ and $(A_1, \ldots, A_n)^\Gamma$ is defined as $A_1^{|\Gamma_1|+1}, A_2^{|\Gamma_2|+1}, \ldots, A_n^{|\Gamma_n|+1}$ if Γ is $\Gamma_0, -, \Gamma_1, -, \Gamma_2, \ldots, -, \Gamma_n$ and Γ_i does not contain $-$.

▶ Example 8.5. The IPA$^-$-proof of $-, A \wedge B \vdash B \wedge A$ in Example 3.1 is translated into the NCIPA-proof given in Figure 8.

▶ Proposition 8.6. $\Gamma \vdash_{\mathrm{IPA}^-} \Delta$ implies $\Gamma^\top \vdash_{\mathrm{NCIPA}} \Delta^\Gamma$.

This is proved by induction on the proof.

Proof of Theorem 7.7. From the left-hand side to the right-hand side.

By Proposition 8.3, we have $\Gamma^{-1} \vdash_{\mathrm{IPA}^-} A$. By Theorem 6.2, we have $\Gamma, \mathrm{EM}_1 \vdash_{\mathrm{IHA}} A$.

From the right-hand side to the left-hand side.

By Theorem 5.2, we have $-, \Gamma \vdash_{\mathrm{IPA}^-} A$. By Proposition 8.6, we get $\top, \Gamma \vdash_{\mathrm{NCIPA}} A^{|\Gamma|+1}$. By $(\top E)$ and $(cont\ R)$ in Proposition 7.5, $\Gamma \vdash_{\mathrm{NCIPA}} A$. ◻

9 Cut Elimination for NCIPA

In this section, we will prove the cut elimination theorem for NCIPA.

▶ Definition 9.1. We give the cut rule in the system NCIPA.

$$\dfrac{\Gamma_1, \top \vdash \Gamma_2, A \quad \Delta_1, A, \Sigma_1 \vdash \Delta_2, \bot, \Sigma_2}{\Gamma_1, \Delta_1, \Sigma_1 \vdash \Gamma_2, \Delta_2, \Sigma_2}(cut)$$

where $|\Pi_1| = |\Pi_2|, |\Delta_1| = |\Delta_2|, |\Sigma_1| = |\Sigma_2|$.

We can eliminate the cut rule in NCIPA.

▶ Theorem 9.2 (Cut Elimination). If $\Gamma \vdash \Delta$ is provable in NCIPA with the rule (cut), then $\Gamma \vdash \Delta$ is provable in NCIPA.

This theorem is proved by using the cut elimination theorem for IPA$^-$ and the next proposition.

▶ Proposition 9.3 (NCIPA to IPA$^-$ to NCIPA). $(\Gamma^{-|\Delta|})^\top \vdash_{\mathrm{NCIPA}} \Delta^{\Gamma^{-|\Delta|}}$ iff $\Gamma \vdash_{\mathrm{NCIPA}} \Delta$.

10 Concluding Remarks

We showed that by removing the exchange rules, Peano arithmetic with the ω-rules becomes Heyting arithmetic with the recursive ω-rules and the Σ^0_1 excluded middle. The equivalence is an open question when the system is Peano arithmetic without the ω-rules.

Future work would be to investigate the computational content of the subclassical systems IPA^- and NCIPA.

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