A Semantic Proof that Reducibility Candidates entail Cut Elimination

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Abstract

Two main lines have been adopted to prove the cut elimination theorem: the syntactic one, that studies the process of reducing cuts, and the semantic one, that consists in interpreting a sequent in some algebra and extracting from this interpretation a cut-free proof of this very sequent.

A link between those two methods was exhibited by studying in a semantic way, syntactical tools that allow to prove (strong) normalization of proof-terms, namely reducibility candidates. In the case of deduction modulo, a framework combining deduction and rewriting rules in which theories like Zermelo set theory and higher order logic can be expressed, this is obtained by constructing a reducibility candidates valued model. The existence of such a *pre*-model for a theory entails strong normalization of its proof-terms and, by the usual syntactic argument, the cut elimination property.

In this paper, we strengthen this gate between syntactic and semantic methods, by providing a full semantic proof that the existence of a pre-model entails the cut elimination property for the considered theory in deduction modulo. We first define a new simplified variant of reducibility candidates *à la* Girard, that is sufficient to prove weak normalization of proof-terms (and therefore the cut elimination property). Then we build, from some model valued on the pre-Heyting algebra of those WN reducibility candidates, a regular model valued on a Heyting algebra on which we apply the usual soundness/strong completeness argument.

Finally, we discuss further extensions of this new method towards normalization by evaluation techniques that commonly use Kripke semantics.

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1 Introduction

The cut elimination theorem [15] is a central result in proof theory and type theory. From a proof theorist's point of view, it implies the consistency of the considered logical framework, as well as other nice results like the disjunction property, the witness property or the subformula property. On the other side, the type theorist is more interested in the cut elimination process itself, and in its termination. Those two different interests led to two distinct main lines of showing cut elimination, namely the semantic and the syntactic methods.

The semantic methods [17, 4, 18] use the soundness/strong completeness paradigm: first show that if we have a proof of A under the hypothesis Γ, then every model of Γ, valued on

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a Heyting algebra, is a model of A, and then show that if the latter holds, then we can build a cut-free proof of A when assuming Γ.

A modern variant of the syntactic method uses the Curry-Howard correspondence and Tait-Girard's reducibility method [16], in order to prove normalization of β-reduction on proof-terms (that syntactically entails cut elimination).

The logical framework we shall work in is Deduction modulo [11]. It is a generic way to integrate computation rules into a deduction system, in our case natural deduction. In this logical framework, theories are expressed via rewrite rules on first order terms and propositions, instead of axioms. One can express, only with rewrite rules, both theories that satisfy the cut elimination property (such as Zermelo set theory [12], Peano's arithmetic [14] or higher-order logic [10, 13]) and theories that do not. One particularity of this framework is that all theories (expressed only with rewrite rules) satisfying the cut elimination property are consistent: if there is no axiom a cut-free proof always ends with an introduction rule, and one cannot prove False with a cut-free proof. Hence, the cut elimination property entails that False is unprovable, which is not true in presence of axioms (consider, for instance, the theory containing the only axiom False. It enjoys cut elimination but it is not consistent).

A first link between the semantic and syntactic methods to prove cut elimination was made by defining reducibility candidates for deduction modulo [13] as a model, and to show that this model has a *pre*-Heyting algebra structure [9] (a Heyting algebra in which the order is replaced by a pre-order). It can be shown, with the usual (syntactic) reducibility arguments, that having such a (*pre*-)model entails strong normalization and therefore cut elimination for a theory in deduction modulo.

In this paper, we strengthen this gate between syntactic and semantic methods, by providing a full semantic proof that the existence of a pre-model entails the cut elimination property for the considered theory in deduction modulo. Since our goal is cut elimination, we consider weak normalization rather than strong normalization. We define a new simplified variant of reducibility candidates à la Girard for weak normalization, which is a first contribution of our work. We give those reducibility candidates a pre-Heyting algebra structure. Then we build, from a given model valued on that pre-Heyting algebra, a regular model valued on a Heyting algebra, that we can use to prove cut elimination with the usual soundness/strong completeness argument. This is the second contribution of our work.

Many results have been obtained in the direction we follow, especially in the normalization by evaluation approach [1, 2, 3, 5, 6], based on a Kripke-like semantic structure. In this paper, the proof of cut elimination generates, in some cases, also a normalization by evaluation algorithm, but with respect to the standard Heyting semantics notion.

We first introduce Deduction modulo in Sec. 2. Sec. 3 is devoted to semantics (Heyting and pre-Heyting algebras) and to the specific pre-Heyting algebra of reducibility candidates à la Girard for weak normalization.Then we prove, in Sec. 4 that we can extract from a model valued on that precise pre-Heyting algebra, a model valued on a Heyting algebra that allows to prove semantically cut elimination (Sec. 5). We finally discuss further extensions of the present work, especially concerning normalization by evaluation algorithms it can raise.

2 Deduction modulo

Natural Deduction modulo [10] is an extension of Natural Deduction with rewrite rules on terms and propositions. As in Natural Deduction, a theory is first defined by a language in first order logic [20], composed of a set of variables, a set of function symbols and a set of predicate symbols (all symbols given with their arities). Formulæ are then built-up from predicates (called *atomic formulæ*), the usual connectives $\Rightarrow$, $\wedge$, $\vee$ and the quantifiers $\forall$ and $\exists$. Given a language in predicate logic, a theory is defined not by axioms but by rewrite rules on terms and formulæ. In this paper, we shall not focus on how to define such a rewrite system, we will only consider the congruence relation on terms and formulæ it generates. The only mandatory property is that the congruence relation has to be non-confusing, *i.e.* two formulæ with different top connectives (or quantifiers) cannot be congruent (cut elimination does not entail consistency for confusing theories, since a cut-free proof does not necessarily end with an introduction rule). The principle of deduction modulo is to adapt the typing/deduction rules of natural deduction, giving the ability to replace a formula by an equivalent one, at each step of a typing derivation, as detailed in Fig. 1. We use the Curry-Howard correspondence to express proof-terms of deduction modulo. Those proof-terms can contain both term variables (written $x, y, \ldots$, given by the language in predicate logic) and proof variables (written $\alpha, \beta, \ldots$). In the same way, terms are written $t, u, \ldots$ while proof-terms are written $\pi, \rho, \ldots$. Typing contexts Γ are sequences of labeled formulæ: $\alpha_1 : A_1, \cdots, \alpha_n : A_n$.

$$\frac{}{\Gamma \vdash \alpha : B}\ \text{axiom, if } \alpha : A \in \Gamma \text{ and } A \equiv B$$

$$\frac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash \lambda\alpha.\pi : C}\ \Rightarrow\text{-intro, if } C \equiv A \Rightarrow B$$

$$\frac{\Gamma \vdash \pi : C \qquad \Gamma \vdash \pi' : A}{\Gamma \vdash (\pi\ \pi') : B}\ \Rightarrow\text{-elim, if } C \equiv A \Rightarrow B$$

$$\frac{\Gamma \vdash \pi : A \qquad \Gamma \vdash \pi' : B}{\Gamma \vdash \langle \pi, \pi' \rangle : A \wedge B}\ \wedge\text{-intro, if } C \equiv A \wedge B$$

$$\wedge\text{-elim1, if } C \equiv A \wedge B\ \frac{\Gamma \vdash \pi : C}{\Gamma \vdash fst(\pi) : A} \qquad \frac{\Gamma \vdash \pi : C}{\Gamma \vdash snd(\pi) : B}\ \wedge\text{-elim2, if } C \equiv A \wedge B$$

$$\vee\text{-intro1, if } C \equiv A \vee B\ \frac{\Gamma \vdash \pi : A}{\Gamma \vdash i(\pi) : C} \qquad \frac{\Gamma \vdash \pi : B}{\Gamma \vdash j(\pi) : C}\ \vee\text{-intro2, if } C \equiv A \vee B$$

$$\frac{\Gamma \vdash \pi_1 : D \qquad \Gamma, \alpha : A \vdash \pi_2 : C \qquad \Gamma, \beta : B \vdash \pi_3 : C}{\Gamma \vdash (\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3) : C}\ \vee\text{-elim, if } D \equiv A \vee B$$

$$\frac{\Gamma \vdash \pi : B}{\Gamma \vdash (\delta_\perp\ \pi) : A}\ \perp\text{-elim, if } B \equiv \perp$$

$$\frac{\Gamma \vdash \pi : A}{\Gamma \vdash \lambda x.\pi : B}\ \forall\text{-intro, if } B \equiv \forall x A, \text{ and } x \notin FV(\Gamma)$$

$$\frac{\Gamma \vdash \pi : B}{\Gamma \vdash (\pi\ t) : C}\ \forall\text{-elim if } B \equiv \forall x A, \text{ and } C \equiv (t/x)A$$

$$\frac{\Gamma \vdash \pi : C}{\Gamma \vdash \langle t, \pi \rangle : B}\ \exists\text{-intro if } B \equiv \exists x A, \text{ and } C \equiv (t/x)A$$

$$\frac{\Gamma \vdash \pi : C \qquad \Gamma, \alpha : A \vdash \pi' : B}{\Gamma \vdash (\delta_\exists\ \pi\ x\alpha\pi') : B}\ \exists\text{-elim, if } C \equiv \exists x A, \text{ and } x \notin FV(\Gamma, B)$$

Figure 1 Intuitionistic natural deduction modulo.

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda\alpha\ \pi$ and $(\pi\ \pi')$ express proofs built respectively with the introduction and elimination rules of the implication,

terms of the form $\langle \pi, \pi' \rangle$ and $fst(\pi)$, $snd(\pi)$ express proofs built with the introduction and elimination rules of the conjunction, terms of the form $i(\pi), j(\pi)$ and $(\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3)$ express proofs built with the introduction and elimination rules of the disjunction, terms of the form $(\delta_\bot\ \pi)$ express proofs built with the elimination rule of the contradiction, terms of the form $\lambda x\ \pi$ and $(\pi\ t)$ express proofs built with the introduction and elimination rules of the universal quantifier and terms of the form $\langle t, \pi \rangle$ and $(\delta_\exists\ \pi\ x\alpha\pi')$ express proofs built with the introduction and elimination rules of the existential quantifier.
We call *neutral* those proof-terms that are formed with an elimination rule or an axiom.

For example, in predicate logic with two 0-ary predicates P and Q, and in a theory defined by a congruence relation $\equiv$ such that $P \equiv (Q \Rightarrow Q)$, the proof-term $\lambda\alpha.\alpha$ is a proof of P in the empty context, with the rules $\Rightarrow$-intro and axiom as proof derivation.

Capture avoiding substitution in proof-terms is defined as usual. Notice that both term-variables and proof-variables can be substituted respectively by terms and proof-terms. The substitution of the variable x (resp. proof-variable α) by the term t (resp. proof-term π') in the proof-term π is written $(t/x)\pi$ (resp. $(\pi'/\alpha)\pi$).

Cut elimination in proof derivations is done, via the Curry-Howard correspondence, by β-reduction on proof-terms. β-reduction is defined as the smallest contextual closure of the following reduction rules (corresponding, respectively, to the elimination of a cut $\Rightarrow$-e/$\Rightarrow$-i, $\wedge$-e1/$\wedge$-i, $\wedge$-e2/$\wedge$-i, $\vee$-e/$\vee$-i1, $\vee$-e/$\vee$-i2, $\forall$-e/$\forall$-i and $\exists$-e/$\exists$-i).

$$\begin{array}{rcl}
(\lambda\alpha.\pi_1\ \pi_2) & \rhd & (\pi_2/\alpha)\pi_1 \\
fst(\langle \pi_1, \pi_2 \rangle) & \rhd & \pi_1 \\
snd(\langle \pi_1, \pi_2 \rangle) & \rhd & \pi_2 \\
(\delta\ i(\pi_1)\ \alpha\pi_2\ \beta\pi_3) & \rhd & (\pi_1/\alpha)\pi_2 \\
(\delta\ j(\pi_1)\ \alpha\pi_2\ \beta\pi_3) & \rhd & (\pi_1/\beta)\pi_3 \\
\lambda x.\pi\ t & \rhd & (t/x)\pi \\
(\delta_\exists\ \langle t, \pi_1 \rangle\ \alpha x\pi_2) & \rhd & (t/x, \pi_1/\alpha)\pi_2
\end{array}$$

Figure 2 Proof-term reduction rules.

Let π be a proof-term. We write $\pi \rhd^* \pi'$ if π β-reduces to π' in zero or more steps. π is said in normal form if no reduction rule applies to π. It is weakly (resp. strongly) normalizing if there exists a finite β-reduction sequence $\pi \rhd^* \pi'$ with π' in normal form (resp. all β-reduction sequences from π are finite). By extension, a theory is weakly (resp. strongly) normalizing if all proof-terms that are proofs of formulæ of that theory, are weakly (resp. strongly) normalizing. Since a step of cut elimination in a proof derivation corresponds to a β-reduction step of the corresponding proof-term, one can prove syntactically that all weakly normalizing theories satisfy the cut elimination property.

Finally, notice that β-reduction is confluent for all theories expressed in deduction modulo, i.e. if π, π_1, π_2 are proof-terms such that $\pi \rhd^* \pi_1$ and $\pi \rhd^* \pi_2$ then there exists a proof-term π' such that $\pi_1 \rhd^* \pi'$ and $\pi_2 \rhd^* \pi'$.

3 Pre-Heyting algebras and pre-models

In [13], Dowek and Werner have generalized the Tait-Girard's reducibility method, by defining the notion of reducibility candidates for deduction modulo, namely *pre-models*, whose existence is a sufficient condition for strong normalization. Later, Dowek exhibited the notion of *pre-Heyting algebras* [9] (also known as pseudo-Heyting algebras or Truth Values Algebras) which is the underlying structure of those pre-models. Let us first recall the definitions of those pre-Heyting algebras and of models valued on them.

3.1 (pre-) Heyting algebras

▶ **Definition 1** (pre-Heyting algebra). Let $\mathcal{B}$ be a set, $\leq$ be a relation on $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{E}$ be subsets of $\wp(\mathcal{B})$, $\tilde{\top}$, $\tilde{\bot}$ be elements of $\mathcal{B}$, $\tilde{\Rightarrow}$, $\tilde{\wedge}$, and $\tilde{\vee}$ be functions from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$, $\tilde{\forall}$ be a function from $\mathcal{A}$ to $\mathcal{B}$ and $\tilde{\exists}$ be a function from $\mathcal{E}$ to $\mathcal{B}$.
The structure $\mathcal{B} = \langle \mathcal{B}, \leq, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ is said to be a *pre-Heyting algebra* if

- the relation $\leq$ is a pre-order,
- $\tilde{\bot}$ is a minimum element,
- $\tilde{\top}$ is a maximum element,
- for all a, b in $\mathcal{B}$, $a \mathrel{\tilde{\wedge}} b$ is a greatest lower bound of a and b and $a \mathrel{\tilde{\vee}} b$ is a least upper bound of a and b,
- $\tilde{\forall}$ and $\tilde{\exists}$ are infinite greatest lower bound and least upper bound, respectively,
- for all a, b, c in $\mathcal{B}$, $a \leq b \mathrel{\tilde{\Rightarrow}} c$ if and only if $a \mathrel{\tilde{\wedge}} b \leq c$.

Compared to [9], we drop the closure conditions that $a \mathrel{\tilde{\Rightarrow}} A$ and $E \mathrel{\tilde{\Rightarrow}} a$ are both in $\mathcal{A}$. This simplification is possible because we do not reason within Truth Values Algebras. See [9] for more detailed definitions and explanations.

▶ **Definition 2** (Heyting algebra). A pre-Heyting algebra is said to be a *Heyting algebra* if the pre-order $\leq$ is antisymmetric and therefore an order.

3.2 Models

Let us define now the notion of model valued on a pre-Heyting algebra.

▶ **Definition 3** ($\mathcal{B}$-valued structure).
Let $\mathcal{L} = \langle f_i, P_j \rangle$ be a language in first order logic and $\mathcal{B}$ be a pre-Heyting algebra, a $\mathcal{B}$*-valued structure* for the language $\mathcal{L}$, $\mathcal{M} = \langle M, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$ is a structure such that $\hat{f}_i$ is a function from M^n to M where n is the arity of the function symbol f_i and $\hat{P}_j$ is a function from M^n to $\mathcal{B}$ where n is the arity of the predicate symbol P_j.

▶ **Definition 4** (Environments).
Given a $\mathcal{B}$-valued structure $\mathcal{M} = \langle M, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$, an *environment* is a function which associates an element of M with each term variable.

▶ **Definition 5** (Denotation). Let $\mathcal{B}$ be a pre-Heyting algebra, $\mathcal{M}$ a $\mathcal{B}$-valued structure and ϕ an environment. The denotation $[\![A]\!]^{\mathcal{M}}_{\phi}$ of a formula A in $\mathcal{M}$ is inductively defined from $\mathcal{B}$ and ϕ as follows:

- $[\![x]\!]^{\mathcal{M}}_{\phi} = \phi(x)$,
- $[\![f(t_1, ..., t_n)]\!]^{\mathcal{M}}_{\phi} = \hat{f}([\![t_1]\!]^{\mathcal{M}}_{\phi}, ..., [\![t_n]\!]^{\mathcal{M}}_{\phi})$,
- $[\![P(t_1, ..., t_n)]\!]^{\mathcal{M}}_{\phi} = \hat{P}([\![t_1]\!]^{\mathcal{M}}_{\phi}, ..., [\![t_n]\!]^{\mathcal{M}}_{\phi})$,
- $[\![\bot]\!]^{\mathcal{M}}_{\phi} = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]^{\mathcal{M}}_{\phi} = [\![A]\!]^{\mathcal{M}}_{\phi} \mathrel{\tilde{\Rightarrow}} [\![B]\!]^{\mathcal{M}}_{\phi}$,
- $[\![A \wedge B]\!]^{\mathcal{M}}_{\phi} = [\![A]\!]^{\mathcal{M}}_{\phi} \mathrel{\tilde{\wedge}} [\![B]\!]^{\mathcal{M}}_{\phi}$,
- $[\![A \vee B]\!]^{\mathcal{M}}_{\phi} = [\![A]\!]^{\mathcal{M}}_{\phi} \mathrel{\tilde{\vee}} [\![B]\!]^{\mathcal{M}}_{\phi}$,

- $[\![\forall x\ A]\!]^{\mathcal{M}}_{\phi} = \tilde{\forall}\ \{[\![A]\!]^{\mathcal{M}}_{\phi+\langle x,e\rangle} \mid e \in M\}$ when it is defined,
- $[\![\exists x\ A]\!]^{\mathcal{M}}_{\phi} = \tilde{\exists}\ \{[\![A]\!]^{\mathcal{M}}_{\phi+\langle x,e\rangle} \mid e \in M\}$ when it is defined.

▶ Remark. We omit $\mathcal{M}$ from $[\![A]\!]^{\mathcal{M}}_{\phi}$ when it is clear from context.
In all the pre-Heyting Algebras we consider in this paper, $\mathcal{A}$ and $\mathcal{E}$ at least contain all the sets of the form $\{[\![A]\!]_{\phi+\langle x,e\rangle} \mid e \in M\}$ so that $[\![A]\!]_{\phi}$ is always defined.
For any formula A, terms t, u and environment ϕ, we have $[\![(t/x)A]\!]_{\phi} = [\![A]\!]_{\phi+\langle x,[\![t]\!]_{\phi}\rangle}$ and $[\![(t/x)u]\!]_{\phi} = [\![u]\!]_{\phi+\langle x,[\![t]\!]_{\phi}\rangle}$

▶ **Definition 6** (Model). The $\mathcal{B}$-valued structure $\mathcal{M}$ is said to be *a model of* a theory $\mathcal{L}, \equiv$ if for all formulæ A and B and terms t and u such that $A \equiv B$ and $t \equiv u$, and for all environments ϕ, we have $[\![A]\!]_{\phi} = [\![B]\!]_{\phi}$ and $[\![t]\!]_{\phi} = [\![u]\!]_{\phi}$.

3.3 Reducibility candidates and pre-models for weak normalization

The main idea of Tait-Girard's reducibility method is to associate with each proposition A, a set R_A of strongly normalizing proof-terms, that verifies some *reducibility* conditions, and then show the so-called *adequacy lemma*: all proof-terms that are proofs of some formula A belong to R_A and are therefore strongly normalizing. We adapt here this notion of reducibility candidates to weak normalization. We shall write *WN-reducibility candidates*. We also write "*WN*" the set of weakly normalizing proof-terms and "π is *WN*" when $\pi \in WN$. A proof-term is said to be *neutral* if it is of the form $\alpha, (\pi\ \pi'), fst(\pi), snd(\pi), (\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3), (\pi\ t)$ or $(\delta_{\exists}\ \pi_1\ \alpha x\pi_2)$. A neutral proof-term is said to be *isolated* if all its reduction sequences end with a neutral proof-term (those proof-terms are called *hereditarily neutral* in [19]).

▶ **Definition 7** (Reducibility candidates for weak normalization).
A set R of proof-terms is a *WN-reducibility candidate* if and only if:
$(\mathbf{P}_1)$ if $\pi \in R$, then π is weakly normalizing,
$(\mathbf{P}_{3a})$ if π is neutral and there exists $\pi' \in R$ such that $\pi \rhd \pi'$, then $\pi \in R$.
$(\mathbf{P}_{3b})$ R contains all isolated weakly normalizing proof-terms.

If we compare this definition to usual reducibility candidates (see [16]), $(\mathbf{CR}_1)$ becomes $(\mathbf{P}_1)$, stability by reduction $(\mathbf{CR}_2)$ is not needed for our particular purpose of proving, via semantic methods, the cut elimination property (albeit we shall see in Section 5.1 that it could be useful to impose this property), and we split the usual property $(\mathbf{CR}_3)$ of reducibility candidates into $(\mathbf{P}_{3a})$ and $(\mathbf{P}_{3b})$: $(\mathbf{P}_{3a})$ is the adaptation of $(\mathbf{CR}_3)$ to weak normalization but this no longer entails the non-emptiness of the considered set (a crucial point in the proof of the adequacy lemma). $(\mathbf{P}_{3b})$ ensures that non-emptiness.

Let us now define the pre-Heyting algebra of *WN*-reducibility candidates.

▶ **Definition 8** (Operations).
If E and F are sets of proof-terms, and $\mathcal{F}$ is a set of sets of proof-terms,

- The sets $\tilde{\top}$ and $\tilde{\bot}$ are both the set of weakly normalizing proof-terms.
- $E \tilde{\Rightarrow} F$ is the set of proof-terms π such that either π is isolated and weakly normalizing, or there exists π_1 such that $\pi \rhd^* \lambda\alpha\ \pi_1$ and for all $\pi' \in E$, $(\pi'/\alpha)\pi_1 \in F$.
- $E\ \tilde{\wedge}\ F$ is the set of proof-terms π such that either π is isolated and weakly normalizing, or there exists π_1, π_2 such that $\pi \rhd^* \langle\pi_1, \pi_2\rangle$, $\pi_1 \in E$ and $\pi_2 \in F$.
- $E\ \tilde{\vee}\ F$ is the set of proof-terms π such that either π is isolated and weakly normalizing, or there exists π_1 such that $\pi \rhd^* i(\pi_1)$ (resp. $j(\pi_1)$) and $\pi_1 \in E$ (resp. F).

- $\tilde{\forall}\ \mathcal{F}$ is the set of proof-terms π such that either π is isolated and weakly normalizing, or there exists π_1 such that $\pi \rhd^* \lambda x\ \pi_1$ and for all terms t and $G \in \mathcal{F}$, $(t/x)\pi_1$ is in G.
- $\tilde{\exists}\ \mathcal{F}$ is the set of proof-terms π such that either π is isolated and weakly normalizing, or there exists π_1, $G \in \mathcal{F}$ and a term t such that $\pi \rhd^* \langle t, \pi_1\rangle$ and $\pi_1 \in G$.

▶ Remark. We could also have left $\tilde{\bot}$ be the smallest (for inclusion) reducibility candidate, *i.e.* the set of isolated weakly normalizing terms. But since we are going to choose the trivial pre-order, there is no particular reason to do so.

▶ **Lemma 9.** *The set of WN-reducibility candidates is closed by the operations of Def. 8.*

We can first remark that the set WN is a WN-reducibility candidate. Moreover, if E, F are WN-reducibility candidates and $\mathcal{F}$ is a set of WN-reducibility candidates, $E \tilde{\Rightarrow} F$, $E \tilde{\wedge} F$, $E \tilde{\vee} F$, $\tilde{\forall}\ \mathcal{F}$ and $\tilde{\exists}\ \mathcal{F}$ contains all isolated weakly-normalizing proof-terms by definition hence those sets satisfy $(\mathbf{P}_{3b})$.

$\tilde{\Rightarrow}$) $(\mathbf{P}_1)$ Let $\pi \in E \tilde{\Rightarrow} F$. Either π is isolated and weakly normalizing. Or there exists π_1 such that $\pi \rhd^* \lambda\alpha\ \pi_1$ and for all $\pi' \in E$, $(\pi'/\alpha)\pi_1 \in F$. Since E satisfies $(\mathbf{P}_{3b})$, we have $\alpha \in E$, hence $\pi_1 \in F \subseteq$WN, since F satisfies P_1 and so do $\lambda\alpha.\pi_1$ and finally π.

$(\mathbf{P}_{3a})$ Let π be a neutral proof-term and $\pi' \in E \tilde{\Rightarrow} F$ such that $\pi \rhd \pi'$. If π' is isolated and WN, then $\pi \in$WN and is also isolated, by confluence. Otherwise $\pi \rhd \pi' \rhd^* \lambda\alpha.\pi_1$ with $(\pi''/\alpha)\pi_1 \in F$ for all $\pi'' \in E$. In both cases, $\pi \in E \tilde{\Rightarrow} F$.

$\tilde{\wedge}$) $(\mathbf{P}_1)$ Let $\pi \in E \tilde{\wedge} F$. Either π is isolated and weakly normalizing. Or there exists π_1, π_2 such that $\pi \rhd^* \langle\pi_1, \pi_2\rangle$ with $\pi \in E \subseteq$WN and $\pi_2 \in F \subseteq$WN, hence $\pi \in$WN.

$(\mathbf{P}_{3a})$ Let π be a neutral proof-term and $\pi' \in E \tilde{\wedge} F$ such that $\pi \rhd \pi'$. If π' is isolated and WN, then $\pi \in$WN and is also isolated, by confluence. Otherwise $\pi \rhd \pi' \rhd^* \langle\pi_1, \pi_2\rangle$ with $\pi_1 \in E$ and $\pi_2 \in F$. In both cases, $\pi \in E \tilde{\wedge} F$.

$\tilde{\vee}$) $(\mathbf{P}_1)$ Let $\pi \in E \tilde{\vee} F$. Either π is isolated and weakly normalizing. Or there exists π_1 such that $\pi \rhd^* i(\pi_1)$ (resp. $j(\pi_1)$) with $\pi_1 \in E$ (resp. F). Since E and F satisfy $(\mathbf{P}_1)$, we have $\pi \in WN$.

$(\mathbf{P}_{3a})$ Let π be a neutral proof-term and $\pi' \in E \tilde{\vee} F$ such that $\pi \rhd \pi'$. If π' is isolated and WN, then $\pi \in$WN and is also isolated, by confluence. Otherwise $\pi \rhd \pi' \rhd^* i(\pi_1)$ (resp. $j(\pi_1)$) with $\pi_1 \in E$ (resp. F). In both cases, $\pi \in E \tilde{\vee} F$.

$\tilde{\forall}$) $(\mathbf{P}_1)$ Let $\pi \in \tilde{\forall}\mathcal{F}$. Either π is isolated and weakly normalizing. Or there exists π_1 such that $\pi \rhd^* \lambda x\ \pi_1$ and for all terms t and sets $G \in \mathcal{F}$, $(t/x)\pi_1 \in \mathcal{F}$. Since each $G \in \mathcal{F}$ satisfies $(\mathbf{P}_1)$, we have, in particular, $\pi_1 = (x/x)\pi_1 \in$WN, and so do $\lambda x.\pi_1$ and finally π.

$(\mathbf{P}_{3a})$ Let π be a neutral proof-term and $\pi' \in \tilde{\forall}\mathcal{F}$ such that $\pi \rhd \pi'$. If π' is isolated and WN, then $\pi \in$WN and is also isolated, by confluence. Otherwise $\pi \rhd \pi' \rhd^* \lambda x.\pi_1$ with $(t/x)\pi_1 \in G$ for all terms t and $G \in \mathcal{F}$. In both cases, $\pi \in \tilde{\forall}\mathcal{F}$.

$\tilde{\exists}$) $(\mathbf{P}_1)$ Let $\pi \in \tilde{\exists}\mathcal{F}$. Either π is isolated and weakly normalizing. Or there exists π_1, $G \in \mathcal{F}$ and a term t such that $\pi \rhd^* \langle t, \pi_1\rangle$ with $\pi_1 \in G$. Hence $\pi_1 \in$WN and so does π.

$(\mathbf{P}_{3a})$ Let π be a neutral proof-term and $\pi' \in \tilde{\exists}\mathcal{F}$ such that $\pi \rhd \pi'$. If π' is isolated and WN, then $\pi \in$WN and is also isolated, by confluence. Otherwise $\pi \rhd \pi' \rhd^* \langle t, \pi_1\rangle$ with π_1 in some $G \in \mathcal{F}$. In both cases, $\pi \in \tilde{\exists}\mathcal{F}$.

▶ **Definition 10** (The algebra of WN-reducibility candidates)**.**
The set $\mathcal{B}$ is the set of WN-reducibility candidates. The sets $\mathcal{A}$ and $\mathcal{E}$ are $\wp(\mathcal{B})$ (the powerset of $\mathcal{B}$). The operations are those of Def. 8. The pre-order is the trivial pre-order, *i.e.* $a \leq b$ for any $a, b \in \mathcal{B}$.

Notice that the existence of a model valued on this pre-Heyting algebra for some theory can be obtained by super-consistency [9] since this pre-Heyting algebra is full, ordered (using the inclusion) and complete.

Syntactic proof of (weak normalization and) cut elimination

In the following, we show how to adapt the usual reducibility method ([13] and [7, 8] for deduction modulo) to prove that the existence of a model valued on the pre-Heyting algebra of WN-reducbility candidates entails weak normalization (and therefore, syntactically, cut elimination) for the considered theory in deduction modulo.
We suppose that the denotation $[\![.]\!]_.$ of a theory forms a model valued on the pre-Heyting algebra of WN-reducibility candidates in the sense of Def. 6.

As usual, we work with proof-terms substitutions adapted to a typing context, that we call *assignments*.

▶ **Definition 11** (Assignments on a typing context). Given a typing context Γ and an environment ϕ, an assignment is a proof-term substitution σ such that for all declarations $\alpha : A$ in Γ, we have $\sigma\alpha \in [\![A]\!]_\phi^{\mathcal{M}}$. When it is clear from context, we may not precise the typing context on which some assignment is defined.

▶ **Lemma 12** (Adequacy lemma).
For all typing contexts Γ, formulæ A, environments ϕ, term substitutions θ, proof-terms π, and assignments σ on Γ,

$$\textit{if } \Gamma \vdash \pi : A \textit{ then } \sigma\theta\pi \in [\![A]\!]_\phi.$$

Proof. By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case analysis on the last rule.

- If the last rule is `axiom`, then π is a variable α and there exists some formula $B \equiv A$ and a declaration $\alpha : B$ in Γ. By hypothesis on the assignment, $\sigma\theta\alpha = \sigma\alpha \in [\![B]\!]_\phi = [\![A]\!]_\phi$ since $[\![.]\!]_.$ is a model.
- If the last rule is $\Rightarrow$`-intro`, then π is an abstraction $\lambda\alpha.\nu$ (we can suppose that α is not in the domain of σ by α-renaming), and there exists formulæ B and C such that $A \equiv B \Rightarrow C$ and $\Gamma, \alpha : B \vdash \nu : C$ (with a smaller derivation). For all $\mu \in [\![B]\!]_\phi$, $(\mu/\alpha)\sigma$ is an assignment on $\Gamma, \alpha : B$ hence by induction hypothesis $(\mu/\alpha)\theta\sigma\nu \in [\![C]\!]_\phi$. Finally, by Def. 8, $\sigma\theta\pi = \lambda\alpha.\sigma\theta\nu \in [\![B]\!]_\phi \tilde{\Rightarrow} [\![C]\!]_\phi = [\![B \Rightarrow C]\!]_\phi = [\![A]\!]_\phi$.
- If the last rule is $\Rightarrow$`-elim`, then π is an application $(\mu\ \nu)$, and there exists formulæ B and C such that $\Gamma \vdash \mu : C$, $\quad \Gamma \vdash \nu : B$ (with smaller derivations), and $C \equiv B \Rightarrow A$. By induction hypothesis, we have $\sigma\theta\mu \in [\![C]\!]_\phi = [\![B \Rightarrow A]\!]_\phi = [\![B]\!]_\phi \tilde{\Rightarrow} [\![A]\!]_\phi$ and $\sigma\theta\nu \in [\![B]\!]_\phi$. Either $\sigma\theta\mu$ is isolated and weakly normalizing (since it belongs to $[\![C]\!]_\phi$), then so is $\sigma\theta\pi = \sigma\theta(\mu\ \nu) = (\sigma\theta\mu\ \sigma\theta\nu)$ which therefore belongs to $[\![A]\!]_\phi$ by $(\mathbf{P}_{3b})$. Or there exists α and π_1 such that $\sigma\theta\mu \rhd^* \lambda\alpha.\pi_1$ and $(\sigma\theta\nu/\alpha)\pi_1 \in [\![A]\!]_\phi$ since $\sigma\theta\mu \in [\![B]\!]_\phi \tilde{\Rightarrow} [\![A]\!]_\phi$. Finally $\sigma\theta\pi = (\sigma\theta\mu\ \sigma\theta\nu) \in [\![A]\!]_\phi$ since it is neutral and it reduces to $(\sigma\theta\nu/\alpha)\pi_1 \in [\![A]\!]_\phi$ (by a repeated use of $(\mathbf{P}_{3a})$).
- Proofs of the other cases follow the same scheme.

◀

In the following, we shall bypass that method and make appear an underlying Heyting algebra structure from WN-reducibility candidates in order to provide a full semantic proof that the existence of a model valued on WN-reducibility candidates entails cut elimination.

4 A Heyting algebra

Pre-Heyting Algebras can easily be turned into Heyting Algebras [9] by a quotient operation, but this is of no help here since with the algebra of Def. 10 we obtain a one-point (*i.e.* trivial) Heyting Algebra. In order to achieve our purpose and extract a non-trivial Heyting algebra from our structure of pre-models for weak normalization, we interpret propositions by sets of contexts (*outer values*) given by this notion of pre-model and we show that it forms a Heyting Algebra with well chosen operations.
For the following, we consider a theory in deduction modulo and we suppose that there exists a model $\mathcal{M}$, valued on the pre-Heyting algebra of *WN*-reducibility candidates, for that theory. We write $[\![.]\!]^{\mathcal{M}}$ for the denotation it defines.

4.1 Outer value

Notice that the algebra we are to build contains sets of contexts (i.e. sequences of unlabeled formulæ) and not typing contexts. We say that a typing context Δ is a *labeling* of a context $\Gamma = A_1, \dots, A_n$ if there exists proof-variables $\alpha_1, \dots, \alpha_n$ such that $\Delta = \alpha_1 : A_1, \dots, \alpha_n : A_n$.

▶ **Definition 13** (Outer Value). Let A be a formula. We define its *weak outer value* $\lfloor A \rfloor$ as the set of contexts Γ such that there exists a labeling Δ of Γ and a proof-term π with:
- $\Delta \vdash \pi : A$
- for any environment ϕ, any term-substitution θ, any assignment σ on Δ, $\sigma\theta\pi \in [\![A]\!]^{\mathcal{M}}_{\phi}$

▶ **Lemma 14.** *For all formulæ A, B and contexts Γ, we have*
- $A \in \lfloor A \rfloor$
- $\Gamma \in \lfloor A \rfloor$ *implies* $\Gamma, B \in \lfloor A \rfloor$
- $\lfloor A \rfloor = \lfloor B \rfloor$ *if* $A \equiv B$.

Proof. Each point is treated separately and is straightforward.
- Because the labeling $\alpha : A$ and the proof $\alpha : A \vdash \alpha : A$ verify Def. 13.
- If Δ is a labeling of Γ, π is a suitable proof-term and β is a fresh proof variable, the typing context $\Gamma, \beta : B$ and the proof $\Gamma, \beta : B \vdash \pi : A$ verify Def. 13.
- From Def. 6, the same proof-terms are $[\![A]\!]^{\mathcal{M}}$ and $[\![B]\!]^{\mathcal{M}}$. Moreover, if $\Gamma \vdash \pi : A$ then $\Gamma \vdash \pi : B$ is an easy property of Deduction modulo [13].

◀

In the following, for all (unlabeled) contexts Γ, proof-terms π and formulæ A, we shall write $\Gamma \vdash \pi : A$ if there exists a labeling Δ of Γ such that $\Delta \vdash \pi : A$.

4.2 The algebra

With the help of Def. 13 we can now define a Heyting algebra:

▶ **Definition 15** (Heyting algebra Ω).
We define Ω to be the set containing all the $\lfloor A \rfloor$ for any formula A. It is ordered by inclusion. $\mathcal{A}$ and $\mathcal{E}$ are both equal to the set of all $\{\lfloor (t/x)A \rfloor \mid t \in \mathcal{T}\}$ for any formula A, where $\mathcal{T}$ is the set of open terms. We define the operations and constants to be:

- $\check{\bot} = \lfloor \bot \rfloor$
- $\check{\top} = \lfloor \bot \Rightarrow \bot \rfloor$
- $\lfloor A \rfloor \check{\wedge} \lfloor B \rfloor = \lfloor A \wedge B \rfloor$
- $\lfloor A \rfloor \check{\vee} \lfloor B \rfloor = \lfloor A \vee B \rfloor$

- $\lfloor A \rfloor \check{\Rightarrow} \lfloor B \rfloor = \lfloor A \Rightarrow B \rfloor$
- $\check{\forall} \{\lfloor (t/x)A \rfloor \mid t \in \mathcal{T}\} = \lfloor \forall x A \rfloor$
- $\check{\exists} \{\lfloor (t/x)A \rfloor \mid t \in \mathcal{T}\} = \lfloor \exists x A \rfloor$

▶ Remark. This definition should formally appear below Lem. 16 and 17, and not just above. Indeed, those lemmata ensure that the introduced operators are well-defined, and do not depend on them. However, we believe that presenting first Def. 15 is more natural.

To show that Def. 15 defines a Heyting algebra, we first prove that $\check{\wedge}$ is set intersection.

▶ **Lemma 16.** *For any formulæ A and B,* $\lfloor A \rfloor \check{\wedge} \lfloor B \rfloor = \lfloor A \rfloor \cap \lfloor B \rfloor$.

Proof.

- $\lfloor A \rfloor \cap \lfloor B \rfloor \subseteq \lfloor A \wedge B \rfloor$: Let $\Gamma \in \lfloor A \rfloor \cap \lfloor B \rfloor$. Let π_1 and π_2 be proof-terms verifying the conditions of Def. 13. Since $\Gamma \vdash \pi_1 : A$ and $\Gamma \vdash \pi_2 : B$, we have $\Gamma \vdash \langle \pi_1, \pi_2 \rangle : A \wedge B$. We claim that $\langle \pi_1, \pi_2 \rangle$ is a suitable proof-term that verifies the conditions of Def. 13: let ϕ be an environment, σ be an assignment and θ be a term-substitution. Then $\sigma\theta\langle \pi_1, \pi_2 \rangle = \langle \sigma\theta\pi_1, \sigma\theta\pi_2 \rangle$ and, since by Def. 13 $\sigma\theta\pi_1 \in [\![A]\!]^{\mathcal{M}}_{\phi}$ and $\sigma\theta\pi_2 \in [\![B]\!]^{\mathcal{M}}_{\phi}$, we get by Def. 8 that $\langle \sigma\theta\pi_1, \sigma\theta\pi_2 \rangle \in [\![A \wedge B]\!]^{\mathcal{M}}_{\phi}$.
- $\lfloor A \wedge B \rfloor \subseteq \lfloor A \rfloor \cap \lfloor B \rfloor$: we show only $\lfloor A \wedge B \rfloor \subseteq \lfloor A \rfloor$ since the other inclusion has exactly the same proof. Let $\Gamma \in \lfloor A \wedge B \rfloor$ and let π be a proof-term that verifies the conditions of Def. 13. Then $\Gamma \vdash fst(\pi) : A$ and we claim that $fst(\pi)$ is a suitable proof-term: let ϕ be an environment, σ be an assignment and θ be a term-substitution. Then $\sigma\theta fst(\pi) = fst(\sigma\theta\pi)$. By hypothesis, $\sigma\theta\pi \in [\![A \wedge B]\!]^{\mathcal{M}}_{\phi}$ and according to Def. 8 we have two choices. If $\sigma\theta\pi$ is *WN* isolated, then so is $fst(\sigma\theta\pi)$ and then $fst(\sigma\theta\pi) \in \lfloor A \rfloor$ by $(\mathbf{P}_{3b})$. Otherwise, $\sigma\theta\pi \rhd^* \langle \pi_1, \pi_2 \rangle$ ($\sigma\theta\pi$ cannot reduce to a non neutral term of another form since the theory is not confusing and the types of β-equivalent proof-terms are equivalent). with $\pi_1 \in [\![A]\!]^{\mathcal{M}}_{\phi}$ and $\pi_2 \in [\![B]\!]^{\mathcal{M}}_{\phi}$. Then we have the following sequence:

  $$fst(\sigma\theta\pi) \rhd^* fst(\langle \pi_1, \pi_2 \rangle) \rhd^1 \pi_1$$

  Since every term but the last in this reduction sequence is neutral, we conclude by a repeated use of $(\mathbf{P}_{3a})$ that $fst(\sigma\theta\pi) \in [\![A]\!]^{\mathcal{M}}_{\phi}$.

◀

We check now that Def. 15 does define a Heyting algebra (since $\mathcal{M}$ is a model valued on the pre-Heyting algebra of *WN*-reducibility candidates).

▶ **Lemma 17.** *The constants* $\check{\bot}$, $\check{\top}$ *and the operators* $\check{\wedge}$, $\check{\vee}$, $\check{\Rightarrow}$, $\check{\forall}$, $\check{\exists}$ *are operators of a Heyting algebra, the order being inclusion.*

Proof. We check one by one each operator:

- $\check{\top}$ is the greatest element. Let $\lfloor C \rfloor \in \Omega$ and $\Gamma \in \lfloor C \rfloor$. Then $\Gamma \vdash \lambda\alpha.\alpha : \bot \Rightarrow \bot$ and to show $\Gamma \in \lfloor \bot \Rightarrow \bot \rfloor$ we claim that $\lambda\alpha.\alpha$ is a suitable proof-term that verifies the conditions of Def. 13. Let ϕ be an environment, σ be an assignment and θ be a term-substitution. $\sigma\theta(\lambda\alpha.\alpha) = \lambda\alpha.\alpha$ and $\lambda\alpha.\alpha \in [\![\bot \Rightarrow \bot]\!]^{\mathcal{M}}_{\phi}$ by Def. 8 since for any $\pi' \in [\![\bot]\!]^{\mathcal{M}}_{\phi}$, $(\pi'/\alpha)\alpha = \pi' \in [\![\bot]\!]^{\mathcal{M}}_{\phi}$.
- $\check{\bot}$ is the least element. Let $\lfloor A \rfloor \in \Omega$, let $\Gamma \in \lfloor \bot \rfloor$ and π be a proof term verifying the conditions of Def. 13. Then $\Gamma \vdash (\delta_\bot\ \pi) : A$ and we claim that $(\delta_\bot\ \pi)$ is a suitable proof-term (Def. 13). Let ϕ be an environment, σ be an assignment and θ be a term-substitution. $\sigma\theta(\delta_\bot\ \pi) = \delta_\bot\ \sigma\theta\pi$ is isolated since there is no $\delta_\bot$ reduction rule and it is *WN* since $\sigma\theta\pi$ is *WN* by hypothesis on π. So $\Gamma \in \lfloor A \rfloor$.

- $\lfloor A \wedge B \rfloor$ is the greatest lower bound of $\lfloor A \rfloor$ and $\lfloor B \rfloor$ follows directly from Lemma 16 and the fact that set intersection is the greatest lower bound for set inclusion.
- $\lfloor A \vee B \rfloor$ is the least upper bound of $\lfloor A \rfloor$ and $\lfloor B \rfloor$:
  - $\lfloor A \rfloor \subseteq \lfloor A \vee B \rfloor$. Let $\Gamma \in \lfloor A \rfloor$ and π a proof-term verifying the conditions of Def. 13. Then $\Gamma \vdash i(\pi) : A \vee B$ and we claim that $i(\pi)$ is a suitable proof-term. Let ϕ be an environment, σ be an assignment and θ be a term-substitution. $\sigma\theta i(\pi) = i(\sigma\theta\pi)$ and we know by hypothesis that $\sigma\theta\pi \in \llbracket A \rrbracket_\phi^\mathcal{M}$. By Def. 8, $i(\sigma\theta\pi) \in \llbracket A \vee B \rrbracket_\phi^\mathcal{M}$.
  - $\lfloor B \rfloor \subseteq \lfloor A \vee B \rfloor$ has exactly the same proof.
  - if $\lfloor A \rfloor \subseteq \lfloor C \rfloor$ and $\lfloor B \rfloor \subseteq \lfloor C \rfloor$ then $\lfloor A \vee B \rfloor \subseteq \lfloor C \rfloor$. Let C be a formula such that $\lfloor A \rfloor \subseteq \lfloor C \rfloor$ and $\lfloor B \rfloor \subseteq \lfloor C \rfloor$, let $\Gamma \in \lfloor A \rfloor$ and let π be the proof-term verifying the conditions of Def. 13 such that $\Gamma \vdash \pi : A \vee B$. We show that $\Gamma \in \lfloor C \rfloor$.
    By Lem. 14 $A \in \lfloor A \rfloor \subseteq \lfloor C \rfloor$ and $\Gamma, A \in \lfloor C \rfloor$. Let π_1 be the proof-term verifying the conditions of Def. 13 such that $\Gamma, \alpha : A \vdash \pi_1 : C$. Analogously let π_2 be the suitable proof-term such that $\Gamma, \beta : B \vdash \pi_2 : C$. We build the proof:
    $$\frac{\Gamma, \alpha : A \vdash \pi_1 : C \qquad \Gamma, \beta : B \vdash \pi_2 : C \qquad \Gamma \vdash \pi : A \vee B}{\Gamma \vdash (\delta\ \pi\ \alpha\pi_1\ \beta\pi_2) : C} \vee\text{-elim}$$
    We claim that $(\delta\ \pi\ \alpha\pi_1\ \beta\pi_2)$ is a suitable proof-term. Let ϕ be an environment, σ be an assignment and θ be a term-substitution. Up to an α-renaming of α, β we have $\sigma\theta(\delta\pi\ \alpha\pi_1\ \beta\pi_2) = (\delta\ \sigma\theta\pi\ \alpha\sigma\theta\pi_1\ \beta\sigma\theta\pi_2)$. $\sigma\theta\pi \in \llbracket A \vee B \rrbracket_\phi^\mathcal{M}$. If $\sigma\theta\pi$ is *WN* isolated, then so is $\sigma\theta(\delta\ \pi\ \alpha\pi_1\ \beta\pi_2)$ and this proof-term belongs to $\llbracket C \rrbracket_\phi^\mathcal{M}$ by $(\mathbf{P}_{3b})$. Otherwise, following Def. 8, assume that $\sigma\theta\pi \rhd^* i(\pi')$ with $\pi' \in \llbracket A \rrbracket_\phi^\mathcal{M}$ (the other case is similar). Then we have the reduction sequence:
    $$\sigma\theta(\delta\ \pi\ \alpha\pi_1\ \beta\pi_2) \rhd^* (\delta\ i(\pi')\ \alpha\sigma\theta\pi_1\ \beta\sigma\theta\pi_2) \rhd (\pi'/\alpha)\sigma\theta\pi_1$$
    Since $\pi' \in \llbracket A \rrbracket_\phi^\mathcal{M}$, $(\pi'/\alpha)\sigma = \sigma + (\pi'/\alpha)$ is an assignment on $\Gamma, \alpha : A$ and by hypothesis on π_1 $(\pi'/\alpha)\sigma\theta\pi_1 \in \llbracket C \rrbracket_\phi^\mathcal{M}$. Every term but the last in the above reduction sequence is neutral, so we conclude by a repeated use of $(\mathbf{P}_{3a})$ that $\sigma\theta(\delta\ \pi\ \alpha\pi_1\ \beta\pi_2) \in \llbracket C \rrbracket_\phi^\mathcal{M}$.
- $\lfloor A \Rightarrow B \rfloor$ is an implication operator. From Def. 1 (see also [21]) we must check:
  $$\lfloor A \rfloor \leq \lfloor B \rfloor \mathrel{\check{\Rightarrow}} \lfloor C \rfloor \text{ iff } \lfloor A \rfloor \mathrel{\check{\wedge}} \lfloor B \rfloor \leq \lfloor C \rfloor$$
  - if part: by Lem. 16 we can assume $\lfloor A \rfloor \cap \lfloor B \rfloor \subseteq \lfloor C \rfloor$. Let $\Gamma \in \lfloor A \rfloor$. By Lem. 14, $\Gamma, B \in \lfloor A \rfloor \cap \lfloor B \rfloor$, so by hypothesis, $\Gamma, B \in \lfloor C \rfloor$. Let π be the proof-term verifying the conditions of Def. 13 such that $\Gamma, \alpha : B \vdash \pi : C$. Then $\Gamma \vdash \lambda\alpha.\pi : B \Rightarrow C$ and we claim that $\lambda\alpha.\pi$ is a suitable proof-term (Def. 13). Let ϕ be an environment, σ be an assignment, θ be a term-substitution and assume without loss of generality that α is fresh, so that $\sigma\theta\lambda\alpha.\pi = \lambda\alpha.\sigma\theta\pi$. Let $\pi' \in \llbracket B \rrbracket_\phi^\mathcal{M}$, then, by Def. 8, we must show that $(\pi'/\alpha)\sigma\theta\pi = (\sigma + (\pi'/\alpha))\theta\pi \in \llbracket C \rrbracket_\phi^\mathcal{M}$ but this is immediate by hypothesis on π.
  - only if part: by Lem. 16, let $\Gamma \in \lfloor A \rfloor \cap \lfloor B \rfloor$. By hypothesis, $\Gamma \in \lfloor B \rfloor \mathrel{\check{\Rightarrow}} \lfloor C \rfloor = \lfloor B \Rightarrow C \rfloor$. Let π and π_B the proof-terms verifying the conditions of Def. 13 such that $\Gamma \vdash \pi : B \Rightarrow C$ and $\Gamma \vdash \pi_B : B$. We have $\Gamma \vdash (\pi\ \pi_B) : C$ and we claim that $(\pi\ \pi_B)$ is a suitable proof-term (Def. 13). Let ϕ be an environment, σ be an assignment and θ be a term-substitution. $\sigma\theta\pi \in \llbracket B \Rightarrow C \rrbracket_\phi^\mathcal{M}$ by hypothesis. If it is *WN* isolated, then so is $\sigma\theta(\pi\ \pi_B)$ and therefore it belongs to $\llbracket C \rrbracket_\phi^\mathcal{M}$. Otherwise, following Def. 8, assume that $\sigma\theta\pi \rhd^* \lambda\alpha\pi_1$ with π_1 verifying the associated hypothesis. Then we have the reduction sequence:
    $$\sigma\theta(\pi\ \pi_B) \rhd^* \lambda\alpha.\pi_1\ \sigma\theta\pi_B \rhd (\sigma\theta\pi_B/\alpha)\pi_1$$

By hypothesis on π_1 and since $\sigma\theta\pi_B \in [\![B]\!]_\phi^\mathcal{M}$, $(\sigma\theta\pi_B/\alpha)\pi_1 \in [\![C]\!]_\phi^\mathcal{M}$, and by a repeated use of $(\mathbf{P}_{3a})$, $\sigma\theta(\pi\ \pi_B) \in [\![C]\!]_\phi^\mathcal{M}$.

- $\lfloor \forall x A \rfloor$ is the greatest lower bound of the set $\{\lfloor (t/x)A \rfloor \mid t \in \mathcal{T}\}$:
  - if for any t, $\lfloor C \rfloor \subseteq \lfloor (t/x)A \rfloor$ then $\lfloor C \rfloor \subseteq \lfloor \forall x A \rfloor$. Let $\Gamma \in \lfloor C \rfloor$. In particular, assuming without loss of generality that x is fresh, $\Gamma \in \lfloor (x/x)A \rfloor$. Let π be the proof-term that verifies the conditions of Def. 13. Then $\Gamma \vdash \lambda x.\pi : \forall x A$, and we claim that $\lambda x.\pi$ is a suitable proof-term. Let ϕ be an environment, σ be an assignment and θ be a term-substitution. Assuming for simplicity that x does not appear in $\sigma\theta$, $\sigma\theta\lambda x.\pi = \lambda x.\sigma\theta\pi$, and we show that for any term t and any $d \in M$, $(t/x)\sigma\theta\pi \in [\![A]\!]_{\phi+(d/x)}^\mathcal{M}$. This is immediate by hypothesis on π, applied to the environment $\phi + (d/x)$, the assignment σ and the term-substitution $\theta + (t/x)$.
  - $\lfloor \forall x A \rfloor \subseteq \lfloor (t/x)A \rfloor$ for any t: let $\Gamma \in \lfloor \forall x A \rfloor$, and let π a proof-term that verifies the conditions of Def. 13. Then $\Gamma \vdash (\pi\ t) : (t/x)A$ and we claim that $(\pi\ t)$ is a suitable proof-term. Let ϕ be an environment, σ be an assignment and θ be a term-substitution. If $\sigma\theta\pi$ is *WN* isolated then so is $\sigma\theta(\pi\ t)$ and this proof-term belongs to $[\![(t/x)A]\!]_\phi^\mathcal{M}$. Otherwise, $\sigma\theta\pi \rhd^* \lambda x.\pi_1$ and $(u/x)\pi_1 \in [\![A]\!]_{\phi+(d/x)}^\mathcal{M}$ for any term u and any $d \in M$, in particular for θt and $[\![t]\!]_\phi^\mathcal{M}$. Thus, by remark 3.2, $(\theta t/x)\pi_1 \in [\![(t/x)A]\!]_\phi^\mathcal{M}$ and by a repeated use of $(\mathbf{P}_{3b})$, $\sigma\theta(\pi\ t) \in [\![(t/x)A]\!]_\phi^\mathcal{M}$.

- $\lfloor \exists x A \rfloor$ is the least upper bound of the set $\{\lfloor (t/x)A \rfloor \mid t \in \mathcal{T}\}$:
  - $\lfloor (t/x)A \rfloor \subseteq \lfloor \exists x A \rfloor$ for any t: let t be a term, $\Gamma \in \lfloor (t/x)A \rfloor$ and let π a proof-term that verifies the conditions of Def. 13. Then $\Gamma \vdash \langle t, \pi \rangle : \exists x A$ and we claim that $\langle t, \pi \rangle$ is a suitable proof-term (Def. 13). Let ϕ be an environment, σ be an assignment and θ be a term-substitution. $\sigma\theta\langle t, \pi \rangle$ respects the condition of Def. 8 since it is such that $\sigma\theta\pi \in [\![(t/x)A]\!]_\phi^\mathcal{M} = [\![A]\!]_{\phi+[\![t]\!]_\phi^\mathcal{M}}^\mathcal{M}$ by hypothesis and remark 3.2.
  - if, for any t, $\lfloor (t/x)A \rfloor \subseteq \lfloor C \rfloor$, then $\lfloor \exists x A \rfloor \subseteq \lfloor C \rfloor$: Let $\Gamma \in \lfloor \exists x A \rfloor$. In particular, assuming without loss of generality that x is fresh, $\lfloor (x/x)A \rfloor \subseteq \lfloor C \rfloor$, and by Lem. 14 we have $\Gamma, A \in \lfloor C \rfloor$. Let π and π' be the proof-terms verifying the conditions of Def. 13 such that $\Gamma \vdash \pi : \exists x A$ and $\Gamma, \alpha : A \vdash \pi' : C$, with α a fresh proof variable. We build the proof:

    $$\frac{\Gamma \vdash \pi : \exists x A \qquad \Gamma, \alpha : A \vdash \pi' : C}{\Gamma \vdash (\delta_\exists\ \pi\ x\alpha\pi') : C}$$

    We claim that $(\delta_\exists\ \pi\ x\alpha\pi')$ is a suitable proof-term (Def. 13). Let ϕ be an environment, σ be an assignment and θ be a term-substitution. If $\sigma\theta\pi$ is *WN* isolated, then so is $\sigma\theta(\delta_\exists\ \pi\ x\alpha\pi')$ and this proof-term belongs to $[\![C]\!]_\phi^\mathcal{M}$. Otherwise $\sigma\theta\pi \rhd^* \langle t_1, \pi_1 \rangle$ and $\pi_1 \in [\![A]\!]_{\phi+(d/x)}^\mathcal{M}$ for some $d \in M$. Assuming for simplicity that x and α do not appear in σ nor in θ:

    $$\sigma\theta(\delta_\exists\ \pi\ x\alpha\pi') \rhd^* \delta_\exists\ \langle t_1, \pi_1 \rangle\ x\alpha\pi' \rhd (t/x, \pi_1/\alpha)\sigma\theta\pi'$$

    Since $\sigma + (\pi_1/\alpha)$ is a suitable assignment (adapted to the environment $\phi + (d/x)$), we have, by hypothesis on π, $(\sigma + (\pi_1/\alpha))(\theta + (t/x))\pi' \in [\![C]\!]_{\phi+(d/x)}^\mathcal{M} = [\![C]\!]_\phi^\mathcal{M}$ so, that by a repeated use of $(\mathbf{P}_{3a})$ we conclude that $\sigma\theta(\delta_\exists\ \pi\ x\alpha\pi') \in [\![C]\!]_\phi^\mathcal{M}$.

◀

We can conclude finally that Ω is a Heyting algebra when $\mathcal{M}$ is a model valued on the pre-Heyting algebra of *WN*-reducibility candidates[1]. We shall see now how it allows to prove the cut elimination property for the considered theory.

5 Model and cut elimination

We continue to suppose that we have a theory, defined by a language $\langle f_i, P_j \rangle$ and a congruence relation $\equiv$, that has a model valued on the pre-Heyting algebra of *WN*-reducibility candidates. We build a model valued on the Heyting algebra Ω of the previous section and we prove that the existence of such a model implies the cut elimination property.

▶ **Definition 18** (Heyting algebra model interpretation).
We define $\mathcal{D}$ as the Ω-valued structure $\langle \mathcal{T}', \Omega, \hat{f}_i, \hat{P}_j \rangle$ where:

- $\mathcal{T}'$ is the set of classes modulo $\equiv$ of open terms (the class of t is denoted $\bar{t}$),
- for any n-ary function symbol f: $\hat{f}^{\mathcal{D}}(\overline{t_1}, ..., \overline{t_n}) = \overline{f(t_1, ..., t_n)}$,
- for any n-ary predicate symbol P: $\hat{P}^{\mathcal{D}}(\overline{t_1}, ..., \overline{t_n}) = \lfloor P(t_1, ..., t_n) \rfloor$.

The last definition is well-formed since the function $t_1, ..., t_n \mapsto \lfloor P(t_1, ..., t_n) \rfloor$ is constant on classes of terms modulo $\equiv$.

As explained in the following lemma, $\lfloor . \rfloor$ is the denotation generated by Def. 18.

▶ **Lemma 19.** *For all terms t, formulæ A, assignments ϕ taking their values in $\mathcal{T}'$, and substitutions σ such that $\overline{\sigma(x)} = \phi(x)$ for any variable x, we have $[\![t]\!]^{\mathcal{D}}_{\phi} = \overline{\sigma t}$ and $[\![A]\!]^{\mathcal{D}}_{\phi} = \lfloor \sigma A \rfloor$.*

Proof. By structural induction on t and then A. There is a little subtlety in the case of quantifiers. Let us process the $\forall$ case, assuming that x is a fresh variable symbol:

$$\begin{array}{rcl} [\![\forall x\ A]\!]^{\mathcal{D}}_{\phi} & = & \tilde{\forall}\{[\![A]\!]^{\mathcal{D}}_{\phi+(\bar{t}/x)} \mid \bar{t} \in \mathcal{T}'\} \\ & = & \tilde{\forall}\{\lfloor(\sigma + (t/x))A\rfloor \mid \bar{t} \in \mathcal{T}'\} \quad = \quad \tilde{\forall}\{\lfloor(\sigma + (t/x))A\rfloor \mid t \in \mathcal{T}\} \\ & = & \lfloor \sigma \forall x\ A \rfloor \end{array}$$

The first equality is the definition of a model interpretation, the second comes by induction hypothesis, the last one is an application of Lem. 17, those steps are standard. Lem. 14 justifies the third equality on the second line: instead of having only one representative of the class $\bar{t}$, we can consider them all. ◀

▶ **Lemma 20.** *Let $t_1 \equiv t_2$ be two terms, $A \equiv B$ be two formulæ and ϕ be an assignment taking its values in $\mathcal{T}'$. Then $[\![t_1]\!]^{\mathcal{D}}_{\phi} = [\![t_2]\!]^{\mathcal{D}}_{\phi}$ and $[\![A]\!]^{\mathcal{D}}_{\phi} = [\![B]\!]^{\mathcal{D}}_{\phi}$.*

Proof. Let σ be a substitution fulfilling the hypothesis of Lem. 19. Then $\sigma t_1 \equiv \sigma t_2$, and $\overline{\sigma t_1} = \overline{\sigma t_2}$ and the result follows by Lem. 19. Similarly, $\sigma A \equiv \sigma B$, $\lfloor \sigma A \rfloor = \lfloor \sigma B \rfloor$ by Lem. 14 and the result follows by Lem. 19. ◀

Hence the model interpretation given by Def. 18 is adapted to the congruence $\equiv$. And we finally get the cut-elimination theorem:

▶ **Theorem 21.** *If the sequent $\Gamma \vdash A$ has a proof, then there exists a weakly normalizable proof of this sequent.*

[1] We can also show that it is not trivial ($\check{\top} \neq \check{\bot}$) but for this we need Thm. 21 that follows, to show, for instance, that the empty context does not belong to $\check{\bot}$

Proof. We let $\llbracket\Gamma\rrbracket^{\mathcal{D}} = \bigcap\{\llbracket A\rrbracket^{\mathcal{D}} \mid A \in \Gamma\}$ and $\lfloor\Gamma\rfloor = \bigcap\{\lfloor A\rfloor \mid A \in \Gamma\}$. By usual soundness w.r.t. Heyting algebras, $\llbracket\Gamma\rrbracket^{\mathcal{D}} \subseteq \llbracket A\rrbracket^{\mathcal{D}}$. Lem. 14 implies $\Gamma \in \lfloor\Gamma\rfloor = \llbracket\Gamma\rrbracket^{\mathcal{D}}$, so $\Gamma \in \llbracket A\rrbracket^{\mathcal{D}} = \lfloor A\rfloor$ and by Def. 13 there exists a proof-term π_0 such that $\Gamma \vdash \pi_0 : A$ and, in particular, $\pi_0 \in \llbracket A\rrbracket^{\mathcal{M}}_{\phi}$ for any ϕ. Therefore, π_0 is weakly normalizable. ◀

5.1 Towards normalization by evaluation

In order to obtain some algorithm of normalization by evaluation, we need to strengthen the result of Thm. 21. First we need to obtain some proof in normal form, instead of, in Thm. 21, a weakly normalizable proof:

$$\text{if } \Gamma \vdash A \text{ has a proof then } \Gamma \vdash A \text{ has a proof in normal form.} \quad (\mathrm{E}_1)$$

Then we need to relate this proof in normal form to the original one:

$$\text{if } \Gamma \vdash A \text{ has a proof } \pi \text{ then } \Gamma \vdash A \text{ has a proof in normal form } \pi_0 \text{ with } \pi \rhd^* \pi_0. \quad (\mathrm{E}_2)$$

Let us see how to obtain (E_1). We shall discuss about how to obtain (E_2) in the conclusion. In order to obtain (E_1) we refine Def. 13 as follows:

▶ **Definition 22** (strong outer value). Let A be a formula. We let $\lfloor A\rfloor$ be the set of contexts Γ such that there exists some proof term π in ***normal form*** such that:

- $\Gamma \vdash \pi : A$
- for any environment ϕ, any assignment σ and any term-substitution θ, $\sigma\theta\pi \in \llbracket A\rrbracket^{\mathcal{M}}_{\phi}$.

We need, in that case, *WN*-reducibility candidates that satisfy stability by β-reduction (the (CR_2) property of usual reducibility candidates), as we shall see in the following. The algebra $\mathcal{B}$ contains *WN*-reducibility candidates (Def. 10) that do not enjoy stability by β-reduction. But we can build a sub-algebra $\mathcal{B}'$ that only contains sets of proof-terms that are stable by β-reduction.

▶ **Lemma 23** (Stability by reduction). *Let $\mathcal{B}'$ be the smallest set of reducibility candidates closed by the operations of Def. 8. Let E be a member of $\mathcal{B}'$, π and π' be proof-terms. If $\pi \in E$ and $\pi \rhd^* \pi'$, then $\pi' \in E$.*

Proof. By induction on the construction of E. If $E = \tilde{\top}$ or $E = \tilde{\bot}$, this is immediate. Assume $E = F \tilde{\Rightarrow} G$. If π is *WN* isolated, then so is π' (it is *WN* by confluence). Otherwise, $\pi \rhd^* \lambda\alpha.\pi_1$. By confluence, $\pi' \rhd^* \lambda\alpha.\pi_1'$ with $\pi_1 \rhd^* \pi_1'$. Let $\pi_2 \in F$. $(\pi_2/\alpha)\pi_1 \in G$ by definition of π_1 and $(\pi_2/\alpha)\pi_1 \rhd^* (\pi_2/\alpha)\pi_1'$. So by induction hypothesis, $(\pi_2/\alpha)\pi_1' \in G$, and $\pi' \in F \tilde{\Rightarrow} G$. The other cases are handled similarly. ◀

This algebra $\mathcal{B}'$ is also full, ordered and complete so that we can also use super-consistency to build a model interpretation in it.
With this new algebra $\mathcal{B}'$, Lem. 14 and 17 hold without effort. We only need to reformulate the cases of Lem. 17.
Let us process a key case as an example: $\lfloor A\rfloor \leq \lfloor B\rfloor \tilde{\Rightarrow} \lfloor C\rfloor$ only if $\lfloor A\rfloor \check{\wedge} \lfloor B\rfloor \leq \lfloor C\rfloor$. Let $\Gamma \in \lfloor A\rfloor \cap \lfloor B\rfloor$. We build the proof:

$$\frac{\Gamma \vdash \pi : B \Rightarrow C \qquad \Gamma \vdash \pi_B : B}{\Gamma \vdash \pi\ \pi_B : C}$$

with $\pi \in [\![B \Rightarrow C]\!]^{\mathcal{M}}$ and $\pi_B \in [\![B]\!]^{\mathcal{M}}$, so that both are in normal form. However, $(\pi\ \pi_B)$ may not be in normal form. If this is not the case it means that $\pi = \lambda\alpha.\pi_1$. π is not isolated and by Def. 8 there exists π_1' such that $\pi \rhd^* \lambda\alpha.\pi_1'$ and $(\pi_B/\alpha)\pi_1' \in [\![C]\!]^{\mathcal{M}}$. Let π_C the normal form of $(\pi_B/\alpha)\pi_1'$.

Since $\sigma\theta(\pi\ \pi') \in [\![C]\!]^{\mathcal{M}}$ (exactly as in Lem. 17), and $\sigma\theta(\pi\ \pi') \rhd^* \sigma\theta\pi_C$, we have by Lem. 23 (so by stability by reduction) $\sigma\theta\pi_C \in [\![C]\!]^{\mathcal{M}}$ and π_C is the proof-term we wanted.

Now, Thm. 21 will produce proof-terms in *normal form*.

6 Conclusion and further work

In this paper, we have defined a notion of pre-models for weak normalization of theories expressed in intuitionistic natural deduction modulo. This notion of pre-model is defined as a model on a specific pre-Heyting algebra. To prove that the existence of such a pre-model implies cut elimination for the considered theory, we do not use the usual syntactic way, but we extract from such a pre-model, a (regular) model valued on some Heyting algebra, which implies cut elimination via the usual soundness/strong completeness paradigm. In the following, we discuss how to extend this result to obtain, in the same way, first weak normalization of the considered theory, and secondly a normalization by evaluation algorithm.

Thm. 21, does not state that all proofs of the sequent $\Gamma \vdash A$ are weakly normalizable, but the fact that if this sequent is provable (by a proof-term π), then there exists a weakly normalizable proof π_0 (or in normal form in Sec. 5.1) of that sequent. We have seen, in the previous section how to obtain that π_0 is not only weakly normalizable but also in normal form. But how to relate π_0 to π in order to obtain, first, a semantic proof that *WN*-reducibility candidates entail weak normalization, and second, a normalization by evaluation algorithm?

Indeed, in our construction, we have split the usual adequacy lemma [13] in two parts: the appeal to the proof-term and an inductive argument is handled by the *soundness* theorem, and the inductive cases, that are handled by Lem. 17. This way, we get rid of the original proof-term π and we have no way to get it back.

But if we formalize this algorithm, in Coq for instance, π will lift from the syntactic level towards the proof-assistant level. So it will be saved, and eventually be re-used to produce π_0 in Def. 13. However, there is still no way to show that π_0 is the normal form of π, unless we examine Coq's proof-term itself.

In order to enforce this, we would have to embed π at a lower level, as a *justification* of the fact that $\lfloor\Gamma\rfloor \subseteq \lfloor A\rfloor$ in the model, and to carry it in every proposition, especially Lem. 17.

It should be instructing to compare this approach to other normalization by evaluation approaches [1, 2, 3, 5, 6], in particular because they are all based on a Kripke-like structure, whereas in this paper we meet a pure Heyting algebra structure. Perhaps transforming Heyting algebras into Kripke structures [21], or a reverse operation like the one of [18] will help in this matter. Also, since Kripke worlds are formed of contexts, the presence of contexts in outer values should be a hint that such a transformation is possible.

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