# Definability and Complexity of Graph Parameters

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**Abstract**

In this talk we survey definability and complexity results of graph parameters which take values in some ring or field $\mathcal{R}$.

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## 1 Overview

Partition functions and related graph parameters have found many applications in computer science, combinatorics, physics, chemistry, biology and even the mathematics of finance.

In this talk we survey definability and complexity results of graph parameters which take values in some ring or field $\mathcal{R}$. For this purpose we introduced the classes $\mathrm{SOLEVAL}_{\mathcal{R}}$ and $\mathrm{MSOLEVAL}_{\mathcal{R}}$ of graph parameters with values in a ring or a field $\mathcal{R}$ which are definable in Second Order Logic SOL and Monadic Second Order Logic MSOL respectively, [6, 12, 13, 9, 10].

Partition functions are special cases of such parameters. They all are in $\mathrm{MSOLEVAL}_{\mathcal{R}}$. Many classes of partition functions have been characterized in a series of papers by M. Freedman, L. Lovasz, A. Schrijver and B. Szegedy [5, 15, 1], Their characterizations all use algebraic properties of connection matrices, which are a generalization of Hankel matrices over words. Partition functions can also be viewed as weighted constraint satisfaction problems (CSP) over relational structures. However, characterizing those functions which map labeled relational structures into $\mathcal{R}$ which are representable as CSP problems, seems harder. It was shown in [6] that for all functions in $\mathrm{MSOLEVAL}_{\mathcal{R}}$ and for a wide class of connection matrices the rank of these connection matrices is finite.

A classical result of J.W. Carlyle and A. Paz [3] used Hankel matrices to characterize functions $f : \Sigma^* \to \mathcal{R}$ of words over a finite alphabet $\Sigma$ recognizable by multiplicity automata (aka weighted automata). We use this to give a complete characterization of $\mathcal{R}$-valued functions over words in terms of their Hankel matrix. We discuss how to extend such characterization to labeled trees, edge-labeled graphs, and, more generally, to relational structures, [14]. This contrasts and complements the approach given in [4], which uses weighted formulas of MSOL rather than functions in $\mathrm{MSOLEVAL}_{\mathcal{R}}$.

Studying the complexity of functions in $\mathrm{SOLEVAL}_{\mathcal{R}}$ and $\mathrm{MSOLEVAL}_{\mathcal{R}}$ poses some problems. To capture the complexity of their combinatorial nature, the Turing model of computation and Valiant's notion of counting complexity classes $\sharp\mathbf{P}$ seem most natural. To capture the algebraic and numeric nature of partition functions as real or complex valued functions, the Blum-Shub-Smale (BSS) model of computation seems more natural.


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However, in BSS there are various analogues of $\sharp\mathbf{P}$ based on discrete counting, but there is no established complexity class suitable for hard to compute graph parameters. As a result most papers use a naive hybrid approach in discussing their complexity or restrict their considerations to sub-fields of $\mathbb{C}$ which can be coded in a way to allow dealing with Turing computability. Polynomial time computability is formulated in BSS but hardness is formulated resorting to $\sharp\mathbf{P}$. Pioneered by F. Jaeger and D.L. Vertigan and D.J.A. Welsh [8], and A. Bulatov and M. Grohe [2], dichotomy theorems for a wide class of partition functions were proven which were all formulated in this hybrid language, see also [7].

In the second part of this talk we discuss a unified natural framework for the study of computability and complexity of partition functions and graph parameters and show how classical results can be cast in this framework, cf. [11].

The emphasis of this talk is conceptual and includes a list of open problems and a discussion further directions of research.

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## References


1. A.Schrijver. Graph invariants in the spin model. *J. Comb. Theory Series B*, 99:502–511, 2009.
2. A. Bulatov and M. Grohe. The complexity of partition functions. *Theoretical Computer Science*, 348:148–186, 2005.
3. J.W. Carlyle and A. Paz. Realizations by stochastic finite automata. *J. Comp. Syst. Sc.*, 5:26–40, 1971.
4. M. Droste and P. Gastin. Weighted automata and weighted logics. In *ICALP 2005*, pages 513–525, 2005.
5. M. Freedman, László Lovász, and A. Schrijver. Reflection positivity, rank connectivity, and homomorphisms of graphs. *Journal of AMS*, 20:37–51, 2007.
6. B. Godlin, T. Kotek, and J.A. Makowsky. Evaluation of graph polynomials. In *34th International Workshop on Graph-Theoretic Concepts in Computer Science, WG08*, volume 5344 of *Lecture Notes in Computer Science*, pages 183–194, 2008.
7. M. Grohe and M. Thurley. Counting homomorphisms and partition functions. In M. Grohe and J.A. Makowsky, editors, *Model Theoretic Methods in Finite Combinatorics*, volume 558 of *Contemporary Mathematics*, pages 243–292. American Mathematical Society, 2011.
8. F. Jaeger, D.L. Vertigan, and D.J.A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Camb. Phil. Soc.*, 108:35–53, 1990.
9. T. Kotek. *Definability of combinatorial functions*. PhD thesis, Technion - Israel Institute of Technology, Haifa, Israel, March 2012. Submitted.
10. T. Kotek and J.A. Makowsky. Connection matrices and the definability of graph parameters. In *CSL 2012*, this volume, 2012.
11. T. Kotek, J.A. Makowsky, and E.V. Ravve. A computational framework for the study of partition functions and graph polynomials. Preprint, 2012.
12. T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In *Computer Science Logic, CSL'08*, volume 5213 of *Lecture Notes in Computer Science*, page 339–353, 2008.
13. T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In M. Grohe and J.A. Makowsky, editors, *Model Theoretic Methods in Finite Combinatorics*, volume 558 of *Contemporary Mathematics*, pages 207–242. American Mathematical Society, 2011.
14. N. Labai and J.A. Makowsky. Recognizable power series and msol-definable functions of words. Preprint, 2012.
15. B. Szegedy. Edge coloring models and reflection positivity. *Journal of the American Mathematical Society*, 20.4:969–988, 2007.