Excluded vertex-minors for graphs of linear rank-width at most k

Jisu Jeong[1], O-joung Kwon[2], and Sang-il Oum*[3]

1,2,3Department of Mathematical Sciences, KAIST
291 Daehak-ro Yuseong-gu Daejeon, 305-701 South Korea
1 `jjisu@kaist.ac.kr`
2 `ilkof@kaist.ac.kr`
3 `sangil@kaist.edu`

Abstract

Linear rank-width is a graph width parameter, which is a variation of rank-width by restricting its tree to a caterpillar. As a corollary of known theorems, for each k, there is a finite set $\mathcal{O}_k$ of graphs such that a graph G has linear rank-width at most k if and only if no vertex-minor of G is isomorphic to a graph in $\mathcal{O}_k$. However, no attempts have been made to bound the number of graphs in $\mathcal{O}_k$ for $k \geq 2$. We construct, for each k, $2^{\Omega(3^k)}$ pairwise locally non-equivalent graphs that are excluded vertex-minors for graphs of linear rank-width at most k. Therefore the number of graphs in $\mathcal{O}_k$ is at least double exponential.

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1 Introduction

Linear rank-width is a width parameter of graphs motivated by rank-width of graphs by Oum and Seymour [11]. A vertex-minor relation is a graph containment relation such that rank-width and linear rank-width cannot increase when taking vertex-minors of a graph. Two graphs G, H are called *locally equivalent* if H is a vertex-minor of G and $|V(H)| = |V(G)|$. The definitions can be found in Section 2.

Oum [10] proved that for every infinite sequence $G_1, G_2, \ldots$ of graphs of bounded rank-width, there exist $i < j$ such G_i is isomorphic to a vertex-minor of G_j. As a corollary, we immediately obtain the following theorem.

▶ **Theorem 1** (Oum [10]). *For every vertex-minor closed class $\mathcal{C}$ of graphs of bounded rank-width, there is a finite list of graphs $G_1, G_2, \ldots, G_m$ such that a graph is in $\mathcal{C}$ if and only if it does not have a vertex-minor isomorphic to G_i for some i.*

Because the rank-width of a graph is less than or equal to the linear rank-width of the graph, we deduce the following.

▶ **Corollary 2.** *For a fixed k, there is a finite set $\mathcal{O}_k$ of graphs $G_1, G_2, \ldots, G_m$ such that a graph has linear rank-width at most k if and only if it does not have a vertex-minor isomorphic to G_i for some $i \in \{1, 2, \ldots, m\}$.*

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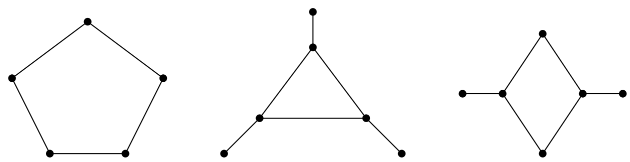

Figure 1 Graphs in $\mathcal{O}_1$.

However, Theorem 1 does not produce an explicit upper or lower bound on the number of graphs in $\mathcal{O}_k$ for Corollary 2. We aim to provide a lower bound on $|\mathcal{O}_k|$.

Our main result is the following.

▶ **Theorem 3.** *Let $k \geq 2$ be an integer. Then $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$. In other words, there are at least $2^{\Omega(3^k)}$ pairwise locally non-equivalent graphs that are vertex-minor minimal with the property that they have linear rank-width larger than k.*

When $\mathcal{C}$ is the set of all graphs having rank-width at most k, Theorem 1 implies that there are finitely many graphs $G_1, G_2, \ldots, G_m$ such that a graph has rank-width at most k if and only if it has no vertex-minor isomorphic to G_i for some i. Again Theorem 1 does not provide a lower or upper bound on m for graphs of rank-width at most k. However, for the upper bound, Oum [9] proved that $|V(G_i)| \leq (6^{k+1} - 1)/5$ for each i. No analogous result is known for linear rank-width.

Characterizing graphs of linear rank-width at most k in terms of forbidden vertex-minors seems hard. So far only 1 case is known. For $k = 1$, Adler, Farley, and Proskurowski [1] characterized the graphs of linear rank-width at most 1 by a set $\mathcal{O}_1$ of three graphs in Figure 1. A structural characterization of graphs of linear rank-width 1 was given by Ganian [6].

There have been similar results on the number of forbidden minors for various graph width parameters; for instance, path-width [12], linear-width [13], tree-width [8], branch-width [7], tree-depth [5].

The paper is organized as follows. We present the definitions of linear rank-width and vertex-minor. In Section 3, we construct a set Δ_k of graphs for every non-negative integer k, and prove that every graph in Δ_k has linear rank-width $k+1$ but every proper vertex-minor has linear rank-width at most k. Roughly speaking, $\Delta_0 = \{K_2\}$ and for $k \geq 1$, the set Δ_k consists of all graphs obtained from a disjoint union of three graphs in Δ_{k-1} by connecting them with a triangle. In Section 4, we show that no two graphs in Δ_k are locally equivalent. At last, we show that the size of Δ_k is $2^{\Omega(3^k)}$ in Section 5, and we conclude that $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$.

2 Preliminaries

In this paper, graphs have no loops and parallel edges. Let G be a graph. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced on S. For $S \subseteq V(G)$, $N_G(S)$ denotes the set of vertices of $V(G) \setminus S$ adjacent to a vertex in S. And for $v \in V(G)$, we let $N_G(v) = N_G(\{v\})$. A vertex v in G is a *leaf* if $|N_G(v)| = 1$. A graph G is a *star* if G is isomorphic to $K_{1,n}$ for some $n \geq 1$.

For an $X \times Y$ matrix M and subsets $A \subseteq X$ and $B \subseteq Y$, $M[A, B]$ denotes the $A \times B$ submatrix $(m_{i,j})_{i \in A, j \in B}$ of M. If $A = B$, then $M[A] = M[A, A]$ is called a *principal submatrix* of M.

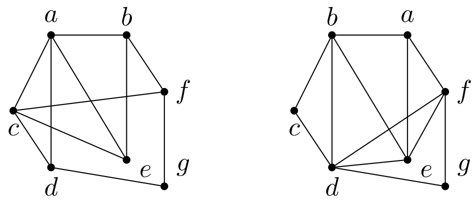

Figure 2 Pivoting an edge ab.

Vertex-minors.

The *local complementation* at a vertex v of a graph $G = (V, E)$ is an operation to obtain a graph $G * v$ from G by replacing the subgraph $G[N_G(v)]$ with the complementary subgraph of $G[N_G(v)]$. The graph obtained from G by *pivoting* an edge uv is defined by $G \wedge uv = G*u*v*u$.

To see how we obtain the resulting graph by pivoting an edge uv, let $V_1 = N_G(u) \cap N_G(v)$, $V_2 = N_G(u) \backslash N_G(v) \backslash \{v\}$, and $V_3 = N_G(v) \backslash N_G(u) \backslash \{u\}$. One can easily verify that $G \wedge uv$ is identical to the graph obtained from G by complementing adjacency of vertices between distinct sets V_i and V_j, and swapping the vertices u and v [9]. See Figure 2 for an example.

A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. A graph H is *locally equivalent* to G if H can be obtained from G by applying a sequence of local complementations.

A vertex-minor H of G is *elementary* if $|V(H)| = |V(G)| - 1$. A vertex-minor H of G is *proper* if $|V(H)| < |V(G)|$. A graph G is an *excluded vertex-minor* for a vertex-minor closed set $\mathcal{C}$ of graphs if $G \notin \mathcal{C}$ and $H \in \mathcal{C}$ for every proper vertex-minor H of G.

Linear rank-width.

The adjacency matrix of a graph G, which is a $(0, 1)$-matrix over the binary field, will be denoted by $A(G)$. The *cut-rank* function $\mathrm{cutrk}_G : 2^V \to \mathbb{Z}$ of a graph $G = (V, E)$ is defined by

$$\mathrm{cutrk}_G(X) = \mathrm{rank}(A(G)[X, V \backslash X]).$$

A *linear layout* L of G is a sequence $(v_1, v_2, \ldots, v_{|V(G)|})$ of $V(G)$. For a linear layout L of G and $a, b \in V(G)$, we denote $a \le_L b$ if $a = b$ or a appears before b in L. For two sequences $L_1 = (v_1, v_2, \ldots, v_n)$ and $L_2 = (w_1, w_2, \ldots, w_m)$, we define $L_1 \oplus L_2 = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)$.

The *width* of a linear layout L in G, denoted by $\mathrm{lrw}_L(G)$, is defined as the maximum over all $\mathrm{cutrk}_G(\{w : w \le_L v\})$ for $v \in V(G)$. The *linear rank-width* of G, denoted by $\mathrm{lrw}(G)$, is the minimum width of all linear layouts of G. The next proposition shows the relation between the cut-rank function and local complementation.

▶ Proposition 4 (Oum [9]). Let G be a graph and $v \in V(G)$. Then for every $X \subseteq V(G)$,

$$\mathrm{cutrk}_G(X) = \mathrm{cutrk}_{G*v}(X).$$

By Proposition 4, $\mathrm{lrw}(G) = \mathrm{lrw}(G * v)$ for every $v \in V(G)$. Thus, we immediately obtain that if H is locally equivalent to G, then $\mathrm{lrw}(H) = \mathrm{lrw}(G)$. And if a graph H is a vertex-minor of a graph G, then $\mathrm{lrw}(H) \le \mathrm{lrw}(G)$.

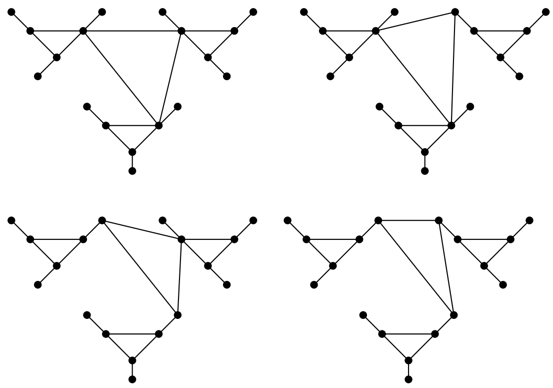

Figure 3 All graphs in Δ_2.

3 Excluded vertex-minors for graphs of bounded linear rank-width

To prove Theorem 3, for each non-negative integer k, we construct a set Δ_k of graphs such that every graph in Δ_k has linear rank-width $k+1$ but every proper vertex-minor has linear rank-width at most k.

A *delta composition* G of graphs G_1, G_2, and G_3 is a graph obtained from the disjoint union of G_1, G_2, and G_3 by adding a triangle $v_1v_2v_3$ where $v_i \in V(G_i)$ for $i = 1, 2, 3$. We call $v_1v_2v_3$ the *main triangle* of G. For a non-negative integer k, we define Δ_k as follows.

1. $\Delta_0 = \{K_2\}$.
2. For $i \geq 1$, Δ_i is the set of all delta compositions of 3 graphs in Δ_{i-1}.

The main theorem of this section is as follows.

▶ **Theorem 5.** *Let k be a non-negative integer. Every graph in Δ_k is an excluded vertex-minor for graphs of linear rank-width at most k.*

First, we prove that every graph in Δ_k has linear rank-width $k+1$.

▶ Proposition 6. Let k be a non-negative integer and $G \in \Delta_k$. Then G has linear rank-width $k+1$. Moreover, for $w \in V(G)$, there is a linear layout of G having width $k+1$ such that the first vertex of the linear layout is w.

Proof. We use induction on k. If $k = 0$, then $G = K_2$. If $V(G) = \{x, y\}$, then both (x, y) and (y, x) are linear layouts of G having width 1. Hence, the statements are true. We may assume that $k \geq 1$. Since $G \in \Delta_k$, G is a delta composition of G_1, G_2, and G_3 in Δ_{k-1}. Let $v_1v_2v_3$ be the main triangle of G such that $v_i \in V(G_i)$ for $i = 1, 2, 3$.

We first show that $\operatorname{lrw}(G) \geq k+1$. Suppose that G has linear rank-width at most k. Since $G_1 \in \Delta_{k-1}$, by induction hypothesis, G_1 has linear rank-width k. Since $\operatorname{lrw}(G) \geq \operatorname{lrw}(G_1) = k$, G has linear rank-width k. Let L be a linear layout of G having width k. And for a vertex v in G, we define $S_v = \{x \in V(G) : x \leq_L v\}$ and $T_v = V(G) \setminus S_v$.

Let a and b be the first and the last vertices in L such that $\operatorname{cutrk}_G(S_a) = \operatorname{cutrk}_G(S_b) = k$. Without loss of generality, we may assume that $\{a, b\} \subseteq V(G_2) \cup V(G_3)$. We want to show that G_1 has linear rank-width at most $k-1$. If it is true, then we obtain a contradiction because $\operatorname{lrw}(G_1) = k$. Let L_{G_1} be the subsequence of L whose elements are the vertices of G_1.

We claim that L_{G_1} is a linear layout of G_1 having width at most $k-1$. Let $v \in V(G_1)$. It is sufficient to show that $\mathrm{cutrk}_{G_1}(S_v \cap V(G_1)) \le k-1$. Note that $v \neq a$ and $v \neq b$. If $v <_L a$ or $v >_L b$, then

$$\begin{aligned}\mathrm{cutrk}_{G_1}(S_v \cap V(G_1)) &\le \mathrm{cutrk}_G(S_v)\\ &\le k-1.\end{aligned}$$

So we may assume that $a <_L v <_L b$. Note that one of $S_v \cap V(G_1)$ and $T_v \cap V(G_1)$ does not have a neighbor in $G \setminus V(G_1)$ because v_1 is the unique vertex in G_1 which has a neighbor in $G \setminus V(G_1)$. And since $G[V(G_2) \cup V(G_3)]$ is connected and $a \in S_v$ and $b \notin S_v$, there is an edge u_1u_2 in $G \setminus V(G_1)$ such that $u_1 \in S_v$ and $u_2 \notin S_v$. So $A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]$ is a non-zero matrix. Depending on whether $v_1 \in S_v \cap V(G_1)$ or $v_1 \in T_v \cap V(G_1)$,

$$\begin{aligned}\mathrm{cutrk}_G(S_v) &= \mathrm{rank}\left(\begin{array}{c} \\ S_v \cap V(G_1) \\ S_v \setminus V(G_1)\end{array}\begin{array}{c|c} T_v \cap V(G_1) & T_v \setminus V(G_1) \\ \hline * & 0 \\ \hline * & * \end{array}\right)\\ &\ge \mathrm{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right) + \mathrm{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right),\end{aligned}$$

or

$$\begin{aligned}\mathrm{cutrk}_G(S_v) &= \mathrm{rank}\left(\begin{array}{c} \\ S_v \cap V(G_1) \\ S_v \setminus V(G_1)\end{array}\begin{array}{c|c} T_v \cap V(G_1) & T_v \setminus V(G_1) \\ \hline * & * \\ \hline 0 & * \end{array}\right)\\ &\ge \mathrm{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right) + \mathrm{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right),\end{aligned}$$

respectively. Thus, we have

$$\begin{aligned}\mathrm{cutrk}_{G_1}(S_v \cap V(G_1)) &= \mathrm{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right)\\ &\le \mathrm{cutrk}_G(S_c) - \mathrm{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right)\\ &\le \mathrm{cutrk}_G(S_v) - 1 \le k-1.\end{aligned}$$

So L_{G_1} is a linear layout of G_1 having width at most $k-1$, which is a contradiction. Hence, $\mathrm{lrw}(G) \ge k+1$.

Now we show that there is a linear layout of G having width $k+1$ with a given starting vertex. Let $v \in V(G)$. Without loss of generality, we assume that $v \in V(G_1)$. By induction hypothesis, there is a linear layout L_1 of G_1 having width k such that the first vertex of L_1 is v. And, for $j = 2, 3$, there is a linear layout L_j of G_j having width k such that the first vertex of L_j is v_j. It is easy to check that $L_1 \oplus L_2 \oplus L_3$ is a linear layout of G having width at most $k+1$. Since this linear layout starts at v, we conclude the result. ◀

Of course, for $v \in V(G)$, there is also a linear layout having width $k+1$ such that the last vertex of the linear layout is v. Let $v \in V(G)$. A vertex w, $w \neq v$, in G is a *twin* of v if $N_G(w) \setminus v = N_G(v) \setminus w$. A twin w of v is a *false twin* if w is not adjacent to v. And a twin w of v is a *true twin* if w is adjacent to v.

Now we prove that every elementary vertex-minor of G in Δ_k has linear rank-width k. To prove it, we will use the following lemmata.

▶ **Lemma 7** (Bouchet [2]). *Let G be a graph, $v \in V(G)$, and H be a vertex-minor of G such that $V(G) \setminus V(H) = \{v\}$. If w is an arbitrary neighbor of v, then H is locally equivalent to either $G \setminus v$, $G * v \setminus v$, or $G \wedge vw \setminus v$.*

▶ **Lemma 8** (Oum [9]). *Let G be a graph and $vv_1, vv_2 \in E(G)$. Then $v_1v_2 \in E(G \wedge vv_1)$ and $G \wedge vv_1 \wedge v_1v_2 = G \wedge vv_2$.*

▶ **Lemma 9.** *Let k be a positive integer. Let G_1, $G_2 \in \Delta_{k-1}$, and let G_3 be a graph having linear rank-width at most $k-1$. Then every delta composition of G_1, G_2, and G_3 has linear rank-width k. Also, if a graph is obtained from the disjoint union of G_1 and G_2 by adding an edge w_1w_2 where $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$, then it has linear rank-width k.*

▶ **Lemma 10.** *Let k be a non-negative integer. Let $G \in \Delta_k$, $v \in V(G)$, and H be a graph obtained from G by adding a twin w of v. Then there is a linear layout L of H having width $k+1$ such that the first vertex of L is v and the last vertex of L is w.*

We are ready to prove the main combinatorial result in this paper.

▶ Proposition 11. Let k be a non-negative integer and $G \in \Delta_k$. Then every elementary vertex-minor of G has linear rank-width k.

Proof. Note that for $v \in V(G)$ and $S \subseteq V(G)$, $\mathrm{cutrk}_{G\setminus v}(S \setminus v) \geq \mathrm{cutrk}_G(S) - 1$ because exactly one column or one row of $A(G)[S, V(G) \setminus S]$ is removed. Thus by Proposition 6, if H is an elementary vertex-minor of G, then $\mathrm{lrw}(H) \geq \mathrm{lrw}(G) - 1 = (k+1) - 1 = k$. Therefore, it is sufficient to prove that every elementary vertex-minor of G in Δ_k has linear rank-width at most k.

We use induction on k. If $k = 0$, then $G = K_2$ and every elementary vertex-minor of G is isomorphic to K_1, so it has linear rank-width 0. We assume that $k \geq 1$. Since $G \in \Delta_k$, G is a delta composition of G_1, G_2, and G_3 in Δ_{k-1}. Let $v_1v_2v_3$ be the main triangle of G such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. Let H be an elementary vertex-minor of G and $V(G) \setminus V(H) = \{v\}$. By Lemma 7, for a neighbor w of v, H is locally equivalent to one of three graphs $G \setminus v$, $G * v \setminus v$, and $G \wedge vw \setminus v$. Without loss of generality, we may assume that $v \in V(G_1)$. Since $G_1 \in \Delta_{k-1}$, by induction hypothesis, $G_1 \setminus v$ has linear rank-width at most $k-1$. Thus, by Lemma 9, $G \setminus v$ has linear rank-width k. What remains to be proved is that for a neighbor w of v, $G * v \setminus v$ and $G \wedge vw \setminus v$ have linear rank-width at most k.

First, suppose that $v \neq v_1$. If $N_G(v) = \{v_1\}$, then $G * v \setminus v = G \setminus v$ and $G \wedge vv_1 \setminus v$ is isomorphic to $G \setminus v_1$. Therefore, by Lemma 9, they have linear rank-width k. If v has a neighbor w other than v_1, then

$$(G*v)[V(G_2) \cup V(G_3) \cup \{v_1\}] = (G \wedge vw)[V(G_2) \cup V(G_3) \cup \{v_1\}] = G[V(G_2) \cup V(G_3) \cup \{v_1\}].$$

Hence, both $G * v \setminus v$ and $G \wedge vw \setminus v$ are delta compositions of two graphs in Δ_{k-1} and one graph having linear rank-width at most $k-1$. Thus, by Lemma 9, they have linear rank-width k.

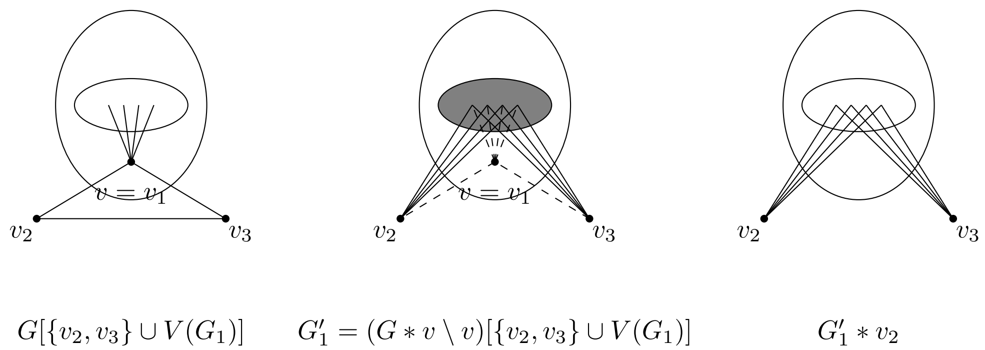

Figure 4 The case $G * v \setminus v$ where $v = v_1$.

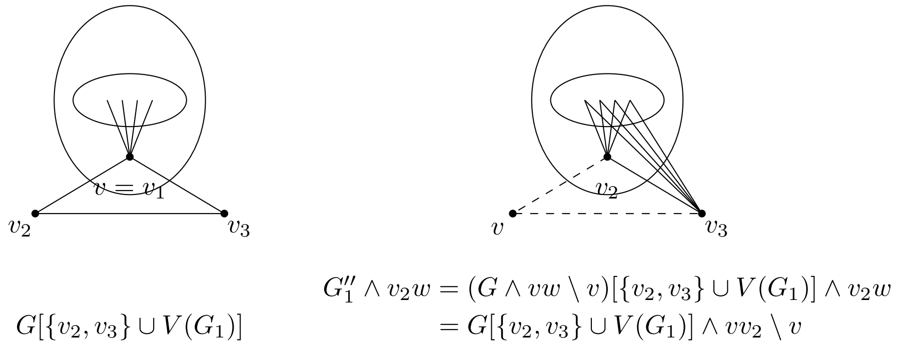

Figure 5 The case $G \wedge vw \setminus v$ where $v = v_1$.

Now we consider $v = v_1$. Let w be a neighbor of v in G_1. By Proposition 6, there is a linear layout L_{G_2} of G_2 having width k such that the end vertex of L_{G_2} is v_2, and there is a linear layout L_{G_3} of G_3 having width k such that the first vertex of L_{G_3} is v_3. We denote $G_1' = (G * v \setminus v)[\{v_2, v_3\} \cup V(G_1)]$ and $G_1'' = (G \wedge vw \setminus v)[\{v_2, v_3\} \cup V(G_1)]$.

We first show that $G * v \setminus v$ has linear rank-width at most k. To prove it, we will find a linear layout L' of G_1' having width k such that the first vertex of L' is v_2 and the last vertex of L' is v_3. In Figure 4, we can observe that $N_{G_1}(v) = N_{G_1' * v_2}(v_2) = N_{G_1' * v_2}(v_3)$ and $A(G)[N_{G_1}(v)] = A(G_1' * v_2)[N_{G_1}(v)]$. Hence, the graph $G_1' * v_2$ is isomorphic to the graph obtained from G_1 by adding a false twin of v. By Proposition 10, there is a linear layout L' of $G_1' * v_2$ having width k such that the first vertex of L' is v_2 and the last vertex of L' is v_3. Let L_{G_1} be the sequence obtained from L' by deleting v_2 and v_3.

We show that $L = L_{G_2} \oplus L_{G_1} \oplus L_{G_3}$ is a linear layout of $G * v \setminus v$ having width at most k. If $x \in V(G_2) \cup V(G_3)$, then clearly $\text{cutrk}_{G*v\setminus v}(\{y : y \leq_L x\}) \leq k$. If $x \in V(G_1) \setminus v$, then by Proposition 4,

$$\begin{aligned}\text{cutrk}_{G*v\setminus v}(\{y : y \leq_L x\}) &= \text{cutrk}_{G_1'}(\{y : y \leq_{L'} x\})\\ &= \text{cutrk}_{G_1' * v_2}(\{y : y \leq_{L'} x\}) \leq k.\end{aligned}$$

Therefore, $G * v \setminus v$ has linear rank-width at most k.

Now we show that $G \wedge vw \setminus v$ has linear rank-width at most k. By the same argument in the previous case, it is sufficient to prove that there is a linear layout L'' of G_1'' having width k such that the first vertex is v_2 and the last vertex is v_3. We claim that $G_1'' \wedge v_2w = G[\{v_2, v_3\} \cup V(G_1)] \wedge vv_2 \setminus v$. Note that

$$\begin{aligned}G_1'' \wedge v_2w &= (G \wedge vw \setminus v)[\{v_2, v_3\} \cup V(G_1)] \wedge v_2w\\ &= G[\{v_2, v_3\} \cup V(G_1)] \wedge vw \setminus v \wedge v_2w\\ &= G[\{v_2, v_3\} \cup V(G_1)] \wedge vw \wedge v_2w \setminus v.\end{aligned}$$

And by Lemma 8,

$$G[\{v_2, v_3\} \cup V(G_1)] \wedge vw \wedge v_2w \setminus v = G[\{v_2, v_3\} \cup V(G_1)] \wedge vv_2 \setminus v.$$

In Figure 5, we can observe that $G_1'' \wedge v_2w$ is isomorphic to the graph obtained from G_1 by adding a true twin of v. Thus, by Proposition 10, there is a linear layout L'' of $G_1'' \wedge v_2w$ having width k such that the first vertex of L'' is v_2 and the last vertex of L'' is v_3. Also, for $x \in V(G_1) \setminus v$,

$$\text{cutrk}_{G_1''}(\{y : y \leq_{L''} x\}) = \text{cutrk}_{G_1'' \wedge vv_2}(\{y : y \leq_{L''} x\}) \leq k.$$

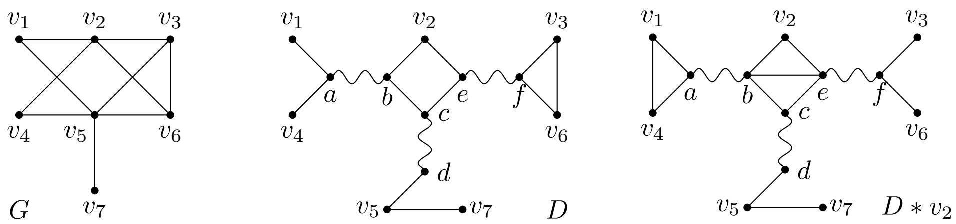

Figure 6 A split-decomposition D of a graph G, and $D * v_2$. The marked edges of D are depicted as wavy edges, and the desendants of the vertex v_2 in D is a and f. Note that $D * v_2$ is a split decomposition of $G * v_2$.

Therefore, we conclude that $G \wedge vw \setminus v$ has linear rank-width at most k. ◀

Proof of Theorem 5. Let $G \in \Delta_k$. By Proposition 6, G has linear rank-width $k+1$. And by Proposition 11, every elementary vertex-minor of G has linear rank-width k. So every proper vertex-minor of G has linear rank-width at most k. Therefore, G is an excluded vertex-minor for graphs of linear rank-width at most k. ◀

4 No two graphs in Δ_k are locally equivalent.

In this section, we show that no two graphs in Δ_k are locally equivalent.

▶ **Theorem 12.** *Let k be a non-negative integer and $G, H \in \Delta_k$. If G and H are locally equivalent, then G and H are isomorphic.*

To prove it, we will use the canonical split-decompositions of graphs in Δ_k.

Split-decomposition.

Let G be a graph. A partition (A, B) of $V(G)$ is a *split* if $|A| \geq 2$, $|B| \geq 2$, and for every $v \in N_G(B)$ and $w \in N_G(A)$, $vw \in E(G)$. If G has no split and $|V(G)| \geq 5$, then we call G a *prime graph*. If G has a split (A, B), then we define a graph G', called a *simple decomposition* of G, as the graph obtained from G by deleting all edges between $N_G(A)$ and $N_G(B)$, and adding two vertices w_1,w_2 and adding edges $\{w_1w_2\} \cup \{vw_1 : v \in N_G(B)\} \cup \{w_2v : v \in N_G(A)\}$. We call w_1w_2 a *marked edge* of G'. A graph is a *marked graph* if it has marked edges, and for a marked graph D, we define $M(D)$ as the set of marked edges of D. A *split-decomposition* of G is recursively defined to be either G or a marked graph obtained from a split-decomposition D by replacing a component H of $D \setminus M(D)$ with a simple decomposition of H. Two components C_1 and C_2 of $D \setminus M(D)$ are *neighbors* if there exist $v_1 \in V(C_1)$, $v_2 \in V(C_2)$ such that $v_1v_2 \in M(D)$. A split-decompositon D of a graph is *canonical* if it satisfies the following:

(i) each component of $D \setminus M(D)$ is either a prime graph or a star or a complete graph,
(ii) no two complete components are neighbors,
(iii) if two star components are neighbors, then two ends of the marked edge are both centers or both leaves of each components.

Two split-decompositions D_1 and D_2 of a graph G are *equivalent* if there is a graph isomorphism f from D_1 to D_2 such that f preserves the marked edges and $f|_{V(G)}$ is an identity function. We need the following result.

▶ **Lemma 13** (Cunningham [4]). *Canonical split-decompositions of a graph are equivalent.*

Let D be the canonical split-decomposition of G and $C(D) = \{C_1, C_2, \ldots, C_n\}$ be the components of $D \setminus M(D)$. A tree T_G is a *canonical tree* of G if $V(T_G) = \{v_{C_1}, v_{C_2}, \ldots, v_{C_n}\}$ and v_{C_i} is adjacent to v_{C_j} if and only if two components C_i and C_j are neighbors in D. We call f the *canonical map* from T_G to D if it is the bijection from $V(T_G)$ to $C(D)$ such that $f(v_{C_k}) = C_k$.

For $v \in V(G) \subseteq V(D)$, a vertex w in D is a *descendant* of v if either $w = v$ or w is the end of a path starting from v, whose successive edges are alternatively non-marked and marked edges, and the last edge is marked. Note that each component of $D \setminus M(D)$ has at most 1 descendant of a vertex because every marked edge in D is a cut-edge. For $v \in V(G)$, we define $D * v$ as the marked graph obtained from D by replacing each component H of $D \setminus M(D)$ having a descendant w of v by $H * w$.

▶ **Lemma 14** (Bouchet [3]). *If D is a canonical split-decomposition of a graph G and $v \in V(G)$, then $D * v$ is a canonical split-decomposition of the graph $G * v$.*

By Lemma 14, if G and H are locally equivalent, then G and H have isomorphic canonical trees. Hence, it is sufficient to prove that for $G, H \in \Delta_k$, if G and H have isomorphic canonical trees, then G is isomorphic to H. To show this, we explicitly describe the canonical decompositions of graphs in Δ_k.

Clearly, K_2 has itself as a canonical split-decomposition. Let $k \geq 1$ and $G \in \Delta_k$. Note that for a non-leaf vertex v in G, v is incident with exactly one cut-edge and meets at least one triangles. For a non-leaf vertex v in G, let l_v be the star on the vertex set $V(l_v) = \{v, a^v, b^v_{C_1}, b^v_{C_2}, \ldots, b^v_{C_m}\}$ with the center v, where v is incident with a cut-edge e and meets trianges $C_1, C_2, \ldots, C_m$. And for each triangle C in G, let t_C be the triangle on the vertex set $\{d^a_C, d^b_C, d^c_C\}$ where $V(C) = \{a, b, c\}$. We define the graph D_G as the graph obtained from the disjoint union of all graphs in $\{l_v : v \text{ is a non-leaf vertex in } G\} \cup \{t_C : C \text{ is a triangle in } G\}$ by adding the marked edge set $M(D_G)$ which consists of

(i) $b^v_C d^v_C$ if v meets a triangle C,
(ii) $a^v a^w$ if vw is a cut-edge of G and both v and w are not leaves of G.

We can verify that the marked graph D_G with $M(D_G)$ of the third graph G in Figure 3 is the first graph in Figure 7. In general, we can show that for $G \in \Delta_k$, D_G with the marked edge set $M(D_G)$ is indeed a canonical split-decomposition of G.

▶ **Lemma 15.** *Let k be a non-negative integer and $G \in \Delta_k$. The graph D_G is a canonical split-decomposition of G with the set $M(D_G)$ of marked edges.*

We can observe the following.

▶ **Lemma 16.** *Let k be a non-negative integer and $G \in \Delta_k$. Let T_G be a canonical tree of G and f be the canonical map from T_G to D_G. Let B be the set of vertices of T_G mapped by f to a complete graph. Then the following are true.*

(i) If $v \in B$, then $N_{T_G}(v) \cap B = \emptyset$ and $|N_{T_G}(v)| = 3$.
(ii) Every component of $T_G[V(T_G) \setminus B]$ has at most 2 vertices.
(iii) If $w \in V(T_G) \setminus B$, then the component $f(w)$ is a star, and the center of $f(w)$ is a non-leaf vertex in G, say u. Suppose that u meets m triangles in G. Then u is adjacent with $m + 1$ vertices in $f(w)$.

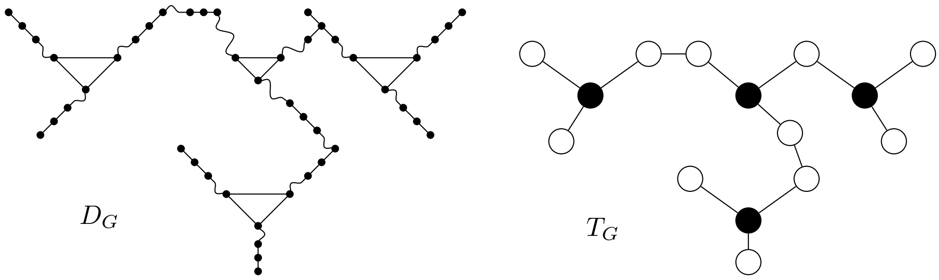

Figure 7 The canonical split-decomposition D_G and the canonical tree T_G of the third graph G in Figure 3. The black vertices in T_G are the vertices mapped by the canonical map to a triangle of D_G.

▶ Proposition 17. Let k be a non-negative integer and $G, H \in \Delta_k$. If G, H have isomorphic canonical trees, then G is isomorphic to H.

Proof. Let T be a canonical tree of both G and H. Let f_G be the canonical map from T to D_G, and let B_G be the set of vertices mapped by f_G to a complete graph of D_G. Similarly, let f_H be the canonical map from T to D_H, and let B_H be the set of vertices mapped by f_H to a complete graph in D_H.

We first show that $B_G = B_H$. Suppose that $B_G \neq B_H$. Since G and H have the same number of triangles, $|B_G| = |B_H|$. So we can choose $v_1 \in B_G \setminus B_H$ and a maximal path $P = v_1v_2 \ldots v_n$ in T such that

(i) P contains vertices from B_G and from $V(T) \setminus B_G$, alternatively, and
(ii) P also contains vertices from $V(T) \setminus B_H$ and from B_H, alternatively.

Suppose v_n is not a leaf. By the symmetry, we assume that $v_n \in B_G$ and $v_n \in V(T) \setminus B_H$. Since $v_n \in B_G$, by Lemma 16, v_n has 3 neighbors in T, which are contained in $V(T) \setminus B_G$. And since $v_n \in V(T) \setminus B_H$, by Lemma 16, v_n has at most 1 neighbor of $V(T) \setminus B_H$. Hence, there exists a vertex in $(N_T(v_n) \setminus V(P)) \cap B_H$, say v_{n+1}. Thus, $v_{n+1} \in V(T) \setminus B_G$ and $v_{n+1} \in B_H$, and $v_1v_2, \ldots, v_nv_{n+1}$ is also a path in T satisfying (i) and (ii). It contradicts to the maximality of P. Thus, v_n is a leaf in T. But if v_n is a leaf in T, neither $f_G(v_n)$ nor $f_H(v_n)$ is a triangle, so it is a contradiction. Therefore, $B_G = B_H$, and we call this set B.

Clearly, for $v \in B$, $f_G(v)$ and $f_H(v)$ are triangles. And by Lemma 16, for $v \in V(T) \setminus B$, the components $f_G(v)$ and $f_H(v)$ are uniquely determined by the neighbors of v in T_G. Therefore, the graphs D_G and D_H are isomorphic, and G is isomorphic to H. ◀

Proof of Theorem 12. Since G and H are locally equivalent, there is a sequence $v_1, v_2, \ldots v_m$ of $V(G)$ such that $G * v_1 * v_2 \ldots * v_m = H$. By Lemma 14, G and $G * v_1 * v_2 \ldots * v_m$ have isomorphic canonical trees. And since $G*v_1*v_2 \ldots *v_m = H$, by Lemma 13, $G*v_1*v_2 \ldots *v_m$ and H have isomorphic canonical trees. Thus G and H have isomorphic canonical trees. Since $G, H \in \Delta_k$, by Proposition 17, G is isomorphic to H. ◀

5 The size of Δ_k is $2^{\Omega(3^k)}$

In this section, we determine the number of graphs in Δ_k for each non-negative integer k. The main theorem of this section is as follows.

▶ **Theorem 18.** *Let $k \geq 2$ be an integer. The size of Δ_k is $2^{\Omega(3^k)}$.*

For graphs G, G' and $v \in V(G)$ and $v' \in V(G')$, two pairs (G, v) and (G', v') are *isomorphic* if there exists a graph isomorphism ϕ from G to G' such that $\phi(v) = v'$. To prove Theorem 18, for a positive integer k, we partition Δ_k into A_k, B_k, and C_k as follows:

(i) $G \in A_k$ if (G_1, v_1), (G_2, v_2), and (G_3, v_3) are isomorphic,
(ii) $G \in B_k$ if only two of (G_1, v_1), (G_2, v_2), (G_3, v_3) are isomorphic,
(iii) $G \in C_k$ otherwise,

where G is a delta composition of G_1, G_2, and G_3 in Δ_{k-1}, and $v_1v_2v_3$ is the main triangle of G such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. If p_k is the number of non-isomorphic pairs (G, v) where $G \in \Delta_k$ and $v \in V(G)$, we can easily verify that

$$|A_k| = p_{k-1}, \quad |B_k| = p_{k-1}(p_{k-1} - 1), \quad |C_k| = \frac{1}{6}p_{k-1}(p_{k-1} - 1)(p_{k-1} - 2).$$

We will give a lower bound of p_k from $|A_k|$, $|B_k|$, $|C_k|$, and obtain a recurrence relation.

For a graph G and $v, w \in V(G)$, we denote $v \simeq_G w$ if (G, v) and (G, w) are isomorphic. We consider the equivalent classes $V(G)/\simeq_G$. We denote $[v]$ as an element of $V(G)/\simeq_G$. For a non-negative integer k, let $P_k = \{(G, [v]) : G \in \Delta_k, [v] \in V(G)/\simeq_G\}$ and $p_k = |P_k|$. Then p_k is exactly the number of all non-isomorphic pairs (G, v) where $G \in \Delta_k$ and $v \in V(G)$. It is obvious that $p_0 = 1$, $p_1 = 2$. And we can see that $p_2 = 24$ in Figure 3. We need the following lemma.

▶ **Lemma 19.** *Let k be a positive integer and $G \in \Delta_k$.*

1. *If $G \in A_k$, then $|V(G)/\simeq_G| \geq 2^k$.*
2. *If $G \in B_k$, then $|V(G)/\simeq_G| \geq 2 \cdot 2^k$.*
3. *If $G \in C_k$, then $|V(G)/\simeq_G| \geq 3 \cdot 2^k$.*

Proof of Theorem 18. By Lemma 19,

$$p_k = \sum_{G \in A_k \cup B_k \cup C_k} |V(G)/\simeq_G| \geq 2^k|A_k| + 2 \cdot 2^k|B_k| + 3 \cdot 2^k|C_k|.$$

Since $|A_k| = p_{k-1}$, $|B_k| = p_{k-1}(p_{k-1} - 1)$ and $|C_k| = \frac{1}{6}p_{k-1}(p_{k-1} - 1)(p_{k-1} - 2)$, we obtain the following recurrence relation,

$$\begin{aligned} |A_{k+1}| = p_k &\geq 2^k|A_k| + 2 \cdot 2^k|B_k| + 3 \cdot 2^k|C_k| \\ &\geq 2^{k-1}|A_k|^3 \end{aligned}$$

and $|A_2| = 2$.

This means $|A_k| = 2^{\Omega(3^k)}$ for $k \geq 3$. Because $|\Delta_2| = 4$ and $|\Delta_k| \geq |A_k| = 2^{\Omega(3^k)}$ for $k \geq 3$, we conclude that $|\Delta_k| = 2^{\Omega(3^k)}$ for $k \geq 2$. ◀

Proof of Theorem 3. By Theorems 5 and 12, $|\mathcal{O}_k| \geq |\Delta_k|$. And by Theorem 18, $|\Delta_k| \geq 2^{\Omega(3^k)}$. Therefore, $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$. ◀

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