The Rank of Tree-Automatic Linear Orderings

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Abstract

A tree-automatic structure is a structure whose domain can be encoded by a regular tree language such that each relation is recognisable by a finite automaton processing tuples of trees synchronously. The finite condensation rank (FC-rank) of a linear ordering measures how far it is away from being dense. We prove that the FC-rank of every tree-automatic linear ordering is below ω^ω. This generalises Delhommé's result that each tree-automatic ordinal is less than ω^{ω^ω}. Furthermore, we show an analogue for tree-automatic linear orderings where the branching complexity of the trees involved is bounded.

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1 Introduction

The fundamental idea of automatic structures can be traced back to the 1960s when Büchi, Elgot, Rabin, and others used finite automata to provide decision procedures for the first-order theory of Presburger arithmetic $(\mathbb{N};+)$ and several other logical problems. Hodgson generalised this idea to the concept of *automaton decidable* first-order theories. Independently of Hodgson and inspired by the successful employment of finite automata and their methods in group theory, Khoussainov and Nerode [8] initiated the systematic investigation of *automatic structures*. Recalling the efforts from the 1960s, Blumensath [2] extended this notion beyond finite automata to finite automaton models recognising, e.g., finite trees.

Basically, a countable relational structure is *tree-automatic* or *tree-automatically presentable* if its elements can be encoded by finite trees in such a way that its domain and its relations are recognisable by finite automata processing either single trees or tuples of trees synchronously. *String-automatic* structures can be regarded as a special case where only specific simple trees—which effectively represent strings—are used. In contrast to the more general concept of *computable structures* and based on the strong closure properties of recognisability, automatic structures provide pleasant algorithmic features. In particular, they possess decidable first-order theories.

Due to this latter circumstance, the concept of automatic structures gained a lot attention which led to noticeable progress (cf. [1, 13]). Automatic presentations were found for many structures and others like the random graph were shown not to be automatic at all. Some structures are provably on an intermediate level, they are tree-automatic but not string-automatic, for instance Skolem arithmetic $(\mathbb{N};\times)$. For the classes of ordinals and Boolean algebras it was even possible to characterise their (string-)automatic members. Certain extensions of first-order logic which preserve decidability of the corresponding theory were detected. The question whether two automatic structures are isomorphic turned out to be highly undecidable in general as well as for some restricted classes of structures. In contrast,

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the isomorphism problem for string-automatic ordinals was proven to be decidable. Recently, some classes of structures for which string-automaticity of their tree-automatic members is decidable were identified. Last but not least, different classes of automatic structures were characterised by means of logical interpretations in universal structures.

The characterisation of automatic ordinals was provided by Delhommé [4]. An ordinal is string-automatic if, and only if, it is less than ω^ω. The respective bound for tree-automatic ordinals is ω^{ω^ω}. To obtain these results, Delhommé developed and employed a *decomposition technique* for automatic structures. Later, Khoussainov, Rubin, and Stephan generalised the only-if-implication of the string-automatic case by proving that the finite condensation rank (FC-rank) of any string-automatic linear ordering is below ω [9]. Roughly speaking, the FC-rank is an ordinal indicating how far a linear ordering is away from being dense. Basically, they applied the decomposition technique for string-automatic structures to the class of (scattered) linear orderings. Since that time, it is presumed that the FC-rank of every tree-automatic linear ordering is below ω^ω. However, this conjecture has not been verified yet.[1] We close this gap by our first main result.

▶ **Theorem 4.6.** *The* FC*-rank of every tree-automatic linear ordering is strictly below* ω^ω*.*

Again, the proof is an application of the decomposition technique to the class of (scattered) linear orderings. Unfortunately, Delhommé never provided a proof of his decomposition theorem for tree-automatic structures. As his wording of the theorem is also too weak for our purposes, we state and prove a refined version (Theorem 3.7).[2] However, the main difficulty in showing Theorem 4.6 is to substantiate that scattered linear orderings are accessible to the decomposition technique for *tree-automatic* structures (Proposition 4.3 and Corollary 4.5).

In the last section, we demonstrate how to adapt the (refined) decomposition technique to finite-rank tree-automatic structures (cf. [1, Section 1.3.7]). Roughly speaking, the rank of a tree-automatic structure describes the branching complexity of the trees involved and is measured in terms of the Cantor-Bendixson rank (cf. [9]). Our second main result is the following analogue of Theorem 4.6 for finite-rank tree-automatic linear orderings.

▶ **Theorem 5.2.** *Let* $k \in \mathbb{N}_+$*. The* FC*-rank of every rank-*k *tree-automatic linear ordering is strictly below* ω^k*.*

In the very end, we briefly sketch how to apply these results to show upper bounds on the Cantor-Bendixson rank of (finite-rank) tree-automatic finitely branching order trees, i.e., partial orderings which happen to be trees.

2 Background

2.1 Tree-Automatic Structures

This section recalls the basic notions of tree-automatic structures (cf. [1, 2]).

Let $\mathbf{2} = \{0, 1\}$ be the binary alphabet. The set of all *strings* over $\mathbf{2}$ is denoted by $\mathbf{2}^\star$ and the *empty string* by ε. A *tree domain* is a non-empty, finite, prefix-closed subset $D \subseteq \mathbf{2}^\star$. The *boundary* of D is the set $\partial D = \{\, ud \mid u \in D, d \in \mathbf{2}, ud \notin D \,\}$. Let Σ be an alphabet. A *finite* Σ*-labelled tree* (or just *tree*) is a map $t \colon D \to \Sigma$ where $\mathrm{dom}(t) = D$ is a tree

[1] Due to personal communication, S. Jain, B. Khoussainov, P. Schlicht, and F. Stephan recently verified the conjecture for scattered linear orderings. This implies that the FC-rank of arbitrary tree-automatic linear orderings is at most ω^ω (including).

[2] A similar refinement was used to bound the ordinal height of well-founded order trees [7].

domain. The set of all finite Σ-labelled trees is denoted by T_Σ. Its subsets are called *(tree) languages*. For $t \in T_\Sigma$ and a node $u \in \mathrm{dom}(t)$ the *subtree* $t{\upharpoonright}u \in T_\Sigma$ rooted at u is defined by $\mathrm{dom}(t{\upharpoonright}u) = \{\, v \in \mathbf{2}^\star \mid uv \in \mathrm{dom}(t) \,\}$ and $(t{\upharpoonright}u)(v) = t(uv)$. For nodes $u_1, \dots, u_n \in \mathrm{dom}(t)$ which form an anti-chain in t, i.e., they are mutually not prefixes of each other, and trees $t_1, \dots, t_n \in T_\Sigma$, we consider the tree $t[u_1/t_1, \dots, u_n/t_n] \in T_\Sigma$ which is obtained from t by simultaneously replacing for each $i \in [1, n]$ the subtree rooted at u_i by t_i.

A *(deterministic bottom-up) tree automaton* $\mathcal{M} = (Q, \iota, \delta, F)$ over Σ consists of a finite set Q of *states*, a *start state* $\iota \in Q$, a *transition function* $\delta \colon \Sigma \times Q \times Q \to Q$, and a set $F \subseteq Q$ of *accepting* states. For all $t \in T_\Sigma$, $u \in \mathrm{dom}(t) \cup \partial\,\mathrm{dom}(t)$, and maps $\rho \colon U \to Q$ with $U \subseteq \partial\,\mathrm{dom}(t)$ a state $\mathcal{M}(t, u, \rho) \in Q$ is defined recursively by

$$\mathcal{M}(t,u,\rho) = \begin{cases} \delta\big(t(u), \mathcal{M}(t,u0,\rho), \mathcal{M}(t,u1,\rho)\big) & \text{if } u \in \mathrm{dom}(t), \\ \rho(u) & \text{if } u \in U, \\ \iota & \text{if } u \in \partial\,\mathrm{dom}(t) \setminus U. \end{cases}$$

We omit the parameter u (resp. U) if $u = \varepsilon$ (resp. $U = \emptyset$). Notice that $\mathcal{M}(t, u) = \mathcal{M}(t{\upharpoonright}u)$. The tree language *recognised* by $\mathcal{M}$ is the set $L(\mathcal{M}) = \{\, t \in T_\Sigma \mid \mathcal{M}(t) \in F \,\}$. A language $L \subseteq T_\Sigma$ is *regular* if it can be recognised by some tree automaton.

Let $\diamond \notin \Sigma$ be a new symbol and $\Sigma_\diamond = \Sigma \cup \{\diamond\}$. The *convolution* of an n-tuple $\bar{t} = (t_1, \dots, t_n) \in (T_\Sigma)^n$ of trees is the tree $\otimes\bar{t} \in T_{\Sigma_\diamond^n}$ defined by

$$\mathrm{dom}(\otimes\bar{t}) = \mathrm{dom}(t_1) \cup \dots \cup \mathrm{dom}(t_n) \quad \text{and} \quad (\otimes\bar{t})(u) = \big(t'_1(u), \dots, t'_n(u)\big)\,,$$

where $t'_i(u) = t_i(u)$ if $u \in \mathrm{dom}(t_i)$ and $t'_i(u) = \diamond$ otherwise. If $n = 2$, we also write $t_1 \otimes t_2$ for $\otimes(t_1, t_2)$. A relation $R \subseteq (T_\Sigma)^n$ is *automatic* if the tree language $\otimes R = \{\, \otimes\bar{t} \mid \bar{t} \in R \,\} \subseteq T_{\Sigma_\diamond^n}$ is regular. We say a tree automaton *recognises* R if it recognises $\otimes R$.

A relational structure $\mathfrak{A} = \big(A; R_1^{\mathfrak{A}}, \dots, R_n^{\mathfrak{A}}\big)$ is called *tree-automatic* if its domain A is a regular tree language and each relation $R_i^{\mathfrak{A}}$ is automatic.[3] In this situation, a *tree-automatic presentation* of $\mathfrak{A}$ is a tuple of tree automata recognising A and the $R_i^{\mathfrak{A}}$. Abusing notation, we sometimes call any structure *tree-automatic* (in a wider sense) which is isomorphic to some tree-automatic structure (in the narrow sense). The following theorem lays out the main motivation for investigating tree-automatic structures.

▶ **Theorem 2.1** (Blumensath [2]). *Let $\mathfrak{A}$ be a tree-automatic structure. For every first-order formula $\phi(\bar{x})$ in the signature of $\mathfrak{A}$ the relation $\phi^{\mathfrak{A}}$ defined by ϕ is automatic and one can compute a tree automaton recognising $\phi^{\mathfrak{A}}$ from a tree-automatic presentation of $\mathfrak{A}$ and the formula ϕ. In particular, the first-order theory of $\mathfrak{A}$ is decidable.*

2.2 Linear Orderings

This section recalls the necessary background on linear orderings (cf. [12]).

A *linear ordering* is a structure $\mathfrak{A} = \big(A; \leq^{\mathfrak{A}}\big)$ where $\leq^{\mathfrak{A}}$ is a *non-strict* linear order on A. The corresponding *strict* linear order is denoted by $<^{\mathfrak{A}}$. If $\mathfrak{A}$ is clear from the context we omit the superscript $\mathfrak{A}$. For $n \in \mathbb{N}$ the (isomorphism type of the) linear ordering with exactly n elements is denoted by $\mathbf{n} = \big(\{0, \dots, n-1\}; \leq\big)$. Let $\mathfrak{I}$ and $\mathfrak{A}_i$ for each $i \in I$ be linear orderings. The $\mathfrak{I}$-*sum* of the $\mathfrak{A}_i$ is the linear ordering $\mathfrak{A} = \sum_{i \in \mathfrak{I}} \mathfrak{A}_i$ defined by $A = \biguplus_{i \in I} A_i$ and $x \leq^{\mathfrak{A}} y$ iff either $x, y \in A_i$ and $x \leq^{\mathfrak{A}_i} y$ for some $i \in I$ or $x \in A_i$ and $y \in A_j$ for some $i, j \in I$ with $i <^{\mathfrak{I}} j$. In case that $\mathfrak{I}$ is finite, say $\mathfrak{I} = \mathbf{n}$, we also write $\mathfrak{A}_0 + \dots + \mathfrak{A}_{n-1}$.

[3] By convention, structures are named in Fraktur and their domains by the same letter in Roman.

A linear ordering $\mathfrak{A}$ is *dense* if for all $x, y \in A$ with $x < y$ there exists a $z \in A$ such that $x < z < y$. Up to isomorphism, there are only five countable dense linear orderings, namely $\mathbf{1}$, η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, and $\mathbf{1} + \eta + \mathbf{1}$, where $\eta = (\mathbb{Q}; \leq)$ are the rational numbers ordered as usual. At the opposite extreme, $\mathfrak{A}$ is *scattered* if η cannot be embedded into $\mathfrak{A}$. Examples of scattered linear orderings include the natural numbers $\omega = (\mathbb{N}; \leq)$, the reversed natural numbers $\omega^* = (\mathbb{N}; \geq)$, the integers $\zeta = (\mathbb{Z}; \leq)$, and the finite linear ordering $\mathbf{n}$ for each $n \in \mathbb{N}$. Moreover, all well-orderings and scattered sums of scattered linear orderings are scattered.

For two subsets $X, Y \subseteq A$ of a linear ordering $\mathfrak{A}$ we write $X \ll Y$ if $x < y$ for all $x \in X$ and $y \in Y$. A *condensation (relation)* on $\mathfrak{A}$ is an equivalence relation $\sim$ on A such that each $\sim$-class is a (possibly non-closed) interval of $\mathfrak{A}$. In this situation, the set of all $\sim$-classes is (strictly) linearly ordered by $\ll$ and we denote this linear ordering by $\mathfrak{A}/{\sim}$. An example of a condensation is given by $x \sim y$ if x and y are at finite distance in $\mathfrak{A}$. Transfinitely iterating this process leads to the inductive definition of a condensation $\sim_\alpha$ on $\mathfrak{A}$ for each ordinal α:

1. $\sim_0$ is the identity relation on $\mathfrak{A}$,
2. for successor ordinals $\alpha = \beta + 1$ let $x \sim_\alpha y$ iff there are only finitely many elements between the $\sim_\beta$-classes of x and y in $\mathfrak{A}/{\sim_\beta}$, and
3. for limit ordinals α let $x \sim_\alpha y$ iff $x \sim_\beta y$ for some $\beta < \alpha$.

There exists an α such that $\sim_\alpha$ and $\sim_\beta$ coincide for each $\beta \geq \alpha$. The least such α is called *finite condensation rank* (FC-rank) of $\mathfrak{A}$ and denoted by $\mathrm{FC}(\mathfrak{A})$. In particular, if $\mathfrak{A}$ is countable then $\mathrm{FC}(\mathfrak{A})$ is also countable [12, Theorem 5.9]. Moreover, each $\sim_\alpha$-class is a scattered interval of $\mathfrak{A}$ and $\mathfrak{A}/{\sim_\alpha}$ is dense, proving the following theorem.

▶ **Theorem 2.2** (Hausdorff [12, Theorem 4.9]). *Every linear ordering $\mathfrak{A}$ is a dense sum of scattered linear orderings, i.e., there are a dense linear ordering $\mathfrak{I}$ and scattered linear orderings $\mathfrak{A}_i$ for each $i \in I$ such that $\mathfrak{A} = \sum_{i \in \mathfrak{I}} \mathfrak{A}_i$.*

Due to Hausdorff, there is a valuable inductive construction of the class of countable scattered linear orderings. For each countable ordinal α a class $\mathcal{VD}_\alpha$ is defined as follows:

1. $\mathcal{VD}_0 = \{\mathbf{0}, \mathbf{1}\}$ and
2. for $\alpha > 0$ the class $\mathcal{VD}_\alpha$ contains all ζ-sums of elements from $\mathcal{VD}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{VD}_\alpha$.

The class $\mathcal{VD}$ of *very discrete* linear orderings is the union of all classes $\mathcal{VD}_\alpha$ and the VD-*rank* of some $\mathfrak{A} \in \mathcal{VD}$, denoted by $\mathrm{VD}(\mathfrak{A})$, is the least ordinal α with $\mathfrak{A} \in \mathcal{VD}_\alpha$.

▶ **Theorem 2.3** (Hausdorff [12, Theorem 5.24]). *A countable linear ordering $\mathfrak{A}$ is scattered if, and only if, it is contained in $\mathcal{VD}$. In case that $\mathfrak{A}$ is scattered,* $\mathrm{FC}(\mathfrak{A}) = \mathrm{VD}(\mathfrak{A})$.

3 Delhommé's Decomposition Technique

3.1 Augmentations and the Decomposition Theorem

In this section, we present the decomposition technique Delhommé developed and employed to show that every tree-automatic ordinal is less than ω^{ω^ω} [4]. As we want to apply this technique to linear orderings, we restrict our attention to structures whose signature contains only a single binary relation symbol $\leq$, i.e., (directed) graphs. First, we introduce the central notions of sum- and box-augmentations. For a graph $\mathfrak{A}$ and a subset $B \subseteq A$ we denote by $\mathfrak{A}{\restriction}B$ the subgraph induced by B.

▶ **Definition 3.1.** A graph $\mathfrak{A}$ is a *sum-augmentation* of graphs $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ if there exists a finite partition $A = \biguplus_{i\in[1,n]} A_i$ of $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}A_i \cong \mathfrak{B}_i$ for each $i \in [1, n]$.

▶ **Example 3.2.** Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ be graphs.

1. Suppose that the $\mathfrak{B}_i$ are linear orderings and let $\mathfrak{A}$ be a linearisation of the partial ordering $\mathfrak{B}_1 \amalg \cdots \amalg \mathfrak{B}_n = \big(\biguplus_{i\in[1,n]} B_i; \preceq\big)$ with $x \preceq y$ iff $x, y \in B_i$ and $x \leq^{\mathfrak{B}_i} y$ for some $i \in [1, n]$. Then $\mathfrak{A}$ is a sum-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$.
2. Let $\mathfrak{A}$ be a linear ordering which is a sum-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$. First, each $\mathfrak{B}_i$ can be embedded into $\mathfrak{A}$ and is hence a linear ordering, which is scattered in case $\mathfrak{A}$ is scattered. Second, $\mathfrak{A}$ is isomorphic to a linearisation of $\mathfrak{B}_1 \amalg \cdots \amalg \mathfrak{B}_n$.

▶ **Definition 3.3.** A graph $\mathfrak{A}$ is a *box-augmentation* of graphs $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ if there exists a bijection $f : B_1 \times \cdots \times B_n \to A$ such that for all $j \in [1, n]$ and $\bar{x} \in \prod_{i\in[1,n],i\neq j} B_i$ the map $f^j_{\bar{x}} : B_j \to A$ defined by $f^j_{\bar{x}}(b) = (x_1, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n)$ is an embedding of $\mathfrak{B}_j$ into $\mathfrak{A}$.

▶ **Example 3.4.** Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ be graphs.

1. Suppose that the $\mathfrak{B}_i$ are linear orderings and let $\mathfrak{A}$ be a linearisation of the partial ordering $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_n = \big(B_1 \times \cdots \times B_n; \preceq\big)$ with $\bar{x} \preceq \bar{y}$ iff $x_i \leq^{\mathfrak{B}_i} y_i$ for all $i \in [1, n]$. The identity map $B_1 \times \cdots \times B_n \to A$ witnesses that $\mathfrak{A}$ is a box-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$.
2. Let $\mathfrak{A}$ be a linear ordering which is a box-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$. First, each $\mathfrak{B}_i$ can be embedded into $\mathfrak{A}$ and is hence a linear ordering, which is scattered in case $\mathfrak{A}$ is scattered. Second, the bijection f from Definition 3.3 above is an isomorphism between a linearisation of $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_n$ and $\mathfrak{A}$.

In order to make the class of linear orderings accessible to the decomposition technique, we have to study the connection between box-augmentations and the FC-rank. More precisely, given some linear orderings $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ we want to establish a bound on the FC-rank of any linear ordering which is a box-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ in terms of the FC-ranks of the $\mathfrak{B}_i$. However, the following example indicates that this is impossible in general.

▶ **Example 3.5.** Consider the partial ordering $\omega \times \omega^* = (\mathbb{N} \times \mathbb{N}, \preceq)$, where $\preceq$ is defined as above. For each $i \in \mathbb{Z}$ the elements $(m, n) \in \mathbb{N} \times \mathbb{N}$ with $m - n = i$ form an anti-chain, i.e., they are mutually incomparable by $\preceq$. Therefore, any ζ-sum of countably infinite linear orderings is (isomorphic to) a linearisation of $\omega \times \omega^*$. In particular, for any countable ordinal $\alpha > 1$ there exists a (scattered) linear ordering $\mathfrak{A}$ with $\mathrm{FC}(\mathfrak{A}) = \alpha$ which is a box-augmentation of ω, ω^*. Compare this to the fact that $\mathrm{FC}(\omega) = \mathrm{FC}(\omega^*) = 1$.

Owing to this observation, we introduce a restricted notion of box-augmentations. Therein, a *finite colouring* of a graph $\mathfrak{A}$ is a map $\sigma : A \times A \to Q$ into a finite set Q such that $\sigma(x, y) = \sigma(x', y')$ and $x \leq y$ imply $x' \leq y'$ for all $x, y, x', y' \in A$.

▶ **Definition 3.6.** The box-augmentation in Definition 3.3 is called *tame* if for each $i \in [1, n]$ there exists a finite colouring $\sigma_i : B_i \times B_i \to Q_i$ of $\mathfrak{B}_i$ such that the map

$$f(\sigma_1, \ldots, \sigma_n) : A \times A \to Q_1 \times \ldots \times Q_n, \big(f(\bar{x}), f(\bar{y})\big) \mapsto \big(\sigma_1(x_1, y_1), \ldots, \sigma_n(x_n, y_n)\big)$$

is a finite colouring of $\mathfrak{A}$.

▶ Remark. In the situation of Definition 3.6, assume that all Q_i are the same set, say $\{1, \ldots, m\}$. For each $i \in [1, n]$ consider the structure $\mathfrak{C}_i = \big(B_i; R_1^{\mathfrak{C}_i}, \ldots, R_m^{\mathfrak{C}_i}\big)$ with $R_q^{\mathfrak{C}_i} = \sigma_i^{-1}(q)$. Due to the definition of a finite colouring, the $R_q^{\mathfrak{C}_i}$ form a finite partition of $B_i \times B_i$ which is compatible with $\leq^{\mathfrak{B}_i}$. Therefore, the graph $\mathfrak{A}$ is a generalised product—in the sense of Feferman and Vaught—of the structures $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ using only atomic formulae.

We will see later, in Lemma 4.4 and its proof, that every linear ordering $\mathfrak{A}$ which is a tame box-augmentation of ω, ω^* is scattered and satisfies $\mathrm{FC}(\mathfrak{A}) \leq 3$. We conclude this section by providing our refined version of Delhommé's decomposition theorem. For a structure

$\mathfrak{A}$, a first-order formula $\phi(x, y_1, \ldots, y_r)$ in the signature of $\mathfrak{A}$, and a tuple $\bar{s} \in A^r$ we let $\phi^{\mathfrak{A}}(\cdot, \bar{s}) = \{\, t \in A \mid \mathfrak{A} \models \phi(t, \bar{s}) \,\}$ and abbreviate $\mathfrak{A}{\upharpoonright}\phi^{\mathfrak{A}}(\cdot, \bar{s})$ by $\mathfrak{A}{\upharpoonright}\phi, \bar{s}$.

▶ **Theorem 3.7.** *Let $\mathfrak{A}$ be a tree-automatic graph and $\phi(x, y_1, \ldots, y_r)$ a first-order formula in the signature of graphs. Then there exists a finite set $\mathcal{S}^{\mathfrak{A}}_{\phi}$ of tree-automatic graphs such that for all tuples $\bar{s} \in A^r$ the graph $\mathfrak{A}{\upharpoonright}\phi, \bar{s}$ is a sum-augmentation of* tame *box-augmentations of elements from $\mathcal{S}^{\mathfrak{A}}_{\phi}$.*

▶ Remark. The only, but very important, difference to Delhommé's decomposition theorem is our addition of the word *tame*. Since by Example 3.5 there is no reasonable connection between the FC-rank and arbitrary box-augmentations, the version without *tame* cannot be used to investigate bounds on the FC-rank of tree-automatic (scattered) linear orderings.

Proof of Theorem 3.7. Suppose that $A \subseteq T_\Sigma$ for some alphabet Σ. Let $\mathcal{M}_\leq$ and $\mathcal{M}_\phi$ be tree automata recognising $\leq^{\mathfrak{A}}$ and $\phi^{\mathfrak{A}}$ and $Q_\leq$ and Q_ϕ be their state sets, respectively. In order to simplify notation, for $t \in T_\Sigma$ we put $t^{\leq} = t \otimes t$ and define $t^\phi \in T_{\Sigma_\diamond^{1+r}}$ by $\mathrm{dom}\big(t^\phi\big) = \mathrm{dom}(t)$ and $t^\phi(u) = \big(t(u), \diamond, \ldots, \diamond\big)$, where the number of $\diamond$-symbols is r.

As a first step, we construct the set $\mathcal{S}^{\mathfrak{A}}_{\phi}$. Therefore, consider the set $\Gamma = Q_\leq \times Q_\phi \times 2^{Q_\leq}$. For each $\gamma = \big(q_\leq, q_\phi, P_\leq\big) \in \Gamma$ we define a graph $\mathfrak{S}_\gamma$ by

$$S_\gamma = \big\{\, t \in T_\Sigma \;\big|\; \mathcal{M}_\leq\big(t^{\leq}\big) = q_\leq \;\&\; \mathcal{M}_\phi\big(t^\phi\big) = q_\phi \,\big\} \quad \text{and} \quad t_1 \leq^{\mathfrak{S}_\gamma} t_2 \text{ iff } \mathcal{M}_\leq(t_1 \otimes t_2) \in P_\leq \,.$$

Clearly, $\mathfrak{S}_\gamma$ is tree-automatic. Finally, we put $\mathcal{S}^{\mathfrak{A}}_{\phi} = \{\, \mathfrak{S}_\gamma \mid \gamma \in \Gamma \,\}$.

We have to show that for each $\bar{s} = (s_1, \ldots, s_r) \in A^r$ the subgraph $\mathfrak{A}{\upharpoonright}\phi, \bar{s}$ is a sum-augmentation of box-augmentations of elements from $\mathcal{S}^{\mathfrak{A}}_{\phi}$. Therefore, we fix a tuple $\bar{s}$, put $\mathfrak{B} = \mathfrak{A}{\upharpoonright}\phi, \bar{s}$, and consider the tree domain $D = \bigcup_{1 \leq i \leq n} \mathrm{dom}(s_i)$. The *$\bar{s}$-type* of a tree $t \in T_\Sigma$ is defined as $\mathrm{tp}_{\bar{s}}(t) = \big(t{\upharpoonright}D, U, \rho_\leq, \rho_\phi\big)$, where $t{\upharpoonright}D \in T_\Sigma$ is the restriction of t to the tree domain $\mathrm{dom}(t) \cap D$, $U = \mathrm{dom}(t) \cap \partial D$, and $\rho_R \colon U \to Q_R, u \mapsto \mathcal{M}_R\big((t{\upharpoonright}u)^R\big)$ for $R \in \{<, \phi\}$. Observe that

$$\mathcal{M}_\phi\big(\otimes(t, \bar{s})\big) = \mathcal{M}_\phi\big(\otimes(t{\upharpoonright}D, \bar{s})\big[\big(u/(t{\upharpoonright}u)^\phi\big)_{u \in U}\big]\big) = \mathcal{M}_\phi\big(\otimes(t{\upharpoonright}D, \bar{s}), \rho_\phi\big) \,. \tag{1}$$

Hence, $\mathrm{tp}_{\bar{s}}(t)$ completely determines whether $t \in B$. Since D is finite, there are only finitely many different $\bar{s}$-types. Thus, the equivalence relation $\sim_{\bar{s}}$ on T_Σ defined by $t_1 \sim_{\bar{s}} t_2$ iff $\mathrm{tp}_{\bar{s}}(t_1) = \mathrm{tp}_{\bar{s}}(t_2)$ has finite index. Due to the mentioned consequence of Eq. (1), B is a union of $\sim_{\bar{s}}$-classes. Say $B = C_1 \uplus \cdots \uplus C_m$ is the corresponding partition of B into $\sim_{\bar{s}}$-classes, then $\mathfrak{B}$ is a sum-augmentation of $\mathfrak{B}{\upharpoonright}C_1, \ldots, \mathfrak{B}{\upharpoonright}C_m$.

As a final step, we fix a single $\sim_{\bar{s}}$-class $C \subseteq B$ and provide a tuple of graphs from $\mathcal{S}^{\mathfrak{A}}_{\phi}$ of whom $\mathfrak{C} = \mathfrak{B}{\upharpoonright}C$ is a box-augmentation. Let $\tau = \big(t_0, U, \rho_\leq, \rho_\phi\big)$ be the $\bar{s}$-type corresponding to C. For $u \in U$ we put $\gamma(u) = \big(\rho_\leq(u), \rho_\phi(u), P_\leq(u)\big) \in \Gamma$, where $P_\leq(u)$ is the set of all $q \in Q_\leq$ for which $\mathcal{M}_\leq\big(t_0^{\leq}, \rho_\leq[u \mapsto q]\big)$ is an accepting state in $\mathcal{M}_\leq$.

It is easy to show that the map $f \colon \prod_{u \in U} S_{\gamma(u)} \to C$ with $f\big((x_u)_{u \in U}\big) = t_0[(u/x_u)_{u \in U}]$ is a bijection witnessing that $\mathfrak{C}$ is a box-augmentation of the collection of the $\mathfrak{S}_{\gamma(u)}$. To see that this box-augmentation is tame, consider for each $u \in U$ the finite colouring σ_u of $\mathfrak{S}_{\gamma(u)}$ which is given by $\sigma_u(x, y) = \mathcal{M}_\leq(x \otimes y)$ and let $\sigma = f\big((\sigma_u)_{u \in U}\big)$. Then $\mathcal{M}_\leq(x \otimes y)$ is completely determined by $\sigma(x, y)$ for all $x, y \in C$ and hence σ is a finite colouring of $\mathfrak{C}$. ◀

3.2 Indecomposability and Tree-Automatic Ordinals

According to Delhommé's approach [4], we introduce the notion of indecomposable ordinals and provide a refined version of his result on indecomposability as a corollary of Theorem 3.7.

Therefore, suppose that $\mathcal{C}$ is a class of graphs and an ordinal $\mathrm{rk}(\mathfrak{A})$ is assigned to each $\mathfrak{A} \in \mathcal{C}$ in an isomorphism invariant way. We say that $\mathcal{C}$ is *ranked* by rk. An ordinal α is rk-*sum-indecomposable* if for any graph $\mathfrak{A} \in \mathcal{C}$ with $\mathrm{rk}(\mathfrak{A}) = \alpha$ and all graphs $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ which $\mathfrak{A}$ is a sum-augmentation of, there exists an $i \in [1, n]$ with $\mathfrak{B}_i \in \mathcal{C}$ and $\mathrm{rk}(\mathfrak{B}_i) = \alpha$. Analogously, rk-*box-indecomposable* and rk-*tame-box-indecomposable* ordinals are defined. Although Example 3.5 shows that neither FC- nor VD-box-indecomposability are useful, Corollary 4.5 indicates that VD-*tame*-box-indecomposability is indeed a reasonable notion.

▶ **Corollary 3.8.** *Let $\mathcal{C}$ be a class of graphs ranked by* rk*, $\mathfrak{A}$ a tree-automatic graph, and $\phi(x, y_1, \dots, y_r)$ a first-order formula in the signature of graphs. Then there are only finitely many ordinals α which are simultaneously* rk*-sum- and* rk*-tame-box-indecomposable and admit a tuple $\bar{s} \in A^r$ with $\mathfrak{A}{\restriction}\phi, \bar{s} \in \mathcal{C}$ and $\mathrm{rk}\big(\mathfrak{A}{\restriction}\phi, \bar{s}\big) = \alpha$.*

Proof. Let $\mathcal{S}_\phi^{\mathfrak{A}}$ be the finite set of graphs which exists by Theorem 3.7. Consider an ordinal α satisfying the condition above, witnessed by $\bar{s} \in A^r$. There exist box-augmentations $\mathfrak{B}_1, \dots, \mathfrak{B}_m$ of elements from $\mathcal{S}_\phi^{\mathfrak{A}}$ such that $\mathfrak{A}{\restriction}\phi, \bar{s}$ is a sum-augmentation of them. Then there is an $i \in [1, m]$ such that $\mathfrak{B}_i \in \mathcal{C}$ and $\mathrm{rk}(\mathfrak{B}_i) = \alpha$. Now, there exist $\mathfrak{C}_1, \dots, \mathfrak{C}_n \in \mathcal{S}_\phi^{\mathfrak{A}}$ of which $\mathfrak{B}_i$ is a tame box-augmentation. Again, there is a $j \in [1, n]$ with $\mathfrak{C}_j \in \mathcal{C}$ and $\mathrm{rk}(\mathfrak{C}_j) = \alpha$. Altogether, α belongs to the finite set $\{\, \mathrm{rk}(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{S}_\phi^{\mathfrak{A}} \,\}$. ◀

To illustrate the general idea behind the proof of Theorem 4.6, we demonstrate how to show Delhommé's characterisation of the tree-automatic ordinals. For the purpose of later reuse, the proof of the if-part is slightly more involved than actually necessary. For the converse implication, let $\mathcal{C}$ be the class of ordinals and $\mathrm{rk}(\alpha) = \alpha$. Results by Caruth [3] imply that ω^α is rk-sum-indecomposable and ω^{ω^α} is rk-box-indecomposable for each ordinal α.

▶ **Corollary 3.9** (Delhommé [4])**.** *An ordinal α is tree-automatic if, and only if, $\alpha < \omega^{\omega^\omega}$.*

Proof. We first show the if-part. There exists a $k \in \mathbb{N}$ such that $\alpha < \omega^{\omega^k}$. For $k = 0$ the claim is trivial. Suppose $k \geq 1$. Then $\alpha < \omega^{\omega^{k-1} n}$ for some $n \in \mathbb{N}$. We show that $\omega^{\omega^{k-1}}$ is tree-automatic by induction on k. The case $k = 1$ is obvious. For $k > 1$ we combine the fact $\omega^{\omega^{k-1}} = \big(\omega^{\omega^{k-2}}\big)^\omega$ with the general idea behind showing that the class of tree-automatic structures is closed under direct ω-sums. Encoding $\bar{\beta} \in \big(\omega^{\omega^{k-1}}\big)^n$ by $\otimes\bar{\beta}$ shows that $\omega^{\omega^{k-1} n}$ is also tree-automatic. Finally, $\big(\omega^{\omega^{k-1} n}\big){\restriction}\phi, \alpha = \alpha$ with $\phi(x, y) = x < y$ proves the claim.

For the sake of a contradiction to the only-if-implication, assume that $\alpha \geq \omega^{\omega^\omega}$ is tree-automatic. For each $d \in \mathbb{N}$ we have $\alpha{\restriction}\phi, \omega^{\omega^d} = \omega^{\omega^d}$, contradicting Corollary 3.8. ◀

4 Tree-Automatic Linear Orderings

The ultimate goal of this section is to prove Theorem 4.6, stating that the FC-rank of every tree-automatic linear ordering is below ω^ω. Owing to the fact that every linear ordering is a dense sum of scattered linear orderings, the strategy is to apply Corollary 3.8 to the class $\mathcal{VD}$ of scattered linear orderings ranked by VD_*, a slight variation of the VD-rank. Since it is already known that every ordinal is VD_*-sum-indecomposable [9], the main difficulty is to identify the VD_*-tame-box-indecomposable ordinals.

▶ **Definition 4.1.** The VD_**-rank* of a scattered linear ordering $\mathfrak{A}$, denoted by $\mathrm{VD}_*(\mathfrak{A})$, is the least ordinal α such that $\mathfrak{A}$ is a finite sum of elements from $\mathcal{VD}_\alpha$.

The VD_*-rank of a scattered linear ordering $\mathfrak{A}$ is closely related to its VD-rank by the inequality $\mathrm{VD}_*(\mathfrak{A}) \leq \mathrm{VD}(\mathfrak{A}) \leq \mathrm{VD}_*(\mathfrak{A}) + 1$. For each $B \subseteq A$ we have $\mathrm{VD}(\mathfrak{A}{\restriction}B) \leq \mathrm{VD}(\mathfrak{A})$

[12, Lemma 5.14]. In particular, this implies $\mathrm{VD}_*(\mathfrak{A}{\restriction}B) \le \mathrm{VD}_*(\mathfrak{A})$ for all $B \subseteq A$. Remember that whenever a scattered linear ordering $\mathfrak{A}$ is a sum- or box-augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ the $\mathfrak{B}_i$ are also scattered linear orderings (cf. Examples 3.2 and 3.4). The following proposition essentially states that every countable ordinal is VD_*-sum-indecomposable.

▶ **Proposition 4.2** (Khoussainov, Rubin, Stephan [9, Proposition 4.4]). *Let a scattered linear ordering $\mathfrak{A}$ be a sum-augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. Then*

$$\mathrm{VD}_*(\mathfrak{A}) = \max\big\{\mathrm{VD}_*(\mathfrak{B}_1), \dots, \mathrm{VD}_*(\mathfrak{B}_n)\big\}\,.$$

Our main tool for identifying the VD_*-tame-box-indecomposable ordinals is Proposition 4.3 below. Let α and β be two ordinals. Due to Cantor normal form, there are ordinal exponents $\gamma_1 > \dots > \gamma_n \ge 0$ and coefficients $k_i, \ell_i \in \mathbb{N}$, which are possibly 0, such that $\alpha = \sum_{i=1}^{i=n} \omega^{\gamma_i} k_i$ and $\beta = \sum_{i=1}^{i=n} \omega^{\gamma_i} \ell_i$. The *natural sum* of α and β is $\alpha \oplus \beta = \sum_{i=1}^{i=n} \omega^{\gamma_i}(k_i + \ell_i)$. Compared to the usual addition of ordinals, this operation is commutative and strictly monotonic in both arguments.

▶ **Proposition 4.3.** *Let the scattered linear ordering $\mathfrak{A}$ be a tame box-augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. Then*

$$\mathrm{VD}_*(\mathfrak{A}) \le \mathrm{VD}_*(\mathfrak{B}_1) \oplus \dots \oplus \mathrm{VD}_*(\mathfrak{B}_n)\,.$$

The proof below reveals the main benefit of tameness: box-augmentations are opened to arguments using *Ramsey's theorem*. It proceeds by induction on n, reducing to case $n = 2$.

▶ **Lemma 4.4.** *Let the scattered linear ordering $\mathfrak{A}$ be a tame box-augmentation of $\mathfrak{B}, \mathfrak{C}$. Then*

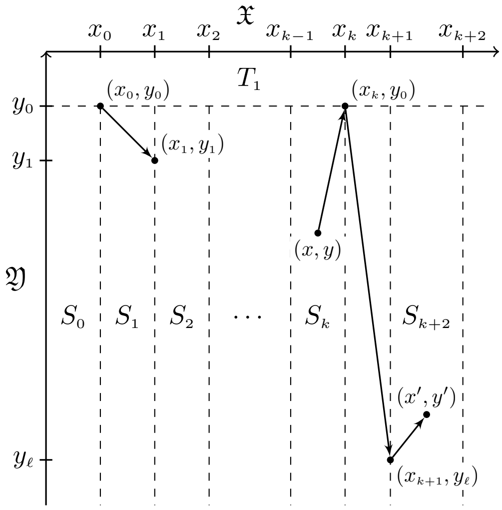
$$\mathrm{VD}_*(\mathfrak{A}) \le \mathrm{VD}_*(\mathfrak{B}) \oplus \mathrm{VD}_*(\mathfrak{C})\,.$$

Proof. We proceed by induction on $\beta = \mathrm{VD}_*(\mathfrak{B})$ and $\gamma = \mathrm{VD}_*(\mathfrak{C})$. If $\beta = 0$, then $\mathfrak{B}$ is finite, $\mathfrak{A}$ a sum-augmentation of $|B|$ many copies of $\mathfrak{C}$, and $\mathrm{VD}_*(\mathfrak{A}) = \mathrm{VD}_*(\mathfrak{C})$ by Proposition 4.2. Similarly, $\mathrm{VD}_*(\mathfrak{A}) = \mathrm{VD}_*(\mathfrak{B})$ whenever $\gamma = 0$. Thus, suppose $\beta > 0$ and $\gamma > 0$.

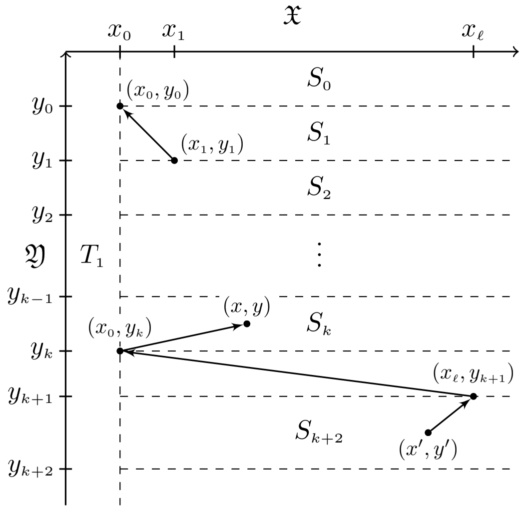
Due to Example 3.4, we may assume that $\mathfrak{A}$ is a linearisation of $\mathfrak{B} \times \mathfrak{C}$. In particular, $A = B \times C$. By definition, $\mathfrak{B} = \mathfrak{B}_1 + \dots + \mathfrak{B}_m$ and $\mathfrak{C} = \mathfrak{C}_1 + \dots + \mathfrak{C}_n$ for some $\mathfrak{B}_i \in \mathcal{VD}_\beta$ and $\mathfrak{C}_j \in \mathcal{VD}_\gamma$. Since every ζ-sum can be split into an ω^*-sum and an ω-sum, we can assume that none of the $\mathfrak{B}_i$ or $\mathfrak{C}_j$ is constructed as a ζ-sum. By Proposition 4.2, it suffices to show $\mathrm{VD}_*\big(\mathfrak{A}{\restriction}(B_i \times C_j)\big) \le \beta \oplus \gamma$ for all $i \in [1, m]$ and $j \in [1, n]$. Therefore, fix i and j, and let $\mathfrak{Z} = \mathfrak{A}{\restriction}(B_i \times C_j)$, $\mathfrak{X} = \mathfrak{B}_i$, and $\mathfrak{Y} = \mathfrak{C}_j$. Notice that $\mathfrak{Z}$ is a tame box-augmentation of $\mathfrak{X}, \mathfrak{Y}$.

If $\mathfrak{X}$ is a finite sum of elements from $\mathcal{VD}_{<\beta}$, then $\mathrm{VD}_*(\mathfrak{X}) < \beta$ and the claim follows by induction. The case of a finite sum $\mathfrak{Y}$ is analogous. Thus, we assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are ω- or ω^*-sums. We only demonstrate the case $\mathfrak{X} = \sum_{k \in \omega} \mathfrak{X}_k$ with $\mathfrak{X}_k \in \mathcal{VD}_{<\beta}$ and $\mathfrak{Y} = \sum_{\ell \in \omega^*} \mathfrak{Y}_\ell$ with $\mathfrak{Y}_\ell \in \mathcal{VD}_{<\gamma}$, for the remaining cases are very similar.

There are finite colourings σ of $\mathfrak{X}$ and σ' of $\mathfrak{Y}$ inducing a finite colouring of $\mathfrak{Z}$. Using *Ramsey's theorem*, we find an unbounded infinite sequence $x_0 < x_1 < x_2 < \cdots$ in $\mathfrak{X}$ which is *monochromatic* w.r.t. σ, i.e., $\sigma(x_k, x_{k'})$ is the same colour for all $k < k'$. Similarly, we find an unbounded infinite sequence $y_0 > y_1 > y_2 > \cdots$ in $\mathfrak{Y}$ which is monochromatic w.r.t. σ'. Depending on how (x_0, y_0) and (x_1, y_1) are ordered in $\mathfrak{Z}$, we distinguish two cases. As they are very similar we only deal with the case $(x_0, y_0) < (x_1, y_1)$, whose treatment is sketched in Figure 1(a). The horizontal axis depicts $\mathfrak{X}$ and increases from left to right, whereas the vertical axis outlines $\mathfrak{Y}$ and grows upwards. Within the grid, arrows point from lesser to greater elements. Figure 1(b) sketches the case $(x_0, y_0) > (x_1, y_1)$.



(a) The case $(x_0, y_0) < (x_1, y_1)$.



(b) The case $(x_0, y_0) > (x_1, y_1)$.

Figure 1 Proof sketch for the inductive step of Lemma 4.4.

We partition the set $Z = X \times Y$ into sets $S_0, S_1, S_2, \ldots$ and T_1 as indicated in Figure 1(a).[4] For each $k \in \mathbb{N}$ there exists an $k' \in \mathbb{N}$ such that $S_k \subseteq (X_0 \cup \cdots \cup X_{k'}) \times Y$. Since $\mathrm{VD}_*(\mathfrak{X}_0 + \cdots + \mathfrak{X}_{k'}) < \beta$, the induction hypothesis yields $\mathrm{VD}_*(\mathfrak{Z}{\restriction}S_k) < \beta \oplus \gamma$. Similarly, we obtain $\mathrm{VD}_*(\mathfrak{Z}{\restriction}T_1) < \beta \oplus \gamma$. The right part of Figure 1(a) sketches how to show $S_k \ll S_{k+2}$ for all $k \in \mathbb{N}$. Therein $(x_k, y_0) < (x_{k+1}, y_\ell)$ follows from $(x_0, y_0) < (x_1, y_1)$ due to *monochromaticity and tameness*. As a consequence, we obtain $\mathfrak{Z}{\restriction}T_2 = \sum_{k\in\omega} \mathfrak{Z}{\restriction}S_{2k}$ for $T_2 = \bigcup_{k\in\mathbb{N}} S_{2k}$. Since every $\mathfrak{Z}{\restriction}S_{2k}$ is a finite sum of elements from $\mathcal{VD}_{<\beta\oplus\gamma}$, $\mathfrak{Z}{\restriction}T_2$ is an ω-sum of elements from $\mathcal{VD}_{<\beta\oplus\gamma}$. Thus, $\mathrm{VD}_*(\mathfrak{Z}{\restriction}T_2) \le \beta \oplus \gamma$. Analogously, $\mathrm{VD}_*(\mathfrak{Z}{\restriction}T_3) \le \beta \oplus \gamma$ for $T_3 = \bigcup_{k\in\mathbb{N}} S_{2k+1}$. Finally, Proposition 4.2 and $Z = T_1 \uplus T_2 \uplus T_3$ imply $\mathrm{VD}_*(\mathfrak{A}) \le \beta \oplus \gamma$. ◀

Proof of Proposition 4.3. We proceed by induction on n. The case $n = 1$ is obvious. Thus, consider $n > 1$. We assume that $\mathfrak{A}$ is a linearisation of $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_n$. There are finite colourings $\sigma_i \colon B_i \times B_i \to Q_i$ of each $\mathfrak{B}_i$ which induce a finite colouring of $\mathfrak{A}$. For each $q \in Q_1$ we put $X_q = \{\, x \in B_1 \mid \sigma_1(x, x) = q \,\}$, fix some $x_q \in X_q$, and let $\mathfrak{C}_q = \mathfrak{A}{\restriction}(\{x_q\} \times Y)$, where $Y = B_2 \times \cdots \times B_n$. Straightforward arguments show that $\mathfrak{C}_q$ is a tame box-augmentation of $\mathfrak{B}_2, \ldots, \mathfrak{B}_n$ and $\mathfrak{A}{\restriction}(X_q \times Y)$ is a tame box-augmentation of $\mathfrak{B}_1{\restriction}X_q, \mathfrak{C}_q$. Lemma 4.4 and the induction hypothesis yield

$$\mathrm{VD}_*\big(\mathfrak{A}{\restriction}(X_q \times Y)\big) \le \mathrm{VD}_*(\mathfrak{B}_1{\restriction}X_q) \oplus \mathrm{VD}_*(\mathfrak{C}_q) \le \mathrm{VD}_*(\mathfrak{B}_1) \oplus \bigoplus_{i\in[2,n]} \mathrm{VD}_*(\mathfrak{B}_i)\,.$$

Finally, $A = \biguplus_{q\in Q_1} X_q \times Y$ and Proposition 4.2 imply the claim. ◀

▶ **Corollary 4.5.** *Every countable ordinal of the shape ω^α is VD_*-tame-box-indecomposable.*

Proof. Let a scattered linear ordering $\mathfrak{A}$ with $\mathrm{VD}_*(\mathfrak{A}) = \omega^\alpha$ be a tame box-augmentation of $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$. Each $\mathfrak{B}_i$ can be embedded into $\mathfrak{A}$ and thus $\mathrm{VD}_*(\mathfrak{B}_i) \le \omega^\alpha$. If this inequality were strict for each $i \in [1, n]$, the definition of $\oplus$ would imply $\mathrm{VD}_*(\mathfrak{B}_1) \oplus \cdots \oplus \mathrm{VD}_*(\mathfrak{B}_n) < \omega^\alpha$, contradicting Proposition 4.3. ◀

[4] It does not matter to which of the adjacent sets the dashed lines belong.

Using the previous results, we prove our main result on tree-automatic linear orderings.

▶ **Theorem 4.6.** *The* FC*-rank of every tree-automatic linear ordering is strictly below* ω^ω*.*

Proof. For the sake of a contradiction, assume there exists a tree-automatic linear ordering $\mathfrak{A}$ with $\mathrm{FC}(\mathfrak{A}) \geq \omega^\omega$. Consider the formula $\phi(x, y_1, y_2) = y_1 \leq x \wedge x \leq y_2$. Due to the proof of [9, Proposition 4.5], for each $d \in \mathbb{N}$ there is a $\bar{s} \in A^2$ such that the closed interval $\mathfrak{I} = \mathfrak{A}{\restriction}\phi, \bar{s}$ is scattered and $\mathrm{VD}(\mathfrak{I}) = \omega^d + 1$. As $\mathfrak{I}$ contains least and greatest element, it is a finite sum of elements from $\mathcal{VD}_{<\omega^d+1} = \mathcal{VD}_{\omega^d}$ and hence $\mathrm{VD}_*(\mathfrak{I}) = \omega^d$. Since ω^d is VD_*-sum- and VD_*-tame-box-indecomposable, this contradicts Corollary 3.8. ◀

5 Finite-Rank Tree-Automatic Linear Orderings

In this section we reintroduce finite-rank tree-automatic structures [1] and investigate the linear orderings among them. The highlight is Theorem 5.2 which states that the FC-rank of every rank-k tree-automatic linear ordering is below ω^k.

5.1 Finite-Rank Tree-Automatic Structures

A *binary tree* is a (possibly empty or infinite) prefix-closed subset $T \subseteq \mathbf{2}^\star$ whose elements are considered to be ordered by the prefix relation $\preceq$. The (isomorphism type of the) *subtree* rooted at $u \in T$ is the binary tree $T{\restriction}u = \{\, v \in \mathbf{2}^\star \mid uv \in T \,\}$. We call T *regular* if it is a regular language, or due to the Myhill-Nerode theorem equivalently, if there are only finitely many distinct subtrees $T{\restriction}u$. To every tree language $L \subseteq T_\Sigma$ we assign the binary tree $T(L) = \bigcup_{t \in L} \mathrm{dom}(t)$, which is effectively regular when L is regular.

An *infinite branch* of T is a prefix-closed infinite subset $P \subseteq T$ which is linearly ordered by $\preceq$. The *derivative* of T is the binary tree $d(T)$ consisting of all $u \in T$ which are contained in at least two distinct infinite branches. This operation effectively preserves regularity. When T is regular, $d^{(n)}(T)$ is finite for some $n \in \mathbb{N}$, where $d^{(n)}$ denotes the n-fold application of d, precisely if the full binary tree $\mathbf{2}^\star$ cannot be embedded into T [9, Section 7]. If these equivalent conditions are satisfied, the *rank* of T is the least such $n \in \mathbb{N}$.[5]

▶ **Definition 5.1.** Let $k \in \mathbb{N}$. A tree-automatic structure $\mathfrak{A}$ is *rank-k tree-automatic* if the rank of $T(A)$ is at most k and *finite-rank tree-automatic* if the rank of $T(A)$ exists.[6]

▶ Remark. For a tree-automatic structure $\mathfrak{A}$ the rank of $T(A)$ is not an isomorphism invariant property, but depends on its specific representation as a tree-automatic structure. The rank of $T(A)$ is computable from a tree automaton recognising A. It can be shown that the rank-1 tree-automatic structures are precisely those which are isomorphic to a string-automatic structure.

5.2 Linear Orderings

Theorem 5.2 is our main result on finite-rank tree-automatic linear orderings. Basically, it is shown by adapting Theorem 4.6's proof. The key idea behind this adaption is provided by Lemma 5.3 below.

[5] This rank is a slight variation of the Cantor-Bendixson rank for trees introduced in [9].
[6] The definition of rank-k tree-automatic structures in [1, Section 1.3.7] is different, but equivalent.

▶ **Theorem 5.2.** *Let $k \in \mathbb{N}_+$. The* FC*-rank of every rank-k tree-automatic linear ordering is strictly below ω^k.*

▶ **Lemma 5.3.** *Let T be a regular binary tree of rank $k \in \mathbb{N}$. Then there exists a constant $K \in \mathbb{N}$ such that any anti-chain (w.r.t. $\preceq$) $A \subseteq T$ contains at most K elements u such that $T{\restriction}u$ has rank k.*

Proof. We proceed by induction on k. If $k = 0$, then T is finite and the claim is obvious. Thus, suppose $k > 0$. Let $n \in \mathbb{N}$ be the *index* of T, i.e., the size of the set $\{\, T{\restriction}u \mid u \in T \,\}$.

For the sake of a contradiction, assume there is an anti-chain A consisting of $2^n + 1$ elements $u \in T$ such that $T{\restriction}u$ has rank k. The set B of all $v \in T$ which are the longest common prefix of two distinct elements from A contains exactly 2^n elements. For every $u \in A$ the binary tree $d^{(k-1)}(T{\restriction}u) = d^{(k-1)}(T){\restriction}u$ is infinite and hence, by König's lemma, there is an infinite branch of $d^{(k-1)}(T)$ containing u. Thus, $B \subseteq d^{(k)}(T)$ and $\left|d^{(k)}(T)\right| \geq 2^n$. Since $d^{(k)}(T){\restriction}v = d^{(k)}(T{\restriction}v)$ for all $v \in d^{(k)}(T)$, the index of $d^{(k)}(T)$ is at most n. Finally, a pumping argument shows that $d^{(k)}(T)$ is infinite, contradicting the choice of k. ◀

Proof of Theorem 5.2. We proceed by induction on $k \geq 1$, using an artificial base case $k = 0$. A rank-0 tree-automatic scattered linear ordering $\mathfrak{A}$ is finite and hence satisfies $\mathrm{VD}_*(\mathfrak{A}) < \omega^0$. Thus, consider $k \geq 1$.

For the sake of a contradiction, assume there exists a rank-k tree-automatic linear ordering $\mathfrak{A}$ with $\mathrm{FC}(\mathfrak{A}) \geq \omega^k$. Let $\mathcal{S}^{\mathfrak{A}}_{\phi}$ be the set constructed in Theorem 3.7's proof from $\mathfrak{A}$ and the formula $\phi(x, y_1, y_2) = y_1 \leq x \wedge x \leq y_2$. We show that $\mathcal{S}^{\mathfrak{A}}_{\phi}$ contains for each $n \in \mathbb{N}$ a scattered linear ordering $\mathfrak{B}$ with $\omega^{k-1}n < \mathrm{VD}_*(\mathfrak{B}) < \omega^k$, contradicting the finiteness of $\mathcal{S}^{\mathfrak{A}}_{\phi}$.

Consider $n \in \mathbb{N}$ and let K be the constant which exists by Lemma 5.3 applied to $T(A)$. Like in Theorem 4.6's proof there is a $\bar{s} \in A^2$ such that $\mathfrak{A}{\restriction}\phi, \bar{s}$ is scattered and $\mathrm{VD}_*(\mathfrak{A}{\restriction}\phi, \bar{s}) = \omega^{k-1}(nK + 1)$. We delve into the details of Theorem 3.7's proof, supposing we have fixed $\bar{s}$. Since $\omega^{k-1}(nK+1)$ is VD_*-sum-indecomposable, there exists a $\sim_{\bar{s}}$-class $C \subseteq B$ such that $\mathrm{VD}_*(\mathfrak{C}) = \omega^{k-1}(nK + 1)$. Let $\tau = \big(t_0, U, q_{\leq}, q_{\phi}\big)$ be the corresponding $\bar{s}$-type. For each $u \in U$ we have $T\big(S_{\gamma(u)}\big) \subseteq T(A){\restriction}u$ and hence the rank of $T\big(S_{\gamma(u)}\big)$ is at most k. Let V be the set of those $u \in U$ for which the rank equals k. Due to Lemma 5.3, $|V| \leq K$. The induction hypothesis yields $\mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big) < \omega^{k-1}$ for $u \in U \setminus V$. If we had $\mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big) \leq \omega^{k-1}n$ for each $u \in V$, this would imply

$$\underbrace{\bigoplus_{u\in V} \mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big)}_{\leq \omega^{k-1}n|V|} \oplus \underbrace{\bigoplus_{u\in U\setminus V} \mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big)}_{<\omega^{k-1}} < \omega^{k-1}n|V| \oplus \omega^{k-1} \leq \omega^{k-1}(nK+1)\,,$$

contradicting Proposition 4.3. Hence, there exists a $u \in V$ with $\mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big) > \omega^{k-1}n$. Since $\mathfrak{S}_{\gamma(u)}$ can be embedded into $\mathfrak{C}$, we also have $\mathrm{VD}_*\big(\mathfrak{S}_{\gamma(u)}\big) \leq \omega^{k-1}(nK+1) < \omega^k$. ◀

For ordinals α and β we have $\mathrm{FC}(\alpha) \leq \beta$ precisely if $\alpha \leq \omega^\beta$. Consequently, Theorem 5.2 implies that every rank-k tree-automatic ordinal is less than ω^{ω^k}. In fact, the construction in Corollary 3.9's proof shows that the converse implication holds as well. Therefore, we obtain the following analogue to Corollary 3.9.

▶ **Corollary 5.4.** *Let $k \in \mathbb{N}$. An ordinal α is rank-k tree-automatic if, and only if, $\alpha < \omega^{\omega^k}$.*

6 Discussion

As an application of the FC-rank of string-automatic linear orderings being finite, it was shown that the Cantor-Bendixson rank of string-automatic order trees is also finite [9]. The proof uses that $\Sigma^\star$ admits an automatic linear order isomorphic to ω, but results in [5] imply this to fail for T_Σ. However, the arguments in [9] carry over for *finitely branching* order trees since T_Σ admits at least some automatic linear order. Thus, the Cantor-Bendixson rank of every (rank-k) tree-automatic finitely branching order tree is below ω^ω respectively ω^k.

Another application of Theorem 4.6 was pointed out by Kuske [10]. Results in [11] can be adapted to show that the isomorphism problem for tree-automatic scattered linear orderings, that he proved to be Π^0_1-hard, belongs to level $\Delta^0_{\omega^\omega}$ of the hyperarithmetical hierarchy.

As a sideline, Corollary 5.4 reproves that the hierarchy of finite-rank tree-automatic structures is strict, a fact whose proof yet depended on deep logical insights [1, Section 1.3.7]. Moreover, it implies that any tree-automatic well-ordering is already finite-rank tree-automatic. However, the respective question for arbitrary linear orderings remains open.

All results present upper bounds on the FC-rank. Unfortunately, for each $k \in \mathbb{N}$ there exists a tree-automatic well-ordering $\mathfrak{A}$ isomorphic to ω^k such that $T(A)$ has rank k. This renders it impossible to provide useful lower bounds in terms of the rank of $T(A)$. Using another approach, a first step in this direction provides a decidable characterisation of the tree-automatic scattered linear orderings of FC-rank at least ω [6].

Finally, it is known that the ordinal height of string-automatic well-founded partial orderings is below ω^ω [4] and of tree-automatic well-founded order trees below ω^{ω^ω} [7]. Regretfully, even the refined decomposition technique seems too weak to verify the resulting conjecture that the height of tree-automatic well-founded partial orderings is below ω^{ω^ω}.

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