Graph Homomorphisms for Quantum Players

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Abstract

A homomorphism from a graph X to a graph Y is an adjacency preserving mapping $f : V(X) \to V(Y)$. We consider a nonlocal game in which Alice and Bob are trying to convince a verifier with certainty that a graph X admits a homomorphism to Y. This is a generalization of the well-studied graph coloring game. Via systematic study of quantum homomorphisms we prove new results for graph coloring. Most importantly, we show that the Lovász theta number of the complement lower bounds the quantum chromatic number, which itself is not known to be computable. We also show that other quantum graph parameters, such as quantum independence number, can differ from their classical counterparts. Finally, we show that quantum homomorphisms closely relate to zero-error channel capacity. In particular, we use quantum homomorphisms to construct graphs for which entanglement-assistance increases their one-shot zero-error capacity.

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1 Graph homomorphism game as a generalization of coloring game

In the (X, c)-coloring game, Alice and Bob are trying to convince a verifier with certainty that the graph $X = (V, E)$ is c-colorable [10, 6]. The verifier sends Alice and Bob vertices $a, b \in V$ respectively and they respond with colors $\alpha, \beta \in [c]$ accordingly. To win Alice an Bob need to respond with $\alpha = \beta$ for $a = b$ and with $\alpha \neq \beta$ for $ab \in E$. Classical Alice and Bob can win with probability 1 if and only if X is c-colorable. In contrast, quantum Alice and Bob using shared entanglement can sometimes win the (X, c)-coloring game even when X is not c-colorable [6, 1, 5, 16].

We introduce a natural generalization of the graph coloring game: the graph homomorphism game. A graph homomorphism is a function $\varphi : V(X) \to V(Y)$ such that $\varphi(x)$ and $\varphi(x')$ are adjacent whenever x and x' are adjacent. When such a map exists we say that X has a homomorphism to Y and write $X \to Y$. A c-coloring of X can be viewed as a homomorphism $\varphi : X \to K_c$, where K_c is the complete graph on c vertices. Graph homomorphisms have been used to prove results about different types of chromatic numbers, graph products etc.; they have applications in areas like complexity theory, statistical physics and others (see [12, 13] for a general reference).

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Our motivation for this work is that a systematic study of quantum homomorphisms can yield

- better understanding of and new results concerning quantum graph coloring (see Section 4);
- new examples of nonlocal games with perfect quantum but not classical strategies (see Section 2);
- new results for zero-error capacity via the connections that we establish in Section 3.

In the (X, Y)-*homomorphism game* the verifier sends Alice and Bob vertices $x, x' \in V(X)$ respectively and they respond with vertices $y, y' \in V(Y)$ accordingly. To win players need to respond with $y = y'$ to questions $x = x'$ and with $yy' \in E(Y)$ to questions $xx' \in E(X)$. Like the coloring game, the (X, Y)-homomorphism game can be won with certainty by classical players if and only if $X \to Y$. If quantum players using shared entanglement can win the (X, Y)-homomorphism game with certainty we say that X has a *quantum homomorphism* to Y and write $X \xrightarrow{q} Y$. As we know from the case of coloring and will see from new examples in the next section, sometimes $X \xrightarrow{q} Y$ even though $X \not\to Y$ (i.e., X does not admit a homomorphism to Y).

It is known that whenever X is quantum c-colorable, the (X, c)-coloring game can be won using projective measurements on maximally entangled state [5]. Moreover, Bob's projectors are the complex conjugates of Alice's. We have verified that the proof of [5] extends to the case of the (X, Y)-homomorphism game. This allows the following combinatorial reformulation:

▶ **Lemma 1.** *We have $X \xrightarrow{q} Y$ if and only if there exists an assignment of projectors P_{xy} to pairs of vertices $(x, y) \in V(X) \times V(Y)$ such that $\sum_y P_{xy} = I$ for all $x \in V(X)$ and*

$$P_{xy}P_{x'y'} = 0 \text{ whenever } (x = x' \;\&\; y \neq y') \text{ or } (x \sim x' \;\&\; y \not\sim y').$$

This reformulation is instrumental in proving many of the results in the coming sections. The other proof technique that we employ only uses the players' ability to win certain homomorphism games to conclude that they can also win some other homomorphism game. For example, this kind of reasoning easily shows that quantum homomorphisms are transitive, i.e., $X \xrightarrow{q} Y$ and $Y \xrightarrow{q} Z$ implies that $X \xrightarrow{q} Z$.

Curiosly, if instead of entanglement Alice and Bob are given access to non-signalling correlations, they can win the (X, K_2)-homomorphism game with certainty for any graph X. This implies that they can win any (X, Y)-homomorphism game for arbitrary graphs X, Y as long as $E(Y) \neq \emptyset$.

2 Quantum parameters

The quantum chromatic number, $\chi_q(X)$, is defined as the smallest c for which quantum players can win the (X, c)-coloring game with certainty [10, 6]. This parameter has been relatively well-studied [1, 5, 9, 17, 16]. In particular, it is known that for the family of graphs Ω_{4n} there is an exponential separation between $\chi(\Omega_{4n})$ and $\chi_q(\Omega_{4n})$. Here, the so-called Hadamard graph Ω_n is the graph with vertex set $\{\pm 1\}^n$ and edge set $\{(v, w) : v^T w = 0\}$. Also, a complete characterization of graphs with $\chi_q(X) < \chi(X)$ has been given [16]. However, many questions remain open. For example, it is not known whether $\chi_q(X)$ is computable, or whether there exists a family of graphs X_n such that $\lim_{n\to\infty} \chi(X_n) = \infty$ but $\lim_{n\to\infty} \chi_q(X_n) < \infty$. A systematic study of quantum homomorphisms could aid in answering these and other questions

Using the framework of quantum homomorphisms, we can introduce a quantum analogue for any graph parameter defined in terms of graph homomorphisms (e.g., clique number, independence number, odd girth, etc.). Here we only consider the following:

- quantum clique number, $\omega_q(X) = \max\{n : K_n \xrightarrow{q} X\}$;
- quantum independence number, $\alpha_q(X) = \omega_q(\overline{X})$ where $\overline{X}$ denotes the complement of X.

Let us remark that by now, the quantum independence number has been further used by many other authors exploring parallel repetition, zero-error communication, binary constraint system games etc.

We are about to see that quantum clique and independence number can be different from their classical counterparts. Moreover, we show how to construct a graph with such a separation using any two graphs X and Y such that $X \xrightarrow{q} Y$ but $X \not\to Y$.

For graphs X and Y, their homomorphic product, $X \ltimes Y$, is the graph with vertex set $V(X) \times V(Y)$, and vertex (x, y) is adjacent to (x', y') if either ($x = x'$ and $y \neq y'$) or ($xx' \in E(X)$ and $yy' \notin E(Y)$). This definition is motivated by the fact that $X \to Y$ if and only if $\alpha(X \ltimes Y) = |V(X)|$. We have proved the quantum version of this fact, i.e., $X \xrightarrow{q} Y$ if and only if $\alpha_q(X \ltimes Y) = |V(X)|$. Combining these two facts gives:

▶ **Theorem 2.** *Let X, Y be graphs such that $X \xrightarrow{q} Y$ but $X \not\to Y$. Then we have that $\alpha(X \ltimes Y) < \alpha_q(X \ltimes Y)$ and $\omega(\overline{X \ltimes Y}) < \omega_q(\overline{X \ltimes Y})$.*

This theorem allows to obtain separations for clique and independence numbers starting from any graph X with $\chi_q(X) < \chi(X)$. For example, the fact that $\Omega_n \xrightarrow{q} K_n$ [1] but $\Omega_{4n} \not\to K_{4n}$ for $n > 2$ [11] implies that $\alpha(\Omega_{4n} \ltimes K_{4n}) < \alpha_q(\Omega_{4n} \ltimes K_{4n})$ for all $n > 2$.

3 Relationship to entanglement-assisted zero-error capacity

The *one-shot zero-error capacity*, $c_0(X)$, of a graph X is the maximum number of different messages that can be sent without error by one use of any classical noisy channel $\mathcal{N}$ with confusability graph X [18, 15]. In the scenario where the communicating parties can use shared entanglement, we speak about *entanglement-assisted zero-error capacity*, $c_0^*(X)$ [7].

The separations between $c_0^*(X)$ and $c_0(X)$ and their asymptotic analogues have been investigated in [7, 14, 16, 3]. It is an open question how large these separations can be. As [16] shows, a separation between the one-shot zero-error capacities can be obtained starting from any graph X with $\chi_q(X) < \chi(X)$.

A somewhat analogous relationship can be shown to hold for quantum homomorphisms in general:

▶ **Theorem 3.** *Let X, Y be graphs such that $X \xrightarrow{q} Y$ but $X \not\to Y$. Then we have that*

$$c_0(X \ltimes Y) < c_0^*(X \ltimes Y).$$

It turns out that the quantum independence number, $\alpha_q(X)$, is closely related to and might equal the one-shot entanglement-assisted zero-error capacity:

▶ **Theorem 4.** *For any graph X we have $\alpha_q(X) \leq c_0^*(X)$ with equality if and only if $c_0^*(X)$ can be achieved using a strategy in which all of Alice's measurements are projective and the shared state is maximally entangled.*

By the above theorem, proving that $\alpha_q(X) = c_0^*(X)$ for all graphs X would settle the open question of whether projective measurements on maximally entangled state suffice to achieve $c_0^*(X)$. If this was the case, the results from [16] would imply a complete characterization of graphs for which $c_0(X) < c_0^*(X)$.

Finally, we show that quantum homomorphisms respect the order of both the one-shot and asymptotic entanglement-assisted zero-error capacities.

▶ **Theorem 5.** *Let Θ^* denote the entanglement-assisted Shannon capacity. For any graphs X, Y we have that $X \xrightarrow{q} Y$ implies both*

$$c_0^*(\overline{X}) \leq c_0^*(\overline{Y}) \text{ and } \Theta^*(\overline{X}) \leq \Theta^*(\overline{Y}).$$

The above theorem can be used to lower bound $\Theta^*(Y)$ in the case when $\overline{X} \xrightarrow{q} \overline{Y}$ and $\Theta^*(X)$ is known for some graph X.

4 Relationship to Lovász ϑ

The Lovász theta number of X, denoted $\vartheta(X)$, was introduced in [15] as an efficiently computable upper bound for the Shannon capacity $\Theta(X)$. It has been shown that $\vartheta(X)$ upper bounds even the entaglement-assisted Shannon capacity $\Theta^*(X)$ [2, 8]. We have established that quantum homomorphisms respect the order of Lovász theta:

▶ **Theorem 6.** *For any graphs X, Y we have that $X \xrightarrow{q} Y$ implies $\vartheta(\overline{X}) \leq \vartheta(\overline{Y})$.*

Applying the above theorem with Y being the complete graph on $\chi_q(X)$ vertices gives the following:

▶ **Corollary 7.** *For any graph X we have $\vartheta(\overline{X}) \leq \chi_q(X)$.*

Corollary 7 gives us an efficiently computable lower bound on the quantum chromatic number $\chi_q(X)$, which itself is not even known to be computable (By now our lower bound on $\chi_q(X)$ has been strengthened by replacing ϑ with ϑ^+ [4]). The lower bound from Corollary 7 can also be used to conclude that the previously established [1] upper bound $\chi_q(\Omega_n) \leq n$ is actually tight for all Hadamard graphs Ω_n with $4|n$. (The other cases are not interesting since Ω_n is either empty or bipartite.)

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