

# Extended Formulation Lower Bounds via Hypergraph Coloring?<sup>\*</sup>

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## Abstract

Exploring the power of linear programming for combinatorial optimization problems has been recently receiving renewed attention after a series of breakthrough impossibility results. From an algorithmic perspective, the related questions concern whether there are compact formulations even for problems that are known to admit polynomial-time algorithms.

We propose a framework for proving lower bounds on the size of extended formulations. We do so by introducing a specific type of extended relaxations that we call *product relaxations* and is motivated by the study of the Sherali-Adams (SA) hierarchy. Then we show that for every approximate extended formulation of a polytope  $P$ , there is a product relaxation that has the same size and is at least as strong. We provide a methodology for proving lower bounds on the size of approximate product relaxations by lower bounding the chromatic number of an underlying hypergraph, whose vertices correspond to gap-inducing vectors.

We extend the definition of product relaxations and our methodology to mixed integer sets. However in this case we are able to show that *mixed product relaxations* are at least as powerful as a special family of extended formulations. As an application of our method we show an exponential lower bound on the size of approximate mixed product relaxations for the metric capacitated facility location problem (CFL), a problem which seems to be intractable for linear programming as far as constant-gap compact formulations are concerned. Our lower bound implies an unbounded integrality gap for CFL at  $\Theta(N)$  levels of the *universal* SA hierarchy which is *independent of the starting relaxation*; we only require that the starting relaxation has size  $2^{o(N)}$ , where  $N$  is the number of facilities in the instance. This proof yields the first such tradeoff for an SA procedure that is independent of the initial relaxation.

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## 1 Introduction

In the past few years there has been an increasing interest in exposing the limitations of compact LP formulations for combinatorial optimization problems. The goal is to show a lower bound on the size of *extended formulations (EFs)* for a particular problem. Extended formulations add extra variables to the natural problem space; the increase in dimension may yield a smaller number of facets. The minimum size over all extended formulations is the *extension complexity* of the corresponding polytope. A superpolynomial lower bound on the extension complexity is of intrinsic interest in polyhedral combinatorics and implies that there is no polynomial-time algorithm relying purely on the solution of a compact linear program. It does not however rule out efficient LP-based algorithms that combine algorithmic steps of arbitrary type, such as preprocessing, primal-dual, etc., with linear programming.

In the seminal paper of Yannakakis [21] the problem of lower bounding the size of extended formulations was considered for the first time: exponential lower bounds were proved for symmetric extended formulations of the matching and TSP polytopes. Yannakakis [21] identified also a crucial combinatorial parameter, the nonnegative rank of the slack matrix of the underlying polytope  $P$ , and he showed that it equals the extension complexity of  $P$ . A strong connection of the extension complexity of a polytope to communication complexity was made in [21], by showing that the nonnegative rank of the slack matrix is at least the size of its minimum rectangle cover. That connection has been exploited in several results on the extension complexity of polytopes.

Fiorini et al. [13] lifted the symmetry condition on the result of [21] regarding the TSP polytope, thus answering a long-standing open problem. The result was obtained by showing that the correlation polytope has exponential extension complexity which in turn was shown using communication complexity tools. Recently, Rothvoß [19] removed the symmetry condition for the matching polytope as well, answering the second long-standing open question of [21]. This was done by a breakthrough in bounding a refined version of the rectangle covering number.

A more general question is that of the size of approximate extended formulations. This problem was first considered in [7] where the methodology of [13] was extended to approximate formulations and an exponential bound for the linear encoding of the  $n^{1/2-\varepsilon}$ -approximate clique problem was given. Subsequently, Braverman and Moitra [10] extended the former bound to  $n^{1-\varepsilon}$ -approximate formulations of the clique, following a new, information theoretic, approach. Braun and Pokutta in [8] further strengthened the lower bounds by introducing the notion of common information. Very recently, Braun and Pokutta [9] extended the result of [19] to approximate formulations of the matching polytope by combining ideas of the latter with the notion of common information.

In [11] it was proved that in terms of approximating maximum constraint satisfaction problems, LPs of size  $O(n^k)$  are exactly as powerful as  $O(k)$ -level relaxations in the Sherali-Adams hierarchy. Their proof differs from previous work in showing that polynomials of low degree can approximate the functional version of the factorization theorem of [21].

The *metric capacitated facility location* problem (CFL) is a well-studied problem for which, while constant-factor approximations are known [6, 2], no efficient LP relaxation with constant integrality gap is known. An instance  $I$  of CFL is defined as follows. We are given a set  $F$  of facilities and a set  $C$  of clients, with each facility  $i$  having a capacity  $U_i$  and each client  $j$  having a demand  $d_j > 0$ . We may open facility  $i$  by paying its opening cost  $f_i$  and we may assign demand from client  $j$  to facility  $i$  by paying the connection cost  $c_{ij}$  per unit of demand. The latter costs satisfy the following variant of the triangle inequality:

$c_{ij} \leq c_{ij'} + c_{i'j} + c_{i'j}$  for any  $i, i' \in F$  and  $j, j' \in C$ . We are asked to open a subset  $F' \subseteq F$  of the facilities and assign all the client demand to the open facilities while respecting the capacities. The goal is to minimize the total opening and connection cost. The question whether an efficient relaxation exists for CFL is among the most important open problems in approximation algorithms [20]. In a recent breakthrough, the first  $O(1)$ -factor LP-based algorithm for CFL was given in [4]. The proposed relaxation is, however, exponential in size and, according to the authors of [4], not known to be separable in polynomial time. To our knowledge, there has not been a compact EF for approximate CFL achieving even an  $o(|F|)$  gap.

In previous work [17, 18], we proved among other results for CFL, unbounded integrality gaps for the Sherali-Adams hierarchy when starting from the natural LP relaxation. We also disqualified, with respect to obtaining a  $O(1)$  gap, valid inequalities from the literature such as the flow-cover inequalities and their generalizations.

## 1.1 Our contribution

In this paper we propose a new approach for proving lower bounds on the size of approximate extended formulations. Our contribution is summarized by the following.

First we introduce a family of extended relaxations of a given polytope which we call *product relaxations*. The product relaxations are inspired by the study of the Sherali-Adams hierarchy. Given a polytope  $K \subseteq [0, 1]^d$  that corresponds to a linear relaxation of the problem at hand, the Sherali-Adams relaxation  $SA^t(K)$  at level  $t$  is produced by a lift-and-project method, where initially every constraint in the description of  $K$  is multiplied with all  $t$ -subsets of variables and their complements. The resulting products of variables are then linearized, i.e., each replaced by a single variable, and finally one projects back to the original variable space of  $K$ . The variable space of the product relaxations is exactly the space of the final  $d$ -level Sherali-Adams relaxation, after linearization and before projection. The variables have the intuitive meaning of corresponding to products over sets of variables from the original space – the "intuitive meaning" of a variable is made precise through the notion of the *section*  $f$  of an extended relaxation  $Q(x, y)$  of a polytope  $P(x)$ . This is a function  $f$  that maps an integer point  $x \in P(x)$  to a vector of values  $(x, y) = f(x)$  such that  $f(x) \in Q(x, y)$ . (See Section 2 for the necessary definitions).

We prove in Theorem 3 that for any  $\rho$ -approximate extended formulation of a 0-1 polytope there is a product relaxation of the same size that is at least as strong. The proof is short and accessible. Theorem 3 reduces lower bounding the size of an extended formulation, which uses some unknown space and encoding, of a polytope  $P$ , to lower bounding the size of product relaxations of  $P$ . In the product space we have the concrete advantage of knowing the section of the target relaxation. We extend the definition of product relaxations and our methodology to mixed integer sets. However in this case we are able to show that *mixed product relaxations* are at least as powerful as a special family of extended formulations (cf. Theorem 5).

We note that our approach does not rely on the notion of the slack matrix introduced by Yannakakis [21]. It differs from that of [11] in which the slack functions of the factorization theorem [21] were shown to be approximable, for max CSPs, by low-degree polynomials and thus SA gaps are transferred to general linear programs.

Then we use a methodology for proving lower bounds for relaxations for which the section is known and in particular for product relaxations. Similar arguments have been used in the context of bounding the number of facets of specific polyhedra, but prior to our work, they seemed inapplicable for lower bounding the size of arbitrary EFs which lift the polytope

in arbitrary variable spaces. The method is the following: first define a set of vectors in the space of the relaxation such that for each one of those vectors there is an admissible objective function witnessing an integrality gap of  $\rho$ . We call that set of vectors the *core*. Then show that, for any partition of the core into fewer than  $\kappa$  parts, there must be some part containing a set of conflicting vectors. A set of infeasible vectors is *conflicting* if its convex hull has nonempty intersection with the convex hull of  $\{f(x) \mid x \in P(x) \cap \{0, 1\}^n\}$ , which is always included in the feasible region of a product relaxation – here  $f(x)$  is the section we associate with product relaxations. Thus, we get that at least  $\kappa$  inequalities are needed to separate the members of the core from the feasible region and so  $\kappa$  is a lower bound on the size of any  $\rho$ -approximate product relaxation. By considering the hypergraph whose set of vertices corresponds to the aforementioned set of vectors and whose set of hyperedges corresponds to the sets of conflicting vectors, the chromatic number of the hypergraph is a lower bound on the size of every  $\rho$ -approximate extended formulation (cf. Theorem 8). Moreover, there is always a core such that the chromatic number of the resulting, possibly infinite, hypergraph equals the extension complexity of the polytope at hand. Thus the characterization of extension complexity in Theorem 8 can be seen as an alternative to the nonnegative rank of the slack matrix. The conflicting vectors are fractional solutions, which are hard to separate from the integer solutions. The method comes closer to standard LP/SDP integrality gap arguments than the existing combinatorial approaches for lower bounding extension complexity.

When arguing about the polyhedral complexity of a specific polytope, i.e., the minimum size of its formulation in the original variable space, the above method can always be simplified to finding a set of gap-inducing vectors with the property that (almost) any pair of them are conflicting. The underlying hypergraph reduces then to a simple graph that is very dense, almost a clique, and thus has high chromatic number. We used this idea in a preliminary version of this work [16] to derive exponential bounds on the polyhedral complexity of approximate metric capacitated facility location, where only the classic variables are used (cf. Corollary 14). A similar idea was independently used by Kaibel and Weltge in [15] to derive lower bounds on the number of facets of a polyhedron which contains a given integer set  $X$  and whose set of integer points is  $\text{conv}(X) \cap \mathbb{Z}^d$ .

We exhibit a concrete application of our methodology by proving in Theorem 12 an exponential lower bound on the size of any  $O(N)$ -approximate mixed product relaxation for the CFL polytope, where  $N$  is the number of facilities in the instance. This result can be shown to imply (cf. Theorem 13) that the  $\Omega(N)$ -level SA relaxation for CFL, which is obtained from any starting LP of size  $2^{o(N)}$  defined on the classic set of variables, has unbounded gap  $\Omega(N)$ . Note, that it is well-known that by lifting only the facility variables, at  $N$  levels the integer polytope is obtained for CFL [5]. This settles the open question of [3] whether there are LP relaxations upon which the application of lift-and-project methods captures the strength of preprocessing steps for CFL. Our result establishes for the first time such a tradeoff for a *universal* SA procedure that is independent of the starting relaxation  $K$ . The proof follows the methodology outlined above and is different from the standard arguments that apply only to the SA lifting of a specific LP. Our earlier SA construction in [18] applied the local-global method [12] that constructs an appropriate distribution of solutions for each explicit constraint of the starting LP.

We leave as an open problem the extension of the equivalence between product and extended relaxations from 0-1 programs to mixed integer sets. We also believe that it would be of interest to revisit known extension-complexity lower bounds using our method, so as to obtain simpler proofs.

## 2 Preliminaries

$X \subseteq \mathbb{R}^d$ , is a *mixed integer set* if there is  $p \in \{1, \dots, d-1\}$  such that  $d = n + p$  and  $X \subseteq \{0, 1\}^n \times [0, 1]^p$ . A *valid relaxation* of the mixed integer set  $X$  is any polyhedron  $P$  such that  $\text{conv}(X) \subseteq P$ . Given a valid relaxation  $P$  of  $X$ , such that  $\text{conv}(X) \cap (\{0, 1\}^n \times [0, 1]^p) = P \cap (\{0, 1\}^n \times [0, 1]^p)$ , the *level  $k$  Sherali-Adams (SA) procedure*,  $k \geq 1$ , is as follows [1]. Let  $P$  be defined by the linear constraints  $Ax - b \leq 0$ . For every constraint  $\pi(x) \leq 0$  of  $P$ , for every set of variables  $U \subseteq \{x_i \mid i = 1, \dots, n\}$  such that  $|U| \leq k$ , and for every  $W \subseteq U$ , consider the *lifted* valid constraint:  $\pi(x) \prod_{x_i \in U-W} x_i \prod_{x_i \in W} (1 - x_i) \leq 0$ . Linearize the system obtained this way by replacing (i)  $x_i^2$  with  $x_i$  for all  $i$  (ii)  $\prod_{x_i \in I \subseteq [n]} x_i$  with  $x_I$  and (iii)  $x_k \prod_{x_i \in I \subseteq [n]} x_i$ , where  $k \in \{n+1, \dots, d\}$  with  $v_{Ik}$ .  $\text{SA}^k(P)$  is the projection of the resulting linear system onto the original variables  $\{x_1, \dots, x_d\}$ . We call  $\text{SA}^k(P)$  the relaxation *obtained from  $P$  at level  $k$*  of the SA hierarchy. It is well-known that  $\text{SA}^n(P) = \text{conv}(X)$  (see, e.g., [5]). If  $X$  is a 0-1 set, i.e.,  $X \subseteq \{0, 1\}^d$ , the above definitions hold mutatis mutandis and  $\text{SA}^d(P) = \text{conv}(X)$ .

Given a polyhedron  $K(x, y) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{d_y} \mid Ax + By \leq b\}$  the *projection to the  $x$ -space* is defined as  $\{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^{d_y} : Ax + By \leq b\}$ , denoted as  $\text{proj}_x(K(x, y))$ . An *extended formulation (relaxation)* of a polyhedron  $P(x) \subseteq \mathbb{R}^d$  is a linear system  $K(x, y) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{d_y} \mid Ax + By \leq b\}$  such that  $\text{proj}_x(K(x, y)) = P(x)$  ( $\text{proj}_x(K(x, y)) \supseteq P(x)$ ). The *size* of a polyhedron  $P(x)$  is the minimum number of inequalities in its halfspace description. The *extension complexity* of  $P(x)$  is the minimum size of an extended formulation of  $P(x)$ .

We define now  $\rho$ -approximate formulations as in [7]. Given a combinatorial optimization problem  $T$ , a *linear encoding* of  $T$  is a pair  $(L, O)$  where  $L \subseteq \{0, 1\}^*$  is the set of *feasible solutions* to the problem and  $O \subset \mathbb{R}^*$  is the set of *admissible objective functions*. An instance of the linear encoding is a pair  $(d, w)$  where  $d$  is a positive integer defining the dimension of the instance and  $w \subseteq O \cap \mathbb{R}^d$  is the set of admissible cost functions for instances of dimension  $d$ . Solving the instance  $(d, w)$  means finding  $x \in L \cap \{0, 1\}^d$  such that  $w^T x$  is either maximum or minimum, according to the type of problem  $T$ . Let  $P = \text{conv}(\{x \in \{0, 1\}^d \mid x \in L\})$  be the corresponding 0-1 polytope of dimension  $d$ . Given a linear encoding  $(L, O)$  of a maximization problem, the corresponding polytope  $P$ , and  $\rho \geq 1$ , a  $\rho$ -approximate extended formulation of  $P$  is an extended relaxation  $Ax + By \leq b$  of  $P$  with  $x \in \mathbb{R}^d, y \in \mathbb{R}^{d_y}$  such that

$$\begin{aligned} \max\{w^T x \mid Ax + By \leq b\} &\geq \max\{w^T x \mid x \in P\} && \text{for all } w \in \mathbb{R}^d \text{ and} \\ \max\{w^T x \mid Ax + By \leq b\} &\leq \rho \max\{w^T x \mid x \in P\} && \text{for all } w \in O \cap \mathbb{R}^d. \end{aligned}$$

For a minimization problem, we require

$$\begin{aligned} \min\{w^T x \mid Ax + By \leq b\} &\leq \min\{w^T x \mid x \in P\} && \text{for all } w \in \mathbb{R}^d \text{ and} \\ \min\{w^T x \mid Ax + By \leq b\} &\geq \rho^{-1} \min\{w^T x \mid x \in P\} && \text{for all } w \in O \cap \mathbb{R}^d. \end{aligned}$$

The  $\rho$ -approximate extension complexity of 0-1 integer polytope  $P(x) \subseteq [0, 1]^d$  is the minimum size of a  $\rho$ -approximate extended formulation of  $P$ . Given an extended formulation  $Q(x, y)$  of  $P(x)$ , a *section* of  $Q$  is defined as a vector-valued boolean function  $g(x): \{0, 1\}^d \rightarrow \mathbb{R}^{d+d_y}$  such that for  $x \in P(x) \cap \{0, 1\}^d$ ,  $g(x)$  belongs to  $Q(x, y)$  and  $\text{proj}_x(g(x)) = x$ . Intuitively, the section extends the encoding of solutions to the auxiliary variables  $y$ . Clearly, if a particular extended formulation  $Q$  has been specified a priori, different such functions can be defined by filling in the last  $d_y$  coordinates of  $g(x_o)$  with a value from  $\{y \in \mathbb{R}^{d_y} \mid Ax_o + By \leq b\}$ .

► **Definition 1.** Given a 0-1 integer polytope  $P(x) \subseteq [0, 1]^d$ , a *product relaxation*  $D(z)$  of  $P(x)$  is an extended relaxation  $D(z)$  of  $P(x)$ , where  $z \in \mathbb{R}^{2^d-1}$  and for every nonempty

subset  $\mathcal{E} \subseteq \{x_1, x_2, \dots, x_d\}$  of the original variables, we have a variable  $z_{\mathcal{E}}$ , (where  $z_{\{x_i\}}$  denotes  $x_i$ ,  $i = 1, \dots, d$ ), and there is a section  $f(x)$  of  $D$  s.t. the corresponding coordinate of  $f$  at  $\mathcal{E}$  is  $f_{\mathcal{E}}(x) = \prod_{x_i \in \mathcal{E}} x_i$ . We refer to this function  $f$  as *the product section*.

Let  $f$  denote the product section. Define the *canonical product relaxation* of  $P$  as  $\hat{D} = \text{conv}\{f(x) \mid x \in P(x) \cap \{0, 1\}^{d_x}\}$ . The polytope  $\hat{D}$  corresponds to the “tightest” possible product relaxation.

For a mixed integer set  $M(x, w) \subseteq \{0, 1\}^{d_x} \times \mathbb{R}^{d_w}$  the corresponding mixed integer polytope  $P(x, w)$  is  $\text{conv}(M(x, w))$ . In case one starts from a mixed integer polytope, the additional  $z$  variables of the product relaxation correspond to sets that contain at most one fractional variable. Including only one fractional variable in each product, mimics the variable space of the final-level SA relaxation.

► **Definition 2.** Let  $P(x, w) \subseteq [0, 1]^{d_x} \times \mathbb{R}^{d_w}$  be a mixed integer polytope. A *mixed product relaxation*  $D(z)$  of  $P(x, w)$  is an extended relaxation  $D(z)$  of  $P(x, w)$ , where  $z \in \mathbb{R}^{(d_w+1)2^{d_x}-1}$ , with  $z_{\{w_j\}} = w_j$ ,  $j = 1, \dots, d_w$ , and

(i) for every set  $\emptyset \neq \mathcal{E} \subseteq \{x_1, x_2, \dots, x_{d_x}\}$  we define  $d_w + 1$  variables: one that we denote  $z_{\mathcal{E}}$  and, for each fractional variable  $w_j$ ,  $j = 1, \dots, d_w$ , one that we denote  $z_{\mathcal{E}w_j}$ . Moreover  $z_{\{x_i\}}$  denotes  $x_i$ ,  $i = 1, \dots, d_x$ .

(ii) there is a section  $f(x, w)$  of  $D$  s.t. the corresponding coordinates of  $f$  are  $f_{\mathcal{E}}(x, w) = (\prod_{x_i \in \mathcal{E}} x_i)$  and, for each variable  $w_j$ ,  $j = 1, \dots, d_w$ ,  $f_{\mathcal{E}w_j}(x, w) = (\prod_{x_i \in \mathcal{E}} x_i) \cdot w_j$ . We refer to this function  $f$  as *the mixed product section*.

The *canonical product relaxation* of  $P(x, w)$  is similarly defined as  $\hat{D} = \text{conv}\{f(x, w) \mid (x, w) \in P(x, w) \cap (\{0, 1\}^{d_x} \times \mathbb{R}^{d_w})\}$ .

Note that the lifted polytope produced by the  $d$ -level ( $d_x$ -level) Sherali-Adams procedure applied on some specific linear relaxation of the 0-1 polytope  $P(x)$  (mixed integer  $P(x, w)$ ), after linearization and before projection to the original variables, is a (mixed) product relaxation.

### 3 The expressive power of product relaxations

In this section we show the following. For every 0-1 polytope  $P(x)$  and every (approximate) extended formulation  $Q(x, y) = \{(x, y) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \mid Ax + By \leq b\}$  of  $P(x)$  there is a product relaxation  $T[Q(x, y)]$  whose size is at most that of  $Q(x, y)$  and is at least as strong.

A *substitution*  $T$  is a linear map of the form  $y = Tz$  where  $T$  is a  $d_y \times (2^{d_x} - 1)$  matrix and  $z$  is a  $2^{d_x} - 1$  dimensional vector having a coordinate  $z_{\mathcal{E}}$  for each nonempty set  $\mathcal{E}$  of the form  $\{x_i \mid i \in S \subseteq \{1, \dots, d_x\}\}$ . For any substitution  $T$ , the *translation* of  $Q(x, y)$ , denoted  $T[Q(x, y)]$ , the formulation resulting by substituting  $T_{(i)}z$ , for  $y_i$ ,  $i = 1, \dots, d_y$ . Here  $T_{(i)}$  denotes the  $i$ th row of  $T$ . If in addition  $T[Q(x, y)]$  is a product relaxation of  $P(x)$  we say that it is a *translation of  $Q$  to product relaxations* (recall that the original variables  $x_i$  coincide with the variables  $z_{\{x_i\}}$ ). Observe that the number of inequalities of  $T[Q(x, y)]$  is the same as in  $Q(x, y)$ . The translation may heighten exponentially the dimension, but since our methodology will give lower bounds on the size of the product relaxations those bounds apply to the size of  $Q(x, y)$  as well.

► **Theorem 3.** Given a 0-1 polytope  $P(x) \subseteq [0, 1]^{d_x}$ , for every polytope  $Q(x, y)$  such that  $P(x) \subseteq \text{proj}_x(Q(x, y))$  there is a translation  $T[Q(x, y)]$  to product relaxations such that  $P(x) \subseteq \text{proj}_x(T[Q(x, y)]) \subseteq \text{proj}_x(Q(x, y))$ .

**Proof.** We shall give a substitution  $T$  for the variables  $y \in \mathbb{R}^{d_y}$  of  $Q(x, y)$  so that the theorem holds. Let  $g(x)$  be a section of  $Q(x, y)$  (recall that a section associates every feasible 0-1 vector  $x$  of  $P(x)$  to a specific  $y$  such that  $(x, y) \in Q(x, y)$ ).

Observe that the coordinates of the product section plus the constant 1 correspond exactly to the monomials of the Fourier basis. We denote by  $(p, 1) \in \mathbb{R}^{n+1}$  the vector resulting from  $p \in \mathbb{R}^n$  by appending the scalar 1 as an extra coordinate. By basic functional analysis (see, e.g., [14]), there is a  $d_y \times 2^{d_x}$  matrix  $A$  such that

$$g(x) = A \cdot (f(x), 1) \tag{1}$$

We define the substitution  $T$  by linearizing the above equation; we replace the sections  $g$  and  $f$  with the corresponding variable vectors  $y$  and  $z$  (recall  $z$  is the product vector) to obtain:

$$y = A \cdot (z, 1).$$

Obviously  $\text{proj}_x(T[Q(x, y)]) \subseteq \text{proj}_x(Q(x, y))$ : from any feasible solution  $(x_0, z_0)$  of  $T[Q(x, y)]$  we can derive a feasible solution  $(x_0, y_0)$  of  $Q(x, y)$  by setting  $y_0$  equal to  $Az_0$ .

We will now show that  $P(x) \subseteq \text{proj}_x(T[Q(x, y)])$ . It suffices to show that for every  $x' \in P(x) \cap \{0, 1\}^{d_x}$  the vector  $f(x')$  is feasible for  $T[Q(x, y)]$  as required by the definition of product relaxations. Observe that by letting the  $z$  vector take the values  $f(x')$ , by (1) we get that the quantities involved in the inequalities of  $T[Q(x, y)]$  are the exact same quantities involved in the corresponding inequalities of  $Q(x, y)$  for  $g(x')$ . But by the definition of section,  $g(x')$  is feasible for  $Q(x, y)$  and thus  $f(x')$  is feasible for  $T[Q(x, y)]$ . ◀

► **Corollary 4.** *A lower bound  $b$  on the size of any product relaxation  $D$  which is a  $\rho$ -approximate extended formulation of the 0-1 polytope  $P(x)$ , for  $\rho \geq 1$ , implies a lower bound  $b$  on the size of any  $\rho$ -approximate extended formulation  $Q(x, y)$  of  $P(x)$ .*

Let  $P(x, w)$  be a mixed integer polytope. The notion of the section of  $P$  for some extended relaxation  $Q(x, w, y)$  of  $P$  is more challenging. Intuitively, the solutions are characterized by two parts – a boolean part of the 0 – 1 assignments on the integer variables  $x$  and a "linear" part of the real variables  $w$  in the following sense: once the boolean part (the "hard" one) is fixed, the linear part can be obtained as the feasible region of a (usually small) system of inequalities, possibly empty.

Motivated by the above we define the following type of sections for an extended formulation  $Q(x, w, y)$  of a mixed-integer polytope. A *mixed-linear section* of EF  $Q$  is a section  $g$  for which at variable  $y_i$  the value  $g_i(x', w)$  for a given integer vector  $x'$  is an affine function on  $w$  denoted  $g_i^{x'}(w)$ . If there is such a mixed-linear section for  $Q(x, w, z)$ , we say that  $Q$  is an *extended formulation with a mixed linear section*. An example of EFs with a mixed linear section are formulations arising from the SA procedure where  $y$  is the vector of the new variables corresponding to the linearized products. The following theorem can be proved similarly to Theorem 3.

► **Theorem 5.** *Given a mixed integer polytope  $P(x, w) \subseteq [0, 1]^{d_x} \times \mathbb{R}^{d_w}$ , for every  $\rho$ -approximate,  $\rho \geq 1$ , extended formulation  $Q(x, w, y)$  with a mixed linear section, there is a translation  $T[Q(x, w, y)]$  to mixed product relaxations such that*

$$P(x, w) \subseteq \text{proj}_{x,w}(T[Q(x, w, y)]) \subseteq \text{proj}_{x,w}(Q(x, w, y)).$$

**Proof.** Let the dimension of  $P(x, w)$  be  $d = d_x + d_w$ . We shall give for the variables  $y \in \mathbb{R}^{d_y}$  of  $Q(x, w, y)$  a substitution  $T$  so that the theorem holds.

Consider a variable  $y_i$  and the corresponding coordinate of the mixed linear section,  $g_i^{x'}(w) = \sum_j b_j^{x'} w_j + c_{x'}$  for each  $x' \in \{0, 1\}^{d_x}$  and  $i = 1, \dots, d_y$ .

First, we will prove a helpful claim which states a fact from elementary Fourier analysis in our setting. For  $x, s \in \text{proj}_x(P(x, w) \cap (\{0, 1\}^{d_x} \times \mathbb{R}^{d_w}))$ , define the boolean indicator operator  $\chi_s(x)$  to be 1 when  $s = x$  and 0 otherwise. First, we will show that this operator can be expressed as a linear combination of the product sections constrained to monomials with only boolean variables. In other words, we determine coefficients  $a_{\mathcal{E}}^s$ ,  $\mathcal{E} \subseteq \{x_1, \dots, x_{d_x}\}$ , such that  $\chi_s(x) = \sum_{\mathcal{E}} a_{\mathcal{E}}^s f_{\mathcal{E}}(x)$ . The translation of the indicator operator  $i_s(x)$  of an integer solution  $s$  is a linear expression of the form  $T_{i_s} = \sum a_{\mathcal{E}}^s P[\mathcal{E}](x)$ . We shall iteratively generate the coefficients  $a_{\mathcal{E}}^s$ . The only nonzero coefficients will be those corresponding to sets of variables that are supersets of the set of variables being 1 in  $s$  – let that set be  $\mathcal{E}_1^s$ . We give the construction iteratively starting from  $|\mathcal{E}_1^s|$  to  $d_x$ , defining in step  $k$  the coefficients of such sets of size  $k$ .

In the first iteration simply set  $a_{\mathcal{E}_1^s} = 1$ . At step  $k > |\mathcal{E}_1^s|$ , for each set  $\mathcal{E}'$  of size  $k$  that is a superset of  $\mathcal{E}_1^s$ , set  $a_{\mathcal{E}'}^s = -\sum_{\mathcal{E} \subset \mathcal{E}'} a_{\mathcal{E}}^s$ . This concludes the definition of the coefficients.

► **Claim 3.1.** For each integer solution  $s' \in \text{proj}_x(P(x, w) \cap (\{0, 1\}^{d_x} \times \mathbb{R}^{d_w}))$ ,  $\chi_s(s') = \sum_{\mathcal{E}} a_{\mathcal{E}}^s f_{\mathcal{E}}(s')$ .

*Proof of the claim.* By overloading the notation, we denote by  $s$  both the integer solution and the support of that integer solution, that is the set  $\{x_i \mid s_i = 1\}$ . If  $s' \supseteq s$  then the nonzero terms of the sum  $\sum_{\mathcal{E}} a_{\mathcal{E}}^s f_{\mathcal{E}}(s')$  are exactly those that correspond to sets  $\mathcal{E}$  such that  $s \subseteq \mathcal{E} \subseteq s'$ . We have that  $\sum_{\mathcal{E}} a_{\mathcal{E}}^s f_{\mathcal{E}}(s') = \sum_{\mathcal{E} \subseteq s'} a_{\mathcal{E}}^s$  which, by the construction of the coefficients, is 1 if  $s = s'$  and 0 if  $s' \supset s$ , as required. Otherwise, if  $s - s' \neq \emptyset$ , then all the  $f_{\mathcal{E}}(s')$  with nonzero coefficients are 0, so  $\sum_{\mathcal{E}} a_{\mathcal{E}}^s f_{\mathcal{E}}(s') = 0$ .

By Claim 3.1 we have that for an integer vector  $s \in \{0, 1\}^{d_x}$  the indicator operator  $\chi_s(x)$  is equal to  $\sum_{\mathcal{E} \subseteq \{x_1, \dots, x_{d_x}\}} a_{\mathcal{E}}^s f_{\mathcal{E}}(x)$ . For each set of integer variables  $\mathcal{E}$  and each fractional variable  $w_j$  let  $z_{\mathcal{E}w_j}$  denote the corresponding mixed product variable and  $f_{\mathcal{E}w_j}(x, w)$  the corresponding coordinate of the mixed product section. It is now easy to show the following.

► **Claim 3.2.** For each mixed integer solution  $(x', w')$ , and for  $i = 1, \dots, d_y$ ,  $g_i^{x'}(w') = \sum_j \sum_{\mathcal{E}} a_{\mathcal{E}}^s b_j^{x'} f_{\mathcal{E}w_j}(x', w') + \sum_{\mathcal{E}} a_{\mathcal{E}}^s c_{x'} f_{\mathcal{E}}(x')$ .

To conclude the definition of  $T$ , set

$$y^i = \sum_{x'} \sum_j \sum_{\mathcal{E}} a_{\mathcal{E}}^s b_j^{x'} z_{\mathcal{E}w_j} + \sum_{x'} \sum_{\mathcal{E}} a_{\mathcal{E}}^s c_{x'} z_{\mathcal{E}}, \quad i = 1, \dots, d_y.$$

which implies

$$y^i = \sum_{x'} \sum_{\mathcal{E}} a_{\mathcal{E}}^s \left( \sum_j b_j^{x'} z_{\mathcal{E}w_j} + c_{x'} z_{\mathcal{E}} \right), \quad i = 1, \dots, d_y$$

By Claim 3.2, using arguments similar to the ones in the proof of Theorem 3, it follows that that  $P(x, w) \subseteq \text{proj}_{x,w}(T[Q(x, w, y)]) \subseteq \text{proj}_{x,w}(Q(x, w, y))$ . ◀

► **Corollary 6.** A lower bound  $b$  on the size of any mixed product relaxation  $D$  which is a  $\rho$ -approximate extended formulation of the 0-1 mixed integer polytope  $P(x, w)$  implies a lower bound  $b$  on the size of any  $\rho$ -approximate extended formulation  $Q(x, w, y)$  of  $P(x, w)$  with a mixed linear section.



#### 4 A method for lower bounding the size of LPs with known sections

Here we present a methodology to lower bound the size of relaxations that achieve a desired integrality gap. For simplicity we do not deal in this section with mixed integer sets.

Our method can be summarized as follows. Let  $G(z) \subseteq [0, 1]^d$  be a 0-1 polytope. We design a family  $\mathcal{I}$  of instances parameterized by the dimension  $d$ . For each instance  $I \in \mathcal{I}$  of dimension  $d$  we define a set of points  $\mathcal{C}_I \subseteq [0, 1]^d \setminus G(z)$  which we call the *core of  $I$  with respect to  $G$* . Note that the points of the core must be infeasible for  $G$ . To prove a lower bound  $r(n)$  on the size of  $G$  it suffices to show that at least that many inequalities are needed to separate  $\mathcal{C}_I$  from  $G$ . Additionally, for a minimization problem with  $O$  being the set of admissible objective functions, if for some  $z \in \mathcal{C}_I$  there is an admissible cost function  $w_z$  such that  $w_z^T z < \rho^{-1} \text{Opt}_{I, w_z}$ ,  $0 < \rho \leq 1$ , where  $\text{Opt}_{I, w_z}$  is the cost of the optimal integer solution with respect to  $w_z$ , we call  $z$   $\rho$ -gap inducing wrt  $O$ . If we design the core so that all its members are  $\rho$ -gap inducing, the lower bound will hold for  $\rho$ -approximate formulations.

To define constructively the core for a specific family of extended formulations of a polytope  $P$  the sections of the variables  $z$  must be known. This requirement is fulfilled by the product relaxations we will focus on. By Theorem 3 above, proving a lower bound on the size for an arbitrary extended relaxation  $Q(x, y)$  of a polytope  $P(x)$  can be reduced to a proof of the same bound on the size of a corresponding product relaxation  $D(z)$ . The following meta-theorem shows that such a proof can always be obtained by proving the existence of a suitable core for the product relaxation. Recall the definition of the “tightest” product relaxation of  $P(x)$ ,  $\hat{D}$ , in Section 2. We say that a set of vectors  $s \subseteq [0, 1]^d \setminus \hat{D}$  is *conflicting* if  $\text{conv}(s) \cap \hat{D} \neq \emptyset$ . Any single valid inequality of  $\hat{D}$  cannot separate all points of a conflicting set. Given a set  $O_d \subseteq \mathbb{R}^d$  of admissible objective functions associated with a 0-1 polytope  $P(x) \subseteq [0, 1]^d$ , we define  $\hat{O}_d \subseteq \mathbb{R}^{2^d-1}$ , to contain the vectors in  $O_d$  extended with zeroes in the coordinates corresponding to the non-singleton product variables.

► **Theorem 7.** *Given a 0-1 polytope  $P(x) \subseteq [0, 1]^d$ , and an associated set of admissible objective functions  $O_d \subseteq \mathbb{R}^d$ , the  $\rho$ -approximate extension complexity,  $\rho \geq 1$ , of  $P(x)$  is at least  $r(n)$ , iff there exists a family of instances  $\mathcal{I}(n)$  and, for every  $I \in \mathcal{I}$ , a core  $\mathcal{C}_I$  wrt  $\hat{D}$ , which consists of  $\rho$ -gap inducing vectors wrt  $\hat{O}_d$ , with the following property: for any partition of  $\mathcal{C}_I$  into less than  $r(n)$  parts there must be a part containing a set of conflicting vectors.*

**Proof.** Assume first that the  $\rho$ -approximate extension complexity is at least  $r(n)$ . Define  $\mathcal{C}_I$  to be the set of all  $\rho$ -gap inducing product vectors. If we can partition  $\mathcal{C}_I$  into less than  $r(n)$  parts so that there is no conflicting subset  $s$  in any part, then we can define an inequality for each part of the partition that separates the vectors of at least that part from  $\hat{D}$ . But we know that less than  $r(n)$  inequalities cannot separate all the  $\rho$ -gap inducing product vectors. Thus we have that for any decomposition of those vectors into less than  $r(n)$  parts there must be a part containing a set of conflicting vectors.

Conversely, assume we can find a core  $\mathcal{C}_I$  wrt  $\hat{D}$  consisting of  $\rho$ -gap inducing vectors such that for any partition of  $\mathcal{C}_I$  into less than  $r(n)$  sets there must be a part containing a set of conflicting vectors. Then the size of  $\hat{D}$  is at least  $r(n)$ . If not, there is a decomposition into less than  $r(n)$  parts where each part consists of the core members separated by each inequality – in case a member is separated by more than one inequality, we arbitrarily include it into just one of the resulting parts. Observe that  $\mathcal{C}_I$  is not only a core wrt  $\hat{D}$  but also is a core wrt any  $\rho$ -approximate product relaxation of  $P$ . By Theorem 3, the lower bound  $r(n)$  applies to the size of any  $\rho$ -approximate extended formulation of  $P$ . ◀

Let  $\mathcal{H}(\mathcal{C}_I)$  be the, possibly infinite, hypergraph with vertices the members of  $\mathcal{C}_I$  and

hyperedges the conflicting subsets of  $\mathcal{C}_I$ . Theorem 7 can be restated more conveniently:

► **Theorem 8.** *Given a 0-1 polytope  $P(x) \subseteq [0, 1]^d$ , and an associated set of admissible objective functions  $O_d \subseteq \mathbb{R}^d$ , the  $\rho$ -approximate extension complexity,  $\rho \geq 1$ , of  $P(x)$  is at least  $r(n)$ , iff there exists a family of instances  $\mathcal{I}(n)$  and, for every  $I \in \mathcal{I}$ , a core  $\mathcal{C}_I$  wrt  $\hat{D}$ , which consists of  $\rho$ -gap inducing vectors wrt  $\tilde{O}_d$ , such that  $\mathcal{H}(\mathcal{C}_I)$  has chromatic number  $r(n)$ .*

Theorem 7 suggests that the best possible lower bound on the extension complexity can always be achieved by proving the existence of an appropriate core in the product space. In the applications in this paper we implement a version of the method that imposes stronger requirements on the decomposition, namely the constructed hypergraph will be a clique.

## 5 Lower bounds for approximate mixed product relaxations for CFL

For CFL, the linear encoding  $\mathcal{N}_{\text{CFL}} = (L, O)$  is defined as follows. For a CFL instance, given the number  $n$  of facilities, the number  $m$  of clients, the capacities  $K \in \mathbb{R}_+^n$  and the demands  $D \in \mathbb{R}_+^m$ , we use the classic variables  $y_i, i = 1, \dots, n, x_{ij}, i = 1, \dots, n, j = 1, \dots, m$  with the usual meaning of facility opening and client assignment respectively. The set of feasible solutions  $(y, x)$  is defined in the obvious manner. Thus for dimension  $d = n + nm, L \cap \{0, 1\}^d$  is completely determined by the quadruple  $(n, m, K, D)$ . The set of admissible objective functions  $O \cap \mathbb{R}^{n+nm}$  is the set of pairs  $(\mathbf{f}, \mathbf{c})$  where  $\mathbf{f} \in \mathbb{R}_+^n$  are the facility opening costs and  $\mathbf{c} = [c_{ij}] \in \mathbb{R}_+^{nm}$  are connection costs that satisfy  $c_{ij} \leq c_{i'j} + c_{ij'} + c_{ij'}$ .

The capacitated facility location problem with general capacities and demands is a mixed integer optimization problem where the facilities are opened integrally but the clients are allowed to be assigned fractionally to the set of opened facilities. In this section, we show an exponential lower bound on the size of any mixed product relaxation of the CFL polytope.

In our proof we will consider a parameterized instance  $I = I(3n, m, U, d)$  with uniform capacities  $U$  and uniform unit demands  $d = 1$ , where  $3n$  is the number of facilities, and  $m$  the number of clients. Furthermore we will have that the number of clients is  $m = n^4 + 1$  and the capacities and demands are such that  $(n^4 + 1) - nU = 2^{-n^2}$ . Observe that  $n^3 < U < (n^3 + 1)$ . In order to define the core  $\mathcal{C}_I$  of the instance  $I$  we first describe a random experiment based on whose outcome we will later define the members of the core. Given disjoint sets  $k, l \subseteq F$  of size  $n$  each, the random experiment defines a distribution  $\mathcal{D}_{k,l}$  over mixed integer vectors in the classic encoding. These vectors correspond in general to pseudo-solutions. The following experiment defines the distribution  $\mathcal{D}_{k,l}$ . The quantities  $\bar{x}_{ij}$  are defined in Lemma 9 below.

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### RANDOM EXPERIMENT

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Facilities in  $k$  are always opened.

*Case 1.* With probability  $1 - \frac{20}{n^2(1+1/n)}$  all facilities in  $F - l$  are opened and those of  $l$  are closed. Distribute evenly the client demand to facilities in  $k$ . Note that this outcome of the experiment does not respect the capacities.

*Case 2.* Otherwise, i.e., with probability  $\frac{20}{n^2(1+1/n)}$ , pick at random a subset  $q$  of the facilities in  $F - k$  with at least one facility from  $l$  and open them. Assign randomly demand to each facility  $i$  in  $q \cap l$  so that  $i$  takes  $\frac{\sum_j \bar{x}_{ij}}{10/n^2}$  units and the rest of the demand is equally distributed to the facilities in  $k$ .

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► **Lemma 9.** *The expected vector  $(\bar{y}, \bar{x})$  wrt  $\mathcal{D}_{k,l}$  is the following:  $\bar{y}_i = 1$  for  $i \in k, \bar{y}_i = 1 - \frac{10}{n^2(1+1/n)}$  for  $i \in F - k - l, \bar{y}_i = \frac{20(2^{n-1})}{n^2(1+1/n)(2^n-1)}$  for  $i \in l$ . For all  $j \in C, \bar{x}_{ij} = \frac{1-n^{-2}}{|k|}$  for  $i \in k, \bar{x}_{ij} = 0$  for  $i \in F - \{k \cup l\}, \bar{x}_{ij} = \frac{n^{-2}}{|l|}$  for  $i \in l$ .*

The distribution  $\mathcal{D}_{k,l}$  will be subsequently used to define the members of the core  $\mathcal{C}_I$ . Let  $\mathcal{E}$  be a subset of integer variables in the original space, i.e.,  $\mathcal{E} \subseteq \{y_1, \dots, y_{3n}\}$ . We denote by  $E_{\mathcal{D}_{k,l}}[\mathcal{E}]$  the expectation of the event where all the variables in  $\mathcal{E}$  have value 1, i.e., the expectation of the product  $\prod_{y_{i_k} \in \mathcal{E}} y_{i_k}$ . Similarly, we denote  $E_{\mathcal{D}_{k,l}}[\mathcal{E}x_{ij}]$  the expectation of the product  $(\prod_{y_{i_k} \in \mathcal{E}} y_{i_k}) \cdot x_{ij}$ . Let  $\chi(\text{case1}), \chi(\text{case2})$  be the 0-1 random variables that indicate whether Case 1 and Case 2 occur, respectively. We denote by  $E_{\mathcal{D}_{k,l}}[\mathcal{E} \cap \text{case1}]$  the expectation of the product  $(\prod_{y_{i_k} \in \mathcal{E}} y_{i_k}) \cdot \chi(\text{case1})$  and by  $E_{\mathcal{D}_{k,l}}[\mathcal{E}x_{ij} \cap \text{case1}]$  the expectation of the product  $(\prod_{y_{i_k} \in \mathcal{E}} y_{i_k}) \cdot x_{ij} \cdot \chi(\text{case1})$ . Similarly for Case 2. Intuitively,  $E_{\mathcal{D}_{k,l}}[\mathcal{E}x_{ij} \cap \text{case1}]$  is the "mass" that  $\mathcal{D}_{k,l}$  assigns to  $x_{ij}$  over all outcomes of case 1 where the variables of  $\mathcal{E}$  have value 1.

To simplify notation, we use  $z(i)$  instead of  $z_i$  to refer to a coordinate of vector  $z$  indexed by  $i$ . From now on,  $P$  denotes the CFL polytope and  $\hat{D}$  its canonical product relaxation.

► **Definition 10.** Fix a set  $k \subset F$  of size  $n$ . The core  $\mathcal{C}_I$  of the instance  $I(3n, n^4 + 1, U, 1)$  wrt  $\hat{D}$  is the following set of product vectors:  $\forall l \subset F$  with  $|l| = n$  and  $k \cap l = \emptyset$  and for every set  $\mathcal{E}$  of integer variables and for every fractional variable  $x_{ij}$  we define  $z_{k,l}(\mathcal{E}) = E_{\mathcal{D}_{k,l}}[\mathcal{E}]$  and  $z_{k,l}(\mathcal{E}x_{ij}) = E_{\mathcal{D}_{k,l}}[\mathcal{E}x_{ij}]$ .

Now we are ready to state the key Lemma 11 from which our main theorem will be derived. The proof of the lemma is quite technical and is deferred to the full version. A sketch is presented in Section 6.

► **Lemma 11.** For any two  $z_{k,l}, z_{k,l'} \in \mathcal{C}_I$  such that  $l - l' \neq \emptyset$  there is some  $z \in \text{conv}(z_{k,l}, z_{k,l'})$  which is feasible for  $\hat{D}$ .

► **Theorem 12.** Given the family of CFL instances  $I(3n, n^4 + 1, U, 1)$ , each member of  $\mathcal{C}_I$  is  $\Omega(n)$ -gap inducing and  $\chi(\mathcal{H}(\mathcal{C}_I)) = 2^{\Omega(n)}$ . Therefore, there is a constant  $c > 0$ , s.t. any  $cN$ -approximate EF for CFL with a mixed linear section has size  $2^{\Omega(N)}$ , where  $N$  is the number of facilities.

**Proof.** Since we proved in Lemma 11 that any two members of the core  $\mathcal{C}_I$  form a conflicting set,  $\mathcal{H}(\mathcal{C}_I)$  is a clique and thus its chromatic number is  $|\mathcal{C}_I| = \binom{2n}{n} = 2^{\Theta(n)}$ . For each member of the core  $z_{k,l}$  there is an admissible cost function  $w_{k,l}$  inducing  $\Theta(n)$  gap: facilities in  $l$  have unit opening costs and every other facility has 0 opening cost. The facilities in  $k \cup l$  and all the clients are co-located, and the rest of the facilities are co-located at distance  $2^{n^2}$  from the former. Observe that each feasible mixed integer solution has a cost of at least 1 since either some facility in  $l$  must be opened integrally or at least  $2^{-n^2}$  client demand has to be assigned to some facility in  $F - k - l$ . On the other hand the cost of  $z_{k,l}$  wrt  $w_{k,l}$  is  $\Theta(n^{-1})$  since the  $(y, x)$  projection of  $z_{k,l}$  is the expected vector  $(\bar{y}, \bar{x})$  of  $\mathcal{D}_{k,l}$ . ◀

For every instance  $I$  of CFL it is easy to see that there is an exact mixed product relaxation of size  $2^N p$  where  $p$  is polynomial in the size of the instance. The idea is to define a formulation for each choice of the opened facilities and then take the convex hull of those polytopes.

► **Observation 5.1.** There is an exact mixed product relaxation of the CFL polytope of size  $2^N p$ , where  $p = \Theta(mN)$ ,  $N$  and  $m$  being the number of facilities and clients respectively.

► **Theorem 13.** Let  $P$  be any linear relaxation of the CFL polytope for the family of instances  $I(3n, n^4 + 1, U, 1)$  that uses the encoding  $\mathcal{N}_{\text{CFL}}$  and has size  $2^{o(n)}$ . There is a constant  $c > 0$ , such that for all  $t \leq cn$ , the integrality gap of  $\text{SA}^t(P)$  is  $\Omega(n)$ .

**Proof.** Observe that for every level of SA there is a suitable projection of  $\mathcal{C}_I$  that yields a legal core with respect to the product variables used in that level. Therefore, the lower bound on the size implied by Theorem 12 holds at all levels. The number of the inequalities of the  $t$ -level SA relaxation after the lifting and linearization stages, and before projection, obtained from any starting relaxation  $P$  of size  $r$  is less than  $r \binom{n}{t} 2^t$ . By choosing  $t \leq cn$ , with  $c$  sufficiently small, we obtain that  $r \binom{n}{t} 2^t \leq r 2^{\delta n}$  for a small  $\delta > 0$ . By Theorem 12 we get that for this value of  $t$ , the integrality gap on the given family of instances is  $\Omega(n)$ . This is asymptotically tight since SA is known to produce an exact formulation after  $3n$  levels  $\blacktriangleleft$

We obtain as a direct consequence a lower bound on the size of formulations that use only the classic variables  $y_i, x_{ij}$ .

► **Corollary 14.** *Let  $P$  be any linear relaxation of the CFL polytope that uses the encoding  $\mathcal{N}_{\text{CFL}}$  and has integrality gap  $o(N)$ ,  $N$  being the number of facilities. Then  $P$  has size  $2^{\Omega(N)}$ .*

## 6 Proof sketch for Lemma 11

In the first part of the proof we will show that by exchanging some measure of some components of the two product vectors  $z_{k,l}, z_{k,l'}$  of the core, we can construct two new product vectors  $z_{k,l}^*, z_{k,l'}^*$  each of which is feasible for  $\hat{D}$ . Consider the two sets of facilities  $l - l'$  and  $l' - l$ . Clearly  $|l - l'| = |l' - l| > 0$ , since  $l \neq l'$  and  $|l| = |l'| = n$ . We construct a product vector  $z_{k,l}^*$  based on  $z_{k,l}$  and making some alterations and, symmetrically, a product vector  $z_{k,l'}^*$  based on  $z_{k,l'}$ . We give below the construction of  $z_{k,l}^*$ .

### Construction of $z_{k,l}^*$

For any set  $\mathcal{E}$  containing only facilities from  $F - l'$  with at least one from  $l - l'$ :  $z_{k,l}^*(\mathcal{E}) = z_{k,l}(\mathcal{E}) + E_{\mathcal{D}_{k,l'}}[\mathcal{E} \cap \text{case1}]$  (Similarly, for any  $i, j$ ,  $z_{k,l}^*(\mathcal{E}x_{ij}) = z_{k,l}(\mathcal{E}x_{ij}) + E_{\mathcal{D}_{k,l'}}[\mathcal{E}x_{ij} \cap \text{case1}]$ ). In the case set  $\mathcal{E}$  contains only facilities from  $F - l$  with at least one from  $l' - l$  we have  $z_{k,l}^*(\mathcal{E}) = z_{k,l}(\mathcal{E}) - E_{\mathcal{D}_{k,l}}[\mathcal{E} \cap \text{case1}]$ . (Similarly, for any  $i, j$ ,  $z_{k,l}^*(\mathcal{E}x_{ij}) = z_{k,l}(\mathcal{E}x_{ij}) - E_{\mathcal{D}_{k,l}}[\mathcal{E}x_{ij} \cap \text{case1}]$ ). In any other case and for any  $i, j$  let  $z_{k,l}^*(\mathcal{E}) = z_{k,l}(\mathcal{E})$  and  $z_{k,l}^*(\mathcal{E}x_{ij}) = z_{k,l}(\mathcal{E}x_{ij})$ .

Next we show, and this is by far the most complicated part of the proof, that the constructed  $z_{k,l}^*$  and  $z_{k,l'}^*$  are indeed the expected vectors of distributions  $\mathcal{D}_{k,l}^*$  and  $\mathcal{D}_{k,l'}^*$ , respectively, over feasible mixed integer product solutions. In the last step of the proof we show the following, which is an easy consequence of the construction of  $z_{k,l}^*$  and  $z_{k,l'}^*$ .

► **Claim 6.1.**  $1/2(z_{k,l}^* + z_{k,l'}^*) \in \text{conv}(z_{k,l}, z_{k,l'})$ .

## 7 Discussion

In the proof of our result for CFL we provided a core whose underlying hypergraph is actually a graph and moreover a clique. For other problems, especially for 0-1 polytopes, we believe that the power of general hypergraphs needs to be exploited, if one wishes to derive a tight bound on the extension complexity. Observe that our methodology requires only the existence of a suitable core, and thus, one could possibly employ probabilistic arguments to prove the existence of suitable hypergraphs of high chromatic number.

In the case of mixed integer polytopes, we believe that the mixed product relaxations can be shown to be strong enough to simulate any extended formulation, as is the case for product relaxations and 0-1 polytopes.

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