1-String B_2-VPG Representation of Planar Graphs*

Therese Biedl and Martin Derka

David R. Cheriton School of Computer Science, University of Waterloo
200 University Ave W, Waterloo, ON N2L 3G1, Canada
`{biedl,mderka}@uwaterloo.ca`

Abstract

In this paper, we prove that every planar graph has a 1-string B_2-VPG representation – a string representation using paths in a rectangular grid that contain at most two bends. Furthermore, two paths representing vertices u, v intersect precisely once whenever there is an edge between u and v.

1998 ACM Subject Classification I.3.5 Computational Geometry and Object Modeling

Keywords and phrases Graph drawing, string graphs, VPG graphs, planar graphs

Digital Object Identifier 10.4230/LIPIcs.SOCG.2015.141

1 Preliminaries

One way of representing graphs is to assign to every vertex a curve so that two curves cross if and only if there is an edge between the respective vertices. Here, two curves $\mathbf{u}, \mathbf{v}$ *cross* means that they share a point s internal to both of them and the boundary of a sufficiently small closed disk around s is crossed by $\mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v}$ (in this order). The representation of graphs using crossing curves is referred to as a *string representation*, and graphs that can be represented in this way are called *string graphs*.

In 1976, Ehrlich, Even and Tarjan showed that every planar graph has a string representation [9]. It is only natural to ask if this result holds if one is restricted to using only some "nice" types of curves. In 1984, Scheinerman conjectured that all planar graphs can be represented as intersection graphs of line segments [12]. This was proved first for bipartite graphs [8, 11] with the strengthening that every segment is vertical or horizontal. The result was extended to triangle-free graphs, which can be represented by line segments with at most three distinct slopes [7].

Since Scheinerman's conjecture seemed difficult to prove for all planar graphs, interest arose in possible relaxations. Note that any two line segments can generally intersect at most once. Define 1-String to be the class of graphs that are intersection graphs of curves (of arbitrary shape) that intersect at most once. We also say that graphs in this class have a 1-*string representation*. The original construction of string representations for planar graphs given in [9] requires curves to cross multiple times. In 2007, Chalopin, Gonçalves and Ochem showed that every planar graph is in 1-String [4, 5]. With respect to Scheinerman's conjecture, while the argument of [4, 5] shows that the prescribed number of intersections can be achieved, it provides no idea on the complexity of curves that is required.

Another way of restricting curves in string representations is to require them to be *orthogonal*, i.e., to be paths in a grid. Call a graph a *VPG-graph* (as in "Vertex-intersection

---

* Research supported by NSERC. The second author was supported by the Vanier CGS.

31st International Symposium on Computational Geometry (SoCG'15).
Editors: Lars Arge and János Pach; pp. 141–155
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

graph of Paths in a Grid") if it has a string representation with orthogonal curves. It is easy to see that all planar graphs are VPG-graphs (e.g. by generalizing the construction of Ehrlich, Even and Tarjan). For bipartite planar graphs, curves can even be required to have no bends [8, 11]. For arbitrary planar graphs, bends are required in orthogonal curves. Recently, Chaplick and Ueckerdt showed that 2 bends per curve always suffice [6]. Let *B_2-VPG* be the graphs that have a string representation where curves are orthogonal and have at most 2 bends; the result in [6] then states that planar graphs are in B_2-VPG. Unfortunately, in Chaplick and Ueckerdt's construction, curves may cross each other repeatedly, and so it does not prove that planar graphs are in 1-String.

The conjecture of Scheinerman remained open until 2009 when it was proved true by Chalopin and Gonçalves [3].

Our Results: In this paper, we show that every planar graph has a string representation that simultaneously satisfies the requirements for 1-String (any two curves cross at most once) and the requirements for B_2-VPG (any curve is orthogonal and has at most two bends). Our result hence re-proves, in one construction, the results by Chalopin et al. [4, 5] and the result by Chaplick and Ueckerdt [6].

▶ **Theorem 1.** *Every planar graph has a* 1*-string B_2-VPG representation.*

Our approach is inspired by the construction of 1-string representations from 2007 [4, 5]. The authors proved the result in two steps. First, they showed that triangulations without separating triangles admit 1-string representations. By induction on the number of separating triangles, they then showed that a 1-string representation exists for any planar triangulation, and consequently for any planar graph.

In order to show that triangulations without separating triangles have 1-string representations, Chalopin et al. [5] used a method inspired by Whitney's proof that 4-connected planar graphs are Hamiltonian [13]. Asano, Saito and Kikuchi later improved Whitney's technique and simplified his proof [1]. Our paper uses the same approach as [5], but borrows ideas from [1] and develops them further to reduce the number of cases.

2 Definitions and Basic Results

Let us begin with a formal definition of a 1*-string B_2-VPG representation.*

▶ **Definition 2** (1-string B_2-VPG representation)**.** A graph G has a *1-string B_2-VPG representation* if every vertex v of G can be represented by a curve $\mathbf{v}$ such that:

1. Curve $\mathbf{v}$ is *orthogonal*, i.e., it consists of horizontal and vertical segments.
2. Curve $\mathbf{v}$ has at most two bends.
3. Curves $\mathbf{u}$ and $\mathbf{v}$ intersect at most once, and $\mathbf{u}$ intersects $\mathbf{v}$ if and only if (u, v) is an edge of G.

We always use $\mathbf{v}$ to denote the curve of vertex v, and write $\mathbf{v}^R$ if the representation R is not clear from the context. We also often omit "1-string B_2-VPG" since we do not consider any other representations.

Our technique for constructing representations of a graph uses an intermediate step referred to as a "partial 1-string B_2-VPG representation of a W-triangulation that satisfies the chord condition with respect to three chosen corners." We define these terms, and related graph terms, first.

A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn so that no edges intersect except at common endpoints. All graphs in this paper are planar. We assume throughout the paper that one combinatorial embedding of the graph has been fixed by specifying the clockwise (CW) cyclic order of incident edges around each vertex. Subgraphs inherit this embedding, i.e., they use the induced clockwise orders. A *facial region* is a connected region of $\mathbb{R}_2 - \Gamma$ where Γ is a planar drawing of G that conforms with the combinatorial embedding. The circuit bounding this region can be read from the combinatorial embedding of G and is referred to as a *face*. The *outer-face* is the one that corresponds to the unbounded region; all others are called *interior faces*. The outer-face cannot be read from the embedding; we assume throughout this paper that the outer-face of G has been specified. Subgraphs inherit the outer-face by using as outer-face the one whose facial region contains the facial region of the outer-face of G. An edge of G is called *interior* if it does not belong to the outer-face.

A *triangulated disk* is a planar graph G for which the outer-face is a simple cycle and every interior face is a triangle. A *separating triangle* is a cycle C of length 3 such that G has vertices both inside and outside the region bounded by C (with respect to the fixed embedding and outer-face of G). Following the notation of [5], a *W-triangulation* is a triangulated disk that does not contain a separating triangle. A *chord* of a triangulated disk is an interior edge for which both endpoints are on the outer-face.

For two vertices X, Y on the outer-face of a connected planar graph, define P_{XY} to be the counter-clockwise (CCW) path on the outer-face from X to Y (X and Y inclusive). We often study triangulated disks with three specified distinct vertices A, B, C called the *corners*. A, B, C must appear on the outer-face in CCW order. We denote $P_{AB} = (a_1, a_2, \dots, a_r)$, $P_{BC} = (b_1, b_2, \dots, b_s)$ and $P_{CA} = (c_1, c_2, \dots, c_t)$, where $c_t = a_1 = A$, $a_r = b_1 = B$ and $b_s = c_1 = C$.

▶ **Definition 3** (Chord condition)**.** A W-triangulation G satisfies the *chord condition* with respect to the corners A, B, C if G has no chord within P_{AB}, P_{BC} or P_{CA}, i.e., no interior edge of G has both ends on P_{AB}, or both ends on P_{BC}, or both ends on P_{CA}.[1]

▶ **Definition 4** (Partial 1-string B_2-VPG representation)**.** Let G be a connected planar graph and $E' \subseteq E(G)$ be a set of edges. An (E')*-1-string* B_2*-VPG representation* of G is a 1-string B_2-VPG representation of the subgraph $(V(G), E')$, i.e., curves $\mathbf{u}, \mathbf{v}$ cross if and only if (u, v) is an edge in E'. If E' consists of all interior edges of G as well as some set of edges F on the outer-face, then we write $(\mathit{int} \cup F)$ *representation* instead.

In our constructions, we use $(\mathit{int} \cup F)$ representations with $F = \emptyset$ or $F = e$, where e is an outer-face edge incident to corner C of a W-triangulation. Edge e is called the *special edge*, and we sometimes write $(\mathit{int} \cup e)$ representation, rather than $(\mathit{int} \cup \{e\})$ representation.

2.1 2-Sided, 3-Sided and Reverse 3-Sided Layouts

To create representations where vertex-curves have few bends, we need to impose geometric restrictions on representations of subgraphs. Unfortunately, no one type of layout seems sufficient for all cases, and we will hence have three different layout types whose existence we will prove in parallel.

---

[1] For readers familiar with [5] or [1]: A W-triangulation that satisfies the chord condition with respect to corners A, B, C is called a *W-triangulation with 3-boundary* P_{AB}, P_{BC}, P_{CA} in [5], and the chord condition is the same as *Condition (W2b)* in [1].

▶ **Definition 5** (2-sided layout). Let G be a connected planar graph and A, B be two distinct outer-face vertices. An $(int \cup F)$ B_2-VPG representation of G has a 2-*sided layout* (with respect to corners A, B) if:

1. There exists a rectangle Θ that contains all intersections of curves and such that the top of Θ is intersected, from right to left in order, by the curves of the vertices of P_{AB}, and the bottom of Θ is intersected, from left to right in order, by the curves of the vertices of P_{BA}.
2. The curve $\mathbf{v}$ of an outer-face vertex v has at most one bend. (By 1., this implies that $\mathbf{A}$ and $\mathbf{B}$ have no bends.)

▶ **Definition 6** (3-sided layout). Let G be a connected planar graph and A, B, C be three distinct vertices in CCW order on the outer-face of G. Let F be a set of exactly one outer-face edge incident to C. An $(int \cup F)$ B_2-VPG representation of G has a 3-*sided layout* (with respect to corners A, B, C) if:

1. There exists a rectangle Θ containing all intersections of curves so that
   (i) the top of Θ is intersected, from right to left in order, by the curves of the vertices on P_{AB};
   (ii) the left side of Θ is intersected, from top to bottom in order, by the curves of the vertices on $P_{Bb_{s-1}}$, possibly followed by $\mathbf{C}$; [2]
   (iii) the bottom of Θ is intersected, from right to left in order, by the curves of vertices on P_{c_2A} in reversed order, possibly followed by $\mathbf{C}$;[2]
   (iv) curve $\mathbf{b_s} = \mathbf{C} = \mathbf{c_1}$ intersects the boundary of Θ exactly once; it is the bottommost curve to intersect the left side of Θ if the special edge in F is (C, c_2), and $\mathbf{C}$ is the leftmost curve to intersect the bottom of Θ if the special edge in F is (C, b_{s-1}).
5. The curve $\mathbf{v}$ of an outer-face vertex v has at most one bend. (By 1., this implies that $\mathbf{B}$ has precisely one bend.)
6. $\mathbf{A}$ and $\mathbf{C}$ have no bends.

We sometimes refer to the rectangle Θ for both 2- and 3-sided representation as a *bounding box*. Figure 1 (which will serve as base case later) shows such layouts for a triangle and varying choices of F. We also need the concept of a *reverse* 3-*sided layout*, which is similar to the 3-sided layout except that B is straight and A has a bend. Formally, it satisfies conditions 1(ii-iv) and (2). 1(i) is replaced by "the right side of Θ is intersected, from bottom to top in order, by the curves of the vertices on P_{AB}" and (3) is replaced by "$\mathbf{B}$ and $\mathbf{C}$ have no bends."

2.2 Private Regions

Our proof starts with constructing representation for triangulations without separating triangles. The construction is then extended to all triangulations by merging representations of subgraphs obtained by splitting at separating triangles. To permit the merge, we apply the technique used in [5] (and re-discovered in [10]): With every triangular face, create a region that intersects the curves of vertices of the face in a predefined way and does not intersect anything else, specifically any other private regions. Following the notation of [10], we call this a *private region* (but we use a different shape).

---

[2] Recall (b_{s-1}, C) and (C, c_2) are the two incident edges of C on the outer-face.

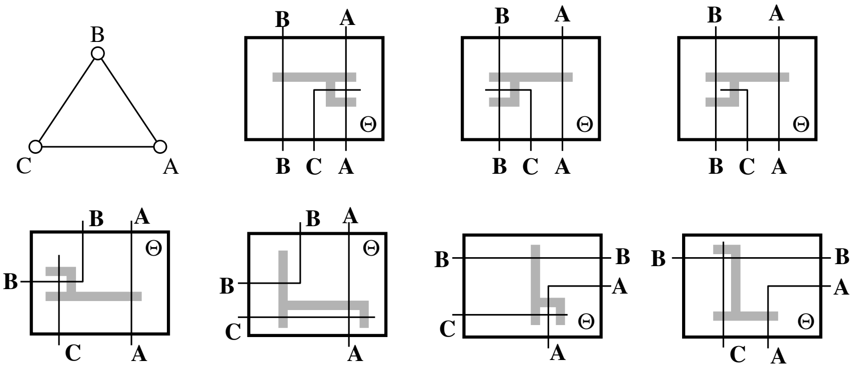

Figure 1 $(int \cup F)$ representations of a triangle: (Top) 2-sided representations for $F \in \{\{(A,C)\},\{(B,C)\},\emptyset\}$. (Bottom) 3-sided and reverse 3-sided representations for $F \in \{\{(A,C)\},\{(B,C)\}\}$. Private regions are shaded in grey.

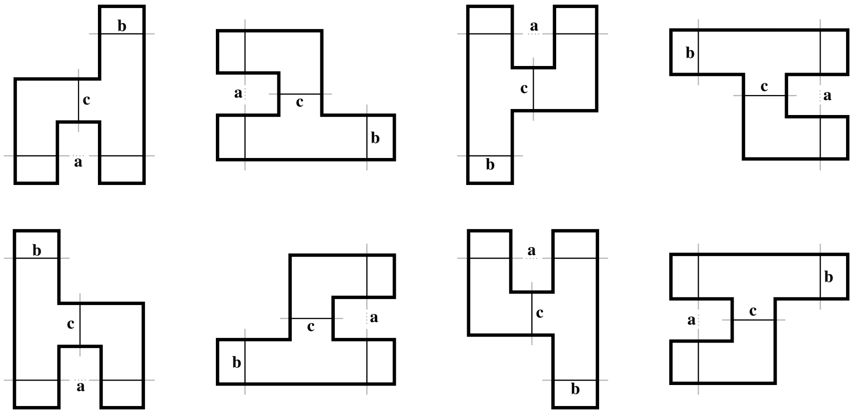

Figure 2 The private region of a triangle a, b, c with possible rotations and flips.

▶ **Definition 7** (Chair-shape)**.** A *chair-shaped area* is a region bounded by a 10-sided orthogonal polygon with CW (clockwise) or CCW (counter-clockwise) sequence of interior angles 90°, 90°, 270°, 270°, 90°, 90°, 90°, 90°, 270°, 90°. See also Figure 2.

▶ **Definition 8** (Private region)**.** Let G be a planar graph with a partial 1-string B_2-VPG representation R and let f be a facial triangle in G. A *private region* of f is a chair-shaped area Φ inside R such that:

1. Φ is intersected by no curves except for the ones representing vertices on f.
2. All the intersections of R are located outside of Φ.
3. For a suitable labeling of vertices of f as $\{a, b, c\}$, Φ is intersected by two segments of $\mathbf{a}$ and one segment of $\mathbf{b}$ and $\mathbf{c}$. The intersections between these segments and Φ occur at the edges of Φ as depicted in Figure 2.

3 Constructions for W-Triangulations

Our key tool for proving Theorem 1 is the following lemma:

▶ **Lemma 9.** *Let G be a W-triangulation that satisfies the chord condition with respect to corners A, B, C. For any $e \in \{(C, b_{s-1}), (C, c_2)\}$, G has an $(int \cup e)$ 1-string B_2-VPG* *representation with* 3*-sided layout and an* ($int \cup e$) 1*-string* B_2*-VPG representation with reverse* 3*-sided layout. Both representations have a chair-shaped private region for every interior face.*

The proof of Lemma 9 will use induction on the number of vertices. To combine the representations of subgraphs, we sometimes need them to have a 2-sided layout, and hence prove the following result:

▶ **Lemma 10.** *Let* G *be a W-triangulation that satisfies the chord condition with respect to corners* A, B, C*. Then* G *has an* ($int \cup F$) 1*-string* B_2*-VPG representation with* 2*-sided layout with respect to* A, B *and for any set* F *of at most one outer-face edge incident to* C*. Furthermore, this representation has a chair-shaped private region for every interior face of* G*.*

Notice that for Lemma 9 the special edge *must* exist (this is needed in Case 1 to find private regions), while for Lemma 10, F is allowed to be empty.

We will prove both lemmas simultaneously by induction on the number of vertices. First, let us make an observation that will greatly help to reduce the number of cases. Define G^{rev} to be the graph obtained from graph G by reversing the combinatorial embedding, but keeping the same outer-face. This effectively switches corners A and B, and replaces special edge (C, c_2) by (C, b_{s-1}) and vice versa. If G satisfies the chord condition with respect to corners (A, B, C), then G^{rev} satisfies the chord condition with respect to corners (B, A, C). (With this new order, the corners are CCW on the outer-face of G^{rev}, as required.)

Presume we have a 2-sided/3-sided/reverse 3-sided representation of G^{rev}. Then we can obtain a 2-sided representation of G by flipping the 2-sided one of G^{rev} horizontally, i.e., along the y-axis. We can obtain a 3-sided/reverse 3-sided representation of G by flipping the reverse 3-sided/3-sided representation of G^{rev} diagonally (i.e., along the line defined by $(x = y)$). Hence for all the following cases, we may (after possibly applying the above flipping operation) either make a restriction on which edge the special edge is, or we only need to give the construction for 3-sided, but not for reverse 3-sided layout.

Now we begin the induction. In the base case, $n = 3$, so G is a triangle, and the three corners A, B, C must be the three vertices of this triangle. The desired ($int \cup F$) representations for all possible choices of F are depicted in Figure 1. The induction step for $n \geq 4$ is divided into three cases which we describe in separate subsections.

3.1 C has degree 2

Since G is a triangulated disk with $n \geq 4$, (b_{s-1}, c_2) is an edge. Define $G' := G - C$ and $F' := \{(b_{s-1}, c_2)\}$. We claim that G' satisfies the chord condition for corners $A' := A, B' := B$ and a suitable choice of $C' \in \{b_{s-1}, c_2\}$, and argue this as follows. If c_2 is not incident to a chord that ends on P_{BC}, then set $C' := c_2$; clearly the chord condition holds for G'. If c_2 is incident to such a chord, then b_{s-1} cannot be incident to a chord by planarity and the chord condition for G. So, in this case with the choice $C' := b_{s-1}$ the chord condition holds for G'. Thus in either case, we can apply induction to G'.

To create a 2-sided representation of G, we use a 2-sided ($int \cup F'$) representation R' of G'. We introduce a new vertical curve $\mathbf{C}$ placed between $\mathbf{b_{s-1}}$ and $\mathbf{c_2}$ below R'. Add a bend at the upper end of $\mathbf{C}$ and extend it leftwards or rightwards. If the special edge e exists, then extend $\mathbf{C}$ until it hits the curve of the other endpoint of e; else extend it only far enough to allow for the creation of the private region.

To create a 3-sided representation of G, we use a 3-sided ($int \cup F'$) representation R' of G'. Note that regardless of which vertex is C', we have $\mathbf{b_{s-1}}$ as bottommost curve on the left and

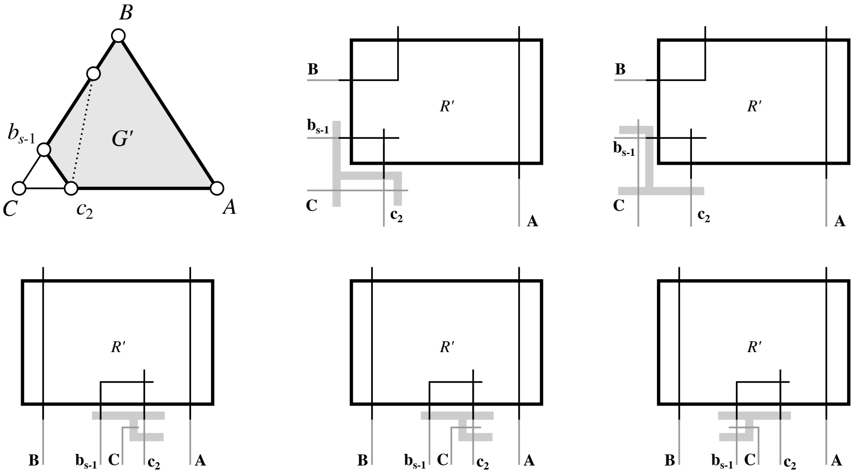

Figure 3 Case 1: C has degree 2. (Top) 3-sided representation. (Bottom) 2-sided representation.

$\mathbf{c_2}$ as leftmost curve on the bottom. Introduce a new horizontal segment representing C which intersects $\mathbf{c_2}$ if $F = \{(C, c_2)\}$, or a vertical segment which intersects $\mathbf{b_{s-1}}$ if $F = \{(C, b_{s-1})\}$.

In both constructions, after suitable lengthening, the curves intersect the bounding box in the required order. One can find the chair-shaped private region for the only new face $\{C, c_2, b_{s-1}\}$ as shown in Figure 3. Observe that no bends were added to the curves of R' and that C has the required number of bends in both representations.

Since we have given the constructions for both possible special edges, we can obtain the reverse 3-sided representation by diagonally flipping a 3-sided representation of G^{rev}.

3.2 G has a chord incident to C

By the chord condition, this chord has the form (C, a_i) for some $1 < i < r$. Select the chord that minimizes i. The graph G can be split along the chord (C, a_i) into two graphs G_1 and G_2. Both G_1 and G_2 are bounded by simple cycles, hence they are triangulated disks. No edges were added, so neither G_1 nor G_2 contains a separating triangle. So, both of them are W-triangulations.

We select (C, A, a_i) as corners for G_1 and (a_i, B, C) as corners for G_2 and can easily verify that G_1 and G_2 satisfy the chord condition with respect to those corners:

- G_1 has no chords on P_{Aa_i} or P_{CA} as they would violate the chord condition in G. There is no chord on P_{a_iC} as it is a single edge.
- G_2 has no chords on P_{a_iB} or P_{BC} as they would violate the chord condition in G. There is no chord on P_{a_iC} as it is a single edge.

So we can apply induction to both G_1 and G_2, obtain representations R_1 and R_2 for them, and combine them suitably. In the 3-sided case, we will do so for all possible choices of special edge, and hence need not give the constructions for reverse 3-sided layout as explained earlier.

Case 2(a): $F = \emptyset$ or $F = \{(C, b_{s-1})\}$. Inductively construct a 2-sided $(\mathit{int} \cup (C, a_i))$ representation R_1 of G_1. Inductively, construct an $(\mathit{int} \cup F)$ representation R_2 of G_2, which should be 2-sided if we want the result to be 2-sided and 3-sided if we want the result to be 3-sided. Note that either way $\mathbf{C}^{R_2}$ and $\mathbf{a_i}^{R_2}$ on the bottom side of R_2 with $\mathbf{C}^{R_2}$ to the left of $\mathbf{a_i}^{R_2}$.

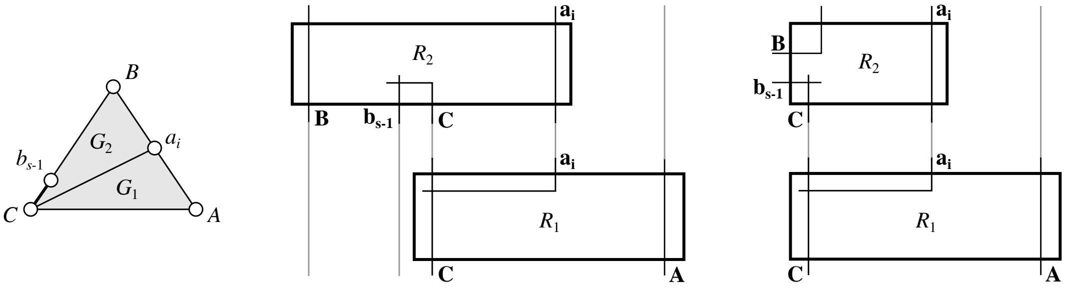

Figure 4 Case 2(a): Constructing an $(int \cup (C, b_{s-1}))$ representation when C is incident to a chord, in 2-sided (middle) and 3-sided (right) layout.

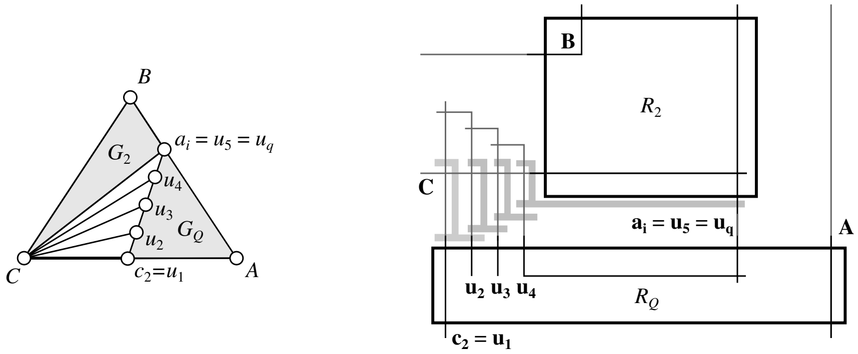

Figure 5 Case 2(b)1: C is incident to a chord, $F = (C, c_2)$, and $c_2 \neq A$.

Rotate R_1 by 180°, and translate it so that it is below R_2 with $\mathbf{a_i}^{R_1}$ in the same column as $\mathbf{a_i}^{R_2}$. Stretch R_1 horizontally as needed until $\mathbf{C}^{R_1}$ is in the same column as $\mathbf{C}^{R_2}$. Then $\mathbf{a_i}^{R}$ and $\mathbf{C}^{R}$ for $R \in \{R_1, R_2\}$ can each be unified without adding bends by adding vertical segments. The curves of outer-face vertices of G then cross (after suitable lengthening) the bounding box in the required order. See also Figure 4.

Every interior face f of G is contained in G_1 or G_2 and hence has a private region in R_1 or R_2. As our construction does not make any changes inside the bounding boxes of R_1 and R_2, the private region of f is contained in R as well.

Case 2(b): $F = \{(C, c_2)\}$. For the 2-sided construction, we apply the reversal trick: Construct a 2-sided representation of G^{rev} with suitable selection of corners (here Case 2(a) then applies) and flip it horizontally.

For the 3-sided construction, we need a different approach, which is quite similar to Case 1 in [1, Proof of Lemma 2]. Let $G_Q = G_1 - C$, and observe that it is bounded by $P_{c_2 A}$, P_{A, a_i}, and the path formed by the neighbours $c_2 = u_1, u_2, \ldots, u_q = a_i$ of C in CCW order. We must have $q \geq 2$, but possibly G_1 is a triangle $\{C, A, a_i\}$ and G_Q then degenerates into an edge. If G_Q contains at least three vertices, then none of $u_2, \ldots, u_{q-1}$ belongs to P_{AB} since chord (C, a_i) was chosen closest to A, and so G_Q is a W-triangulation.

We divide the proof into two subcases, depending on whether $A \neq c_2$ or $A = c_2$. As the constructions are sufficiently simple, we refer the reader to Figures 5 and 6 here. A detailed description is in [2, Case 2(b)].

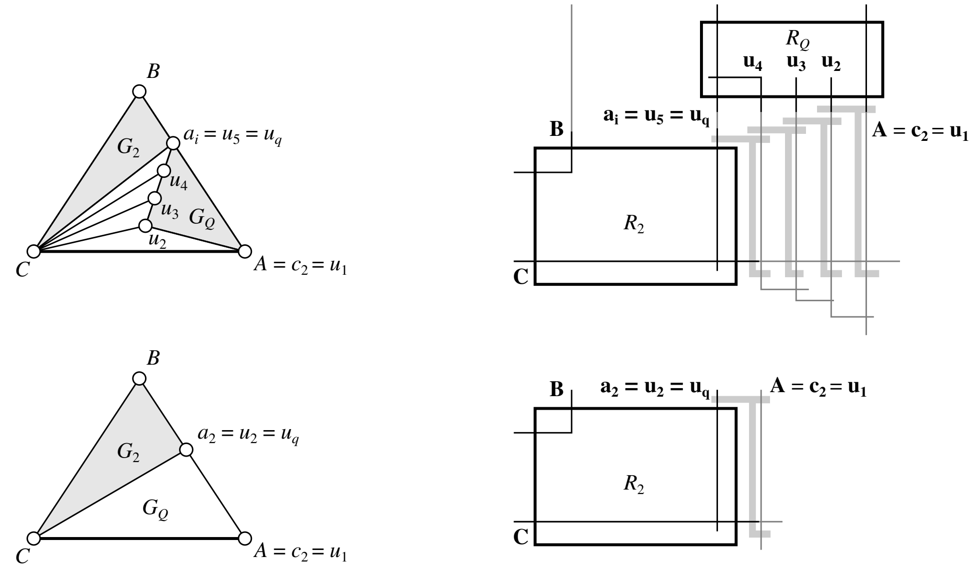

Figure 6 Case 2(b)2: Construction when C is incident to a chord, $c_2 = A$, and (A, a_i, C) is not a face (top), and when (A, a_i, C) is a face (bottom). We only show the 3-sided constructions.

3.3 G has no chords incident with C and $deg(C) \geq 3$

We will give explicit constructions for 2-sided, 3-sided and reverse 3-sided layout, and may hence (after applying the reversal trick) assume that the special edge, if it exists, is (C, c_2).

Let $u_1, \dots, u_q$ be the neighbours of vertex C in clockwise order, starting with b_{s-1} and ending with c_2. We know that $q = deg(C) \geq 3$ and that $u_2, \dots, u_{q-1}$ are not on the outer-face, since C is not incident to a chord. Let u_j be a neighbour of C that has at least one neighbour other than C on P_{CA}, and among all those, choose j to be minimal. Such a j exists because G is triangulated and therefore u_{q-1} is adjacent to both C and u_q. We distinguish two sub-cases.

Case 3(a): $j \neq 1$. Denote the neighbours of u_j on $P_{c_2 A}$ by $t_1, \dots, t_x$ in the order in which they appear on $P_{c_2 A}$. Separate G into subgraphs as follows (see also Figure 7):

- The *right* graph G_R is bounded by $(A, \overset{P_{AB}}{\dots}, B, \overset{P_{Bu_1}}{\dots}, u_1, u_2, \dots, u_j, t_x, \overset{P_{t_x A}}{\dots}, A)$.
- Let G_B be the graph bounded by $(u_j, t_1, \overset{P_{t_1 t_x}}{\dots}, t_x, u_j)$. We are chiefly interested in its subgraph $G_Q := G_B - u_j$.
- Let G_L be the graph bounded by $(C, \overset{P_{Ct_1}}{\dots}, t_1, u_j, C)$. We are chiefly interested in its subgraph $G_0 := G_L - \{u_j, C\}$.

The idea is to obtain representations of these subgraphs and then to combine them suitably. We first explain how to obtain the representation R_R used for G_R. Clearly G_R is a W-triangulation, since $u_2, \dots, u_j$ are interior vertices of G, and hence the outer-face of G_R is a simple cycle. Set $A_R := A$ and $B_R := B$. If $B \neq u_1$ then set $C_R := u_1$ and observe that G_R satisfies the chord condition with respect to these corners:

- G_R does not have any chords with both ends on $P_{A_R B_R} = P_{AB}$, $P_{B_R u_1} \subseteq P_{BC}$, or $P_{t_x A_R} \subseteq P_{CA}$ since G satisfies the chord condition.
- If there were any chords between a vertex in $u_1, \dots, u_j$ and a vertex on $P_{C_R A_R}$, then by $C_R = u_1$ the chord would either connect two neighbours of C (hence give a separating triangle of G), or connect some u_i for $i < j$ to P_{CA} (contradicting the minimality of j),

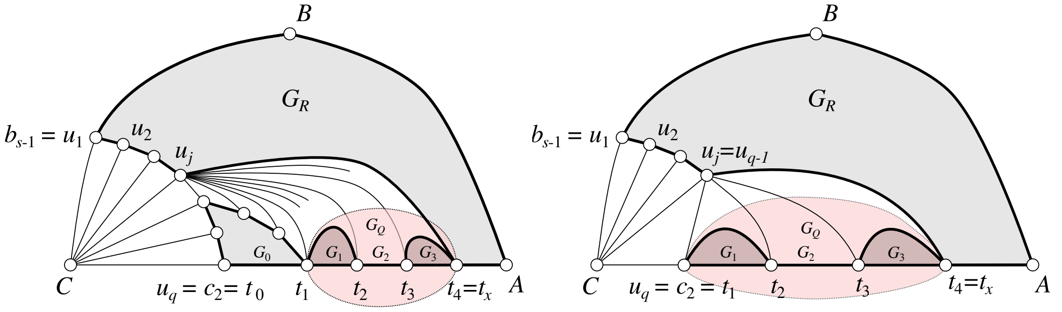

Figure 7 Case 3: Splitting the graph when $deg(C) \geq 3$ and no chord is incident to C. (Left) $j < q-1$; G_0 is non-trivial. (Right) $j = q-1$; $G_0 = \{c_2\}$.

or connect u_j to some other vertex on $P_{t_x A}$ (contradicting that t_x is the last neighbour of u_j on P_{CA}). Hence no such chord can exist either.

If $B = u_1$, then set $C_R := u_2$ (which exists by $q \geq 3$) and similarly verify that it satisfies the chord condition as $P_{B_R C_R}$ is the edge (B, u_2). Since $C_R \in \{u_1, u_2\}$ in both cases, we can apply induction on G_R and obtain an $(int \cup (u_1, u_2))$ representation R_R. We use as layout for R_R the type that we want for G, i.e., use a 2-sided/3-sided/reverse 3-sided layout if we want G to have a 2-sided/3-sided/reverse 3-sided representation.

Next consider the graph G_0, which is bounded by $u_{j+1}, \ldots, u_q, P_{c_2 t_1}$ and the neighbours of u_j in CCW order between t_1 and u_{j+1}. We distinguish two cases:

(1) $j = q-1$, and hence $t_1 = u_q = c_2$ and G_0 consists of only c_2. In this case, the representation of R_0 consists of a single vertical line segment $\mathbf{c_2}$.

(2) $j < q-1$, so G_0 contains at least three vertices u_{q-1}, u_q and t_1. Then G_0 is a W-triangulation since C is not incident to a chord and by the choice of t_1. Also, it satisfies the chord condition with respect to corners $A_0 := c_2, B_0 := t_1$ and $C_0 := u_{j+1}$ since the three paths on its outer-face are sub-paths of P_{CA} or contained in the neighbourhood of C or u_j. In this case, construct a 2-sided $(int \cup (u_{j+1}, u_{j+2}))$ representation R_0 of G_0 with respect to these corners inductively.

Finally, we create a representation R_Q of $G_Q = G_B - u_j$. If G_Q is a single vertex or a single edge, then simply use vertical segments for the curves of its vertices. Otherwise, we can show:

▶ **Claim 11.** *G_Q has a 2-sided $(int \cup \emptyset)$ 1-string B_2-VPG representation with respect to corners t_1 and t_x.*

Proof. G_Q is not necessarily 2-connected, so we cannot apply induction directly. Instead we break it into $x-1$ graphs $G_1, \ldots, G_{x-1}$, where for $i = 1, \ldots, x-1$ graph G_i is bounded by $P_{t_i t_{i+1}}$ as well as the neighbours of u_j between t_i and t_{i+1} in CCW order. Note that G_i is either a single edge, or it is bounded by a simple cycle since u_j has no neighbours on P_{CA} between t_i and t_{i+1}. In the latter case, use $B_i := t_i$, $A_i := t_{i+1}$, and C_i an arbitrary third vertex on $P_{t_i t_{i+1}} \subseteq P_{CA}$, which exists since the outer-face of G_i is a simple cycle and (t_i, t_{i+1}, u_j) is not a separating triangle. Observe that G_i satisfies the chord condition since all paths on the outer-face of G_i are either part of P_{CA} or in the neighbourhood of u_j. Hence by induction there exists a 2-sided $(int \cup \emptyset)$ representation R_i of G_i. If G_i is a single edge (t_i, t_{i+1}), then let R_i consists of two vertical segments $\mathbf{t_i}$ and $\mathbf{t_{i+1}}$.

Since each representation R_i has at its leftmost end a vertical segment $\mathbf{t_i}$ and at its rightmost end a vertical segment $\mathbf{t_{i+1}}$, we can combine all these representations by aligning

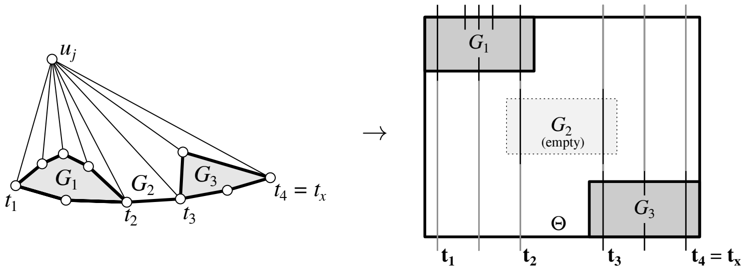

Figure 8 Left: Graph G_B. The boundary of G_Q is shown bold. Right: Merging 2-sided $(int \cup \emptyset)$ representations of $G_i, 1 \leq i \leq 3$, into a 2-sided $(int \cup \emptyset)$ representation of G_Q.

$\mathbf{t_i}^{R_i}$ and $\mathbf{t_i}^{R_{i+1}}$ horizontally and filling in the missing segment. See also Figure 8. One easily verifies that the result is a 2-sided $(int \cup \emptyset)$ representation of G_Q. ◀

We now explain how to combine these three representations R_R, R_Q and R_0; see also Figure 9. Translate R_Q so that it is below R_R with $\mathbf{t_x}^{R_R}$ and $\mathbf{t_x}^{R_Q}$ in the same column; then connect these two curves with a vertical segment. Rotate R_0 by 180° and translate it so that it is below R_R and to the left and above R_Q, and $\mathbf{t_1}^{R_0}$ and $\mathbf{t_1}^{R_Q}$ are in the same column; then connect these two curves with a vertical segment. Notice that the vertical segments of $\mathbf{u_2}^{R_R}, \ldots, \mathbf{u_j}^{R_R}$ are at the bottom left of R_R. Horizontally stretch R_0 and/or R_R so that $\mathbf{u_2}^{R_R}, \ldots, \mathbf{u_j}^{R_R}$ are to the left of the vertical segment of $\mathbf{u_{j+1}}^{R_0}$, but to the right (if $j < q-1$) of the vertical segment of $\mathbf{u_{j+2}}^{R_0}$. There are such segments by $j > 1$.

Introduce a new horizontal segment $\mathbf{C}$ and place it so that it intersects curves $\mathbf{u_q}, \ldots, \mathbf{u_{j+2}}$, $\mathbf{u_2}, \ldots, \mathbf{u_j}, \mathbf{u_{j+1}}$ (after lengthening them, if needed). For a 2-sided layout also attach a vertical segment to $\mathbf{C}$. If $j < q-1$ then top-tangle $\mathbf{u_q}, \ldots, \mathbf{u_{j+2}}$ leftwards (see [2, Section 2.2] for a precise definition of this operation). Bottom-tangle $\mathbf{u_2}, \ldots, \mathbf{u_j}$ rightwards. The construction hence creates intersections for all edges in the path $u_1, \ldots, u_q$, except for (u_{j+2}, u_{j+1}) (which was represented in R_0) and (u_2, u_1) (which was represented in R_R).

Bend and stretch $\mathbf{u_j}^{R_R}$ rightwards so that it crosses the curves of all its neighbours in $G_0 \cup G_Q$. Finally, consider the path between the neighbours of u_j CCW from u_{j+1} to t_x. Create intersections for any edge on this path that is interior in G by top-tangling their curves rightwards.

One verifies that the curves intersect the bounding boxes as desired. The constructed representations contain private regions for all interior faces of G_R, G_Q and G_0 by induction. The remaining faces are of the form $(C, u_i, u_{i+1}), 1 \leq i < q$, and (u_j, w_k, w_{k+1}) where w_k and w_{k+1} are two consecutive neighbours of u_j on the outer-face of G_0 or G_Q. Private regions for those faces are shown in Figure 9.

Case 3(b): $j = 1$, i.e., there exists a chord (b_{s-1}, c_i). In this case we cannot use the above construction directly since $\mathbf{b_{s-1}}$ ends on the left (in the 3-sided construction) while we need $\mathbf{u_j}$ to end at the bottom and not to be on the outer-face. However, if we use a different vertex as u_j (and argue carefully that the chord condition then holds), then the same construction works.

Recall that $u_1, \ldots, u_q$ are the neighbours of corner C in CW order starting with b_{s-1} and ending with c_2. We know that $q \geq 3$ and $u_2, \ldots, u_{q-1}$ are not on the outer-face. Now define j' as follows: Let $u_{j'}$, $j' > 1$ be a neighbour of C that has at least one neighbour on P_{CA} other than C, and choose $u_{j'}$ so that j' is minimal while satisfying $j' > 1$. Such a j' exists since u_{q-1} has another neighbour on P_{CA}, and by $q \geq 3$ we have $q - 1 > 1$. Now,

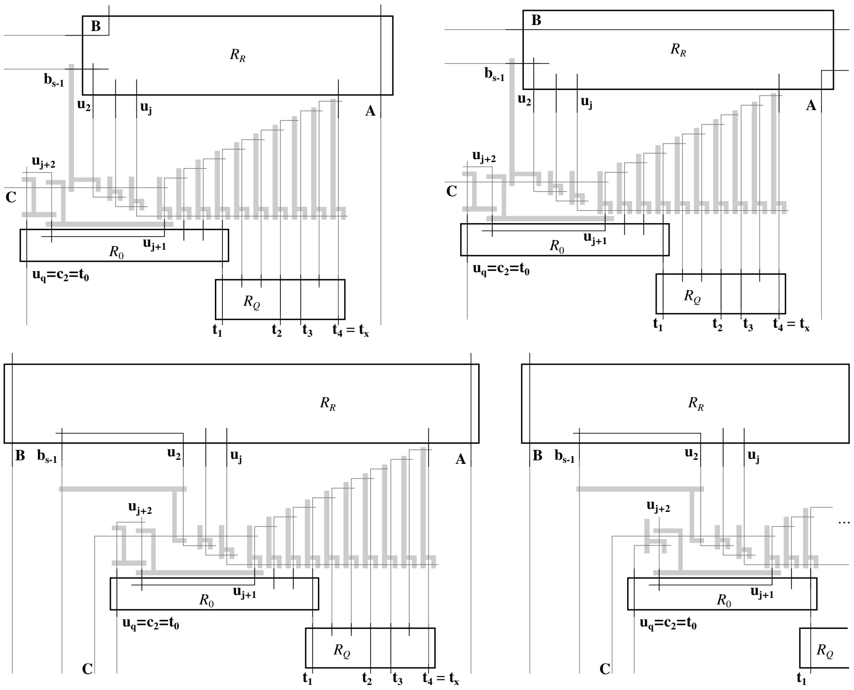

Figure 9 Case 3: Combining subgraphs when $deg(C) \geq 3$, there is no chord incident with C, and $F \subseteq \{(C, c_2)\}$. (Top left) 3-sided and (top right) reverse 3-sided construction. (Bottom) 2-sided construction for the case $F = \{(C, c_2)\}$ and $F = \emptyset$. The construction matches the graph depicted in Figure 7 left.

separate G as in the previous case, except use j' in place of j. Thus, define $t_1, \ldots, t_x$ to be the neighbours of $u_{j'}$ on P_{c_2A}, in order, and separate G into three graphs as follows:

- The *right* graph G_R is bounded by $(A, \overset{P_{AB}}{\ldots}, B, \overset{P_{Bu_1}}{\ldots}, u_1, u_2, \ldots, u_{j'}, t_x, \overset{P_{t_xA}}{\ldots}, A)$.
- Let G_B be the graph bounded by $(u_{j'}, t_1, \overset{P_{t_1t_x}}{\ldots}, t_x, u_{j'})$. Define $G_Q := G_B - u_{j'}$.
- Let G_L be the graph bounded by $(C, \overset{P_{Ct_1}}{\ldots}, t_1, u_{j'}, C)$. Define $G_0 := G_L - \{u_{j'}, C\}$.

Observe that the boundaries of all the graphs are simple cycles, and thus they are W-triangulations. Select $(A_R := A, B_R := B, C_R := u_2)$ to be the corners of G_R and argue the chord condition as follows:

- G_R does not have any chords on $P_{C_RA_R}$ as such chords would either contradict minimality of j', or violate the chord condition in G.
- G_R does not have any chords of $P_{A_RB_R} = P_{AB}$.
- G_R does not have any chords on $P_{Bb_{s-1}}$ as it is a sub-path of P_{BC} and they would violate the chord condition in G. It also does not have any chords in the form $(C_R = u_2, b_\ell), 1 \leq \ell < s-1$ as they would have to intersect the chord (b_{s-1}, c_i), violating the planarity of G. Hence, G_R does not have any chords on $P_{C_RA_R}$.
- Notice in particular that the chord (u_1, c_i) of G_R is *not* a violation of the chord condition since we chose u_2 as a corner.

Hence, we can obtain a representation R_R of G_R with 2-sided, 3-sided and reverse 3-sided layout and special edge $(u_1 = b_{s-1}, u_2)$. For graphs G_Q and G_0 the corners are chosen, the chord condition is verified, and the representations are obtained exactly as in Case 3a. Since the special edge of G_R is (u_1, u_2) as before, curves $\mathbf{u_1}$ and $\mathbf{u_2}$ are situated precisely as in Case 3a, and we merge representations and find private regions as before.

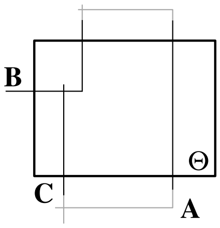

Figure 10 Completing a 3-sided $(int \cup (B,C))$ representation by adding intersections for (A,B) and (A,C).

This ends the description of the construction in all cases, and hence proves Lemma 9 and Lemma 10.

4 From 4-Connected Triangulations to All Planar Graphs

In this section, we prove Theorem 1. Observe that Lemma 9 essentially proves it for 4-connected triangulations. As in [5] we extend it to all triangulations by induction on the number of separating triangles.

▶ **Theorem 12.** *Let G be a triangulation with outer-face (A,B,C). G has a 1-string B_2-VPG representation with a chair-shaped private region for every interior face f of G.*

Proof. Our approach is exactly the same as in [5], except that we must be careful not to add too many bends when merging subgraphs at separating triangles, and hence must use 3-sided layouts. Formally, we proceed by induction on the number of separating triangles. In the base case, G has no separating triangle, i.e., it is 4-connected. As the outer-face is a triangle, G clearly satisfies the chord condition. Thus, by Lemma 9, it has a 3-sided $(int \cup (B,C))$ representation R with private region for every face. R has an intersection for every edge except for (A,B) and (A,C). These intersection can be created by tangling $\mathbf{B},\mathbf{A}$ and $\mathbf{C},\mathbf{A}$ suitably (see Figure 10). Recall that $\mathbf{A}$ initially did not have any bends, so it has 2 bends in the constructed representation of G. The existence of private regions is guaranteed by Lemma 9.

Now assume for induction that G has $k+1$ separating triangles. Let $\Delta = (a,b,c)$ be an inclusion-wise minimal separating triangle of G. It was shown in [5] that the subgraph G_2 induced by the vertices inside Δ is either an isolated vertex, or a W-triangulation (A,B,C) such that the vertices on P_{AB} are adjacent to b, the vertices on P_{BC} are adjacent to c, and the vertices on P_{CA} are adjacent to a. Furthermore, G_2 satisfies the chord condition. Also, graph $G_1 = G - G_2$ is a W-triangulation that satisfies the chord condition and has k separating triangles. By induction, G_1 has a representation R_1 with a chair-shaped private region for every interior face f. Let Φ be the region for face Δ. Permute a,b,c, if needed, so that the naming corresponds to the one needed for the private region.

Case 1: G_2 is a single vertex v. Represent v by inserting into Φ an orthogonal curve $\mathbf{v}$ with 2 bends that intersects $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$. The construction, together with private regions for the newly created faces (a,b,v), (a,c,v) and (b,c,v), is shown in Figure 11 (left).

Case 2: G_2 is a W-triangulation. Recall that G_2 satisfies the chord condition with respect to corners (A,B,C). Apply Lemma 9 to construct a 3-sided $(int \cup (C, b_{s-1}))$ representation R_2 of G_2. Let us assume that (after possible rotation) Φ has the orientation shown in

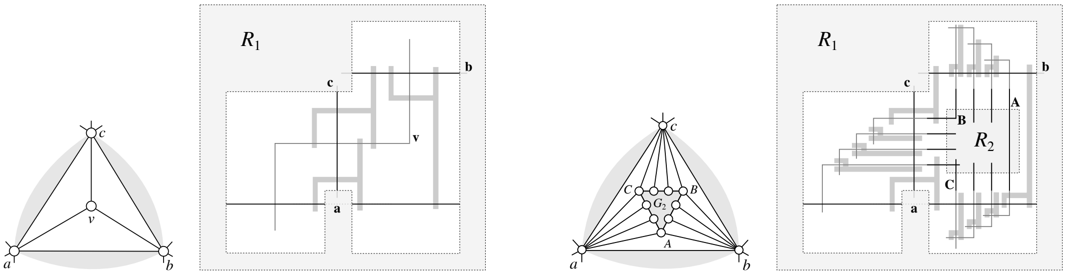

Figure 11 Separating triangle with one vertex and the construction (left), and separating triangle enclosing a W-triangulation and the construction (right).

Figure 11 (right); if it had the symmetric orientation then we would do a similar construction using a reverse 3-sided representation of G_2. Place R_2 inside Φ as shown in Figure 11 (right). Stretch the curves representing vertices on P_{CA}, P_{AB} and $P_{Bb_{s-1}}$ downwards, upwards and leftwards respectively so that they intersect $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Top-tangle leftwards the curves $\mathbf{A} = \mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_r} = \mathbf{B}$. Left-tangle downwards the curves $\mathbf{B} = \mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_{s-1}}$ and bend and stretch $\mathbf{C}$ downwards so that it intersects $\mathbf{a}$. Bottom-tangle leftwards the curves $\mathbf{C} = \mathbf{c_1}, \ldots, \mathbf{c_t} = \mathbf{A}$. It is easy to verify that the construction creates intersection for all the edges between vertices of Δ and the outer-face of G_2. The tangling operation then creates intersections for all the outer-face edges of G_2 except edge (C, b_{s-1}), which is already represented in R_2.

Every curve that receives a new bend represents a vertex on the outer-face of G_2, which means that it initially had at most 1 bend. Curve $\mathbf{A}$ is the only curve that receives 2 new bends, but this is allowed as $\mathbf{A}$ does not have any bends in R_2. Hence, the number of bends for every curve does not exceed 2.

Private regions for faces formed by vertices a, b, c and vertices on the outer-face of G_2 can be found as shown in Figure 11 right. ◀

With Theorem 12 in hand, we can show our main result: every planar graph has a 1-string B_2-VPG representation.

Proof of Theorem 1. If G is a planar triangulated graph, the claim holds by Theorem 12. So, assume that G is a planar graph. Then *stellate* the graph, i.e., insert a vertex into each non-triangulated face and connect it to all vertices on that face. It is well known that after at most 3 repetitions, the construction produces a 3-connected triangulated graph G' such that G is an induced subgraph of G'. Apply Theorem 12 to construct a 1-string B_2-VPG representation R' of G'. By removing curves representing vertices that are not in G, we obtain a 1-string B_2-VPG representation of G. ◀

5 Conclusions and Outlook

We showed that every planar graph has a 1-string B_2-VPG representation, i.e., a representation as an intersection graph of strings where strings cross at most once and each string is orthogonal with at most two bends. One advantage of this is that the coordinates to describe such a representation are small, since orthogonal drawings can be deformed easily such that all bends are at integer coordinates. Every vertex curve has at most two bends and hence at most 3 segments, so the representation can be made to have coordinates in an $O(n) \times O(n)$-grid with perimeter at most $3n$. Note that none of the previous results provided an intuition of the required size of the grid.

Following the steps of our proof, it is not hard to see that our representation can be found in linear time, since the only non-local operation is to test whether a vertex has a neighbour on the outer-face. This can be tested by marking such neighbours whenever they become part of the outer-face. Since no vertex ever is removed from the outer-face, this takes overall linear time.

The representation constructed in this paper uses curves of 8 possible shapes for planar graphs. One can in fact verify that the 2-sided layout (which only uses 2-sided layouts in its recursions) uses only 4 possible shapes: C, Z and their horizontal mirror images. Hence for triangulations without separating triangles (and, after stellating, all 4-connected planar graphs) 4 shapes suffice. A natural question is if one can restrict the number of shapes required to represent all planar graphs.

Bringing this effort further, is it possible to restrict the curves even more? Felsner et al. [10] asked the question whether every planar graph is the intersection graph of only two shapes, namely $\{L, \Gamma\}$. As they point out, a positive result would provide a different proof of Scheinerman's conjecture. Somewhat inbetween: is every planar graph the intersection graph of xy-monotone orthogonal curves, preferably in the 1-string model and with few bends?

References

1 Takao Asano, Shunji Kikuchi, and Nobuji Saito. A linear algorithm for finding Hamiltonian cycles in 4-connected maximal planar graphs. *Discr. Applied Mathematics*, 7(1):1 – 15, 1984.

2 Therese C. Biedl and Martin Derka. 1-string B_2-VPG representation of planar graphs. *CoRR*, abs/1411.7277, 2014.

3 Jérémie Chalopin and Daniel Gonçalves. Every planar graph is the intersection graph of segments in the plane: extended abstract. In *ACM Symposium on Theory of Computing, STOC 2009*, pages 631–638. ACM, 2009.

4 Jérémie Chalopin, Daniel Gonçalves, and Pascal Ochem. Planar graphs are in 1-string. In *ACM-SIAM Symposium on Discrete Algorithms*, SODA '07, pages 609–617. SIAM, 2007.

5 Jérémie Chalopin, Daniel Gonçalves, and Pascal Ochem. Planar graphs have 1-string representations. *Discrete & Computational Geometry*, 43(3):626–647, 2010.

6 Steven Chaplick and Torsten Ueckerdt. Planar graphs as VPG-graphs. *J. Graph Algorithms Appl.*, 17(4):475–494, 2013.

7 Natalia de Castro, Francisco Javier Cobos, Juan Carlos Dana, Alberto Márquez, and Marc Noy. Triangle-free planar graphs and segment intersection graphs. *J. Graph Algorithms Appl.*, 6(1):7–26, 2002.

8 Hubert de Fraysseix, Patrice Ossona de Mendez, and János Pach. Representation of planar graphs by segments. *Intuitive Geometry*, 63:109–117, 1991.

9 Gideon Ehrlich, Shimon Even, and Robert Endre Tarjan. Intersection graphs of curves in the plane. *J. Comb. Theory, Ser. B*, 21(1):8–20, 1976.

10 Stefan Felsner, Kolja B. Knauer, George B. Mertzios, and Torsten Ueckerdt. Intersection graphs of L-shapes and segments in the plane. In *Mathematical Foundations of Computer Science (MFCS'14), Part II*, volume 8635 of *Lecture Notes in Computer Science*, pages 299–310. Springer, 2014.

11 Irith Ben-Arroyo Hartman, Ilan Newman, and Ran Ziv. On grid intersection graphs. *Discrete Mathematics*, 87(1):41–52, 1991.

12 Edward R. Scheinerman. *Intersection Classes and Multiple Intersection Parameters of Graphs.* PhD thesis, Princeton University, 1984.

13 Hassler Whitney. A theorem on graphs. *The Annals of Mathematics*, 32(2):387–390, 1931.