# Riemannian Simplices and Triangulations*

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## Abstract

We study a natural intrinsic definition of geometric simplices in Riemannian manifolds of arbitrary finite dimension, and exploit these simplices to obtain criteria for triangulating compact Riemannian manifolds. These geometric simplices are defined using Karcher means. Given a finite set of vertices in a convex set on the manifold, the point that minimises the weighted sum of squared distances to the vertices is the Karcher mean relative to the weights. Using barycentric coordinates as the weights, we obtain a smooth map from the standard Euclidean simplex to the manifold. A Riemannian simplex is defined as the image of the standard simplex under this barycentric coordinate map. In this work we articulate criteria that guarantee that the barycentric coordinate map is a smooth embedding. If it is not, we say the Riemannian simplex is degenerate. Quality measures for the "thickness" or "fatness" of Euclidean simplices can be adapted to apply to these Riemannian simplices. For manifolds of dimension 2, the simplex is non-degenerate if it has a positive quality measure, as in the Euclidean case. However, when the dimension is greater than two, non-degeneracy can be guaranteed only when the quality exceeds a positive bound that depends on the size of the simplex and local bounds on the absolute values of the sectional curvatures of the manifold. An analysis of the geometry of non-degenerate Riemannian simplices leads to conditions which guarantee that a simplicial complex is homeomorphic to the manifold.

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## 1 Introduction

The standard definition of a Euclidean simplex as the convex hull of its vertices is not useful for defining simplices in general Riemannian manifolds. Besides the problem that convex hulls are difficult to compute, the resulting objects could not be used as building blocks for triangulations: a minimising geodesic between two points on a shared facet would have to lie within the facet, which is not a realisable constraint in general. A more detailed discussion and references can be found in the full version [9] of this work.

Given the vertices, a geometric Euclidean simplex can also be defined as the domain on which the barycentric coordinate functions are non-negative. This definition *does* extend to general Riemannian manifolds in a natural way. The construction is based on the fact that the barycentric coordinate functions can be defined by a "centre of mass" construction.

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Suppose $\{v_0, \dots, v_n\} \subset \mathbb{R}^n$, and $(\lambda_i)_{0 \le i \le n}$ is a set of non-negative weights that sum to 1. If $u$ is the unique point that minimises the function

$$y \mapsto \sum_{i=0}^{n} \lambda_i d_{\mathbb{R}^n}(y, v_i)^2, \tag{1}$$

where $d_{\mathbb{R}^n}(x, y) = |x - y|$ is the Euclidean distance, then $u = \sum \lambda_i v_i$, and if the $v_i$ are affinely independent, then the $\lambda_i$ are the barycentric coordinates of $u$ in the simplex $[v_0, \dots, v_n]$.

We can view a given set of barycentric coordinates $\lambda = (\lambda_0, \dots, \lambda_n)$ as a point in $\mathbb{R}^{n+1}$. The set $\Delta^n$ of all points in $\mathbb{R}^{n+1}$ with non-negative coefficients that sum to 1 is called the *standard Euclidean n-simplex*. Thus the minimisation of the function (1) defines a map from the standard Euclidean simplex to the Euclidean simplex $[v_0, \dots, v_n] \subset \mathbb{R}^n$.

If instead the points $\{v_i\}$ lie in a sufficiently small neighbourhood $W$ in a Riemannian manifold $M$, then, by using the metric of the manifold instead of $d_{\mathbb{R}^n}$ in Equation (1), we obtain a function $\mathcal{E}_\lambda : W \to \mathbb{R}$ that has a unique minimum $x_\lambda \in W$. In this way we obtain a mapping $\lambda \mapsto x_\lambda$ from $\Delta^n$ to $W$. We call the image of this map an *intrinsic simplex*, or a *Riemannian simplex*.

Karcher [10] studied such centre of mass constructions extensively in the Riemannian setting, and this kind of averaging technique is often called “Karcher means”. More recently, Rustamov [13] introduced barycentric coordinates on a surface via Karcher means. Sander [14] used the method in arbitrary dimensions to define Riemannian simplices as described above. We are not aware of any published work exploiting this notion of Riemannian simplices prior to that of Rustamov [13] and Sander [14], although the idea was known much earlier [1, § 6.1.5].

Our work is motivated by a desire to develop general sampling density criteria for triangulations of manifolds. To this end we need to establish a property that Sander did not consider. We need to ensure that the map from the Euclidean simplex to the manifold is a smooth embedding. This ensures that the barycentric coordinates mapped to the manifold do in fact provide a local system of coordinates. If the map is not an embedding, we call the Riemannian simplex *degenerate*. Independently, von Deylen [16] has also treated the question of degeneracy of Riemannian simplices. His work includes a detailed analysis of the geometry of the barycentric coordinate map, and several applications. He does not address the problem of sampling density criteria for triangulation.

A Euclidean simplex is non-degenerate if and only if its vertices are affinely independent. We show that a Riemannian simplex is non-degenerate if and only if, for every point in the simplex, the vertices are affinely independent when they are lifted by the inverse of the exponential map to the tangent space at that point.

In a two dimensional manifold this condition is satisfied for a triangle as long as the vertices do not lie on a common geodesic. Similar to the Euclidean case, such a configuration can be avoided by applying an arbitrarily small perturbation to the vertices. However, when the dimension is greater than two, a non-trivial constraint on simplex quality is required; one that cannot be attained by an arbitrarily small perturbation of the vertices.

In order to define a Riemannian simplex, we need the vertices to lie in a geodesically convex set, and this imposes a bound on the edge lengths with respect to an upper bound on the local sectional curvatures. For a surface, this is the only real constraint needed to ensure a non-degenerate simplex. In higher dimensions, we require the simplex size to be constrained also by a lower bound on the sectional curvatures.

## Outline and main results

In Section 2 we present the framework for centre of mass constructions, and introduce the barycentric coordinate map and Riemannian simplices. Riemannian simplices are defined (Definition 2) as the image of the barycentric coordinate map, so they are "filled in" geometric simplices. Each of the three subsequent sections is devoted to presenting one of our three main results: conditions for non-degeneracy of Riemannian simplices, Theorem 6; conditions for triangulation, Theorem 11; and the geometric fidelity of the resulting triangulation, Theorem 14.

In Section 3 we establish criteria to ensure that a Riemannian simplex is non-degenerate. In the tangent space at any point in a Riemannian simplex $\boldsymbol{\sigma}_M$, there is a Euclidean simplex $\sigma(x)$ that is a natural approximation of $\boldsymbol{\sigma}_M$. We give a characterisation of non-degeneracy of $\boldsymbol{\sigma}_M$ in terms of these Euclidean simplices: $\boldsymbol{\sigma}_M$ is non-degenerate if and only if $\sigma(x)$ is a non-degenerate Euclidean simplex for every $x \in \boldsymbol{\sigma}_M$ (Proposition 4).

The *thickness* of a Euclidean simplex, defined in Section 3.1, is a measure of its quality, i.e., how far it is from being degenerate. We choose a representative $\sigma(p)$ for some $p \in \boldsymbol{\sigma}_M$ and observe that all the $\sigma(x)$ are geometrically small perturbations of $\sigma(p)$. We then exploit previous results on the stability of Euclidean simplex quality [2, Lemma 8] to establish a simple inequality, relating the thickness of $\sigma(p)$ to the edge lengths of $\boldsymbol{\sigma}_M$ and a bound on the absolute value of the local sectional curvatures, which when satisfied guarantees that all the $\sigma(x)$ are non-degenerate. It then follows, from the above-mentioned Proposition 4, that $\boldsymbol{\sigma}_M$ is non-degenerate, and this is Theorem 6.

In Section 4 we develop our criteria for triangulating manifolds. A *triangulation* of a manifold $M$ is a homeomorphism $H\colon |\mathcal{A}| \to M$, where $\mathcal{A}$ is an abstract simplicial complex, and $|\mathcal{A}|$ is its carrier (topological realisation).[1] We establish properties of maps whose differentials are all small perturbations of a fixed linear isometry, and use these properties to reveal conditions under which the star of a vertex in a manifold complex will be embedded into $M$. This allows us to express, in Proposition 8, generic conditions that ensure that a simplicial complex is homeomorphic to $M$. We then demonstrate that the differential of the barycentric coordinate map can be bounded as required by Proposition 8, and thus arrive at our triangulation criteria expressed in Theorem 11.

The triangulation $H\colon |\mathcal{A}| \to M$ is defined by the barycentric coordinate map on each of the simplices. The quantative aspect of the triangulation criteria is expressed in terms of a scale parameter $h$ which bounds the edge lengths of the Riemannian simplices defined by the triangulation. This bound on $h$ is of the same character as the non-degeneracy criteria: it depends on a *thickness bound* $t_0$ governing the quality of the simplices involved, and also on a bound on the absolute value of the sectional curvatures.

The complex $\mathcal{A}$ in Theorem 11 naturally admits a piecewise linear metric by assigning edge lengths to the simplices given by the geodesic distance in $M$ between the endpoints. In Section 5 we observe that in order to ensure that this does in fact define a piecewise-flat metric, we need to employ slightly stronger constraints on the scale parameter $h$. In this case, the complex $\mathcal{A}$ becomes a good geometric approximation of the original manifold, as expressed in Theorem 14, which states that the metric distortion of $H$ is proportional to $h^2$.

[1] In fact the triangulations of interest to us have the property that the restriction of $H$ to each simplex in $|\mathcal{A}|$ is a smooth embedding, and also the star of each simplex admits a piecewise linear embedding into $\mathbb{R}^n$. These additional properties ensure that $\mathcal{A}$ represents the unique piecewise linear structure associated with $M$. See Thurston [15, Thm 3.10.2], or Munkres [12, Cor. 10.13] for details.

## 2 Riemannian simplices

In this section we summarise the essential properties of Karcher means, and define Riemannian simplices. We work with an $n$-dimensional Riemannian manifold $M$. The centre of mass construction developed by Karcher [10] hinges on the notion of convexity in a Riemannian manifold. A set $B \subseteq M$ is *convex* if any two points $x, y \in B$ are connected by a minimising geodesic $\gamma_{xy}$ that is unique in $M$, and contained in $B$. For $c \in M$, the geodesic ball of radius $r$ centred at $c$ is the set $B_M(c; r)$ of points in $M$ whose distance from $c$ is less than $r$, and we denote its closure by $\overline{B}_M(c; r)$. If $r$ is smaller than $\rho_0$, defined below (4), then $\overline{B}_M(c; r)$ will be convex [5, §6.4].

Recall that the exponential map at $p \in M$ sends a vector $v$ in the tangent space $T_pM$ to the point $\exp_p(v)$ defined by the geodesic of length $|v|$ emanating from $p$ in the direction $v$. The exponential map is a diffeomorphism when restricted to a ball whose radius is smaller than the *injectivity radius*.

In our context, we are interested in finding a weighted centre of mass of a finite set $\{p_0, \dots p_j\} \subset B \subset M$, where the containing set $B$ is open, and its closure $\overline{B}$ is convex. The centre of mass construction is based on minimising the function $\mathcal{E}_\lambda : \overline{B} \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, p_i)^2, \tag{2}$$

where the $\lambda_i \geq 0$ are non-negative weights that sum to 1, and $d_M$ is the geodesic distance function on $M$. Karcher's first simple observation is that the minima of $\mathcal{E}_\lambda$ must lie in the interior of $\overline{B}$, i.e., in $B$ itself. This follows from considering the gradient of $\mathcal{E}_\lambda$:

$$\operatorname{grad} \mathcal{E}_\lambda(x) = -\sum_i \lambda_i \exp_x^{-1}(p_i). \tag{3}$$

At any point $x$ on the boundary of $\overline{B}$, the gradient vector lies in a cone of outward pointing vectors. It follows that the minima of $\mathcal{E}_\lambda$ lie in $B$. The more difficult result that the minimum is unique, Karcher showed by demonstrating that $\mathcal{E}_\lambda$ is convex. If $B \subseteq M$ is a convex set, a function $f : B \to \mathbb{R}$ is *convex* if for any geodesic $\gamma : I \to B$, the function $f \circ \gamma$ is convex. If $f$ has a minimum in $B$, it must be unique. By Equation (3), it is the point $x$ where

$$\sum_i \lambda_i \exp_x^{-1}(p_i) = 0.$$

Our results will require a bound $\Lambda$ on the absolute value of the sectional curvatures in $M$. However, the definition of Riemannian simplices only requires an upper bound on the sectional curvatures, which we denote by $\Lambda_+$. We denote the injectivity radius of $M$ by $\iota$. We have the following result [10, Thm 1.2]:

▶ **Lemma 1** (Unique centre of mass). *If $\{p_0, \dots, p_j\} \subset B_\rho \subset M$, and $B_\rho$ is an open ball of radius $\rho$ with*

$$\rho < \rho_0 = \min\left\{\frac{\iota}{2}, \frac{\pi}{4\sqrt{\Lambda_+}}\right\} \tag{4}$$

*(if $\Lambda_+ \leq 0$ we take $1/\sqrt{\Lambda_+}$ to be infinite), then on any geodesic $\gamma : I \to B_\rho$, we have*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}_\lambda(\gamma(t)) \geq C(\Lambda_+, \rho) > 0, \tag{5}$$

*where $C(\Lambda_+, \rho)$ is a positive constant depending only on $\Lambda_+$ and $\rho$. In particular, $\mathcal{E}_\lambda$ is convex and has a unique minimum in $B_\rho$, characterised by the vanishing of the gradient* (3)*.*

▶ **Definition 2** (Riemannian simplex)**.** If a finite set $\sigma^j = \{p_0, \ldots, p_j\} \subset M$ in an $n$-manifold is contained in an open geodesic ball $B_\rho$ whose radius, $\rho$, satisfies Equation (4), then $\sigma^j$ is the set of vertices of a geometric *Riemannian simplex*, denoted $\boldsymbol{\sigma}^j_M$, and defined to be the image of the map

$$\begin{aligned} \mathcal{B}_{\sigma^j} :& \Delta^j \to M \\ & \lambda \mapsto \underset{x \in \overline{B}_\rho}{\operatorname{argmin}} \, \mathcal{E}_\lambda(x). \end{aligned}$$

We say that $\boldsymbol{\sigma}^j_M$ is *non-degenerate* if $\mathcal{B}_{\sigma^j}$ is a smooth embedding; otherwise it is *degenerate*.

▶ Remark. Lemma 1 demands that a Riemannian simplex be contained in a ball whose radius is constrained by $\rho_0$. Thus Riemannian simplices always have edge lengths less than $2\rho_0$. If the longest edge length, $L(\boldsymbol{\sigma}_M)$, of $\boldsymbol{\sigma}_M$ is less than $\rho_0$, then $\boldsymbol{\sigma}_M$ must be contained in the closed ball of radius $L(\boldsymbol{\sigma}_M)$ centred at a vertex. Indeed, any open ball centred at a vertex whose radius is larger than $L(\boldsymbol{\sigma}_M)$, but smaller than $\rho_0$, must contain the vertices and have a convex closure. The simplex is thus contained in the intersection of these balls. If $L(\boldsymbol{\sigma}_M) \geq \rho_0$, then a ball of radius $L(\boldsymbol{\sigma}_M)$ need not be convex. In this case we claim only that $\boldsymbol{\sigma}_M$ is contained in a ball of radius $2\rho_0$ centred at any vertex.

Define an $i$-face of $\boldsymbol{\sigma}^j_M$ to be the image of an $i$-face of $\Delta^j$. Since an $i$-face of $\Delta^j$ may be identified with $\Delta^i$ (e.g., by an order preserving map of the vertex indices), the $i$-faces of $\boldsymbol{\sigma}^j_M$ are themselves Riemannian $i$-simplices. In particular, if $\tau$ and $\mu$ are the vertices of Riemannian simplices $\boldsymbol{\tau}_M$ and $\boldsymbol{\mu}_M$, and $\sigma^i = \tau \cap \mu$, then the Riemannian $i$-simplex $\boldsymbol{\sigma}^i_M$ is a face of both $\boldsymbol{\tau}_M$ and $\boldsymbol{\mu}_M$. The *edges* of a Riemannian simplex are the Riemannian 1-faces. We observe that these are geodesic segments. We will focus on full dimensional simplices. Unless otherwise specified, $\boldsymbol{\sigma}_M$ will refer to a Riemannian simplex defined by a set $\sigma$ of $n+1$ vertices in our $n$-dimensional manifold $M$.

The barycentric coordinate map $\mathcal{B}_\sigma$ is differentiable. This follows from the implicit function theorem, as is shown by Buser and Karcher [5, §8.3.3], for example.

A Riemannian simplex is not convex in general, but by Karcher's observation it is contained in any open set that contains the vertices and has a convex closure. Thus the simplex is contained in the intersection of such sets.

Equation (4) gives an upper bound on the size of a Riemannian simplex that depends only on the injectivity radius and an *upper* bound on the sectional curvature. For example, in a non-positively curved manifold, the size of a well defined Riemannian simplex is constrained only by the injectivity radius. However, if the dimension $n$ of the manifold is greater than 2, we will require also a *lower* bound on the sectional curvatures in order to ensure that the simplex is non-degenerate.

## 3 Non-degeneracy criteria

In this section we establish geometric criteria that ensure that a Riemannian simplex is non-degenerate. We first review the properties of Euclidean simplices, including the thickness quality measure, which parameterises how far a simplex is from being degenerate. We observe that we can bound the change in the thickness of a simplex if the edge lengths are perturbed a small amount.

Next we examine the differential of the barycentric coordinate map, and arrive at a characterisation of non-degenerate Riemannian simplices in terms of affine independence (Proposition 4). The Rauch comparison theorem is a central result in Riemannian geometry which allows us to bound the metric distortion of the exponential map. Combined with the stability of the thickness of Euclidean simplices, this bound on the metric distortion yields conditions which ensure that a Riemannian simplex meets the affine independence characterisation of non-degeneracy, resulting in Theorem 6.

## 3.1 The stability of Euclidean simplex quality

A Euclidean simplex $\sigma$ of dimension $j$ is defined by a set of $j+1$ points in Euclidean space $\sigma = \{v_0, \dots, v_j\} \subset \mathbb{R}^n$. In general we work with abstract simplices, even though we attribute geometric properties to the simplex, inherited from the embedding of the vertices in the ambient space. When we wish to make the dimension explicit, we write it as a superscript, thus $\sigma^j$ is a $j$-simplex. Traditional "filled in" geometric simplices are denoted by boldface symbols; $\boldsymbol{\sigma}_{\mathbb{E}} = \mathrm{conv}(\sigma)$ is the convex hull of $\sigma$.

A Euclidean simplex $\sigma = \{v_0, \dots, v_j\} \subset \mathbb{R}^n$ has a number of geometric attributes. An $i$-face of $\sigma$ is a subset of $i+1$ vertices, and a $(j-1)$ face of a $j$-simplex is a *facet*. The facet of $\sigma$ that does not have $v_i$ as a vertex is denoted $\sigma_{v_i}$. The *altitude* of $v_i \in \sigma$ is the distance from $v_i$ to the affine hull of $\sigma_{v_i}$, denoted $a_{v_i}(\sigma)$. The longest edge length is denoted $L(\sigma)$. When there is no risk of confusion, we will omit explicit reference to the simplex, and ignore the distinction between the vertices and their labels. Thus we write $L$, and $a_i$ instead of $L(\sigma)$ and $a_{v_i}(\sigma)$.

The *thickness* of $\sigma^j$, defined as

$$t(\sigma^j) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{v \in \sigma^j} \frac{a_v}{jL} & \text{otherwise.} \end{cases}$$

If $t(\sigma^j) = 0$, then $\sigma^j$ is *degenerate.* We say that $\sigma^j$ is $t_0$-thick, if $t(\sigma^j) \geq t_0$. If $\sigma^j$ is $t_0$-thick, then so are all of its faces. We write $t$ for the thickness if the simplex in question is clear.

The *barycentric coordinate functions* $\lambda_i$ associated to $\sigma^j$ are affine functions on the affine hull of the simplex $\lambda_i \colon \mathrm{aff}(\sigma^j) \to \mathbb{R}$ that satisfy $\lambda_i(v_j) = \delta_{ij}$ and $\sum_{i=0}^{j} \lambda_i = 1$. It is often convenient to choose one of the vertices, $v_0$ say, of $\sigma$ to be the origin. We let $P$ be the matrix whose $i^{\text{th}}$ column is $v_i - v_0$. Then the barycentric coordinate functions $\lambda_i$ are linear functions for $i > 0$, and they are dual to the basis defined by the columns of $P$. This means that if we represent the function $\lambda_i$ as a row vector, then the matrix $Q$ whose $i^{\text{th}}$ *row* is $\lambda_i$ satisfies $QP = I_{j \times j}$.

A full dimensional Euclidean simplex $\sigma$ is non-degenerate, if and only if the corresponding matrix $P$ is non-degenerate. In particular, if $\sigma$ is full dimensional (i.e., $j = n$), then $Q = P^{-1}$. Suppose $\sigma \subset \mathbb{R}^n$ is an $n$-simplex. If $\xi \in \mathbb{R}^n$, let $\lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))^{\mathsf{T}}$. Then $\lambda(\xi)$ is the vector of coefficients of $\xi - v_0$ in the basis defined by the columns of $P$. I.e., $\xi - v_0 = P\lambda(\xi)$.

The quality of a simplex $\sigma$ is closely related to the quality of $P$, which can be quantified by means of its *singular values*. In fact, we are only interested in the smallest and largest singular values. The smallest singular value, $s_k(P) = \inf_{|x|=1} |Px|$, vanishes if and only if the matrix $P$ does not have full rank. The largest singular value is the same as the operator norm of $P$, i.e., $s_1(P) = \|P\| = \sup_{|x|=1} |Px|$. The thickness of $\sigma$ provides a lower bound [3, Lem. 2.4] on the smallest singular value of $P$. Specifically, for a $j$-simplex, we have $s_j(P) \geq \sqrt{j}tL$.

The crucial property of thickness for our purposes is its stability. If two Euclidean simplices with corresponding vertices have edge lengths that are almost the same, then their thicknesses will be almost the same. This allows us to quantify a bound on the smallest singular value of the matrix associated with one of the simplices, given a bound on the other, as shown in the following Lemma [2, Lem. 8]:

▶ **Lemma 3** (Thickness under distortion). *Suppose that* $\sigma = \{v_0, \ldots, v_k\}$ *and* $\tilde{\sigma} = \{\tilde{v}_0, \ldots, \tilde{v}_k\}$ *are two* $k$*-simplices in* $\mathbb{R}^n$ *such that*

$$||v_i - v_j| - |\tilde{v}_i - \tilde{v}_j|| \leq C_0 L(\sigma) \quad \text{with} \quad C_0 = \frac{\eta t(\sigma)^2}{4} \quad \text{and} \quad 0 \leq \eta \leq 1,$$

*for all* $0 \leq i < j \leq k$*. Let* $P$ *be the matrix whose* $i^{th}$ *column is* $v_i - v_0$*, and define* $\tilde{P}$ *similarly. Then*

$$s_k(\tilde{P}) \geq (1-\eta) s_k(P) \quad \text{and} \quad t(\tilde{\sigma}) \geq \frac{4}{5\sqrt{k}}(1-\eta)t(\sigma).$$

## 3.2 The affine independence criterion for non-degeneracy

In this subsection we show that a Riemannian simplex $\boldsymbol{\sigma}_M$ is non-degenerate if and only if, for any $x \in \boldsymbol{\sigma}_M$, the lift of the vertices by the inverse exponential map yields a non-degenerate Euclidean simplex. The expression for the differential of the barycentric coordinate map obtained in Equation (7) below is the result of a particular case of an argument presented by Buser and Karcher [5, §8.3] in a more general setting.

A Riemannian simplex $\boldsymbol{\sigma}_M$ is defined by its vertices $\sigma = \{p_0, \ldots, p_n\} \subset M$, which are constrained to lie in a convex ball $B_\rho \subseteq M$. For any $x \in B_\rho$ we define a Euclidean simplex $\sigma(x) \subset T_x M$ by $\sigma(x) = \{v_0(x), \ldots, v_n(x)\}$, where $v_i(x) = \exp_x^{-1}(p_i)$. The vertices $p_i \in B_\rho$ are considered fixed, but $x \in B_\rho$ is a variable. We continue to use a boldface symbol when we are referring to a simplex as a set of non-negative barycentric coordinates, and normal type refers to the finite vertex set; the convex hull of $\sigma(x)$ is $\boldsymbol{\sigma}_{\mathbb{E}}(x)$.

We work in a domain $U \subseteq \mathbb{R}^n$ defined by a chart $\phi : W \to U$ with $B_\rho \subseteq W \subseteq M$. Let $\tilde{\sigma} = \phi(\sigma)$ be the image of the vertices of a Riemannian $n$-simplex $\boldsymbol{\sigma}_M \subset B_\rho$. Label the vertices of $\tilde{\sigma} = \{\tilde{v}_0, \ldots, \tilde{v}_n\}$ such that $\tilde{v}_i = \phi(p_i)$, and assume $\tilde{v}_0$ is at the origin. The affine functions $\lambda_i : u \mapsto \lambda_i(u)$ are the barycentric coordinate functions of $\tilde{\sigma}$. We consider $\operatorname{grad} \mathcal{E}_\lambda$, introduced in Equation (3), now to be a vector field that depends on both $u \in U$ and $x \in B_\rho$. Specifically, we consider the vector field $\nu : U \times B_\rho \to TM$ defined by

$$\nu(u, x) = -\sum_{i=0}^{n} \lambda_i(u) v_i(x). \tag{6}$$

Let $b : \tilde{\boldsymbol{\sigma}}_{\mathbb{E}} \to \boldsymbol{\sigma}_M$ be defined by $b = \mathcal{B}_\sigma \circ \mathcal{L}$, where $\mathcal{L}$ is the canonical linear isomorphism that takes the vertices of $\tilde{\sigma}$ to those of $\Delta^n$, and $\mathcal{B}_\sigma$ is the barycentric coordinate map introduced in Definition 2. This map is differentiable, by the arguments presented by Buser and Karcher, and $\nu(u, b(u)) = 0$ for all $u \in \tilde{\boldsymbol{\sigma}}_{\mathbb{E}}$. Regarding $\nu$ as a vector field along $b$, its derivative may be expanded as

$$\partial_u \nu + (\nabla \nu)\, db = 0,$$

where $\partial_u \nu$ denotes the differential of $\nu(u, x)$ with $x$ fixed, $\nabla \nu$ is the covariant differential of $\nu(u, x)$ with $u$ fixed, and $db$ is the differential of $b$, our barycentric coordinate map onto $\boldsymbol{\sigma}_M$, i.e., $db_u : T_u \mathbb{R}^n \to T_x M$.

Our objective is to exhibit conditions that ensure that $db$ is non-degenerate. It follows from the strict convexity condition (5) of Lemma 1 that the map $\nabla \nu \colon w \mapsto \nabla_w \nu$ is non-degenerate. Indeed, if $w \in T_x M$ for some $x \in B_\rho$, there is a geodesic $\gamma : I \to B_\rho$ with $\gamma'(0) = w$, and $\frac{d^2}{dt^2} \mathcal{E}_\lambda(\gamma(t))\big|_{t=0} = \langle \nabla_w \nu, w \rangle_{\gamma(0)} > 0$. Therefore, we have that

$$db = -\left(\nabla \nu\right)^{-1} \partial_u \nu, \tag{7}$$

and thus $db$ is full rank if and only if $\partial_u \nu$ is full rank.

From (6) we observe that when $x$ is fixed, $\nu$ is the unique affine map $\mathbb{R}^n \supset U \to T_xM$, that sends the vertices of $\tilde{\sigma}$ to the corresponding vertices of $\sigma(x)$. In particular, $(\partial_u \nu)_v = (\partial_u \nu)_w$ for all $v, w \in U$. Thus $\partial_u \nu$ is the unique linear map that sends the basis $\{\tilde{v}_i\}$ to $\{\big(v_i(x) - v_0(x)\big)\}$.

We choose an arbitrary linear isometry to establish a coordinate system on $T_xM$, and let $P$ be the matrix whose $i^{\text{th}}$ column is $\big(v_i(x) - v_0(x)\big)$. Then, if $\tilde{P}$ is the matrix whose $i^{\text{th}}$ column is $\tilde{v}_i$, we obtain [9] the matrix expression for $\partial_u \nu$:

$$\partial_u \nu = -P\tilde{P}^{-1}. \tag{8}$$

From Equation (8) we conclude that $\partial_u \nu$ is full rank if and only if $P$ is of full rank, and this is the case if and only if $\sigma(x)$ is a non-degenerate Euclidean simplex, i.e., its vertices $\{v_i(x)\}$ are affinely independent.

We observe that if $db$ is non-degenerate on $\boldsymbol{\sigma}_M$, then $b$ must be injective. Indeed, if $x = b(u)$, then $\{\lambda_i(u)\}$, the barycentric coordinates of $u$ with respect to $\tilde{\sigma}$, are also the barycentric coordinates of the origin in $T_xM$, with respect to the simplex $\sigma(x)$. Thus if $b(u) = x = b(\tilde{u})$, then $\lambda_i(u) = \lambda_i(\tilde{u})$, and we must have $\tilde{u} = u$ by the uniqueness of the barycentric coordinates.

In summary, we have

▶ **Proposition 4.** *A Riemannian simplex $\boldsymbol{\sigma}_M \subset M$ is non-degenerate if and only if $\sigma(x) \subset T_xM$ is non-degenerate for every $x \in \boldsymbol{\sigma}_M$.*

## 3.3 Metric distortion of exponential transition

Now we choose the coordinate chart $\phi$ to be the inverse of the exponential map at some fixed point $p \in B_\rho$. Specifically, we set $\phi = \mathbf{u} \circ \exp_p^{-1} : W \to \mathbb{R}^m$, where $\mathbf{u} : T_pM \to \mathbb{R}^n$ is an arbitrary linear isometry that defines the $u$-coordinate functions in $U = \phi(W)$. The Euclidean simplex $\tilde{\sigma}$ in the coordinate domain can now be identified with $\sigma(p)$.

Our goal now is to estimate the metric distortion incurred when we map a simplex from one tangent space to another via the exponential maps. This will enable us to establish conditions ensuring that $\sigma(x)$ is non-degenerate, based on quality assumptions on $\sigma(p)$. Specifically, we want to bound the difference in the corresponding edge lengths of $\sigma(p)$ and $\sigma(x)$, and since the exponential transition function

$$\exp_x^{-1} \circ \exp_p : T_pM \to T_xM, \tag{9}$$

maps $\sigma(p)$ to $\sigma(x)$, it suffices to bound the metric distortion of $\exp_x^{-1}$ and $\exp_p$. This is accomplished by the bounds on the norm of the differential of the exponential map obtained from the Rauch Comparison Theorem (c.f. Buser and Karcher [5, §6.4]). For our purposes the theorem can be stated [9] as:

▶ **Lemma 5** (Rauch Theorem)**.** *Suppose the sectional curvatures in $M$ are bounded by $|K| \leq \Lambda$. If $v \in T_pM$ satisfies $|v| = r < \frac{\pi}{2\sqrt{\Lambda}}$, then for any vector $w \in T_v(T_pM) \cong T_pM$, we have*

$$\left(1 - \frac{\Lambda r^2}{6}\right)|w| \leq \left|(d\exp_p)_v w\right| \leq \left(1 + \frac{\Lambda r^2}{2}\right)|w|.$$

If $x, p, y \in B_\rho$, with $y = \exp_p(v)$, then $|v| < 2\rho$, and $\left|\exp_x^{-1}(y)\right| < 2\rho$, and Lemma 5 tells us that

$$\left\|d\left(\exp_x^{-1} \circ \exp_p\right)_v\right\| \leq \left\|\left(d\exp_x^{-1}\right)_y\right\| \left\|\left(d\exp_p\right)_v\right\| \leq 1 + 5\Lambda\rho^2.$$

The image of the line between $v_i(p)$ and $v_j(p)$ in $T_pM$, under the map $\exp_x^{-1} \circ \exp_p$, is a curve between $v_i(x)$ and $v_j(x)$ in $T_xM$, whose length is bounded by

$$|v_i(x) - v_j(x)| \leq (1 + 5\Lambda\rho^2)\,|v_i(p) - v_j(p)|\,.$$

We can do the same argument the other way, so

$$|v_i(p) - v_j(p)| \leq (1 + 5\Lambda\rho^2)\,|v_i(x) - v_j(x)|\,,$$

and we find

$$\begin{aligned}\big|\,|v_i(x) - v_j(x)| - |v_i(p) - v_j(p)|\,\big| &\leq 5\Lambda\rho^2(1 + 5\Lambda\rho^2)\,|v_i(p) - v_j(p)| \\ &\leq 21\Lambda\rho^2\,|v_i(p) - v_j(p)| \qquad \text{when } \rho < \rho_0.\end{aligned}$$

Letting $P$ be the matrix associated with $\sigma(p)$, and using $C_0 = 21\Lambda\rho^2$, in Lemma 3, we find that the matrix $\tilde{P}$ associated with $\sigma(x)$ in Proposition 4 is non-degenerate if $\sigma(p)$ satisfies a thickness bound of $t_0 > 10\sqrt{\Lambda}\rho$, and we have

▶ **Theorem 6** (Non-degeneracy criteria). *Suppose $M$ is a Riemannian manifold with sectional curvatures bounded by $|K| \leq \Lambda$, and $\boldsymbol{\sigma}_M$ is a Riemannian simplex, with $\boldsymbol{\sigma}_M \subset B_\rho \subset M$, where $B_\rho$ is an open geodesic ball of radius $\rho$ with*

$$\rho < \rho_0 = \min\left\{\frac{\iota}{2}, \frac{\pi}{4\sqrt{\Lambda}}\right\}.$$

*Then $\boldsymbol{\sigma}_M$ is non-degenerate if there is a point $p \in B_\rho$ such that the lifted Euclidean simplex $\sigma(p)$ has thickness satisfying*

$$t(\sigma(p)) > 10\sqrt{\Lambda}\rho.$$

*The ball $B_\rho$ may be chosen so that this inequality is necessarily satisfied if*

$$t(\sigma(p)) > 10\sqrt{\Lambda}L(\boldsymbol{\sigma}_M), \tag{10}$$

*where $L(\boldsymbol{\sigma}_M)$ is the geodesic length of the longest edge in $\boldsymbol{\sigma}_M$.*

The last assertion follows from the remark following Definition 2: If $L(\boldsymbol{\sigma}_M) < \rho_0$, then $\boldsymbol{\sigma}_M$ is contained in a closed ball of radius $L(\boldsymbol{\sigma}_M)$ centred at one of the vertices.

## 4 Triangulation criteria

Suppose we have a finite set of points $S$ in a compact Riemannian manifold $M$, and an (abstract) simplicial complex $\mathcal{A}$ whose vertex set is $S$, and such that every simplex in $\mathcal{A}$ defines a non-degenerate Riemannian simplex. When can we be sure that $\mathcal{A}$ triangulates $M$? Consider a convex ball $B_\rho$ centred at $p \in S$. We require that, when lifted to $T_pM$, the simplices near $p$ triangulate a neighbourhood of the origin. If we require that the simplices be small relative to $\rho$, and triangulate a region that extends to near the boundary of the lifted ball, then Riemannian simplices outside of $B_\rho$ cannot have points in common with the simplices near the centre of the ball, and it is relatively easy to establish a triangulation.

Instead, we aim for finer local control on the geometry. We establish conditions (Lemma 7) that ensure that the complex consisting of simplices incident to $p$, (i.e., the star of $p$) is embedded. In order to achieve this, we require finer control on the differential of the map into the manifold than bounds on its singular values.

We are interested in smooth maps from non-degenerate closed Euclidean simplices of dimension $n$ into an $n$-dimensional manifold $M$. We will work within coordinate charts, so our primary focus will be on maps of the form $F : \boldsymbol{\sigma}^n_{\mathbb{E}} \to \mathbb{R}^n$. Requiring that $F$ be smooth on the closed set $\boldsymbol{\sigma}^n_{\mathbb{E}}$ means that its partial derivatives are continuous on $\boldsymbol{\sigma}^n_{\mathbb{E}}$. Equivalently, $F$ can be extended to a smooth map on an open neighbourhood of $\boldsymbol{\sigma}^n_{\mathbb{E}}$. We demand that $dF_u$ is always close to the *same* linear isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ for all $u \in \boldsymbol{\sigma}_{\mathbb{E}}$:

$$\|dF_u - T\| \leq \eta. \tag{11}$$

This is a stronger constraint than can be obtained by a bound of the form $(1-\eta)\,|w| \leq |dF_u w| \leq (1+\eta)\,|w|$, as in the Rauch theorem (Lemma 5). In this latter case we can only say that $\|dF_u - T_u\| \leq \eta$, where $T_u$ is a linear isometry that depends on $u$.

A simplicial complex $\mathcal{C}$ is embedded in $\mathbb{R}^n$ if the vertices lie in $\mathbb{R}^n$ and the convex hulls of any two simplices in $\mathcal{C}$ either do not interesect, or their intersection is the convex hull of a simplex in $\mathcal{C}$. We identify $|\mathcal{C}|$, the carrier of $\mathcal{C}$, with the union of these geometric simplices; the complex naturally inherits a piecewise flat metric from the embedding.

If $p$ is a vertex in $\mathcal{C}$, we define the *star of* $p$ to be the subcomplex $\mathrm{star}(p)$ of $\mathcal{C}$ consisting of all simplices that contain $p$, together with the faces of these simplices. We say that $\mathrm{star}(p)$ is a *full star* if $|\mathrm{star}(p)|$ is a closed topological ball of dimension $n$ with $p$ in its interior, and $\mathcal{C}$ contains no simplices of dimension greater than $n$.

The *scale* of $\mathcal{C}$ is an upper bound on the length of the longest edge in $\mathcal{C}$, and is denoted by $h$. We say that $\mathcal{C}$ is $t_0$*-thick* if each simplex in $\mathcal{C}$ has thickness greater than $t_0$. The *dimension* of $\mathcal{C}$ is the largest dimension of the simplices in $\mathcal{C}$. We call a complex of dimension $n$ an $n$*-complex*. If every simplex in $\mathcal{C}$ is the face of an $n$-simplex, then $\mathcal{C}$ is a *pure* $n$*-complex*.

A map $F : |\mathcal{C}| \to \mathbb{R}^n$ is *smooth on* $\mathcal{C}$ if for each $\sigma \in \mathcal{C}$ the restriction $F\big|_{\boldsymbol{\sigma}_{\mathbb{E}}}$ is smooth. This means that $d(F\big|_{\boldsymbol{\sigma}_{\mathbb{E}}})$ is well defined, and even though $dF$ is not well defined, we will use this symbol when the particular restriction employed is either evident or unimportant. When the underlying complex on which $F$ is smooth is unimportant, we simply say that $F$ is *piecewise smooth*.

The strong constraint on the differential allows us to ensure that thick stars are embedded:

▶ **Lemma 7** (Embedding a star). *Suppose $\mathcal{C} = \mathrm{star}(p)$ is a $t_0$-thick, pure $n$-complex embedded in $\mathbb{R}^n$ such that all of the $n$-simplices are incident to a single vertex, $p$, and $p \in \mathrm{int}(|\mathcal{C}|)$ (i.e., $\mathrm{star}(p)$ is a full star). If $F : |\mathcal{C}| \to \mathbb{R}^n$ is smooth on $\mathcal{C}$, and satisfies*

$$\|dF - \mathrm{Id}\| < nt_0 \tag{12}$$

*on each $n$-simplex of $\mathcal{C}$, then $F$ is an embedding.*

The proof [9, Lem. 14] hinges on the fact that thickness provides a lower bound on the angle between a radial ray from $p$ and a facet on the boundary of $\mathrm{star}(p)$. Together with the bound on the differential of $F$, this enables us to demonstrate that the boundary of $\mathrm{star}(p)$ is embedded by $F$. Then, since each simplex individually is embedded by $F$, topological considerations imply that $\mathrm{star}(p)$ itself is embedded by $F$.

We use this observation to establish conditions that ensure that a map $H : |\mathcal{A}| \to M$ is a homeomorphism. If $H$ is such that for every vertex in $\mathcal{A}$, the restriction of $H$ to $|\mathrm{star}(p)|$ is an embedding, then $H$ is a covering map. So if $H$ is injective, it is a triangulation. Injectivity is established by constraining the size of the simplices relative to the injectivity radius of $M$, and by implicitly constraining the metric distortion associated with $H$. We obtain the following proposition, which generically models the situation we will work with when we describe a triangulation by Riemannian simplices:

▶ **Proposition 8** (Triangulation). *Let $\mathcal{A}$ be a manifold simplicial $n$-complex with finite vertex set $S$, and $M$ a compact Riemannian manifold with an atlas $\{(W_p, \phi_p)\}_{p\in S}$ indexed by $S$. Suppose $H : |\mathcal{A}| \to M$ satisfies:*

1. *For each $p \in S$ the secant map of $\phi_p \circ H$ restricted to $|\mathrm{star}(p)|$ is a piecewise linear embedding $\mathcal{L}_p : |\mathrm{star}(p)| \to \mathbb{R}^n$ such that each simplex $\sigma \in \mathcal{C}_p = \mathcal{L}_p(\mathrm{star}(p))$ is $t_0$-thick, and $|\mathcal{C}_p| \subset B_{\mathbb{R}^n}(\mathcal{L}_p(p); h)$, with $\mathcal{L}_p(p) \in \mathrm{int}(|\mathcal{C}_p|)$. The scale parameter $h$ must satisfy $h < \frac{\iota}{4}$, where $\iota$ is the injectivity radius of $M$.*
2. *For each $p \in S$, $\phi_p : W_p \xrightarrow{\cong} U_p \subset \mathbb{R}^n$ is such that $\overline{B} = \overline{B}_{\mathbb{R}^n}(\mathcal{L}_p(p); \frac{3}{2}h) \subseteq U_p$, and $\left\|(d\phi_p^{-1})_u\right\| \leq \frac{4}{3}$, for every $u \in \overline{B}$.*
3. *The map*
$$F_p = \phi_p \circ H \circ \mathcal{L}_p^{-1} : |\mathcal{C}_p| \to \mathbb{R}^n$$
*satisfies*
$$\|(dF_p)_u - \mathrm{Id}\| \leq \frac{nt_0}{2}$$
*on each $n$-simplex $\sigma \in \mathcal{C}_p$, and every $u \in \boldsymbol{\sigma}_{\mathbb{E}}$.*

*Then $H$ is a smooth triangulation of $M$.*

**Proof.** By Lemma 7, $F_p$ is a homeomorphism onto its image. It follows then that $H\big|_{|\mathrm{star}(p)|}$ is an embedding for every $p \in S$. Therefore, since $|\mathcal{A}|$ is compact, $H : |\mathcal{A}| \to M$ is a covering map.

Given $x \in |\mathcal{A}|$, with $x \in \boldsymbol{\sigma}_{\mathbb{E}}$, and $p$ a vertex of $\boldsymbol{\sigma}_{\mathbb{E}}$, let $\tilde{x} = \mathcal{L}_p(x) \in |\mathcal{C}_p|$. Then the bound on $dF$ implies that $|F_p(\tilde{x}) - \mathcal{L}_p(p)| \leq \left(1 + \frac{nt_0}{2}\right) h \leq \frac{3}{2}h$, so $F_p(\tilde{x}) \in \overline{B}$. Since $\phi_p^{-1} \circ F_p(\tilde{x}) = H(x)$, and
$$\left|(d\phi_p^{-1})_{F(u)}(dF_p)_u\right| \leq \frac{4}{3}\left(1 + \frac{nt_0}{2}\right) \leq 2$$
for any $u \in \boldsymbol{\sigma}_{\mathbb{E}} \subset |\mathcal{C}_p|$, we have that $d_M(H(p), H(x)) \leq 2h$.

Suppose $y \in |\mathcal{A}|$ with $H(y) = H(x)$. Let $\tau \in \mathcal{A}$ with $y \in \boldsymbol{\tau}_{\mathbb{E}}$, and $q \in \tau$ a vertex. Then $d_M(H(p), H(q)) \leq 4h < \iota$. Thus there is a path $\gamma$ from $H(x)$ to $H(p)$ to $H(q)$ to $H(y) = H(x)$ that is contained in the topological ball $B_M(H(p); \iota)$, and is therefore null-homotopic. Since $H$ is a covering map, this implies that $x = y$. Thus $H$ is injective, and therefore defines a smooth triangulation. ◀

In the context of the barycentric coordinate mapping defining Riemannian simplices, we obtain the desired strong bound on the differential by means of a refinement of the Rauch theorem due to Buser and Karcher [5, §6.4], which for our purposes may be stated as:

▶ **Lemma 9** (Strong Rauch Theorem). *Assume the sectional curvatures on $M$ satisfy $|K| \leq \Lambda$, and suppose there is a unique minimising geodesic between $x$ and $p$. If $v = \exp_p^{-1}(x)$, and*
$$|v| = d_M(p, x) = r \leq \frac{\pi}{2\sqrt{\Lambda}},$$
*then*
$$\left\|(d\exp_p)_v - T_{xp}\right\| \leq \frac{\Lambda r^2}{2},$$
*where $T_{xp}$ denotes the parallel transport operator along the unique minimising geodesic from $p$ to $x$.*

Given three points $x, y, p \in B_\rho$ in a convex ball, we use further results of Buser and Karcher [5, §6] to obtain a bound on $\|T_{xp} - T_{xy}T_{yp}\|$ with respect to $\rho$ and a bound on the absolute value of the sectional curvatures. This result together with Lemma 9 yields a bound of the desired form (11) on the differential of exponential transition functions:

▶ **Proposition 10** (Strong exponential transition bound)**.** *Suppose the sectional curvatures on $M$ satisfy $|K| \leq \Lambda$. Let $v \in T_pM$, with $y = \exp_p(v)$. If $x, y \in B_M(p;\rho)$, with*

$$\rho < \frac{1}{2}\rho_0 = \frac{1}{2}\min\left\{\frac{\iota}{2}, \frac{\pi}{4\sqrt{\Lambda}}\right\},$$

*then*

$$\left\|d(\exp_x^{-1} \circ \exp_p)_v - T_{xp}\right\| \leq 6\Lambda\rho^2.$$

Proposition 10 in turn allows us to obtain the desired form of bound on the differential (7) of the barycentric coordinate map so that we can exploit Proposition 8 to obtain sampling criteria for triangulating a Riemannian manifold, our main result:

▶ **Theorem 11.** *Suppose $M$ is a compact $n$-dimensional Riemannian manifold with sectional curvatures $K$ bounded by $|K| \leq \Lambda$, and $\mathcal{A}$ is an abstract simplicial complex with finite vertex set $S \subset M$. Fix a thickness bound $t_0 > 0$, and let*

$$h = \min\left\{\frac{\iota}{4}, \frac{\sqrt{n}t_0}{6\sqrt{\Lambda}}\right\}. \tag{13}$$

*If*

1. *for every $p \in S$, the vertices of $\mathrm{star}(p)$ are contained in $B_M(p;h)$, and the balls $\{B_M(p;h)\}_{p\in S}$ cover $M$;*
2. *for every $p \in S$, the restriction of the inverse of the exponential map $\exp_p^{-1}$ to the vertices of $\mathrm{star}(p) \subset \mathcal{A}$ defines a piecewise linear embedding of $|\mathrm{star}(p)|$ into $T_pM$, realising $\mathrm{star}(p)$ as a full star such that every simplex $\sigma(p)$ has thickness $t(\sigma(p)) \geq t_0$,*

*then $\mathcal{A}$ triangulates $M$, and the triangulation is given by the barycentric coordinate map on each simplex.*

## 5 The piecewise flat metric

The complex $\mathcal{A}$ described in Theorem 11 naturally inherits a piecewise flat metric from the construction. The length assigned to an edge $\{p, q\} \in \mathcal{A}$ is the geodesic distance in $M$ between its endpoints: $\ell_{pq} = d_M(p, q)$. We first describe conditions which ensure that this assignment of edge lengths does indeed make each $\sigma \in \mathcal{A}$ isometric to a Euclidean simplex. With this piecewise flat metric on $\mathcal{A}$, the barycentric coordinate map is a bi-Lipschitz map between metric spaces $H : |\mathcal{A}| \to M$, and we estimate the metric distortion of this map.

If $G$ is a symmetric positive definite $n \times n$ matrix, then it can be written as a Gram matrix, $G = P^\mathsf{T} P$ for some $n \times n$ matrix $P$. Then $P$ describes a Euclidean simplex with one vertex at the origin, and the other vertices defined by the column vectors. The matrix $P$ is not unique, but if $G = Q^\mathsf{T} Q$, then $Q = OP$ for some linear isometry $O$. Thus a symmetric positive definite matrix defines a Euclidean simplex, up to isometry.

If $\sigma = \{p_0, \ldots, p_n\} \subset B_\rho$, is the vertex set of a Riemannian simplex $\boldsymbol{\sigma}_M$, we define the numbers $\ell_{ij} = d_M(p_i, p_j)$. These are the edge lengths of a Euclidean simplex $\boldsymbol{\sigma}_\mathbb{E}$ if and only if the matrix $G$ defined by

$$G_{ij} = \frac{1}{2}(\ell_{0i}^2 + \ell_{0j}^2 - \ell_{ij}^2) \tag{14}$$

is positive definite.

The same kind of argument that bounds the thickness of a simplex subjected to small distortions of its edge lengths, Lemma 3, allows us to ensure that the numbers $\ell_{ij}$ do define a Euclidean simplex $\boldsymbol{\sigma}_{\mathbb{E}}$ if they are close enough to the edge lengths of a Euclidean simplex, $\sigma(p)$ whose thickness is bounded below. Then, again exploiting the Rauch Theorem 5, we find we need a slightly tighter bound on the scale parameter in order to ensure that $\mathcal{A}$ admits a piecewise flat metric:

▶ **Proposition 12.** *If the requirements of Theorem 11 are satisfied when the scale parameter* (13) *is replaced with*

$$h = \min\left\{\frac{\iota}{4}, \frac{t_0}{6\sqrt{\Lambda}}\right\},$$

*then the geodesic distances between the endpoints of the edges in $\mathcal{A}$ define a piecewise flat metric on $\mathcal{A}$ such that each simplex $\sigma \in \mathcal{A}$ satisfies*

$$t(\sigma) > \frac{3}{4\sqrt{n}} t_0.$$

In the context of Theorem 11 the barycentric coordinate map on each simplex defines a piecewise smooth homeomorphism $H : |\mathcal{A}| \to M$. If the condition of Proposition 12 is also met, then $\mathcal{A}$ is naturally endowed with a piecewise flat metric. We wish to compare this metric with the Riemannian metric on $M$. It suffices to consider an $n$-simplex $\sigma \in \mathcal{A}$, and establish bounds on the singular values of the differential $dH$. If $p \in \sigma$, then we can write $H\big|_{\boldsymbol{\sigma}_{\mathbb{E}}} = b \circ \mathcal{L}_p$, where $\mathcal{L}_p : \boldsymbol{\sigma}_{\mathbb{E}} \to \boldsymbol{\sigma}_{\mathbb{E}}(p)$ is the linear map that sends $\sigma \in \mathcal{A}$ to $\sigma(p) \in T_pM$.

A bound on the metric distortion of a linear map that sends one Euclidean simplex to another is a consequence of the following (reformulation of [2, Lemma 9]):

▶ **Lemma 13** (Linear distortion bound)**.** *Suppose that $P$ and $\tilde{P}$ are non-degenerate $k \times k$ matrices such that*

$$\tilde{P}^{\mathsf{T}}\tilde{P} = P^{\mathsf{T}}P + E. \tag{15}$$

*Then there exists a linear isometry $\Phi : \mathbb{R}^k \to \mathbb{R}^k$ such that*

$$\left\|\tilde{P}P^{-1} - \Phi\right\| \leq \frac{s_1(E)}{s_k(P)^2}.$$

Taking $P$ and $\tilde{P}$ to represent $\boldsymbol{\sigma}_{\mathbb{E}}(p)$ and $\boldsymbol{\sigma}_{\mathbb{E}}$, we can bound $|\mathcal{L}_p|$ and $\left|\mathcal{L}_p^{-1}\right|$, and combined with the bounds on $db$ that we have already estimated, we obtain the desired bounds on $dH$, and we find:

▶ **Theorem 14** (Metric distortion)**.** *If the requirements of Theorem 11, are satisfied with the scale parameter* (13) *replaced by*

$$h = \min\left\{\frac{\iota}{4}, \frac{t_0}{6\sqrt{\Lambda}}\right\},$$

*then $\mathcal{A}$ is naturally equipped with a piecewise flat metric $d_{\mathcal{A}}$ defined by assigning to each edge the geodesic distance in $M$ between its endpoints.*

*If $H : |\mathcal{A}| \to M$ is the triangulation defined by the barycentric coordinate map in this case, then the metric distortion induced by $H$ is quantified as*

$$|d_M(H(x), H(y)) - d_{\mathcal{A}}(x, y)| \leq \frac{50\Lambda h^2}{t_0^2} d_{\mathcal{A}}(x, y),$$

*for all $x, y \in |\mathcal{A}|$.*

## 6 Discussion

Traditional demonstrations that smooth manifolds can be triangulated [6, 17, 18] involve establishing a lower bound on simplex quality that is invariant under some kind of refinement operation, and showing that a triangulation will be achieved when the scale parameter is sufficiently small. Theorem 11 provides a means to explicitly quantify "sufficiently small" in this context. Similarly, an analysis of more recent triangulation algorithms in computational geometry [8, 4] could exploit Theorem 11 to quantify a sufficient sampling density.

We refer to the criteria of Theorem 11 as sampling criteria, even though they require a simplicial complex for their definition. Although there is no explicit constraint on the minimal distance between points of $S$, one is implicitly imposed by the quality constraint on the Riemannian simplices. The required sampling density depends on the quality of the Riemannian simplices, which leaves open the question of what kind of quality of simplices can we hope to attain. A Delaunay complex conforming to the requirements of Theorem 11 can be constructed [2] with the thickness $t_0$ bounded by $2^{-\mathcal{O}(n^3)}$, and even in flat manifolds, e.g., Euclidean space, the situation is not better in general [7], but in this case, at least in dimension 3, dramatic improvements can be made if the placement of sample points can be structured according to a lattice [11].

More work needs to be done to understand the limitations imposed by the thickness bound $t_0$ that appears in the density contraint (13), but there is another aspect to the bound that merits more attention. The non-degeneracy criterion established in Theorem 6 demands that the Riemannian simplices be "almost flat". In other words, if the bound on the absolute value of the sectional curvatures in the neighbourhood is very large, then the simplex must be very small. However, we know that in spaces of constant curvature, where the Riemannian simplex coincides with the usual definition of a simplex as the convex hull of its vertices, the simplices are not constrained to be small. In hyperbolic space the edge lengths of a non-degenerate simplex can be arbitrarily large. It seems that a more refined bound on the scale should depend on the amount the sectional curvatures deviate from some fixed constant, that need not be 0. Given upper and lower bounds $\Lambda_+$ and $\Lambda_-$ on the sectional curvatures, our preliminary unpublished calculations demonstrate a bound on simplex quality for non-degeneracy involving $\Lambda_+ - \Lambda_-$ when $\Lambda_- > 0$. The same analysis in the hyperbolic setting ($\Lambda_+ < 0$) yields a more complicated expression.

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