Multi-Focusing on Extensional Rewriting with Sums

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Abstract

We propose a logical justification for the rewriting-based equivalence procedure for simply-typed lambda-terms with sums of Lindley [8]. It relies on maximally multi-focused proofs, a notion of canonical derivations introduced for linear logic. Lindley's rewriting closely corresponds to *preemptive rewriting* [5], a technical device used in the meta-theory of maximal multi-focus.

1998 ACM Subject Classification F.4.1 Lambda calculus and related systems

Keywords and phrases Maximal multi-focusing, extensional sums, rewriting, natural deduction

Digital Object Identifier 10.4230/LIPIcs.TLCA.2015.317

1 Introduction

Deciding observational equality of pure typed lambda-terms in presence of sum types is a difficult problem. After several solutions based on complex syntactic [6] or semantic [1, 2] techniques, Sam Lindley presented a surprisingly simple rewriting solution [8]. While the underlying intuition (extrude contexts to move pattern-matchings as high as possible in the term) makes sense, the algorithm is still mysterious in many aspects: even though they were synthesized from the previous highly-principled approach, the rewriting rules may feel strangely ad-hoc.

In this paper, we will propose a *logical* justification of this algorithm. It is based on recent developments in proof search, maximally multi-focused proofs [5]. The notion of *preemptive rewriting* was introduced in the meta-theory of multi-focusing as a purely technical device; we claim that it is in fact strongly related to Lindley's rewriting, and formally establish the correspondence.

The reference work on multi-focused systems [5] has been carried in a sequent calculus for linear logic. We will first establish the meta-theory of maximal multi-focusing for intuitionistic logic (Section 2). We start from a sequent calculus presentation, which is closest to the original system. Our first contribution is to propose an equivalent multi-focusing system in natural deduction 2.2. We then define preemptive rewriting in this natural deduction 2.4 and establish canonicity of maximally multi-focused proofs 2.6.

In Section 3, we transpose the preemptive rewrite rules into a relation on proof terms. We can then formally study the correspondence between rewriting a multi-focused proof into a canonical maximally multi-focused one, and Lindley's γ-reduction on lambda-terms. We demonstrate that they compute the same normal forms, modulo a form of redundancy elimination that is missing in the multi-focused system.

We finally introduce redundancy-elimination rewriting and equivalence for the proof terms of the multi-focused natural deduction (Section 4). The resulting notion of canonical proofs, *simplified maximal proofs*, precisely corresponds to normal forms of Lindley's rewriting relation. The natural notion of local equivalence between simplified maximal proofs therefore captures extensional equality.

13th International Conference on Typed Lambda Calculi and Applications (TLCA'15).
Editor: Thorsten Altenkirch; pp. 317–331
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

2 Intuitionistic multi-focusing

The space of proofs in sequent calculus or natural deduction exhibits a lot of redundancy: many proofs that are syntactically distinct really encode the same semantics. In particular, it is often possible to permute two inference rules in a way that preserves the validity of proofs, but also the reduction semantics of the corresponding proof terms. If a permutation transforms a proof with rule A applied above rule B into a proof with rule B applied above rule A, we say that it is an A/B permutation (A is above the slash, as in the source proof).

Focusing is a general discipline that can be imposed upon proof system, based on the separation of inference rules into two classes. *Invertible* rules (called as such because their inverse is admissible) always preserve provability, and can thus be applied as early as possible. *Non-invertible* rules may result in dead ends if they are applied too early (consider proving $A+B \vdash A+B$ by first introducing the sum on the right-hand side)). In focusing calculi, derivations are structured in "sequences" or "phases", that either only apply invertible rules or only non-invertible rules. Focusing imposes that phases be as long as possible. During invertible phases, one must apply any valid invertible rule. During non-invertible phases, one *focuses* on a set of formulas, and applies non-invertible operations on those formulas as long as possible – if the phase is started too early, this may result in a dead end.

Invertibility determines a notion of polarity of logical connectives: we call *positive* those whose right-hand-side rule is non-invertible (they are "only interesting in positive position"), and *negative* those whose left-hand-side rule is non-invertible. In single-succedent intuitionistic logic, $(\to)$ is negative, $(+)$ is positive, and the product $(\times)$ may actually be assigned either polarity.

In single-sided calculi, non-invertible rules are those that introduce positive connectives, and are called "positive". For continuity of vocabulary, we will also call non-invertible rules *positive*, and invertible rules *negative*. In particular, a permutation that moves a non-invertible rule below an invertible rule is a "pos/neg permutation".

2.1 Multi-focused sequent calculus

Multi-focusing ([9, 5]) is an extension of focusing calculi where, instead of focusing on a single formula of the sequent (either on the left or on the right), we allow to simultaneously focus on several formulas at once. The multiple foci do not interact during the focusing phase, and this allows to express the fact that several focusing sequences are in fact independent and can be performed in parallel, condensing several distinct focused proofs into a single multi-focused derivation.

We start with a multi-focused variant of the intuitionistic sequent calculus, presented in Fig. 1. We denote focus using brackets: the rules with no brackets are invertible. This notation will change in natural deduction calculi.

In particular, we write A_n or Δ_p for formula or contexts that must be all negative or positive, and X, Y or Z for atoms. We write B_{pa} and Γ_{na} when either a positive (resp. negative) or an atom is allowed. For readability reasons, we only add polarity annotations when necessary; if we consider only derivations whose end conclusion is unfocused, then the invariant holds that the unfocused left-hand-side context is always all-negative, while the unfocused right-hand-side formula is always positive.

Our intuitionistic calculi are, as is most frequent, single-succedent. The notation $A \mid B$ on the right does not denote a real disjunction but a single formula, one of the two variables being empty. The focusing rule SEQ-FOCUS with conclusion $\Gamma, \Delta \vdash A \mid B$ can be instantiated in two ways, one when A is empty, and the premise is $\Gamma, [\Delta] \vdash [B]$ (the succedent is part of

$$\dfrac{X \text{ atomic}}{\Gamma_n, X \vdash X}\ \textsc{seq-atom} \qquad \dfrac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A + B \vdash C}\ \textsc{seq-inv-sum-l} \qquad \dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B}\ \textsc{seq-inv-prod-r} \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \textsc{seq-inv-arr-r}$$

$$\dfrac{\Gamma_{na}, [\Delta_n] \vdash A_{pa} \mid [B_{pa}]}{\Gamma_{na}, \Delta_n \vdash A_{pa} \mid B_{pa}}\ \textsc{seq-focus} \qquad \dfrac{\Gamma, \Delta_{pa} \vdash A \mid B_{na}}{\Gamma, [\Delta_{pa}] \vdash A \mid [B_{na}]}\ \textsc{seq-release} \qquad \dfrac{\Gamma \vdash [A] \qquad \Gamma, [\Delta, B] \vdash C \mid [D]}{\Gamma, [\Delta, A \to B] \vdash C \mid [D]}\ \textsc{seq-foc-arr-l}$$

$$\dfrac{\Gamma, [\Delta, A_i] \vdash B \mid [C]}{\Gamma, [\Delta, A_1 \times A_2] \vdash B \mid [C]}\ \textsc{seq-foc-prod-l} \qquad \dfrac{\Gamma, [\Delta] \vdash [A_i]}{\Gamma, [\Delta] \vdash [A_1 + A_2]}\ \textsc{seq-foc-sum-l}$$

Figure 1 Multifocused sequent calculus for intuitionistic logic.

the multi-focus), and one when B is empty, and the premise is $\Gamma, [\Delta] \vdash A$ (the succedent is not part of the multi-focus). Note that Δ is a set and may be empty, in which case the focus only happens on the right.

As a minor presentation difference to the reference work on multi-focusing [5], our contexts are unordered multi-sets, and all the formulas under focus are released at once – by SEQ-RELEASE, which releases positives (resp. negatives) or atoms.

This multi-focused calculus proves exactly the same formulas as the singly-focused sequent calculus. The latter is trivially included in the former, and conversely one can turn a multi-focus into an arbitrarily ordered sequence of single foci. As a corollary, relying on non-trivial proofs from the literature (e.g., [11]), it is equivalent in provability to the (non-focused) sequent calculus for intuitionistic logic.

2.2 Multi-focused natural deduction

While the multi-focusing sequent calculus closely corresponds to existing focused presentations, its natural deduction presentation in Fig. 2 is new. We took inspiration from the presentation of focused linear logic in natural deduction of [3], in particular the $\Uparrow$ and $\Downarrow$ notations coming from intercalation calculi.

There are three main judgments. $\Gamma \vdash A$ is the unfocused judgment with the invertible rules. $\Gamma; A \Downarrow B$ is the "elimination-focused" judgment, and $A \Uparrow B$ is the "introduction-focused" judgment (focused on A). $\Gamma; A \Downarrow B$ means that the assertion B can be produced from the hypothesis A by non-invertible elimination rules; the context Γ is used in any non-focused subgoal. $A \Uparrow B$ means that proving the goal A can be reduced, by applying non-invertible introduction rules, to proving the goal B. Those two judgments do not come separately, they are introduced by the focusing rule NAT-FOCUS.

In Fig. 2, we used auxiliary rules (NAT-START-INTRO, NAT-START-NO-INTRO, NAT-START-ELIM) to present the focusing compactly (this is important when rewriting proofs); those rules can only happen immediately above NAT-FOCUS, and can thus be considered definitional syntactic sugar – we used a double bar to reflect this. If we inlined these auxiliary rules, the focusing rule would read (equivalently):

$$\dfrac{(A_n^i)^{i \in I} \subseteq \Gamma_{na} \qquad (\Gamma_{na}; A_n^i \Downarrow A_{pa}'^i)^{i \in I} \qquad (B_p \Uparrow B_{na}' \mid B = B') \qquad \Gamma_{na}, (A_{pa}'^i)^{i \in I} \vdash B'}{\Gamma_{na} \vdash B_{pa}}$$

This rule can only be used when all invertible rules have been performed: the context must be negative or atomic, and the goal positive or atomic. It selects set of foci on the

$$\frac{X \text{ atomic}}{\Gamma_{na}, X \vdash X}\;\textsc{nat-atom} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A + B \vdash C}\;\textsc{nat-inv-sum-l} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B}\;\textsc{nat-inv-prod-r} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\;\textsc{nat-inv-arr-r}$$

$$\frac{\Gamma_{na} \Downarrow \Gamma' \qquad A_{pa} \Uparrow^{?} A' \qquad \Gamma_{na}, \Gamma' \vdash A'}{\Gamma_{na} \vdash A_{pa}}\;\textsc{nat-focus} \qquad \frac{}{\Gamma; A_{na} \Downarrow A_{na}}\;\textsc{nat-end-elim} \qquad \frac{}{A_{na} \Uparrow A_{na}}\;\textsc{nat-end-intro}$$

$$\frac{\Gamma; A \Downarrow B \to C \qquad B \Uparrow B' \qquad \Gamma \vdash B'}{\Gamma; A \Downarrow C}\;\textsc{nat-elim-arr} \qquad \frac{\Gamma; A \Downarrow B_1 \times B_2}{\Gamma; A \Downarrow B_i}\;\textsc{nat-elim-prod} \qquad \frac{A_i \Uparrow B}{A_1 + A_2 \Uparrow B}\;\textsc{nat-intro-sum}$$

$$\frac{}{A \Uparrow^{?} A}\;\textsc{nat-start-no-intro} \qquad \frac{A_p \Uparrow B_{na}}{A_p \Uparrow^{?} B_{na}}\;\textsc{nat-start-intro} \qquad \frac{(A_n^i)^{i \in I} \subseteq \Gamma \qquad (\Gamma; A_n^i \Downarrow A'^{i}_{pa})^{i \in I}}{\Gamma \Downarrow (A'^{i}_{pa})^{i \in I}}\;\textsc{nat-start-elim}$$

Figure 2 Multifocused natural deduction for intuitionistic logic.

left, the family of strictly negative assumptions $(A_n^i)^{i \in I}$ (we consistently use the superscript notation for family indices), and optionally a focus on the right; if the goal is focused it must be strictly positive. All foci must be as long as possible: elimination foci go from a variable down to a positive or atomic A'^{i}_{pa}, and the introduction focus goes up until it encounters a negative or atomic B'_{na}.

In comparison to the sequent calculus, the positive or atomic formulas $(A'^{i}_{pa})^{i \in I}$ appearing at the *start* of the elimination-focus correspond to the formulas *released* at the *end* of a multi-focus in a sequent proof; natural deduction, when compared to the sequent calculus, has elimination rules "upside down". Also characteristic of natural deduction is the horizontal parallelism between eliminations and introductions; for example, the following two partial derivations correspond to the same natural deduction:

$$\cfrac{\cfrac{\cfrac{\cfrac{A_{pa} \times B, A_{pa} \vdash C_{na}}{A_{pa} \times B, [A_{pa}] \vdash [C_{na}]}}{A_{pa} \times B, [A_{pa}] \vdash [C_{na} + D]}}{A_{pa} \times B, [A_{pa} \times B] \vdash [C_{na} + D]}}{A_{pa} \times B \vdash C_{na} + D} \qquad \cfrac{\cfrac{\cfrac{\cfrac{A_{pa} \times B, A_{pa} \vdash C_{na}}{A_{pa} \times B, [A_{pa}] \vdash [C_{na}]}}{A_{pa} \times B, [A_{pa} \times B] \vdash [C_{na}]}}{A_{pa} \times B, [A_{pa} \times B] \vdash [C_{na} + D]}}{A_{pa} \times B \vdash C_{na} + D}$$

$$\frac{\dfrac{\dfrac{}{A_{pa} \times B; A_{pa} \times B \Downarrow A_{pa} \times B}}{A_{pa} \times B; A_{pa} \times B \Downarrow A_{pa}} \qquad \dfrac{\dfrac{}{C_{na} \Uparrow C_{na}}}{C_{na} + D \Uparrow C_{na}} \qquad A_{pa} \times B, A_{pa} \vdash C_{na}}{A_{pa} \times B \vdash C_{na} + D}$$

On the other hand, we kept the less important invertible rules in sequent style: the sum elimination is a left introduction. Invertible rules being morally "automatically" applied, the sequent-style left introduction, which is directed by the type of its conclusion, is more natural in this context. Ironically, this brings us rather close to the sequent calculus of Krishnaswami [7] which, for presentation purposes, preserved a function-elimination rule in natural deduction style.

▶ **Lemma 1.** *The multi-focused natural deduction system proves exactly the same non-focused judgments as the multi-focused sequent calculus.*

$$\frac{\Gamma_{npa} \Downarrow \Delta'_{pa} \qquad B_p \Uparrow^? B'_{na} \qquad \Gamma_{npa}, \Delta'_{pa} \vdash A_{npa} \mid B'_{na}}{\Gamma_{npa} \vdash A_{npa} \mid B_p} \text{ PREEMPT-FOCUS} \qquad \frac{\Gamma_{npa} \Downarrow \Delta'_{pa} \qquad \Gamma_{npa}, \Delta'_{pa}; A \Downarrow A'}{\Gamma_{npa}; A \Downarrow A'} \text{ PREEMPT-ELIM}$$

Figure 3 Preemptive rules for intuitionistic multifocused natural deduction.

2.3 A preemptive variant of multi-focused natural deduction

Multi-focusing was introduced to express the idea of *parallelism* between non-invertible rules on several independent foci. A proof has more parallelism than another if two sequential foci of the latter are merged (through rule permutations) in a single multi-focus in the former. A natural question is whether there exists "maximally parallel proofs". To answer it (affirmatively), the original article on multi-focusing ([5]) introduced a rewriting relation that permutes non-invertible phases down in proof derivations, until they cannot go any further without losing provability – neighboring phases can then be merged into a maximally focused proof.

In the process of moving down, a non-invertible phase will traverse invertible phases below. The intermediary states of this reduction sequence may break the invariant that invertible rules must be applied as early as possible; we say that the non-invertible phase *preempts* (a part of) the invertible phase. As this intermediary state is not a valid proof in off-the-shelf multi-focusing systems, the original article introduced a relaxed variant called a preemptive system, in which the phase-sinking transformation, called *preemptive rewriting*, can be defined following [5].

We present in Fig. 3 a preemptive variant of multi-focused natural deduction, except for the invertible and focused-introduction rules that are strictly unchanged from the previous multi-focusing rules in Fig. 2. There are two important differences:

- Preemption of invertible phases. To allow the start of a focusing phase when some invertible rules could still be applied, we lifted the polarity constraints for starting focusing. In the rule PREEMPT-FOCUS, the goal $\Gamma_{npa} \vdash A_{npa}$ may be of any polarity. We use a tautological Γ_{npa} annotation to emphasize this change.
- Preemption of non-invertible phases. This is expressed by the rule PREEMPT-ELIM, where an ongoing focus on A is preempted by a complete focus on Δ'_{pa}. Note that stored contexts are *not* available during the current elimination phase (they are unused in NAT-END-ELIM); they are only available to non-focused phases that appear as subgoals (in the arrow elimination rule). This preserves the central idea that the simultaneous foci of a single focusing rule are *independent*.

2.4 Preemptive rewriting

We can then define in Fig. 4 the rewriting relation on the preemptive calculus, that lets any non-invertible phase move as far as possible down the derivation tree. Maximally multi-focused proofs, which can be characterized on permutation-equivalence classes of multi-focused proofs, correspond to normal forms of this rewriting relation.

A focused phase cannot move below an inference rule if some of the foci depend on this inference rule. Instead of expressing the non-dependency requirement by implicit absence of the foci, we have explicitly canceled out the foci that *must be absent* to improve readability. In the first rule for example, $\Gamma, \cancel{A} \Downarrow \Delta$ means that the A hypothesis must be weakened (not used) in the derivation of $\Gamma \Downarrow \Delta$, orelse it cannot move below the introduction of A.

$$\dfrac{\dfrac{\Gamma,\cancel{A}\Downarrow\Delta \quad \cancel{B\Uparrow B'} \quad \Gamma,A,\Delta\vdash B}{\Gamma,A\vdash B}}{\Gamma\vdash A\to B} \quad\to\quad \dfrac{\Gamma\Downarrow\Delta \quad \dfrac{\Gamma,\Delta,A\vdash B}{\Gamma,\Delta\vdash A\to B}}{\Gamma\vdash A\to B}$$

$$\dfrac{\dfrac{\Gamma\Downarrow\Delta \quad \cancel{A\Uparrow A'} \quad \Gamma,\Delta\vdash A}{\Gamma\vdash A} \quad \Gamma\vdash B}{\Gamma\vdash A\times B} \quad\to\quad \dfrac{\Gamma\Downarrow\Delta \quad \dfrac{\Gamma,\Delta\vdash A \quad \Gamma\vdash B}{\Gamma,\Delta\vdash A\times B}}{\Gamma\vdash A\times B}$$

$$\dfrac{\begin{matrix}\left(\dfrac{\Gamma,\cancel{A}\Downarrow\Delta \quad C\Uparrow^{?} D \quad \Gamma,A,\Delta\vdash D}{\Gamma,A\vdash C}\right)\\ \left(\dfrac{\Gamma,\cancel{B}\Downarrow\Delta \quad C\Uparrow^{?} D \quad \Gamma,B,\Delta\vdash D}{\Gamma,B\vdash C}\right)\end{matrix}}{\Gamma,A+B\vdash C} \quad\to\quad \dfrac{\Gamma\Downarrow\Delta \quad C\Uparrow^{?} D \quad \dfrac{\Gamma,A,\Delta\vdash D \quad \Gamma,B,\Delta\vdash D}{\Gamma,A+B,\Delta\vdash D}}{\Gamma,A+B\vdash C,D}$$

$$\dfrac{\dfrac{\Gamma\Downarrow\Gamma' \quad \Gamma,\Gamma';A\Downarrow B_1\times B_2}{\Gamma;A\Downarrow B_1\times B_2}}{\Gamma,\Gamma';A\Downarrow B_i} \quad\to\quad \dfrac{\Gamma\Downarrow\Gamma' \quad \dfrac{\Gamma,\Gamma';A\Downarrow B_1\times B_2}{\Gamma,\Gamma';A\Downarrow B_i}}{\Gamma;A\Downarrow B_i}$$

$$\dfrac{\dfrac{\Gamma\Downarrow\Gamma' \quad \Gamma,\Gamma';A\Downarrow B\to C}{\Gamma;A\Downarrow B\to C} \quad B\Uparrow B' \quad \Gamma\vdash B'}{\Gamma;A\Downarrow C} \quad\to\quad \dfrac{\Gamma\Downarrow\Gamma' \quad \dfrac{\Gamma,\Gamma';A\Downarrow B\to C \quad B\Uparrow B' \quad \Gamma\vdash B'}{\Gamma,\Gamma'\Downarrow B\to C}}{\Gamma;A\Downarrow C}$$

$$\dfrac{\Gamma;A\Downarrow B\to C \quad B\Uparrow B'_{na} \quad \dfrac{\Gamma\Downarrow\Gamma' \quad \Gamma,\Gamma'\vdash B'_{na}}{\Gamma\vdash B'_{na}}}{\Gamma;A\Downarrow C} \quad\to\quad \dfrac{\Gamma\Downarrow\Gamma' \quad \dfrac{\Gamma;A\Downarrow B\to C \quad B\Uparrow B'_{na} \quad \Gamma,\Gamma'\vdash B'_{na}}{\Gamma,\Gamma';A\Downarrow C}}{\Gamma;A\Downarrow C}$$

$$\dfrac{\dfrac{\Gamma\Downarrow\Gamma' \quad A_n\in\Gamma \quad \dfrac{\Gamma\Downarrow\Delta \quad \Gamma,\Delta;A_n\Downarrow A'}{\Gamma;A_n\Downarrow A'}}{\Gamma\Downarrow\Gamma',A'} \quad B\Uparrow^{?} B' \quad \Gamma,\Gamma',A'\vdash B'}{\Gamma\vdash B}$$

$$\to\quad \dfrac{\Gamma\Downarrow\Delta \quad \dfrac{\dfrac{\Gamma\Downarrow\Gamma' \quad A_n\in\Gamma \quad \Gamma,\Delta;A_n\Downarrow A'}{\Gamma,\Delta\Downarrow\Gamma',A'} \quad B\Uparrow^{?} B' \quad \Gamma,\Gamma',A'\vdash B'}{\Gamma,\Delta\vdash B}}{\Gamma\vdash B}$$

$$\dfrac{\Gamma\Downarrow\Delta \quad A\Uparrow^{?} B \quad \dfrac{\Gamma\Downarrow\Delta' \quad B\Uparrow^{?} C \quad \Gamma,\Delta,\Delta'\vdash C}{\Gamma,\Delta\vdash B}}{\Gamma\vdash A} \quad\leftrightarrow\quad \dfrac{\Gamma\Downarrow\Delta,\Delta' \quad A\Uparrow^{?} C \quad \Gamma,\Delta,\Delta'\vdash C}{\Gamma\vdash A}$$

Figure 4 Preemptive rewriting for multifocused natural deduction for intuitionistic logic.

In this situation, it may be the case that other parts of the multi-focus do not depend on the rule below, and those should not be blocked. To allow rewriting to continue, the last rewrite of our system is bidirectional. It allows to separate the foci of a multi-focus, in particular separate the foci that depend on the rule below from those that do not – and can thus permute again. This corresponds to the first rule of the original preemptive rewriting system [5], which splits a multi-focus in two. We only need to apply this rule when the result can make one more unidirectional rewrite step – this strategy ensures termination.

In the left-to-right direction, this rule relies on the possibility of merging together two elimination-focused derivations, or two optional introduction-focused derivations, with the implicit requirement that at least one of them is empty.

2.5 Reinversion

After the preemptive rewriting rules have been applied, the result is not, in general, a valid derivation in the non-preemptive system. Consider for example the following rewriting process:

$$\begin{bmatrix} & \nu_3 \\ & \pi_3 \\ \nu_2 & \\ \pi_2 & \\ \nu_1 & \\ \pi_1 & \end{bmatrix} \to^* \begin{bmatrix} & \nu_3 \\ \nu_2 & \\ & \pi_3;\nu_2 \\ \pi_2 & \\ \nu_1 & \\ \pi_1 & \end{bmatrix} \to^* \begin{bmatrix} \nu_2 & \nu_3 \\ \pi_2 & \pi_3 \\ \nu_1 & \\ \pi_1 & \end{bmatrix} \to^* \begin{bmatrix} \nu_2 & \nu_3 \\ \pi_2;\nu_3 & \\ \nu_1 & \\ & \pi_3;\nu_1 \\ \pi_1 & \end{bmatrix} \to^* \begin{bmatrix} \nu_2 & \nu_3 \\ \pi_2;\nu_3 & \\ \nu_1 & \\ \pi_1 & \pi_3 \end{bmatrix}$$

We are here representing derivations from a high-level point of view, by naming complete sequences of rules of the same polarity. Sequences of positive (non-invertible) are named π_n, and sequences of negative (invertible) rules ν_m. We use horizontal position to denote parallelism, or dependencies between phases: each dipole (π_k, ν_k) is vertically aligned as the invertibles of ν_k have been produced by the foci of π_k, but we furthermore assume that the second dipole depends on formulas released by the first, while the third dipole is independent.

The third dipole is independent from the others, and its foci in π_3 move downward in the derivation as expected in the preemptive system. After the first step, its negative phase has preempted the invertible phase ν_2, and it is thus written $\pi_3;\nu_2$ to emphasize that any rule of this sequence will have all the invertible formulas of ν_2 in non-focused positions (positives in the hypotheses, and negatives in the succedent). It can then be merged with the foci of π_2, in which case it does not see the invertibles of ν_2 anymore. When it moves further down, the invertible formulas in its topmost sequent, those consumed by ν_3, are present/preempted by all the non-invertible rules of π_2. It is eventually merged with π_1.

The normal form of this rewrite sequence could be considered a maximally multi-focused proof, in the sense that the foci happen as soon as possible in the derivation – which was not the case in the initial proof, where π_3 was delayed. However, while the initial proof is a valid proof in the non-preemptive system, the last derivation is not: the invertible formulas produced by π_3 are not consumed as early as possible, but only at the very end of the derivation, and the foci of π_2 therefore happen while there are still invertible rules to be applied.

We introduce a *reinversion* relation between proofs, written $\mathcal{D} \triangleright \mathcal{E}$, that turns the proof $\mathcal{D}$ with possible preemption into a proof $\mathcal{E}$ valid in the non-preemptive system, by doing the inversions where they are required, without changing the structure of the negative phases –

the foci are exactly the same. In our example, we have:

$$\begin{bmatrix} \nu_2 & \nu_3 \\ \pi_2;\nu_3 & \\ \nu_1 & \\ \pi_1 & \pi_3 \end{bmatrix} \triangleright \begin{bmatrix} \nu_2 & \\ \pi_2 & \\ \nu_1 & \nu_3 \\ \pi_1 & \pi_3 \end{bmatrix}$$

▶ **Definition 2** (Rewriting relation)**.** If $\mathcal{D}$ and $\mathcal{E}$ are proofs of the non-preemptive system, we write $\mathcal{D} \Rightarrow \mathcal{E}$ if there exists a $\mathcal{E}'$ such that $\mathcal{D} \rightarrow^* \mathcal{E}' \triangleright \mathcal{E}$.

Reinversion was not discussed directly in the original multi-focusing work [5], but it plays an important role and can be described and understood in several fairly different ways. For lack of space, we omit this discussion from this short article, and will only formally define reinversion as a relation on the (more concise) proof terms in Section 3.1, Definition 4.

2.6 Maximal multi-focusing and canonicity

Now that we have defined the focusing-lowering rewrite ($\Rightarrow$) between non-preemptive proof, we can define the notion of *maximal* multi-focusing and its meta-theory. It is defined by looking at the width of multi-focus phases in equivalence classes of rule permutations; but it can also be characterized as the normal forms of the ($\Rightarrow$) relation.

For lack of space, we have omitted this development (which is a mere adaptation of the previous work [5]) from this short article. The central result is summarized below.

▶ **Definition.** *We say that two proofs $\mathcal{D}$ and $\mathcal{E}$ are* locally equivalent*, or* iso-polar*, written $\mathcal{D} \approx_{loc} \mathcal{E}$, if one can be rewritten into the other using only local positive/positive and negative/negative permutations, preserving their initial sequents.*

▶ **Definition.** *We say that two proofs $\mathcal{D}$ and $\mathcal{E}$ are* globally equivalent*, or* iso-initial*, written $\mathcal{D} \simeq_{glob} \mathcal{E}$, when one can be rewritten into the other using local permutations of any polarity (so when seen as proofs in a non-focused system), preserving their initial sequents.*

▶ **Fact.** *Two multi-focused proofs are globally equivalent if and only if they are rewritten by ($\Rightarrow$) in locally equivalent maximal proofs.*

3 On the side of proof terms

3.1 Preemption and reinversion as term rewriting

Now that we have a notion of maximally multi-focused proofs in natural deduction, we can cross the second bridge between multi-focusing and Lindley's work by moving to a term system. We define in Figure 5 a term syntax for multi-focused derivations in natural deduction.

As the distinction between the preemptive and the non-preemptive systems are mostly about invariants of the focusing rule, the same term calculus is applicable to both. The only syntactic difference is that preemptive terms allow a multi-focusing $f[n]$ to preempt an ambient elimination focus n'.

Structural constraints on the multi-focusing system (preemptive or not) guarantee that strong typing invariants are verified. In particular, in a focused term ($\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? t$), the $\bar{n}$ are typed by the formulas in Δ at the end of a $\Gamma \Downarrow \Delta$ elimination phase: by our release discipline they have a positive or atomic type, so the `let`-introduced $\bar{x}$ are always bound to positive types. The rewriting rules corresponding to the preemptive rewriting relation are defined in Figure 6.

$$\begin{array}{lll} t ::= & & \text{terms} \\ & \mid x, y, z & \text{variable} \\ & \mid \lambda(x)\, t & \text{lambda} \\ & \mid (t, t) & \text{pair} \\ & \mid \delta(x, x.t, x.t) & \text{case} \\ & \mid f[t] & \text{focusing} \end{array} \qquad \frac{X \text{ atomic}}{\Gamma_{na}, x : X \vdash x : X}$$

$$\frac{\Gamma, x : A \vdash t : C \qquad \Gamma, x : B \vdash u : C}{\Gamma, x : A + B \vdash \delta(x, x.t, x.u) : C} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash (t, u) : A \times B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda(x)\, t : A \to B}$$

$$f[\square] ::= \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? \square \quad \text{multi-focusing} \qquad \begin{array}{lll} p^? ::= & & \text{optional introduction focus} \\ & \mid \emptyset & \text{no introduction} \\ & \mid p & \text{introduction focus} \end{array}$$

$$\frac{\Gamma_{na} \Downarrow \mathtt{let}\ \bar{x} = \bar{n} : \Gamma' \qquad A_{pa} \Uparrow^? p^? : A' \qquad \Gamma_{na}, \Gamma' \vdash t : A'}{\Gamma_{na} \vdash \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? t : A_{pa}}$$

$$\frac{}{A \Uparrow^? \emptyset : A} \qquad \frac{A_p \Uparrow p : B_{na}}{A_p \Uparrow^? p : B_{na}} \qquad \frac{(x^i : A_n^i)^{i \in I} \subseteq \Gamma \qquad (\Gamma; x^i : A_n^i \Downarrow n^i : A'^i_{pa})^{i \in I}}{\Gamma \Downarrow \mathtt{let}\ (x^i)^{i \in I} = (n^i)^{i \in I} : (A'^i_{pa})^{i \in I}}$$

$$\begin{array}{lll} n ::= & & \text{negatives} \\ & \mid x, y, z & \text{variable} \\ & \mid \pi_i\ n & \text{pair projection} \\ & \mid n\ p(t) & \text{function application} \\ & \mid \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n & \text{focusing (only in the preemptive calculus)} \end{array} \qquad \frac{}{\Gamma; x : A_{na} \Downarrow x : A_{na}}$$

$$\frac{\Gamma; A \Downarrow n : B \to C \qquad B \Uparrow p : B' \qquad \Gamma \vdash t : B'}{\Gamma; A \Downarrow n\ p(t) : C} \qquad \frac{\Gamma; A \Downarrow n : B_1 \times B_2}{\Gamma; A \Downarrow \pi_i\ n : B_i}\ i \in \{1, 2\}$$

$$\frac{\Gamma_{npa} \Downarrow \mathtt{let}\ (x^i)^{i \in I} = (n^i)^{i \in I} : \Delta'_{pa} \qquad \Gamma_{npa}, \Delta'_{pa}; A \Downarrow n' : A'}{\Gamma_{npa}; A \Downarrow \mathsf{let}\ (x^i)^{i \in I} = (n^i)^{i \in I}\ \mathsf{in}\ n' : A'}$$

$$\begin{array}{lll} p ::= & & \text{positives} \\ & \mid \star & \text{identity} \\ & \mid \sigma_i\ p & \text{sum injection} \end{array} \qquad \frac{}{A_{na} \Uparrow \star : A_{na}} \qquad \frac{A_i \Uparrow p : B}{A_1 + A_2 \Uparrow \sigma_i\ pB}\ i \in \{1, 2\}$$

Figure 5 Preemptive term calculus.

▶ **Lemma 3.** *If t is a proof term for the preemptive derivation $\mathcal{D}$, then $t \to u$ if and only if u is a proof term for a preemptive derivation $\mathcal{E}$ with $\mathcal{D} \to \mathcal{E}$.*

The reinversion relation also has a corresponding term-rewriting interpretation. To perform each invertible rule as early as it should be, it suffices to let any invertible rule skip over a non-invertible phase it does not depend on. Depending on the order of the invertible rules after this phase, the invertible rule we want to move may be after a series of invertible rules that cannot be moved.

We "skip" over invertible contexts, we reduce invertible rules happening inside contexts of

$$
\begin{array}{rcl}
\lambda(y)\ \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ t & \overset{y \notin \bar{n}}{\to} & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ \lambda(y)\, t \\
((\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ t_1), t_2) & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ (t_1, t_2) \\
(t_1, (\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ t_2)) & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ (t_1, t_2) \\
\delta(y, y_1.\, (\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? t_1), y_2.\, (\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? t_2)) & \overset{y_1, y_2 \notin \bar{n}}{\to} & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? \delta(y, y_1.t_1, y_2.t_2) \\
\pi_i\ (\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n') & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ \pi_i\ n' \\
(\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n')\ t & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n'\ t \\
n'\ p(\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ t) & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n'\ p(t) \\
\mathsf{let}\ y = (\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ n')\ \mathsf{in}\ p^? t & \to & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ \mathsf{let}\ y = n'\ \mathsf{in}\ p^? t \\
\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? (\mathsf{let}\ \bar{y} = \bar{n'}\ \mathsf{in}\ q^?\ t) & \overset{\bar{x} \notin \bar{n'}}{\leftrightarrow} & \mathsf{let}\ \bar{x}, \bar{y} = \bar{n}, \bar{n'}\ \mathsf{in}\ (p^?.q^?) t
\end{array}
$$

$$
\begin{array}{rclcrcl}
p^?.\emptyset & = & p^? & \quad & \star.q & = & q \\
\emptyset.q^? & = & q^? & & (\sigma_i\ p).q & = & \sigma_i\ (p.q)
\end{array}
$$

Figure 6 Preemptive rewriting on proof terms.

$$
\begin{array}{rcl}
F_i[\square] & ::= & \lambda(x)\,\square \\
& | & \delta(x, x_1.\square, x_2.t) \\
& | & \delta(x, x_1.t, x_2.\square) \\
& | & (t, \square) \\
& | & (\square, t)
\end{array}
\qquad
\begin{array}{rcl}
C_{neg}[\square] & ::= & n\ p(\square) \\
& | & C_{neg}[\square]\ p(t) \\
& | & \pi_i\ C_{neg}[\square] \\
& | & C_{ni}[C_{neg}[\square]]
\end{array}
$$

$$
C_i[\square] \ ::= \ \square \ \mid \ F_i[C_i[\square]]
\qquad
\begin{array}{rcl}
C_{ni}[\square] & ::= & \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? \square \\
& | & \mathsf{let}\ \bar{x}, y = \bar{n}, C_{neg}[\square]\ \mathsf{in}\ p^? t
\end{array}
$$

Figure 7 Invertible frames and contexts, non-invertible contexts and elimination contexts.

the form $C_{ni}[C_i[\ ]]$, where $C_i[t]$ is a notation for invertible contexts (defined using invertible frames $F_i[t]$), and $C_{ni}[t]$ for non-invertible contexts. Defining the latter requires describing negative/elimination contexts $C_{neg}[t]$, with holes where a term may appear in a series of elimination-focused terms.

▶ **Definition 4.** Reinversion can be precisely defined as the transitive congruence closure of the rewrite rules listed in Figure 8.

The rewrite conditions are expressed in terms of a $C[\square] \prec c$ relation (read "context C *blocks* term-constructor c") that indicates a dependency of an invertible construction c on a given context $C[\square]$. For example, it would make no sense to extrude a λ in argument position in a destructor, or move a sum-elimination $\delta(x)$ across the frame that defined the variable x. This blocking relation is defined in Figure 9 – $(A \mid B)$ in a rule means that the rule holds with either A or B in place of $(A \mid B)$.

It may at first seem surprising that reinversion rules have instances that are the *opposite* of some of the preemptive rewriting rules – those about pos/neg permutations. But that is precisely one of the purposes of reinversion: after preemptive rewriting rules have been fully applied, we undo those that have gone "too far", in the sense that they let a non-invertible phase preempt a portion of an invertible phase below, but were blocked by dependencies without reaching the next non-invertible phase. This blocked phase does not increase the parallelism of multi-focusing in the proof, but stops the derivation from being valid in the original multi-focusing system, so reinversion undoes its preemption.

$$C_{ni}[C_i[\lambda(x)\,t]] \overset{C_{ni}[C_i[\Box]] \not\prec \lambda}{\triangleright} \lambda(x)\,C_{ni}[C_i[t]]$$

$$C_{ni}[C_i[(t_1, t_2)]] \overset{C_{ni}[C_i[\Box]] \not\prec (,)}{\triangleright} (C_{ni}[C_i[t_1]], C_{ni}[C_i[t_2]])$$

$$C_{ni}[C_i[\delta(x, x_1.t_1, x_2.t_2)]] \overset{C_{ni}[C_i[\Box]] \not\prec \delta(x)}{\triangleright} \delta(x, x_1.C_{ni}[C_i[t_1]], x_2.C_{ni}[C_i[t_2]])$$

Figure 8 Reinversion rewrite rules.

$$c ::= (,) \mid \lambda \mid \delta(x)$$

$$\frac{y \in \bar{x}}{\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^?\Box \prec \delta(y)} \qquad \frac{p \neq \emptyset}{\mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p\Box \prec (,) \mid \lambda} \qquad \frac{C_{neg}[\Box] \prec c}{\mathsf{let}\ \bar{x}, y = \bar{n}, C_{neg}[\Box]\ \mathsf{in}\ p^?t \prec c}$$

$$((\Box, t) \mid (t, \Box) \mid \lambda(x)\,\Box) \prec ((,) \mid \lambda) \qquad \lambda(x)\,\Box \prec \delta(x) \qquad (\delta(x, y.\Box, z.t) \mid \delta(x, z.t, y.\Box)) \prec \delta(y)$$

$$n\ p(\Box) \prec (,) \mid \lambda \qquad \frac{C_{neg}[\Box] \prec c}{C_{neg}[\Box]\ p(t) \mid \pi_i\ C_{neg}[\Box] \mid C_{ni}[C_{neg}[\Box]] \prec c}$$

$$\frac{C_{ni}[\Box] \prec c \mid C_{neg}[\Box] \prec c}{C_{ni}[C_{neg}[\Box]] \prec c} \qquad \frac{F_i[\Box] \prec c \mid C_i[\Box] \prec c}{F_i[C_i[\Box]] \prec c} \qquad \frac{C_{ni}[\Box] \prec c \mid C_i[\Box] \prec c}{C_{ni}[C_i[\Box]] \prec c}$$

Figure 9 Reinversion blocking relation.

Remark, in relation to this situation, that preemptive rewriting cannot be easily defined on equivalence classes of neg/neg permutations (or other presentations of focusing that crush the invertible phase in one not-so-interesting step, such as higher-order focusing), as the order of the invertible rules in a single phase may determine where a non-invertible phase stops its preemption and is blocked in the middle of the invertible phase. Reinversion restores this independence on invertible ordering. This explains why the meta-theory of maximal multi-focusing was conducted in the non-preemptive system, using the relation between proofs that always applies reinversion after preemptive rewriting.

The other interesting case is a non-invertible phase π_0 having traversed a family of non-invertible phases $(\pi'_i)_{i\in I}$, before merging into some non-invertible phase π_1. Reinversion will move its negative phase ν_0, reverting the preemption of the (π'_i) on the invertible formulas introduced by π_0. But the important preemptions that happened, namely the traversal by π_0 of each of the invertible phases $(\nu'_i)_{i\in I}$, are not reverted: each ν'_i is blocked by the π'_i below and thus cannot be reverted below π_0. As π_0 traversed *both* the ν'_i and the π'_i, it does not have the corresponding invertible formulas in its context anymore, and is well-positioned even in a non-preemptive proof.

▶ **Lemma 5.** *If t is the proof term of the preemptive derivation $\mathcal{D} : \Gamma \vdash A$, and u is such that $t \triangleright u$, then u is a valid (preemptive) proof term for $\Gamma \vdash A$.*

▶ **Lemma 6.** *If u is a valid proof term in the preemptive system, and a normal form of the relation $(\triangleright)$, then u is also a valid proof term for the non-preemptive system.*

▶ **Theorem 7.** *If t is a proof term for $\mathcal{D}$ and u for $\mathcal{E}$, then $\mathcal{D} \Rightarrow \mathcal{E}$ if and only there is a u' such that $t \to^* u' \triangleright u$, and u is a normal form for $(\triangleright)$.*

3.2 Multi-focused terms as lambda-terms

There is a natural embedding $\lfloor t \rfloor$ of a multi-focused term t into the standard lambda-calculus, generated by the following transformation, where $t[\bar{x} := \bar{u}]$ represents simultaneous substitution:

$$\lfloor \mathsf{let}\ \bar{x} = \bar{n}\ \mathsf{in}\ p^? t \rfloor := \lfloor p^? \rfloor (\lfloor t \rfloor [\bar{x} := \bar{\lfloor n \rfloor}])$$

$$\lfloor \emptyset \rfloor (t) := t \qquad \lfloor \star \rfloor (t) := t \qquad \lfloor \sigma_i\ p \rfloor (t) := \sigma_i\ \lfloor p \rfloor (t)$$

The substitutions break the invariant that the scrutinee of a sum-elimination construct is always a variable. However, as only negative terms are substituted, sum-elimination scrutinee are always neutrals – embedding of negative terms. In particular, this embedding does not create any β-redex. Proof terms coming from non-preemptive multi-focusing are also always in η-long form, and this is preserved by the embedding; with the restriction present in Lindley's work that only neutral terms (eliminations) are expanded – this avoids issues of commuting conversions. We mean here the *weak* η-long form, determined by the weak equation $(m : A + B) =_{\text{weak-}\eta} \delta(m, x_1.\sigma_1\ x_1, x_2.\sigma_2\ x_2)$.

▶ **Lemma 8.** *If $\Gamma \vdash t : A$ in the preemptive multi-focused system, then $\Gamma \vdash \lfloor t \rfloor : A$ in simply-typed lambda-calculus, and $\lfloor t \rfloor$ is in β-normal form. If t is valid in the non-preemptive system, then the* pure neutral subterms *of $\lfloor t \rfloor$ are also in* weak *η-long form.*

3.3 Lindley's rewriting relation

The strong η-equivalence for sums makes lambda-term equivalence a difficult notion. For any term $m : A+B$ and well-typed context $C[\square]$, it dictates that $C[m] \approx \delta(m, x_1.C[x_1], x_2.C[x_2])$. In his article [8], Sam Lindley breaks it down in four simpler equations, including in particular the "weak", non-local η-rule (where F represents a frame, that is a context of term-size exactly 1):

$$m \approx \delta(m, x_1.\sigma_1\ x_1, x_2.\sigma_2\ x_2) \quad (+.\eta)$$

$$F[\delta(p, x_1.t_1, x_2.t_2)] \approx \delta(p, x_1.F[t_1], x_2.F[t_2]) \quad (\textit{move-case})$$

$$\delta\left(p, \begin{array}{l} x_1.\delta(p, y_1.t_1, y_2.t_2), \\ x_2.\delta(p, z_1.u_1, z_2.u_2) \end{array}\right) \approx \delta(p, x_1.t_1[y_1 := x_1], x_2.u_2[z_2 := x_2]) \quad (\textit{repeated-guard})$$

$$\delta(p, x_1.t, x_2.t) \overset{x_1, x_2 \notin t}{\approx} t \quad (\textit{redundant-guard})$$

Lindley further refines the *move-case* equivalence into a less-local *hoist-case* rule. Writing D for a frame that is either $\delta(p, x_1.\square, x_2.t)$ or $\delta(p, x_1.t, x_2.\square)$, D^* for an arbitrary (possibly empty) sequence of them, and H any frame that *is not* of this from, *hoist-case* is defined as:

$$H[D^*[\delta(t, x_1.t_1, x_2.t_2)]] \rightarrow \delta(t, x_1.H[D^*[t_1]], x_2.H[D^*[t_2]])$$

Lindley's equivalence algorithm (Theorem 36, p. 13) proceeds in three steps: rewriting terms in $\beta\eta\gamma_E$-normal forms (using the weak $(+.\eta)$ on sums), then rewriting them in γ-normal form, and finally using a decidable redundancy-eliminating equivalence relation called $\sim$. The rewriting relation γ is defined as the closure of *repeated-guard*, *redundant-guard* (when read left-to-right) and *hoist-case*; γ_E is a weak restriction of it defined below. The equivalence $\sim$ is the equivalence closure of the equivalence *repeated-guard*, *redundant-guard*, and *move-case* restricted to D-frames – clauses of a sum elimination.

We discuss redundancy elimination, that is aspects related to *repeated-guard* and *redundant guard*, in Section 4, and focus here on explanation of the other rewriting processes ($\beta\eta\gamma_E$ and *hoist-case*) in logical terms. We show that multi-focused terms in ($\Rightarrow$)-normal form embed into $\beta\eta\gamma_E\gamma$-normal forms. As we ignore redundancy elimination, this is modulo $\sim$.

The β and η rewriting rules are standard – for sums, this is the weak, local η-relation, and not the strong η-equivalence. As explained in the previous subsection, embeddings of proof terms valid in the non-preemptive system – as are ($\Rightarrow$)-normal forms – are in $\beta\eta$-normal form. The rewriting γ_E is defined as the extrusion of a sum-elimination out of an elimination context: $\square\ t \mid \pi_i\ \square \mid \delta(\square, x_1.t, x_2.t)$.

▶ **Lemma 9.** *Terms for valid preemptive multi-focusing derivations are in γ_E-normal form.*

This rigid structure of focused proofs is well-known, just as $\beta\eta$-normality or commuting conversions are not the interesting points of Lindley's work. The crux of the correspondence is between the transformation to maximal proofs, computed by ($\Rightarrow$), and his γ-rewriting relation. There is an interesting dichotomy:

- Preemptive rewriting, which merges non-invertible phases, is where most of the work happens from a logical point of view. Yet this transformation, on the embeddings of the multi-focused proof terms, corresponds to the identity!
- Reinversion, which is obvious logically as it only concerns invertible rules which commute easily, corresponds to γ-rewriting on the embeddings.

Of course, preemptive rewriting is in fact crucial for γ-rewriting. It is the one that determines upto where negative terms can move in the derivation, and in particular the scrutinees of sum eliminations. Reinversion would not work without the first preemptive rewriting step, and applying reinversion on a proof term that is not in preemptive-normal form may not give a γ-normal embedding. Note that the proof of the last theorem in this section makes essential use of the confluence of γ-rewriting, one of Lindley's key results.

▶ **Lemma 10.** *If $t \to u$, then $\lfloor t \rfloor =_\alpha \lfloor u \rfloor$.*

▶ **Lemma 11.** *If $t \rhd u$, then $\lfloor t \rfloor \to_\gamma^* \lfloor u \rfloor$.*

▶ **Lemma 12.** *If u is in ($\Rightarrow$)-normal form, then for some $u' \approx_{loc} u$, $\lfloor u' \rfloor$ is in γ-normal form modulo $\sim$.*

▶ **Theorem 13** (γ-normal forms are embeddings of maximally-focused proofs)**.** *If $\lfloor t \rfloor \to_\gamma^* n$ and n is γ-normal, then there are $u \approx_{loc} u'$ such that $t \Rightarrow u$ and $\lfloor u' \rfloor \sim n$. In particular, u is maximally multi-focused.*

4 Redundancy elimination

In the previous section, we have glossed over the fact that Lindley's γ-reduction also simplifies redundant and duplicated sum-eliminations. Those simplifications are *not* implied by multi-focusing – they are not justified by proof theory alone. Our understanding is that they correspond to purity assumptions that are stronger than the natural equational theory of focused proofs. On the other hand, starting from maximally multi-focused forms is essential to being able to define those extra simplifications. We do so in this section, to obtain a system that is completely equivalent to Lindley's.

We simply have to add the following simplifications on proof terms:

REDUNDANT-FOCUS
$$\text{let } \bar{x}, y, z = \bar{n}, n', n' \text{ in } p^? t \rightarrow_s \text{let } \bar{x}, y = \bar{n}, n' \text{ in } p^? t[z := y]$$

REDUNDANT-GUARD
$$\delta(x, x_1.t, x_2.t) \overset{x_1,x_2 \notin t}{\approx_{\text{loc}}} t$$

REPEATED-CASE-1
$$\delta(x, x_1.\delta(x, y_1.u_1, y_2.u_2), x_2.t_2) \approx_{\text{loc}} \delta(x, x_1.u_1[y_1 := x_1], x_2.t_2)$$

REPEATED-CASE-2
$$\delta(x, x_1.t_1, x_2.\delta(x, y_1.u_1, y_2.u_2)) \approx_{\text{loc}} \delta(x, x_1.y_1, x_2.u_2[y_2 := x_2])$$

While those rules are not implied by focusing, they are reasonable in a focused setting, as they respect the phase separation. As the redundancy-elimination rules test for equality of subterms, they have an unpleasant non-atomic aspect (repeated cases only test variables), but this seems unavoidable to handle sum equivalence (Lindley [8], or Balat, Di Cosmo and Fiore [2], have a similar test in their normal form judgments), and have also been used previously in the multi-focusing literature, for other purposes; in Alexis Saurin's PhD thesis [10], an equality test is used to give a convenient $\otimes/\&$ permutation rule (p. 231).

▶ **Definition 14.** We define the relation $t \Rightarrow_s u$ between proof terms of the (preemptive) multi-focusing calculus as follows, where t_1 is a preemptive normal form, t_2 is a redundant-foci normal form, and u_0 is a ($\triangleright$)-normal form: $t \rightarrow^* t_1 \rightarrow_s^* t_2 \triangleright^* u_0 \approx_{\text{loc}} u$

▶ **Definition 15.** We call the u in the target of the $(\Rightarrow_s)$ relation *simplified maximal forms.*

▶ **Theorem 16** (Simplified maximal forms are γ-normal)**.** *Given a multi-focused term t, there exists some u such that $t \Rightarrow_s u$, $\lfloor t \rfloor \rightarrow_\gamma^* \lfloor u \rfloor$, and $\lfloor u \rfloor$ is in γ-normal form. This u is unique modulo local equivalence.*

▶ **Corollary 17.** *Two multi-focused proof terms are extensionally equivalent if their maximally multi-focused normal forms are locally equivalent (modulo redundancy elimination).*

Related and Future work

Maximally multi-focused proofs were previously used to bridge the gap between sequent calculus, as a rather versatile way of defining proof systems, and specialized proof structures designed to minimize redundancy for a fixed logic. The original paper on multi-focusing [5] demonstrated an isomorphism between maximal proofs and proof nets for a subset of linear logic. In recent work [4], maximally multi-focused proof of a sequent calculus for first-order logic have been shown isomorphic to *expansion proofs*, a compact description of first-order classical proofs.

There are some recognized design choices in the land of equivalence-checking presentation that can now be linked to design choices of focused system. For example, Altenkirch et al. [1] proposed to make the syntax more canonical with respect to redundancy-elimination by using a n-ary sum elimination construct, while Lindley prefers to quotient over local reorderings of unary sum-eliminations. This sounds similar to the choice between higher-order focusing ([12]), where all invertible rules are applied at once, or quotienting of concrete proofs by neg/neg permutations as used here.

When we started this work, we planned to also study the proof-term presentation of preemptive rewriting, in a term language for sequent calculus. We have been collaborating with Guillaume Munch-Maccagnoni to study the normal forms of an intuitionistic restriction of System L, with sums. In this untyped calculus, syntactic phases appear that closely resemble a focusing discipline, and equivalence relations can be defined in a more uniform way, thanks to the symmetric status of the (non)-invertible rules that "change the type of the result" (terms, values) and those that only manipulate the environments (co-terms, stacks).

Conclusion

We propose a multi-focused calculus for intuitionistic logic in natural deduction, and establish the canonicity of maximally multi-focused proofs by transposing the preemptive rewriting technique [5] in our intuitionistic, natural deduction setting. By studying the computational effect of preemptive rewriting on proof terms, we demonstrate the close correspondence with the rewriting on lambda-terms with sums proposed by Lindley [8] to compute extensional equivalence. Adding a notion of redundancy elimination to our multi-focused system makes preemptive rewriting precisely equivalent to Lindley's γ-rules. In particular, the resulting canonical forms, *simplified maximal proofs*, capture extensional equality.

Acknowledgements. The author thanks his advisor, Didier Rémy for the freedom of making an overly long detour through the proof search literature; Alexis Saurin for his advice, and Pierre-Évariste Dagand, Sam Lindley, and anonymous reviewers for their helpful comments.

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