

Infinite and Bi-infinite Words with Decidable Monadic Theories*

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Abstract

We study word structures of the form (D, \leq, P) where D is either \mathbb{N} or \mathbb{Z} , \leq is a linear ordering on D and $P \subseteq D$ is a predicate on D . In particular we show:

- (a) The set of recursive ω -words with decidable monadic second order theories is Σ_3 -complete.
- (b) We characterise those sets $P \subseteq \mathbb{Z}$ that yield bi-infinite words (\mathbb{Z}, \leq, P) with decidable monadic second order theories.
- (c) We show that such “tame” predicates P exist in every Turing degree.
- (d) We determine, for $P \subseteq \mathbb{Z}$, the number of predicates $Q \subseteq \mathbb{Z}$ such that (\mathbb{Z}, \leq, P) and (\mathbb{Z}, \leq, Q) are indistinguishable.

Through these results we demonstrate similarities and differences between logical properties of infinite and bi-infinite words.

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1 Introduction

The decision problem for logical theories of linear structures and their expansions has been an important question in theoretical computer science. Büchi in [2] proved that the monadic second order theory of the linear ordering (\mathbb{N}, \leq) is decidable. Expanding the structure (\mathbb{N}, \leq) by unary functions or binary relations typically leads to undecidable monadic theories. Hence many works have been focusing on structures of the form (\mathbb{N}, \leq, P) where P is a unary predicate. Elgot and Rabin [5] showed that for many natural unary predicates P , such as the set of factorial numbers, the set of powers of k , and the set of k th powers (for fixed k), the structure (\mathbb{N}, \leq, P) has decidable monadic second order theory; on the other hand, there are structures (\mathbb{N}, \leq, P) whose monadic theory is undecidable [3]. Numerous subsequent works further expanded the field [13, 4, 11, 10, 9, 8].

1. Semenov generalised periodicity to a notion of “almost periodicity”. While periodicity implies that certain patterns are repeated through a fixed period, almost periodicity captures the fact that certain patterns occur before the expiration of some period. This led him to consider “recurrent structures” within an infinite word. Such a recurrent structure is captured by a certain function, which he called “indicator of recurrence”. In [11], he provided a full characterisation: (\mathbb{N}, \leq, P) has decidable monadic theory if and only if P is recursive and there is a recursive indicator of recurrence for P .

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2. Rabinovich and Thomas generalised periodicity to a notion of “uniform periodicity”. Such a uniform periodicity condition is captured by a *homogeneous set* which exists by Ramsey’s theorem. More precisely, a k -homogeneous set for (\mathbb{N}, \leq, P) partitions the natural numbers into infinitely many finite segments that all have the same k -type. A uniformly homogeneous set specifies an ascending sequence of numbers that ultimately becomes k -homogeneous for any $k > 0$. In [9], Rabinovich and Thomas provided a full characterisation: (\mathbb{N}, \leq, P) has a decidable monadic theory if and only if P is recursive and there is a recursive uniformly homogeneous set.

Note that a recursive uniformly homogeneous set describes *how to* divide (\mathbb{N}, \leq, P) such that the factors all have the same k -type. If P is recursive, this implies that the recurring k -type can be computed. A weakening of the existence of a recursive uniformly homogeneous set is therefore the requirement that one can compute a k -type such that (\mathbb{N}, \leq, P) *can, in some way*, be divided. Nevertheless, Rabinovich and Thomas also showed that the monadic second order theory of (\mathbb{N}, \leq, P) is decidable if and only if P is recursive and there is a “recursive type-function” (see below for precise definitions).

This paper has three general goals: The first is to compare these characterisations in some precise sense. The second is to investigate the above results in the context of *bi-infinite words*, which are structures of the form (\mathbb{Z}, \leq, P) . The third is to compare the logical properties of infinite words and bi-infinite words. More specifically, the paper discusses:

- (a) In Section 4, we analyze the recursion-theoretical bound of the set of all computable predicates $P \subseteq \mathbb{N}$ where (\mathbb{N}, \leq, P) has a decidable monadic theory. The second characterisation by Rabinovich and Thomas turns out to be a Σ_5 -statement. In contrast, the characterisation by Semenov and the 1st characterisation by Rabinovich and Thomas both consist of Σ_3 statements, and hence deciding if a given (\mathbb{N}, \leq, P) has decidable monadic theory is in Σ_3 . We show that the problem is in fact Σ_3 -complete. Hence these two characterisations are optimal in terms of their recursion-theoretical complexity.
- (b) In Section 5, we then investigate which of the three characterisations can be lifted to bi-infinite words, i.e., structures of the form (\mathbb{Z}, \leq, P) with $P \subseteq \mathbb{Z}$. It turns out that this is nicely possible for Semenov’s characterisation and for the second characterisation by Rabinovich and Thomas, but not for their first one.
- (c) If the monadic second order theory of (\mathbb{N}, \leq, P) is decidable, then P is recursive. For bi-infinite words of the form (\mathbb{Z}, \leq, P) , this turns out not to be necessary. In Section 6, we actually show that every Turing degree contains a set $P \subseteq \mathbb{Z}$ such that the monadic second order theory of (\mathbb{Z}, \leq, P) is decidable.
- (d) The final Section 7 investigates how many bi-infinite words are indistinguishable from (\mathbb{Z}, \leq, P) . It turns out that this depends on the periodicity properties of P : if P is periodic, there are only finitely many equivalent bi-infinite words, if P is recurrent and non-periodic, there are 2^{\aleph_0} many, and if P is not recurrent, then there are \aleph_0 many.

2 Preliminaries

2.1 Words

We use \mathbb{N} , $\tilde{\mathbb{N}}$ and \mathbb{Z} to denote the set of natural numbers (including 0), negative integers (not containing 0), and integers, respectively. A *finite word* is a mapping $u: \{0, 1, \dots, n-1\} \rightarrow \{0, 1\}$ with $n \in \mathbb{N}$, it is usually written $u(0)u(1)u(2) \cdots u(n-1)$. The set of positions of u is $\{0, 1, \dots, n-1\}$, its length $|u|$ is n . The unique finite word of length 0 is denoted ε . The set of all (resp. non-empty) finite words is $\{0, 1\}^*$ (resp. $\{0, 1\}^+$). An ω -*word* is a mapping $\alpha: \mathbb{N} \rightarrow \{0, 1\}$; it is usually written as the sequence $\alpha(0)\alpha(1)\alpha(2) \cdots$. Its set of positions

is \mathbb{N} ; $\{0, 1\}^\omega$ is the set of ω -words. An ω^* -word is a mapping $\alpha: \tilde{\mathbb{N}} \rightarrow \{0, 1\}$; it is usually written as the sequence $\cdots \alpha(-3)\alpha(-2)\alpha(-1)$. Its set of positions is $\tilde{\mathbb{N}}$ and $\{0, 1\}^{\omega^*}$ is the set of ω^* -words. Finally, a *bi-infinite word* ξ is a mapping from \mathbb{Z} into $\{0, 1\}$, written as the sequence $\cdots \xi(-2)\xi(-1)\xi(0)\xi(1)\xi(2)\cdots$ (this notation has to be taken with care since, e.g., the bi-infinite words $\xi_i: \mathbb{Z} \rightarrow \{0, 1\}: n \mapsto (|n| + i) \bmod 2$ with $i \in \{0, 1\}$ are both described as $\cdots 0101010\cdots$, but they are different). The set of positions of a bi-infinite word is \mathbb{Z} . When saying “word”, we mean “a finite, an ω -, an ω^* - or a bi-infinite word”, “infinite word” means “ ω - or ω^* -word”.

The concatenation uv of two finite words u, v has its usual meaning. More generally, and in a similar way, we can also concatenate a finite or ω^* -word u and a finite or ω -word v giving rise to some word uv . Similarly, we can concatenate infinitely many finite words u_i giving an ω -word $u_0u_1u_2\cdots$, an ω^* -word $\cdots u_{-2}u_{-1}u_0$, and a bi-infinite word $\cdots u_{-2}u_{-1}u_0u_1u_2\cdots$ (where the position 0 is the first position of u_0). As usual, u^ω denotes the ω -word $uuuu\cdots$ for $u \in \{0, 1\}^+$, analogously, $u^{\omega^*} = \cdots uuu$.

Let w be some word and i, j be two positions with $i \leq j$. Then we write $w[i, j]$ for the finite word $w(i)w(i+1)\cdots w(j) \in \{0, 1\}^+$. A finite word u is a *factor* of w if $u = w[i, j]$ for some i, j or if u is the empty word ε . The set of factors of w is $F(w)$. If w is an ω - or a bi-infinite word, then $w[i, \infty)$ is the ω -word $w(i)w(i+1)w(i+2)\cdots$. If w is an ω^* - or a bi-infinite word, then $w(-\infty, i]$ is the ω^* -word $\cdots w(i-2)w(i-1)w(i)$. A bi-infinite word β is *recurrent* if for all $u \in F(\beta)$ and all $i \in \mathbb{Z}$, $u \in F(\beta[i, \infty)) \cap F(\beta(-\infty, i])$.

Let u be some finite word. Then u^R is the *reversal* of u , i.e., the finite word of length $|u|$ with $u^R(i) = u(|u| - i - 1)$ for all $0 \leq i < |u|$. The reversal of an ω -word (resp. ω^* -word) α is the ω^* -word (resp. ω -word) α^R with $\alpha^R(i) = \alpha(-i - 1)$ for all positions i . Finally, the reversal of a bi-infinite word ξ is the bi-infinite word ξ^R with $\xi^R(i) = \xi(-i)$ for all $i \in \mathbb{Z}$.

2.2 Logic

With any word w , we associate a relational structure $M_w = (D, \leq, P)$ where $D \subseteq \mathbb{Z}$ is the set of positions of w , \leq is the restriction of the natural linear order on \mathbb{Z} to D , and $P = \{n \in D \mid w(n) = 1\} = w^{-1}(1)$. Structures of this form are called *labeled linear orders*. The word w is *recursive* (resp. *recursively enumerable*) if so is the set P .

We use the standard logical system over the signature of labeled linear orders. Hence first order logic FO has relational symbols \leq and P . The monadic second order logic MSO extends FO by allowing unary second order variables X, Y, \dots , their corresponding atomic predicates (e.g. $X(y)$), and quantification over set variables. By **Sent**, we denote the set of sentences of the logic MSO. For a word w and an MSO-sentence φ , we write $w \models \varphi$ for “the sentence φ holds in the relational structure M_w ”. The *MSO-theory* of the word w is the set $\text{MTh}(M)$ of all MSO-sentences φ that are true in w .

► **Example 1.** Let $n \in \mathbb{N}$ and consider the following formula:

$$\varphi(x, y) = \exists X: \forall z: (X(z) \Leftrightarrow z = x \vee (x < z \wedge X(z - n))) \wedge X(y)$$

If w is a word with positions i, j , then $w \models \varphi(i, j)$ if and only if $i \leq j$ and $n \mid j - i$.

With any MSO-formula φ , we associate its *quantifier rank* $\text{qr}(\varphi) \in \mathbb{N}$: the atomic formulas have quantifier rank 0; $\text{qr}(\varphi_1 \wedge \varphi_2) = \text{qr}(\varphi_1 \vee \varphi_2) = \max\{\text{qr}(\varphi_1), \text{qr}(\varphi_2)\}$; $\text{qr}(\neg\varphi) = \text{qr}(\varphi)$; and $\text{qr}(\exists X: \varphi) = \text{qr}(\forall X: \varphi) = \text{qr}(\varphi_1) + 1$ where X is a first- or second-order variable.

► **Definition 2.** Let $k \in \mathbb{N}$. Two words w_1 and w_2 are *k-equivalent* (denoted $w_1 \equiv_k w_2$) if $w_1 \models \varphi$ iff $w_2 \models \varphi$ for all MSO-sentences φ with $\text{qr}(\varphi) \leq k$. Equivalence classes of this equivalence relation are called *k-types*. The words w_1 and w_2 are *MSO-equivalent* (denoted $w_1 \equiv w_2$) if $w_1 \equiv_k w_2$ for all $k \in \mathbb{N}$. Equivalence classes of \equiv are called *types*.

Let $k \geq 2$ and u, v be two words with $u \equiv_k v$. If u is finite, then it satisfies the sentence $(\exists x \forall y: x \leq y) \wedge (\exists x \forall y: x \geq y)$. Consequently, also v is finite. Analogously, u is an ω -word iff v is an ω -word etc. We will therefore speak of a “ k -type of finite words” when we mean a k -type that contains some finite word (and analogously for ω -, ω^* -, bi-infinite words etc).

Often, we will use the following known results without mentioning them again. They follow from the well-understood relation between MSO and automata (cf. [15, 6]).

► **Theorem 3.**

1. Let $k \geq 2$.
 - For any ω -word (ω^* -word) α , there exist finite words x and y with $xy \equiv_k x$ ($yx \equiv_k x$), $yy \equiv_k y$ and $\alpha \equiv_k xy^\omega$ ($\alpha \equiv_k y^{\omega^*}x$). Any such pair (x, y) is a representative of the k -type of α .
 - For any bi-infinite word ξ , there exist finite words x, y and z with $xy \equiv_k yz \equiv_k y$, $xx \equiv_k x$, $zz \equiv_k z$, and $\xi \equiv_k x^{\omega^*}yz^\omega$. Any such triple (x, y, z) is a representative of the k -type of ξ .
2. The following sets are decidable:
 - $\{\varphi \in \text{Sent} \mid \forall u \in \{0, 1\}^*: u \models \varphi\}$ and $\{(u, \varphi) \mid u \in \{0, 1\}^*, \varphi \in \text{Sent}, u \models \varphi\}$
 - $\{(u, v, \varphi) \mid u, v \in \{0, 1\}^*, v \neq \varepsilon, \varphi \in \text{Sent}, uv^\omega \models \varphi\}$
 - $\{(u, v, w, \varphi) \mid u, v, w \in \{0, 1\}^*, u, w \neq \varepsilon, \varphi \in \text{Sent}, u^{\omega^*}vw^\omega \models \varphi\}$
 - $\{(u, v, k) \mid u, v \in \{0, 1\}^*, k \in \mathbb{N}, u \equiv_k v\}$. This means in particular that it is decidable whether u and v represent the same k -type of finite words.
 - Similarly, it is decidable whether two pairs of finite words represent the same k -type of ω -words (of ω^* -words, resp). It is also decidable whether two triples of finite words represent the same k -type of bi-infinite words.
3. If $u, v \in \{0, 1\}^* \cup \{0, 1\}^{\omega^*}$ and $u', v' \in \{0, 1\}^* \cup \{0, 1\}^\omega$ with $u \equiv_k v$ and $u' \equiv_k v'$, then $uu' \equiv_k vv'$. From representatives of the k -types of u and v , one can compute a representative of the k -type of uv .
4. If $u_i, v_i \in \{0, 1\}^+$ with $u_i \equiv_k v_i$ for all $i \in \mathbb{Z}$, then we have

$$u_0u_1 \cdots \equiv_k v_0v_1 \cdots, \text{ and } \cdots u_{-1}u_0 \equiv_k \cdots v_{-1}v_0, \text{ and } \cdots u_{-1}u_0u_1 \cdots \equiv_k \cdots v_{-1}v_0v_1 \cdots$$

5. If u is a finite or ω^* -word and v is a finite or ω -word such that $\text{MTh}(u)$ and $\text{MTh}(v)$ are both decidable, then $\text{MTh}(uv)$ is decidable [12].

2.3 Recursion theoretic notions

This paper makes use of standard notions in recursion theory; the reader is referred to [14] for a thorough introduction. We assume a canonical effective enumeration $\Phi_0, \Phi_1, \Phi_2, \dots$ of all partial recursive functions on the natural numbers. The set W_e is the domain $\text{dom}(\Phi_e)$ and is the e th recursively enumerable set. Let TOT be the set $\{e \in \mathbb{N} \mid \Phi_e \text{ is total}\}$ and REC be the set $\{e \in \mathbb{N} \mid W_e \text{ is decidable}\}$.

A set $A \subseteq \mathbb{N}$ belongs to the level Π_2 of the arithmetical hierarchy if there exists a decidable set $P \subseteq \mathbb{N}^{m+n+1}$ such that A is the set of natural numbers a satisfying $\forall x_1, \dots, x_m \exists y_1, \dots, y_n: P(a, \bar{x}, \bar{y})$. A set $B \subseteq \mathbb{N}$ is Π_2 -hard if, for every $A \in \Pi_2$, there exists a m -reduction from A to B ; the set B is Π_2 -complete if, in addition, $B \in \Pi_2$. Similarly, $A \subseteq \mathbb{N}$ belongs to Σ_3 if there exists a decidable set $P \subseteq \mathbb{N}^{\ell+m+n+1}$ such that A is the set of natural numbers a satisfying $\exists x_1, \dots, x_\ell \forall y_1, \dots, y_m \exists z_1, \dots, z_n: P(a, \bar{x}, \bar{y}, \bar{z})$. The notions Σ_3 -hard and Σ_3 -complete are defined similarly. For our purposes, it is important that the set TOT is Π_2 -complete and the set REC is Σ_3 -complete [14].

3 When is the MSO-theory of an ω -word decidable?

In this section, we recall the answers by Semenov [11] and by Rabinovich and Thomas [9]. Semenov defined a form of “*periodic words*” in which words from certain regular sets recur.

► **Definition 4.** Let α be some ω -word. An *indicator of recurrence* for α is a function $\text{rec}: \text{Sent} \rightarrow \mathbb{N} \cup \{\top\}$ such that, for every MSO-sentence φ , the following hold:

- if $\text{rec}(\varphi) = \top$, then $\forall k \exists j \geq i \geq k: \alpha[i, j] \models \varphi$
- if $\text{rec}(\varphi) \neq \top$, then $\forall j \geq i \geq \text{rec}(\varphi): \alpha[i, j] \models \neg\varphi$

► **Theorem 5 (Semenov’s Characterisation [11]).** *Let α be an ω -word. Then $\text{MTh}(\alpha)$ is decidable if and only if the ω -word α is recursive and there exists a recursive indicator of recurrence for α .*

Note that an ω -word can have many recursive indicators of recurrence: if rec is such an indicator, then so is $\varphi \mapsto 2 \cdot \text{rec}(\varphi)$.

Two other characterisations are given by Rabinovich and Thomas in [9]. The idea is to decompose an infinite word into infinitely many finite sections all of which (except possibly the first one) have the same k -type.

► **Definition 6.** Let $\alpha \in \{0, 1\}^\omega$, $u, v \in \{0, 1\}^+$, $k \in \mathbb{N}$, and $H \subseteq \mathbb{N}$ be infinite.

- The set H is a *k -homogeneous factorisation of α into (u, v)* if $\alpha[0, i-1] \equiv_k u$ and $\alpha[i, j-1] \equiv_k v$ for all $i, j \in H$ with $i < j$. The set H is *k -homogeneous for α* if it is a k -homogeneous factorisation of α into some finite words (u, v) .
- Let $H = \{h_i \mid i \in \mathbb{N}\}$ with $h_0 < h_1 < \dots$. The set H is *uniformly homogeneous for α* if, for all $k \in \mathbb{N}$, the set $\{h_i \mid i \geq k\}$ is k -homogeneous for α .

As with indicators of recurrence, any ω -word has many uniformly homogeneous sets: the existence of at least one follows by a repeated and standard application of Ramsey’s theorem, and there are infinitely many since any infinite subset of a uniformly homogeneous set is again uniformly homogeneous.

► **Theorem 7 (1st Rabinovich-Thomas’ Characterisation [9]).** *Let α be an ω -word. Then $\text{MTh}(\alpha)$ is decidable if and only if the ω -word α is recursive and there exists a recursive uniformly homogeneous set for α .*

Suppose $h_0 < h_1 < h_2 < \dots$ is an enumeration of some uniformly homogeneous set for α . This sequence determines finite words u_k and v_k such that $w \equiv_k u_k(v_k)^\omega$, $u_k v_k \equiv_k u_k$, and $v_k v_k \equiv_k v_k$: simply set $u_k = \alpha[0, h_k - 1]$ and $v_k = \alpha[h_k, h_{k+1} - 1]$. If the ω -word α is recursive, we can therefore, from $k \in \mathbb{N}$, compute a representative of the k -type of α .

► **Definition 8.** Let α be some ω -word and $\text{tp}: \mathbb{N} \rightarrow \{0, 1\}^+ \times \{0, 1\}^+$. The function tp is a *type-function* if, for all $k \in \mathbb{N}$, α has a k -homogeneous factorisation into $\text{tp}(k) = (u, v)$.

Let tp be a type-function for the ω -word α and let $k \in \mathbb{N}$. Then there exists a k -homogeneous factorisation H of α into $\text{tp}(k) = (u, v)$. Let $H = \{h_0 < h_1 < h_2 < \dots\}$. Then we have $\alpha = \alpha[0, h_0 - 1] \alpha[h_0, h_1 - 1] \alpha[h_1, h_2 - 1] \dots \equiv_k uv^\omega$. Furthermore, $v \equiv_k \alpha[h_0, h_2 - 1] = \alpha[h_0, h_1 - 1] \alpha[h_1, h_2 - 1] \equiv_k vv$. Consequently, $\text{tp}(k)$ is a representative of the k -type of α .

► **Theorem 9 (2nd Rabinovich-Thomas’ Characterisation [9]).** *Let α be an ω -word. Then $\text{MTh}(\alpha)$ is decidable if and only if α has a recursive type-function.*

Note that, differently from Thm. 7 this theorem does not mention that α is recursive. But this recursiveness is implicit: Let tp be a recursive type-function and $k \in \mathbb{N}$. Then one can write a FO sentence of quantifier-depth $k+2$ expressing that $\alpha(k) = 1$. Let $\text{tp}(k+2) = (u, v)$. Then $\alpha \equiv_{k+2} uv^\omega$ implies $\alpha(k) = uv^k(k)$, hence $\alpha(k)$ is computable from k .

4 How hard is it to tell if the MSO-theory of an ω -word is decidable?

In this section, we determine the recursion-theoretical complexity of the question whether the MSO-theory of a recursive ω -word is decidable. Technically, we will consider the following two sets:

$$\text{DecTh}_{\mathbb{N}}^{\text{MSO}} = \{e \in \text{REC} \mid \text{MTh}(\mathbb{N}, \leq, W_e) \text{ is decidable}\} \quad \text{UndecTh}_{\mathbb{N}}^{\text{MSO}} = \text{REC} \setminus \text{DecTh}_{\mathbb{N}}^{\text{MSO}}$$

Recall that $W_e \subseteq \mathbb{N}$ denotes the e^{th} recursively enumerable set.

But first note the following: Let α be some recursive word. Then, by Büchi's and McNaughton's theorems, $\text{MTh}(\alpha)$ is decidable iff the set of deterministic parity automata accepting α is decidable. Recall that "the deterministic parity automaton no. n accepts α " (where we assume any computable enumeration of all deterministic parity automata) is a Boolean combination of Σ_2 -statements, cf. [15, Prop. 5.3]. It follows that $e \in \text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ if and only if the following holds:

$$\exists f \in \text{TOT} \forall n: \Phi_f(n) = 1 \Leftrightarrow \text{the deterministic parity automaton no. } n \text{ accepts } (\mathbb{N}, \leq, W_e)$$

Hence $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ belongs to Σ_4 . The following lemma improves this by one level in the arithmetical hierarchy:

► **Lemma 10.** *The set $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ belongs to Σ_3 .*

We present two proofs of this lemma, one based on the first Rabinovich-Thomas characterisation, the second one based on the Semenov characterisation.

Proof. (based on Thm. 7) Let α be some recursive ω -word. Recall that a set $H \subseteq \mathbb{N}$ is infinite and recursive if there exists a total computable and strictly monotone function f such that $H = \{f(n) \mid n \in \mathbb{N}\}$. Now consider the following:

$$\begin{aligned} \exists e \forall k, i, j, i', j': e \in \text{TOT} \wedge (i < j \Rightarrow \Phi_e(i) < \Phi_e(j)) \wedge \\ (k \leq i < j \wedge k \leq i' < j' \Rightarrow \alpha[\Phi_e(i), \Phi_e(j) - 1] \equiv_k \alpha[\Phi_e(i'), \Phi_e(j') - 1]) \end{aligned}$$

It expresses that there exists a total recursive function (namely Φ_e) that is strictly monotone. Its image then consists of the numbers $\Phi_e(0) < \Phi_e(1) < \Phi_e(2) < \dots$. The last line expresses that this image is uniformly homogeneous for α . Hence this statement says that there exists a recursive uniformly homogeneous set for α , i.e., that $\text{MTh}(\alpha)$ is decidable by Thm. 7.

From $k, i, i', j, j' \in \mathbb{N}$ with $k \leq i < j$, and $k \leq i' < j'$ we can compute the finite words $\alpha[\Phi_e(i), \Phi_e(j) - 1]$ and $\alpha[\Phi_e(i'), \Phi_e(j') - 1]$ since α is recursive. Hence it is decidable whether $\alpha[\Phi_e(i), \Phi_e(j) - 1] \equiv_k \alpha[\Phi_e(i'), \Phi_e(j') - 1]$. The whole statement is in Σ_3 as $\text{TOT} \in \Pi_2$. ◀

Proof. (based on Thm. 5) We enumerate the set Sent of MSO-sentences in any effective way as $\varphi_0, \varphi_1, \dots$. Let $e \in \text{TOT}$. Then the function $\text{rec}: \text{Sent} \rightarrow \mathbb{N}: \varphi_i \mapsto \Phi_e(i)$ is an indicator of recurrence for the ω -word α if and only if the following holds for all $\varphi \in \text{Sent}$

$$(\text{rec}(\varphi) \neq \top \Rightarrow \forall k \geq j \geq \text{rec}(\varphi): \alpha[j, k] \models \neg \varphi) \wedge (\text{rec}(\varphi) = \top \Rightarrow \forall j \exists \ell \geq k \geq j: \alpha[k, \ell] \models \varphi)$$

Given the definition of rec , this is equivalent to requiring (for all $i \in \mathbb{N}$)

$$(\Phi_e(i) \neq \top \Rightarrow \forall k \geq j \geq \Phi_e(i): \alpha[j, k] \models \neg \varphi_i) \wedge (\Phi_e(i) = \top \Rightarrow \forall j \exists \ell \geq k \geq j: \alpha[k, \ell] \models \varphi_i)$$

If α is recursive, this is a Π_2 -statement. Prefixing it with $\exists e \in \text{TOT} \forall i$ yields a Σ_3 -statement that expresses the existence of a recursive indicator of recurrence. ◀

► **Remark.** From the 2nd characterisation by Rabinovich and Thomas (Thm. 9), we can only infer that $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ is in Σ_5 : Let α be some recursive ω -word and $u, v \in \{0, 1\}^+$. Then, by the proof of [9, Prop. 7], there exists a k -homogeneous factorisation of α into (u, v) , if the following Σ_3 -statement $\varphi(u, v)$ holds: $\exists x \forall y \exists z, z': (\alpha[0, x-1] \equiv_k u \wedge y < z < z' \wedge \alpha[x, z-1] \equiv_k \alpha[z, z'-1] \equiv_k v)$. Hence the function $\text{tp}: \mathbb{N} \rightarrow \{0, 1\}^+ \times \{0, 1\}^+$ is a type-function if the Π_4 -statement $\forall k \in \mathbb{N}: \varphi(\text{tp}(k))$ holds. Consequently, there is a recursive type-function if we have $\exists e: e \in \text{TOT} \wedge \forall k: \varphi(\Phi_e(k))$ which is a Σ_5 -statement.

The above raises the natural question whether these characterisations are “optimal”. Namely, if one can separate $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ from $\text{UndecTh}_{\mathbb{N}}^{\text{MSO}}$ using a simpler statement. We now prepare a negative answer to this last question (which is an affirmative answer to the optimality question posed first).

We now construct an m-reduction from the set REC to any separator of $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ and $\text{UndecTh}_{\mathbb{N}}^{\text{MSO}}$: Let $e \in \mathbb{N}$. One can compute $f \in \mathbb{N}$ such that Φ_f is total and injective and $\{\Phi_f(i) \mid i \in \mathbb{N}\} = \{2a \mid a \in W_e\} \cup (2\mathbb{N} + 1)$. For $i \in \mathbb{N}$, set $x_i = 2^{\Phi_f(i)} \times \prod_{0 \leq j \leq i} (2j + 1)$ and consider the ω -word $\alpha_e = 10^{x_0} 10^{x_1} 10^{x_2} \dots$. Since Φ_f is total, this ω -word is recursive.

► **Lemma 11.** *Let $e \in \mathbb{N}$. The MSO-theory of the ω -word α_e is decidable if and only if the e^{th} recursively enumerable set W_e is recursive, i.e., $e \in \text{REC}$.*

Proof. First suppose that the MSO-theory of α_e is decidable. For $a \in \mathbb{N}$, we have $a \in W_e$ iff there exists $i \geq 0$ with $2a = \Phi_f(i)$ iff there exists $i \geq 0$ such that 2^{2a} is the greatest power of 2 that divides x_i . Consequently, $a \in W_e$ if the ω -word α_e satisfies

$$\exists x, y \in P: (x < y \wedge \forall z: (x < z < y \Rightarrow z \notin P)) \wedge (2^{2a} \mid y - x - 1 \wedge 2^{2a+1} \nmid y - x - 1) \quad (1)$$

Recall that $n \mid y - x - 1$ is expressible by an MSO-formula. Since validity in α_e of the resulting MSO-sentence is decidable, the set W_e is recursive.

Conversely, let W_e be recursive. To show that the MSO-theory of α_e is decidable, let φ be some MSO-sentence. Let $k = \text{qr}(\varphi)$ be the quantifier-rank of φ . To decide whether $\alpha_e \models \varphi$, we proceed as follows:

- Using standard semigroup arguments, compute $\ell > 0$ such that $0^\ell \equiv_k 0^{2\ell}$ and determine $a, b \in \mathbb{N}$ with $\ell = 2^a(2b + 1)$.
- Compute $i \geq b$ such that $\Phi_f(j) > a$ for all $j > i$: to this aim, first determine $A = \{n \leq a \mid n \in W_e \text{ or } a \text{ odd}\}$ which is possible since W_e is decidable. Then compute the least $i \geq b$ such that $A \subseteq \{\Phi_f(j) \mid j \leq i\}$. Since Φ_f is injective, $\Phi_f(j) > a$ for all $j > i$.
- Decide whether $10^{x_0} 10^{x_1} \dots 10^{x_i} (10^\ell)^\omega$ satisfies φ which is possible since this ω -word is ultimately periodic.

Let $j > i$. Then $\Phi_f(j) > a$ and $j > i \geq a$ imply that x_j is a multiple of ℓ . Thus $0^{x_j} \equiv_k 0^\ell$. We therefore obtain $\alpha_e \equiv_k 10^{x_0} 10^{x_1} 10^{x_2} \dots 10^{x_i} (10^\ell)^\omega$. Hence the above algorithm is correct. ◀

Lemmas 11 and 10 imply that the problem of deciding whether a recursive ω -word has a decidable MSO-theory is Σ_3 -complete:

► **Theorem 12.** ■ $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ is in Σ_3 .

■ Any set containing $\text{DecTh}_{\mathbb{N}}^{\text{MSO}}$ and disjoint from $\text{UndecTh}_{\mathbb{N}}^{\text{MSO}}$ is Σ_3 -hard.

► **Remark.** Thm. 7 also holds for the weaker logics FO and FO+MOD that extends FO by modulo-counting quantifiers [9]. Consequently, Lemma 10 also holds, *mutatis mutantis*, for these logics.

Conversely, Lemma 11 also holds for FO+MOD since (1) is easily expressible in this logic. To also handle FO, replace the definition of x_i by $x_i = \Phi_f(j)$. A similar argument as in Lemma 11 proves that W_e is recursive iff the ω -word α_e obtained this way has a decidable FO-theory. Thus, Thm. 12 also holds for the logics FO and FO+MOD.

5 When is the MSO-theory of a bi-infinite word decidable?

In this section, we investigate whether the characterisations from Theorems 5, 7, and 9 can be lifted from ω - to bi-infinite words.

5.1 A characterisation à la Semenov

► **Definition 13.** Let ξ be a bi-infinite word. A pair of functions $(\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow})$ with $\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow}: \text{Sent} \rightarrow \mathbb{Z} \cup \{\top\}$ is an *indicator of recurrence* for ξ if for any $\varphi \in \text{Sent}$:

- if $\text{rec}_{\leftarrow}(\varphi) = \top$, $\forall k \in \mathbb{Z} \exists i \leq j \leq k: \xi[i, j] \models \varphi$; otherwise, $\forall i \leq j \leq \text{rec}_{\leftarrow}(\varphi): \xi[i, j] \models \neg\varphi$
- if $\text{rec}_{\rightarrow}(\varphi) = \top$, $\forall k \in \mathbb{Z} \exists j \geq i \geq k: \xi[i, j] \models \varphi$; otherwise, $\forall j \geq i \geq \text{rec}_{\rightarrow}(\varphi): \xi[i, j] \models \neg\varphi$

A bi-infinite word ξ “consists” of an ω^* -word ξ_{\leftarrow} and an ω -word ξ_{\rightarrow} . Then, roughly speaking, an indicator of recurrence for the *bi-infinite* word ξ consists of a pair of indicators of recurrence, one for ξ_{\leftarrow} and one for ξ_{\rightarrow} . Therefore, the following is similar to Thm. 5.

► **Theorem 14.** *Let ξ be a bi-infinite word. Then $\text{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence and the bi-infinite word ξ is recursive or recurrent.*

This theorem is an immediate consequence of Propositions 15 and 16 below. If ξ is non-recurrent, there is a finite word u that has a leftmost or a rightmost occurrence in ξ , say at a position $x \in \mathbb{Z}$. Then x is definable in MSO. Consequently, also the position 0 is definable. This allows one to reduce the decidability of $\text{MTh}(\xi)$ to the decidability of both $\text{MTh}(\xi(-\infty, -1])$ and $\text{MTh}(\xi[0, \infty))$. Hence Prop. 15 is a consequence of Thm. 5.

► **Proposition 15.** *Let ξ be a non-recurrent bi-infinite word. Then $\text{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence and the bi-infinite word ξ is recursive.*

► **Proposition 16.** *Let ξ be a recurrent bi-infinite word. Then $\text{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence.*

Proof. First suppose $\text{MTh}(\xi)$ is decidable. We have to construct a recursive indicator of recurrence $(\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow})$ for ξ . Let $\varphi \in \text{Sent}$. Set $\text{rec}_{\leftarrow}(\varphi) = \text{rec}_{\rightarrow}(\varphi) = \top$ if there exist integers $i \leq j$ with $\xi[i, j] \models \varphi$, otherwise set $\text{rec}_{\leftarrow}(\varphi) = \text{rec}_{\rightarrow}(\varphi) = 0$.

It remains to be shown that these functions are recursive and that they form an indicator of recurrence. Regarding the recursiveness, note that there are $i \leq j$ with $\xi[i, j] \models \varphi$ iff $\xi \models \exists x, y: x \leq y \wedge \varphi_{x,y}$ where $\varphi_{x,y}$ is obtained from φ by restricting all quantifiers to the interval $[x, y]$. Since $\text{MTh}(\xi)$ is decidable, the functions rec_{\leftarrow} and rec_{\rightarrow} are recursive.

Next we show that $(\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow})$ is an indicator of recurrence for ξ : If $\text{rec}_{\leftarrow}(\varphi) = \top$, then (by the definition of rec_{\leftarrow}) there are $i \leq j$ with $\xi[i, j] \models \varphi$. Since ξ is recurrent, it follows that there are arbitrary small and large integers $a \leq b$ with $\xi[a, b] = \xi[i, j] \models \varphi$. If, in the other case, $\text{rec}_{\leftarrow}(\varphi) = 0$, then there are no integers $i \leq j$ with $\xi[i, j] \models \varphi$, in particular, there are no integers $i \leq j \leq \text{rec}_{\leftarrow}(\varphi)$ with $\xi[i, j] \models \varphi$.

Conversely, suppose $(\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow})$ is a recursive indicator of recurrence for ξ . Then, for $\varphi \in \text{Sent}$, we can decide whether there are integers $i \leq j$ with $\xi[i, j] \models \varphi$ (since ξ is recurrent, this is the case if and only if $\text{rec}_{\leftarrow}(\varphi) = \top$). In [1, Thm. 3.1(2)] and in [11, 7], it is stated that then $\text{MTh}(\xi)$ is decidable (a proof can be extracted from [6, Section IX.6]). ◀

Thm. 14 connects the decidability of the MSO theory of a recurrent bi-infinite word ξ with a decidability question on its set of factors $F(\xi)$. It follows that, if $\text{MTh}(\xi)$ is decidable, then $F(\xi)$ is decidable. We now show that the converse implication does not hold.

► **Lemma 17.** *A set of finite words F containing at least one non-empty word is the factor set of a recurrent bi-infinite word if and only if it satisfies the following conditions:*

- (a) If $uvw \in F$, then $v \in F$.
 (b) For any $u, w \in F$, there is a word $v \in F$ such that $uvw \in F$

Proof. Necessity of (a) and (b) is obvious. So suppose $F \subseteq \{0, 1\}^*$ contains at least one non-empty word u and satisfies (a) and (b). We construct a bi-infinite recurrent word ξ such that $F(\xi) = F$. Since F is non-empty, (b) implies that F is infinite. Let $F = \{u_i \mid i \in \mathbb{N}\}$. Inductively, we define two sequences $(x_i)_{i>0}$ and $(y_i)_{i>0}$ of words from F such that, for all $i \in \mathbb{N}$, the finite word $w_i = u_i x_i u_{i-1} x_{i-1} \dots u_1 x_1 u_0 y_1 u_1 y_2 u_2 \dots y_i u_i$ belongs to F .

Let $i > 0$ and suppose we already defined the words x_j and y_j for $j < i$ such that $w_{i-1} \in F$. Then, by (b), there exists $x_i \in F$ such that $u_i x_i w_{i-1} \in F$. Again by (b), there exists $y_i \in F$ such that $u_i x_i w_{i-1} y_i u_i \in F$. Now set $\xi = \dots u_3 x_3 u_2 x_2 u_1 x_1 u_0 y_1 u_1 y_2 u_2 y_3 u_3 \dots$. Let $v \in \{0, 1\}^*$ be some factor of ξ . Then there is $i \in \mathbb{N}$ such that v is a factor of w_i . Since $w_i \in F$, condition (a) implies $v \in F$. Hence $F(\xi) = F$.

Now let $v \in F(\xi) = F$. By (b), there are infinitely many $i \in \mathbb{N}$ such that v is a factor of u_i . Hence ξ is recurrent. \blacktriangleleft

► **Theorem 18.** *There exists a recurrent bi-infinite word ξ whose set of factors is decidable, but $\text{MTh}(\xi)$ is undecidable.*

Proof. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be some recursive and total function such that $\{f(i) \mid i \in \mathbb{N}\}$ is not recursive. Let $F \subseteq \{0, 1\}^*$ be the set of all finite words u with the following property: If $10^{2i+1}10^{2j}1$ is a factor of u , then $j = f(i)$. This set is clearly recursive, contains a non-empty word, and satisfies conditions (a) and (b) from Lemma 17. Hence there exists a bi-infinite word ξ with $F(\xi) = F$. For $j \in \mathbb{N}$, consider the following sentence:

$$\exists x < y: P(x) \wedge P(y + 2j) \wedge \neg 2 \mid y - x - 1 \wedge \forall z: (x < z < y + 2j \wedge P(z) \rightarrow z = y)$$

It expresses that the language $1(00)^*010^{2j}1$ contains a factor of ξ . But this is the case iff it contains a factor of some word from F iff there exists $i \in \mathbb{N}$ with $j = f(i)$. Since this is undecidable, the MSO-theory of ξ is undecidable by Thm. 14. \blacktriangleleft

5.2 A characterisation à la Rabinovich-Thomas I

We return to the question when the MSO-theory of a recurrent bi-infinite word is decidable. We will see that Thm. 7 naturally extends to *recursive* bi-infinite words. We will then demonstrate that it does not extend to non-recursive bi-infinite words.

► **Definition 19.** Let $\xi \in \{0, 1\}^{\mathbb{Z}}$, $u, v, w \in \{0, 1\}^+$, $k \in \mathbb{N}$, and let $H_{\leftarrow} = \{h_i^- \mid i \in \mathbb{N}\}$ and $H_{\rightarrow} = \{h_i^+ \mid i \in \mathbb{N}\}$ with $h_0^- > h_1^- > \dots$ and $h_0^+ < h_1^+ < \dots$.

- The pair $(H_{\leftarrow}, H_{\rightarrow})$ is a k -homogeneous factorisation of ξ into (u, v, w) if
 - $\xi[i, j - 1] \equiv_k u$ for all $i, j \in H_{\leftarrow}$ with $i < j$,
 - $\xi[i, j - 1] \equiv_k v$ for all $i \in H_{\leftarrow}$ and $j \in H_{\rightarrow}$ with $i < j$ and
 - $\xi[i, j - 1] \equiv_k w$ for all $i, j \in H_{\rightarrow}$ with $i < j$.
- The pair $(H_{\leftarrow}, H_{\rightarrow})$ is k -homogeneous for ξ if it is a k -homogeneous factorisation of ξ into some finite words (u, v, w) .
- The pair $(H_{\leftarrow}, H_{\rightarrow})$ is uniformly homogeneous for ξ if, for all $k \in \mathbb{N}$, the pair $(\{h_i^- \mid i \geq k\}, \{h_i^+ \mid i \geq k\})$ is k -homogeneous for ξ .

Let ξ be a bi-infinite word split into an ω^* -word ξ_{\leftarrow} and an ω -word ξ_{\rightarrow} . As for any ω -word, there exists a uniformly homogeneous set H_{\rightarrow} for ξ_{\rightarrow} . Symmetrically, there exists a set $H_{\leftarrow} \subseteq \tilde{\mathbb{N}}$ that is “uniformly homogeneous” for ξ_{\leftarrow} . Then the pair $(H_{\leftarrow}, H_{\rightarrow})$ is a uniformly homogeneous pair for $\xi = \xi_{\leftarrow} \xi_{\rightarrow}$.

► **Lemma 20.** *Let ξ be a recursive bi-infinite word with a decidable MSO-theory. Then the MSO-theories of $\xi_{\leftarrow} = \xi(-\infty, -1]$ and of $\xi_{\rightarrow} = \xi[0, \infty)$ are both decidable.*

Proof. We handle the cases of recurrent and non-recurrent words separately.

First let ξ be non-recurrent. Then some word $u \in F(\xi)$ has a leftmost or a rightmost occurrence, at some position $x \in \mathbb{Z}$ which is definable in FO. Hence, also the positions -1 and 0 are definable. Hence the MSO-theories of ξ_{\leftarrow} and of ξ_{\rightarrow} can be reduced to that of ξ and are therefore decidable.

Now let ξ be recurrent. By Thm. 14, ξ has a recursive indicator of recurrence $(\text{rec}_{\leftarrow}, \text{rec}_{\rightarrow})$. Define the functions $f, g: \text{Sent} \rightarrow \mathbb{N} \cup \{\top\}$ as follows:

$$f(\varphi) = \begin{cases} \top & \text{if } \text{rec}_{\leftarrow}(\varphi) = \top \\ 0 & \text{if } \text{rec}_{\leftarrow}(\varphi) \geq 0 \text{ and } g(\varphi) = \begin{cases} \top & \text{if } \text{rec}_{\rightarrow}(\varphi) = \top \\ 0 & \text{if } \text{rec}_{\rightarrow}(\varphi) < 0 \\ \text{rec}_{\rightarrow}(\varphi) & \text{otherwise} \end{cases} \\ |\text{rec}_{\leftarrow}(\varphi)| - 1 & \text{otherwise} \end{cases}$$

Exploiting the properties of rec_{\leftarrow} and rec_{\rightarrow} , it is then routine to check that f, g are indicators of recurrences for the two ω -words ξ_{\leftarrow}^R and ξ_{\rightarrow} . Note that ξ_{\leftarrow}^R and ξ_{\rightarrow} are recursive ω -words. Hence, by Thm. 5, the MSO-theories of ξ_{\leftarrow}^R and of ξ_{\rightarrow} are both decidable. ◀

► **Theorem 21.** *A recursive bi-infinite word ξ has a decidable MSO-theory if and only if there exists a recursive uniformly homogeneous pair for ξ .*

Proof. Suppose $\text{MTh}(\xi)$ is decidable. By Lemma 20, the MSO-theories of $\xi_{\leftarrow}^R = \xi(-\infty, -1]^R$ and of $\xi_{\rightarrow} = \xi[0, \infty)$ are both decidable. Consequently, by Thm. 7, there are recursive uniformly homogeneous factorisations $H_{\leftarrow}^R, H_{\rightarrow} \subseteq \mathbb{N}$ for ξ_{\leftarrow}^R and ξ_{\rightarrow} into (x^R, y^R) and (y', z) , respectively. Deleting, if necessary, the minimal element from H_{\leftarrow}^R , we can assume $0 \notin H_{\leftarrow}^R$. We set $H_{\leftarrow} = \{-n \mid n \in H_{\leftarrow}^R\} \subseteq \tilde{\mathbb{N}}$ and show that $(H_{\leftarrow}, H_{\rightarrow})$ is a uniformly homogeneous pair for ξ : Let $H_{\leftarrow} = \{h_i^- \mid i \in \mathbb{N}\}$ and $H_{\rightarrow} = \{h_i^+ \mid i \in \mathbb{N}\}$ such that $h_0^- > h_1^- > \dots$ and $h_0^+ < h_1^+ < \dots$.

- Let $j > i \geq k$. Then $\xi[h_i^- + 1, h_j^-] = \xi_{\leftarrow}[h_i^- + 1, h_j^-] = (\xi_{\leftarrow}^R[-h_j^-, -h_i^- - 1])^R \equiv_k y^R$.
- Let $i, j \geq k$. Then $\xi[h_i^-, h_j^+ - 1] = \xi_{\leftarrow}[h_i^- + 1, 0] \xi_{\rightarrow}[0, h_j^+ - 1] \equiv_k xy'$.
- Let $j > i \geq k$. Then $\xi[h_i^+, h_j^+ - 1] = \xi_{\rightarrow}[h_i^+, h_j^+ - 1] \equiv_k z$.

Hence the pair $(\{h_i^- \mid i \geq k\}, \{h_i^+ \mid i \geq k\})$ is a k -homogeneous factorisation of ξ into (y^R, xy', z) . Since k is arbitrary, $(H_{\leftarrow}, H_{\rightarrow})$ is uniformly homogeneous for ξ . Since these two sets are clearly recursive, this proves the first implication.

Conversely, suppose there exists a recursive uniformly homogeneous pair $(H_{\leftarrow}, H_{\rightarrow})$ for ξ . Then the sets $H_{\leftarrow}^R = \{|n| \mid n \in H_{\leftarrow} \cap \tilde{\mathbb{N}}\}$ and $H_{\rightarrow} \cap \mathbb{N}$ are recursive and uniformly homogeneous for ξ_{\leftarrow}^R and ξ_{\rightarrow} , resp. Since ξ_{\leftarrow} and ξ_{\rightarrow} are both recursive, we can apply Thm. 7. Hence the infinite words ξ_{\leftarrow} and ξ_{\rightarrow} both have decidable MSO-theories. Since $\xi = \xi_{\leftarrow} \xi_{\rightarrow}$, the MSO-theory of ξ is decidable. ◀

We next show that we cannot hope to extend Thm. 21 to non-recursive words:

► **Theorem 22.** *There exists a recurrent r.e. bi-infinite word ξ with decidable MSO-theory such that there is no r.e. uniformly homogeneous pair for ξ .*

Proof. We prove this theorem by constructing a recurrent bi-infinite word ξ such that the set $F(\xi)$ of factors is $\{0, 1\}^*$. Hence ξ has decidable MSO-theory by Thm. 14.

There is a computable function $f: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that the following hold:

- $\Phi_{f(e,s)}$ is total and $W_{f(e,s)} \subseteq \{0, 1, \dots, s\}$ for any $e, s \in \mathbb{N}$.
- $W_e = \bigcup_{s \in \mathbb{N}} W_{f(e,s)}$ for any $e \in \mathbb{N}$.

In the following, we fix the function f and write $W_{e,s}$ for $W_{f(e,s)}$. Furthermore, we fix some recursive enumeration u_0, u_1, \dots of the set $\{0, 1\}^+$ of non-empty finite words.

5.2.1 Construction

By induction on $s \in \mathbb{N}$, we construct tuples

$$t_s = (w_s, m_{0,s}, m_{1,s}, \dots, m_{s,s}, P_s) \in \{0, 1\}^* \times \mathbb{N}^{s+1} \times 2^{\{0, \dots, s\}} \quad \text{such that}$$

- $m_{i,s} + |u_i| \leq m_{i+1,s}$ for all $0 \leq i < s$ and $m_{s,s} + |u_s| \leq |w_s|$ (in particular, $|w_s| > s$),
- $w_s[m_{i,s}, m_{i,s} + |u_i| - 1] = u_i$ for all $0 \leq i \leq s$, and
- for all $e \in P_s$, there exist $a, b \in W_e$ with $a < b < |w_s|$ and $w_s[a, b - 1] \in 1^*$.

Set $w_0 = u_0$, $m_{0,0} = 0$, and $P_0 = \emptyset$. Then the inductive invariant holds for the tuple $t_0 = (w_0, m_0, P_0)$.

Now suppose the tuple t_s has been constructed. Let H_{s+1} denote the set of indices $0 \leq e \leq s+1$ with $e \notin P_s$ such that $W_{e,s}$ contains at least two numbers $a > b \geq m_{e,s}$. In the construction of the tuple t_{s+1} , we distinguish two cases:

- 1st case: $H_{s+1} = \emptyset$. Then set $w_{s+1} = w_s u_{s+1}$, $m_{i,s+1} = m_{i,s}$ for $0 \leq i \leq s$, $m_{s+1,s+1} = |w_s|$, and $P_{s+1} = P_s$. Since the inductive invariant holds for the tuple t_s , it also holds for the newly constructed tuple t_{s+1} .
- 2nd case: $H_{s+1} \neq \emptyset$. Let e_{s+1} be the minimal element of H_{s+1} and let a_{s+1} and b_{s+1} be the minimal elements of $W_{e_{s+1},s}$ satisfying $m_{e,s} \leq a_{s+1} < b_{s+1}$. Then set
 - $w_{s+1} = w_s[0, a_{s+1} - 1] 1^{b_{s+1} - a_{s+1}} w_s[b_{s+1}, |w_s| - 1] u_{e_{s+1}} u_{e_{s+1}+1} \dots u_{s+1}$ (in other words, the words $u_{e_{s+1}}$ up to u_{s+1} are appended to w_s and the positions between a_{s+1} and $b_{s+1} - 1$ are set to 1).
 - $m_{i,s+1} = \begin{cases} m_{i,s} & \text{if } i < e_{s+1} \\ |w_s u_{e_{s+1}} u_{e_{s+1}+1} \dots u_{i-1}| & \text{if } e_{s+1} \leq i \leq s+1 \end{cases}$
 - $P_{s+1} = P_s \cup \{e_{s+1}\}$

The first two conditions of the inductive invariant are obvious. Regarding the last one, let $e \in P_{s+1}$. If $e \neq e_{s+1}$, then $e \in P_s$ and therefore there exist $a, b \in W_e$ with $a < b < |w_s| < |w_{s+1}|$ such that $w_s[a, b - 1] \in 1^*$. Note that any position in w_s that carries 1 also carries 1 in w_{s+1} . Hence $w_{s+1}[a, b - 1] \in 1^*$ as well. It remains to consider the case $e = e_{s+1}$. But then, by the very construction, $a_{s+1} < b_{s+1}$ belong to $W_{e_{s+1},s} \subseteq W_e$ and satisfy $w_{s+1}[a_{s+1}, b_{s+1} - 1] \in 1^*$.

This finishes the construction of the sequence of tuples t_s .

5.2.2 Verification

Let ξ_{\rightarrow} be the ω -word with $\xi_{\rightarrow}(i) = 1$ iff there exists $s \in \mathbb{N}$ with $w_s(i) = 1$. Since the tuple t_{s+1} is computable from the tuple t_s , the word ξ_{\rightarrow} is clearly recursively enumerable.

Furthermore, let $u \in \{0, 1\}^+$. Then there exists $e \in \mathbb{N}$ with $u = u_e$. Note that $m_{e,s} \leq m_{e,s+1}$ for all $e, s \in \mathbb{N}$. Furthermore, $m_{e,s} < m_{e,s+1}$ iff $H_{s+1} \neq \emptyset$ and $e_{s+1} \leq e$. Since the numbers $e_{s'+1}$ for $s' \in \mathbb{N}$ (if defined) are mutually distinct, there exists $s \in \mathbb{N}$ such that $e_{t+1} > e$ and therefore $m_{e,s} = m_{e,t}$ for all $t \geq s$. Consequently, $\xi_{\rightarrow}[m_{e,s}, m_{e,s} + |u_e| - 1] = w_s[m_{e,s}, m_{e,s} + |u_e| - 1] = u_e = u$. This means that $F(\xi_{\rightarrow}) = \{0, 1\}^*$. It follows that ξ_{\rightarrow} is recurrent.

Claim 1. If W_e is infinite, then $e \in \bigcup_{s \in \mathbb{N}} P_s$.

Proof of Claim 1. By contradiction, suppose this is not the case. Let $e \in \mathbb{N}$ be minimal with W_e infinite and $e \notin \bigcup_{s \in \mathbb{N}} P_s$. Since W_e is infinite, we get $e \in H_{s+1}$ for almost all $s \in \mathbb{N}$. By minimality of e , there is $s \in \mathbb{N}$ with $e = \min H_{s+1}$. But then $e_{s+1} = e$ and $e \in P_{s+1}$. **q.e.d.**

Claim 2. No recursively enumerable set W is uniformly homogeneous for the ω -word ξ_{\rightarrow} .

Proof of Claim 2. Suppose W is recursively enumerable and uniformly homogeneous for ξ_{\rightarrow} . Then W is infinite and there exists $e \in \mathbb{N}$ with $W = W_e$. By claim 1, there exists $s \in \mathbb{N}$ with $e \in P_s$. Hence there are $a, b \in W_e$ with $w_s[a, b-1] \in 1^*$ and therefore $\xi_{\rightarrow}[a, b-1] = w_s[a, b-1]$. There are $d > c > b$ in W_e such that $\xi_{\rightarrow}[c, d-1] \notin 1^*$. But then $\xi_{\rightarrow}[a, b-1]$ and $\xi_{\rightarrow}[c, d-1]$ do not have the same 1-type. Hence the set W_e is not 1- and therefore not uniformly homogeneous for ξ_{\rightarrow} . **q.e.d.**

Finally, let ξ_{\leftarrow} be the reversal of ξ_{\rightarrow} and consider the bi-infinite word $\xi = \xi_{\leftarrow} \xi_{\rightarrow}$. By Thm. 14, $\text{MTh}(\xi)$ is decidable since ξ is recurrent and contains every finite word as a factor. Finally, suppose $(H_{\leftarrow}, H_{\rightarrow})$ is uniformly homogeneous for ξ . Then $H_{\rightarrow} \cap \mathbb{N}$ is uniformly homogeneous for ξ_{\rightarrow} . By claim 2, this set cannot be recursively enumerable. Hence $(H_{\leftarrow}, H_{\rightarrow})$ is not recursively enumerable either. ◀

5.3 A characterisation à la Rabinovich-Thomas II

We next extend the 2nd characterisation by Rabinovich and Thomas to bi-infinite words. Differently from the 1st characterisation, this also covers non-recursive bi-infinite words.

► **Definition 23.** Let ξ be some bi-infinite word and $\text{tp}: \mathbb{N} \rightarrow \{0, 1\}^+ \times \{0, 1\}^+ \times \{0, 1\}^+$. The function tp is a *type-function* for ξ if, for all $k \in \mathbb{N}$, the bi-infinite word ξ has a k -homogeneous factorisation into $\text{tp}(k)$.

► **Theorem 24.** Let ξ be a bi-infinite word. Then $\text{MTh}(\xi)$ is decidable if and only if ξ has a recursive type-function.

Proof. First suppose that $\text{MTh}(\xi)$ is decidable. We have to construct a recursive type-function $\text{tp}: \mathbb{N} \rightarrow (\{0, 1\}^+)^3$. To this aim, let $k \in \mathbb{N}$. Then one can compute a finite sequence $\varphi_1, \dots, \varphi_n$ of MSO-sentences of quantifier-rank k such that, for all finite words u and v , we have $u \equiv_k v$ if and only if $\forall 1 \leq i \leq n: u \models \varphi_i \iff v \models \varphi_i$. For finite words u, v , and w , consider the following statement:

$$\begin{aligned} \exists H_{\leftarrow}, H_{\rightarrow}: \quad & \forall y \exists x, z: (x < y < z \wedge H_{\leftarrow}(x) \wedge H_{\rightarrow}(z)) \\ & \wedge \forall x, y: (x < y \wedge H_{\leftarrow}(x) \wedge H_{\leftarrow}(y) \rightarrow \xi[x, y-1] \equiv_k u) \\ & \wedge \forall x, y: ((H_{\leftarrow}(x) \wedge H_{\rightarrow}(y) \wedge x < y \rightarrow \xi[x, y-1] \equiv_k v) \\ & \wedge \forall x, y: (x < y \wedge H_{\rightarrow}(x) \wedge H_{\rightarrow}(y) \rightarrow \xi[x, y-1] \equiv_k w) \end{aligned}$$

This statement holds for a bi-infinite word ξ iff ξ has a k -homogeneous factorisation into (u, v, w) . Using $\varphi_1, \dots, \varphi_n$, the statements $\xi[x, y-1] \equiv_k u$ etc. can be expressed as MSO-formulas with free variables x and y . Since $\text{MTh}(\xi)$ is decidable, we can decide (given k, u, v , and w) whether ξ has a k -homogeneous factorisation into (u, v, w) . Since some k -homogeneous factorisation always exist, this allows to compute, from k , a tuple $\text{tp}(k)$ such that ξ has a k -homogeneous factorisation into $\text{tp}(k)$; tp is the wanted type function.

Conversely suppose that tp is a recursive type-function for ξ . To show that $\text{MTh}(\xi)$ is decidable, let $\varphi \in \text{Sent}$ be any MSO-sentence. Let k denote the quantifier-rank of φ . First, compute $\text{tp}(k) = (u, v, w)$. Then $\xi \models \varphi$ iff $u^{\omega^*} v w^{\omega} \models \varphi$ which is decidable since this bi-infinite word is ultimately periodic on the left and on the right. ◀

6 How complicated are bi-infinite words with decidable MSO-theories?

By Thm. 14, non-recurrent bi-infinite words with decidable MSO-theory are recursive. In this section, we will show in a strong sense that this does not hold for recurrent bi-infinite words: there are “arbitrarily complicated” bi-infinite words with decidable MSO-theories.

► **Definition 25.** Let $L \subseteq \{0, 1\}^*$ be a language. A word $u \in L$ is *left-determined in L* if for any $k \in \mathbb{N}$ there is exactly one word $vu \in L$ with $|v| = k$. Similarly, u is *right-determined in L* if for any $k \in \mathbb{N}$ there is exactly one word $uv \in L$ with $|v| = k$. The word $u \in L$ is *determined in L* if it is both left- and right-determined.

Intuitively, a word $w \in L$ is left-determined (right-determined) in L if it can be extended on the left (right) in a unique way.

► **Lemma 26.** Let ξ be a recurrent bi-infinite word. The following are equivalent:

- (1) ξ is periodic
- (2) $F(\xi)$ contains a determined word
- (3) $F(\xi)$ contains a right-determined word
- (3') $F(\xi)$ contains a left-determined word

Proof. For (1)→(2), let $\xi = u^\omega u^\omega$ be a periodic word. Then u is determined in $F(\xi)$. The direction (2)→(3) is trivial by the very definition.

For (3)→(1), suppose u is a right-determined word in $F(\xi)$. Choose $i < j$ such that $\xi[i, i+|u|-1] = \xi[j, j+|u|-1] = u$ (such a pair $i < j$ exists since ξ is recurrent). With $p = j - i$, we claim $\xi(n) = \xi(n+p)$ for all $n \in \mathbb{Z}$: First let $n \geq j + |u|$. Then $\xi[i, n]$ and $\xi[j, n+p]$ are two words from $F(\xi)$ that both start with u . We have $|\xi[i, n]| = n - i - 1 = n + p - j - 1 = |\xi[j, n+p]|$. Since u is right-determined, this implies $\xi[i, n] = \xi[j, n+p]$ and therefore $\xi(n) = \xi(n+p)$. Consequently, $\xi[j + |u|, \infty) = \xi[j + |u|, j + |u| + p]^\omega$. Next let $n < j + |u|$. Since ξ is recurrent, there is $k < n$ with $\xi[k, k + |u| - 1] = u$. Since u is right-determined, this implies $\xi[k, \infty) = \xi[j + |u|, \infty) = \xi[j + |u|, j + |u| + p]^\omega$ and therefore in particular $\xi(n) = \xi(n+p)$. The implications (2)→(3')→(1) are shown analogously. ◀

Lemma 26 states that a recurrent non-periodic bi-infinite word does not contain any left-determined or right-determined factor, and thus can be extended in both directions (left and right) in at least two ways. This observation allows to prove the following:

► **Lemma 27.** Let ξ be a recurrent non-periodic bi-infinite word. For any set $A \subseteq \mathbb{N}$, there is a recurrent bi-infinite word ξ_A such that $F(\xi) = F(\xi_A)$, $(A, F(\xi)) \leq_T \xi_A$, and $\xi_A \leq_T (A, F(\xi))$.

Proof. Let w_0, w_1, \dots be the enumeration of $F(\xi)$ in length-lexicographic order. Note that this is recursive in $F(\xi)$. There is also an effective enumeration of all pairs of words of the same length, say $(\ell_0, r_0), (\ell_1, r_1), \dots$. Now let $A \subseteq \mathbb{N}$ be arbitrary. We will construct a sequence of tuples $t_s = (u_s, v_s, x_s, y_s) \in (\{0, 1\}^*)^4$ such that, for all $s \in \mathbb{N}$, the finite word

$$\begin{aligned} z_s &= w_s y_s v_s z_{s-1} u_s x_s w_s \\ &= w_s y_s v_s w_{s-1} y_{s-1} v_{s-1} \dots w_0 y_0 v_0 u_0 x_0 w_0 \dots u_{s-1} x_{s-1} w_{s-1} u_s x_s w_s \end{aligned}$$

belongs to $F(\xi)$ (the bi-infinite word ξ_A will be the “limit” of these words).

To start with $s = 0$ note the following: since ξ is recurrent and $w_0 \in F(\xi)$, the bi-infinite word ξ contains a factor of the form $w_0 x w_0$. Set $y_0 = x$ and $u_0 = v_0 = x_0 = \varepsilon$.

For the induction step, assume that we constructed the tuple t_s and that z_s is a factor of ξ . Since ξ is recurrent but not periodic, the word z_s is not right-determined in $F(\xi)$ by Lemma 26. Hence there are two distinct finite words u and u' of the same length such that $z_s u, z_s u' \in F(\xi)$. For (u, u') , choose the first such pair in the effective enumeration $(\ell_i, r_i)_{i \in \mathbb{N}}$. If $s \in A$, then set $u_{s+1} = u$, otherwise set $u_{s+1} = u'$. Now the word $z_s u_{s+1}$ is a factor of ξ . Since ξ is recurrent, there is $x_{s+1} \in \{0, 1\}^*$ such that $z_s u_{s+1} x_{s+1} w_{s+1}$ is a factor of ξ – choose x_{s+1} length-lexicographically minimal among all possible such words.

To choose v_{s+1} and y_{s+1} , we proceed symmetrically to the left: $z'_s = z_s u_{s+1} x_{s+1} w_{s+1}$ is a factor of ξ that is not left-determined. Hence there exists a pair of distinct words v and v' of the same length with $v z'_s, v' z'_s \in F(w)$. Choose this pair minimal in the effective enumeration. If $s \in A$, then set $v_{s+1} = v$, otherwise set $v_{s+1} = v'$. Now there is $y_{s+1} \in \{0, 1\}^*$ with $w_{s+1} y_{s+1} v_{s+1} z'_s \in F(\xi)$ since ξ is recurrent. Choosing y_{s+1} length-lexicographically minimal completes the construction of the tuple t_{s+1} and therefore the inductive construction of all the tuples t_s . Now set $\xi_A = \dots w_1 y_1 v_1 w_0 y_0 v_0 u_0 x_0 w_0 u_1 x_1 w_1 \dots$. Observe the following:

- If $u \in F(\xi)$, then there exists $s \in \mathbb{N}$ such that $u \in F(z_s)$. Hence $F(\xi) \subseteq F(\xi_A)$.
- Let $u \in F(\xi_A)$. There exists $s \in \mathbb{N}$ such that $u \in F(z_s)$. In particular, $F(\xi_A) \subseteq F(\xi)$. Since z_s is a factor of ξ , there are infinitely many $i \in \mathbb{N}$ such that z_s (and therefore u) is a factor of w_i . Hence the word ξ_A is recurrent.

Since the above describes how to compute the bi-infinite word ξ_A using the oracles A and $F(w)$, we get $\xi_A \leq_T (A, F(\xi))$.

It remains to be shown that $A \leq_T (\xi_A, F(\xi))$ holds: To determine whether $s \in A$ suppose we already know which of the natural numbers $i < s$ belong to A . Then the construction of ξ_A above allows to build t_s using the oracle $F(\xi)$. Now construct t_{s+1} assuming $s \in A$ again using the oracle $F(\xi)$. If the resulting word z_{s+1} is an initial segment of ξ_A , then $s \in A$. Otherwise, $s \notin A$. ◀

From this lemma and Thm. 14, we get immediately that indeed, every decidable theory of some recurrent bi-infinite word is represented in every Turing-degree:

► **Theorem 28.** *Let ξ be a recurrent non-periodic bi-infinite word and \mathbf{a} a Turing-degree above the degree of $\text{MTh}(\xi)$. Then \mathbf{a} contains a bi-infinite word ξ_A with $\text{MTh}(\xi_A) = \text{MTh}(\xi)$.*

7 How many indistinguishable bi-infinite words are there?

If α and β are MSO-equivalent ω -words, then $\alpha = \beta$. In this final section we study this question for bi-infinite words. Shift-equivalence and period will be important notions in this context: two bi-infinite words ξ and ζ are *shift-equivalent* if there is $p \in \mathbb{N}$ with $\xi(n) = \zeta(n+p)$ for all $n \in \mathbb{Z}$. Furthermore, the period of the bi-infinite word ξ is the least natural number $p > 0$ with $\xi(n) = \xi(n+p)$ for all $n \in \mathbb{Z}$ – clearly, the period need not exist. To count the number of MSO-equivalent bi-infinite words, we need a characterisation when two bi-infinite words are MSO-equivalent.

► **Theorem 29** ([6, Chp. 9, Thm. 6.1]). *Two bi-infinite words ξ and ζ are MSO-equivalent if and only if one of the following conditions is satisfied:*

1. ξ and ζ are shift-equivalent.
2. ξ and ζ are recurrent and have the same set of factors.

This characterisation is the central ingredient in the proof of the following result:

► **Theorem 30.** *Let ξ be a bi-infinite word.*

- (a) *If ξ is periodic, then the cardinality of the type of ξ is finite and equals the period of ξ .*
- (b) *If ξ is non-recurrent, then the cardinality of the type of ξ is \aleph_0 .*
- (c) *If ξ is recurrent and non-periodic, then the cardinality of the type of ξ is 2^{\aleph_0} .*

Proof.

- (a) Let p be the period of ξ . Since p is minimal, there are precisely p distinct bi-infinite words that are shift-equivalent with ξ . Since shift-equivalent words are MSO-equivalent, the type of ξ contains at least p elements. It remains to be shown that no further MSO-equivalent word exists. So let ζ be some MSO-equivalent word. Then ζ is p -periodic since ξ (and therefore ζ) satisfies $\forall x: (P(x) \Leftrightarrow P(x+p))$ and does not satisfy $\forall x: (P(x) \Leftrightarrow P(x+q))$ for any $1 \leq q < p$. Furthermore $u = \xi[1, p]$ is a factor of ξ and therefore of ζ of length p . Hence $\zeta = u^{\omega^*} u^{\omega}$.
- (b) This claim follows immediately from Thm. 29.
- (c) This follows from Thm. 28 as there are 2^{\aleph_0} Turing-degree above any Turing-degree. ◀

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