Are Short Proofs Narrow? QBF Resolution is *not* Simple

Olaf Beyersdorff[1], Leroy Chew[2], Meena Mahajan[3], and Anil Shukla[4]

1 **School of Computing, University of Leeds, United Kingdom**
2 **School of Computing, University of Leeds, United Kingdom**
3 **The Institute of Mathematical Sciences, Chennai, India**
4 **The Institute of Mathematical Sciences, Chennai, India**

Abstract

The groundbreaking paper 'Short proofs are narrow – resolution made simple' by Ben-Sasson and Wigderson (J. ACM 2001) introduces what is today arguably *the* main technique to obtain resolution lower bounds: to show a lower bound for the width of proofs. Another important measure for resolution is space, and in their fundamental work, Atserias and Dalmau (J. Comput. Syst. Sci. 2008) show that space lower bounds again can be obtained via width lower bounds.

Here we assess whether similar techniques are effective for resolution calculi for quantified Boolean formulas (QBF). A mixed picture emerges. Our main results show that both the relations between size and width as well as between space and width drastically *fail* in Q-resolution, even in its weaker tree-like version. On the other hand, we obtain positive results for the expansion-based resolution systems ∀Exp+Res and IR-calc, however only in the weak tree-like models.

Technically, our negative results rely on showing width lower bounds together with simultaneous upper bounds for size and space. For our positive results we exhibit space and width-preserving simulations between QBF resolution calculi.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems: Complexity of proof procedures

Keywords and phrases proof complexity, QBF, resolution, lower bound techniques, simulations

Digital Object Identifier 10.4230/LIPIcs.STACS.2016.15

1 Introduction

The main objective in *proof complexity* is to obtain precise bounds on the size of proofs in various formal systems; and this objective is closely linked to and motivated by foundational questions in computational complexity (Cook's program), first-order logic (separating theories of bounded arithmetic), and SAT solving. In particular, resolution is one of the best studied and most important propositional proof systems, as it forms the backbone of modern SAT solvers based on conflict-driven clause learning (CDCL). Complexity bounds for resolution proofs directly translate into bounds on the performance of SAT solvers.

What is arguably even more important than showing the actual bounds is to develop *general techniques* that can be applied to obtain lower bounds for important proof systems. A number of ingenious techniques have been designed to show lower bounds for the *size of resolution proofs*, among them feasible interpolation [22], which applies to many further systems. In their pioneering paper [7], Ben-Sasson and Wigderson showed that resolution size lower bounds can be elegantly obtained by showing lower bounds to the *width* of resolution proofs. Indeed, the discovery of this relation between width and size of resolution proofs was

33rd Symposium on Theoretical Aspects of Computer Science (STACS 2016).
Editors: Nicolas Ollinger and Heribert Vollmer; Article No. 15; pp. 15:1–15:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany
SYMPOSIUM ON THEORETICAL ASPECTS OF COMPUTER SCIENCE

a milestone in our understanding of resolution, and today many if not most lower bounds for resolution are obtained via the size-width technique.

Another important measure for resolution is *space* [18], as it corresponds to memory requirements of solvers in the same way as resolution size relates to their running time. In their fundamental work [1], Atserias and Dalmau demonstrated that also space is tightly related to width. Indeed, showing lower bounds for width serves again as the primary method to obtain space lower bounds. Since these discoveries the relations between resolution size, width, and space have been subject to intense research (cf. [14]), and in particular sharp trade-off results between the measures have been obtained (cf. e.g. [4, 6, 24]).

In this paper we initiate the study of width and space in resolution calculi for quantified Boolean formulas (QBF) and address the question whether similar relations between size, width, and space as for classical resolution hold in QBF. Before explaining our results we sketch recent developments in QBF proof complexity.

QBF proof complexity is a relatively young field studying proof systems for quantified Boolean logic. Similarly as in the propositional case, one of the main motivations for the field comes via its intimate connection to solving. Although QBF solving is at an earlier state than SAT solving, due to its PSPACE completeness, QBF even applies to further fields such as formal verification or planning [25, 8, 17]. Each successful run of a solver on an unsatisfiable instance can be interpreted as a proof of unsatisfiability; and this connection turns proof complexity into the main theoretical tool to understand the performance of solving. As in SAT, QBF solvers are known to correspond to the resolution proof system and its variants.

However, compared to SAT, the QBF picture is more complex as there exist two main solving approaches utilising CDCL and expansion-based solving. To model the strength of these QBF solvers, a number of resolution-based QBF proof systems have been developed. Q-resolution (Q-Res) by Kleine Büning, Karpinski, and Flögel [21] forms the core of the CDCL-based systems. To capture further ideas from CDCL solving, Q-Res has been augmented to long-distance resolution [28, 2], universal resolution [27], and their combinations [3]. Powerful proof systems for expansion-based solving were recently developed in the form of ∀Exp+Res [20], and the stronger IR-calc and IRM-calc [10].

In this paper we concentrate on the three QBF resolution systems Q-Res, ∀Exp+Res, and IR-calc. This choice is motivated by the fact that Q-Res and ∀Exp+Res form the base systems for CDCL and expansion-based solving, respectively, and IR-calc unifies both approaches in a natural way, as it simulates both Q-Res and ∀Exp+Res [10]. Recent findings show that CDCL and expansion are indeed orthogonal paradigms as Q-Res and ∀Exp+Res are incomparable with respect to simulations [11].

Understanding which lower bound techniques are effective in QBF proof complexity is important for progress in the field. In [12], the feasible interpolation technique was shown to apply to all QBF resolution systems. Another successful transfer of a classical technique was obtained in [13] for a game-theoretic characterisation of proof size in tree-like Q-Res.

Our Contributions

The central question we address here is whether *lower bound techniques via width*, which have revolutionised classical proof complexity, are also effective for QBF resolution systems.

Though space and width have not been considered in QBF before, these notions straightforwardly apply to QBF resolution systems. However, due to the ∀-reduction rule in Q-Res handling universal variables, it is relatively easy to enforce that universal literals accumulate in clauses of Q-Res proofs, thus always leading to large width, irrespective of size and space requirements (Lemma 4). This prompts us to consider *existential width* – counting only existential literals – as an appropriate width measure in QBF. This definition aligns both with Q-Res, resolving only on existential variables, as well as with ∀Exp+Res and IR-calc, which like all expansion systems only operate on existential literals.

1. Negative results. Our main results show that the size-width relation of [7] as well as the space-width relation of [1] dramatically *fail* for Q-Res, even when considering the tighter existential width. We first notice that the proof establishing the size-width result in [7] almost fully goes through, except for some very inconspicuous step that fails in QBF (Proposition 5). But not only the technique fails: we prove that Tseitin transformations of formulas expressing a natural completion principle from [20] have small size and space, but require large existential width in tree-like Q-Res (Theorem 6), thus refuting the size-width relation for tree-like Q-Res as well as the space-width relation for general dag-like Q-Res.

As the formulas for the completion principle have $O(n^2)$ variables, they do not rule out size-width relations in general Q-Res. However, we show that different formulas, hard for tree-like Q-Res [20], provide counterexamples for size-width relations in full Q-Res (Theorem 7).

Technically, our main contributions are width lower bounds for the above formulas, which we show by careful counting arguments. We complement these results by existential width lower bounds for parity-formulas from [11], providing an optimal width separation between Q-Res and ∀Exp+Res (Theorem 17).

2. Positive results and width-space-preserving simulations. Though the negative picture above prevails, we prove some positive results for size-width-space relations for tree-like versions of the expansion resolution systems ∀Exp+Res and IR-calc. Proofs in ∀Exp+Res can be decomposed into two clearly separated parts: an expansion phase followed by a classical resolution phase. This makes it easy to transfer almost the full spectrum of the classical relations to ∀Exp+Res (Theorem 18).

To lift these results to IR-calc (Theorem 19), we show a series of careful space and width-preserving simulations between tree-like Q-Res, ∀Exp+Res, and IR-calc. In particular, we show the surprising result that tree-like ∀Exp+Res and tree-like IR-calc are equivalent (Lemma 14), thus providing a rare example of two proof systems that coincide in the tree-like, but are separated in the dag-like model [11]. The only other such example that we are aware of is regular resolution vs. full resolution (although this is perhaps slightly less natural as regular resolution is just a sub-system of resolution). In addition, our simulations provide a simpler proof for the simulation of tree-like Q-Res by ∀Exp+Res (Corollary 16), shown in [20] via a very involved argument.

Our last positive result is a size-space relation in tree-like Q-Res (Theorem 19), which we show by a pebbling game analogous to the classical relation in [18]. Not surprisingly, this only positive result for Q-Res avoids any reference to the notion of width.

As the bottom line we can say that QBF proof complexity is not just a replication of classical proof complexity: it shows quite different and interesting effects as we demonstrate here. Especially for lower bounds it requires new ideas and techniques. We remark that in this direction, a new and 'genuine QBF technique' based on strategy extraction was recently developed, showing lower bounds for Q-Res [11] and indeed much stronger systems [9].

Organisation of the paper. We start by reviewing background information on classical and QBF resolution systems (Sect. 2), including definitions of size, space, and width together with their main classical relations (Sect. 3). In Sect. 4 we prove our main negative results on the failure of the transfer of the classical size-width and space-width results to QBF.

Section 5 contains the simulations between tree-like versions of Q-Res, ∀Exp+Res, and IR-calc, paying special attention to width and space. This enables us to show in Sect. 6 the positive results for relations between size, width, and space in these systems. We conclude in Sect. 7 with a discussion and directions for future research.

2 Notations and Preliminaries

Quantified Boolean formulas. A (closed prenex) *quantified Boolean formula* (QBF) is a formula in quantified propositional logic where each variable is quantified at the beginning of the formula, using either an existential or universal quantifier. We denote such formulas as $\mathcal{Q} \,.\, \phi$, where ϕ is a propositional Boolean formula in conjunctive normal form (CNF), called *matrix*, and $\mathcal{Q}$ is its *quantifier prefix*. The *quantification level* $\mathrm{lv}(y)$ of a variable y in $\mathcal{Q} \,.\, \phi$ is the number of alternations of quantifiers y has on its left in the quantifier prefix of $\mathcal{Q} \,.\, \phi$.

Classical resolution. *Resolution* (Res), introduced by Blake [15] and Robinson [26], is a refutational proof system manipulating unsatisfiable CNFs as sets of clauses. The only inference rule is $\dfrac{C \vee x \qquad D \vee \neg x}{C \vee D}$ where C, D denote clauses and x is a variable. A Res refutation derives the empty clause $\Box$. If we only allow proofs in form of a tree, i.e., each derived clause can be used at most once, we speak of *tree-like resolution*, denoted Res_T.

QBF resolution calculi. *Q-resolution* (Q-Res) [21] is a resolution-like calculus that operates on QBFs in prenex form where the matrix is a CNF. It uses the propositional resolution rule above with the side conditions that variable x is existential and if $z \in C$, then $\neg z \notin D$. In addition Q-Res has a universal reduction rule $\dfrac{C \vee u}{C}$ (∀-Red) where variable u is universal and all other existential variables $x \in C$ are left of u in the quantifier prefix.

In addition to Q-Res we consider two further QBF resolution calculi that have been introduced to model *expansion-based QBF solving*. These calculi are based on *instantiation* of universal variables: ∀Exp+Res [20], and IR-calc [10]. Both calculi operate on clauses that comprise only existential variables from the original QBF, which are additionally *annotated* by a substitution to some universal variables, e.g. $\neg x^{u/0,v/1}$. For any annotated literal l^σ, the substitution σ must not make assignments to variables right of l, i.e. if $u \in \mathsf{dom}(\sigma)$, then u is universal and $\mathrm{lv}(u) < \mathrm{lv}(l)$. To preserve this invariant, we use the *auxiliary notation* $l^{[\sigma]}$, which for an existential literal l and an assignment σ to the universal variables filters out all assignments that are not permitted, i.e. $l^{[\sigma]} = l^{\{u/c \in \sigma \,\mid\, \mathrm{lv}(u) < \mathrm{lv}(l)\}}$. We say that an assignment is complete if its domain is all universal variables. Likewise, we say that a literal x^τ is fully annotated if all universal variables u with $\mathrm{lv}(u) < \mathrm{lv}(x)$ in the QBF are in $\mathsf{dom}(\tau)$, and a clause is fully annotated if all its literals are fully annotated.

The calculus ∀Exp+Res from [20] works with fully annotated clauses on which resolution is performed. For each clause C from the matrix and an assignment τ to all universal variables, ∀Exp+Res can use the axiom $\left\{l^{[\tau]} \mid l \in C, l \text{ existential}\right\} \cup \{\tau(l) \mid l \in C, l \text{ universal}\}$. As its only rule it uses the resolution rule on annotated variables

$$\frac{C \vee x^\tau \qquad D \vee \neg x^\tau}{C \vee D} \text{ (Res)}.$$

In contrast, the system IR-calc from [10] is more flexible. It uses 'delayed' expansion and can mix instantiation with resolution steps. Formally, IR-calc works with partial assignments on which we use auxiliary operations of *completion* and *instantiation*. For assignments τ and μ, we write $\tau \veebar \mu$ for the assignment σ defined as $\sigma(x) = \tau(x)$ if $x \in \mathsf{dom}(\tau)$, otherwise

$\sigma(x) = \mu(x)$ if $x \in \mathsf{dom}(\mu)$. The operation $\tau \veebar \mu$ is called *completion* as μ provides values for variables not defined in τ. For an assignment τ and an annotated clause C, the function $\mathsf{inst}(\tau, C)$ returns the annotated clause $\{l^{[\sigma \veebar \tau]} \mid l^{\sigma} \in C\}$.

Axioms in IR-calc allow to infer $\{x^{[\tau]} \mid x \in C, x \text{ is existential}\}$ for each non-tautological clause C from the matrix and $\tau = \{u/0 \mid u \text{ is universal in } C\}$, where the notation $u/0$ for literals u is shorthand for $x/0$ if $u = x$ and $x/1$ if $u = \neg x$. Rules in IR-calc comprise the (Res) rule above together with the instantiation rule $\dfrac{C}{\mathsf{inst}(\tau, C)}$ for a (partial) assignment τ to universal variables.

Simulations. Given two proof systems P and Q for the same language (TAUT or QBF), P *p-simulates* Q if each Q-proof can be transformed in polynomial time into a P-proof of the same formula. Two systems are called *p-equivalent* if they p-simulate each other.

In [10] it was shown that IR-calc p-simulates both Q-Res and ∀Exp+Res, while [11] shows that Q-Res and ∀Exp+Res are incomparable, i.e., IR-calc is exponentially stronger than both Q-Res and ∀Exp+Res. However, ∀Exp+Res can p-simulate $\mathsf{Q\text{-}Res_T}$ [20].

3 Size, Width, and Space in Resolution Calculi

The purpose of the section is twofold: first to review the measures size, width, and space and their relations in classical resolution; and second to explain how to apply these measures to QBF resolution systems. While this is straightforward for size and space, we need a more elaborate discussion on what constitutes a good notion of width for QBF resolution systems.

3.1 Defining Size, Width, and Space for Resolution

For a CNF F, $|F|$ is the number of clauses in it, and $w(F)$ denotes the maximum number of literals in any clause of F. We extend the same notation to QBFs with a CNF matrix.

For P one of the resolution calculi Res, Q-Res, ∀Exp+Res, IR-calc, let $\pi \vdash_{\mathsf{P}} F$ (resp. $\pi \vdash_{\mathsf{P_T}} F$) denote that π is an P-proof (tree-like P-proof, respectively) of the formula F. For a proof π of F in system P, its size $|\pi|$ is defined as the number of clauses in π. The *size complexity* $S(\vdash_{\mathsf{P}} F)$ of deriving F in P is defined as $\min\{|\pi| \ : \ \pi \vdash_{\mathsf{P}} F\}$. The tree-like size complexity, denoted $S(\vdash_{\mathsf{P_T}} F)$, is $\min\{|\pi| \ : \ \pi \vdash_{\mathsf{P_T}} F\}$.

The *width* of a clause C is the number of literals in C, denoted $w(C)$. The width $w(F)$ of a CNF F is the maximum width of a clause in F. The width $w(\pi)$ of a proof π is the maximum width of any clause appearing in π, and the width $w(\vdash_{\mathsf{P}} F)$ of refuting a CNF F in P is defined as $\min\{w(\pi) \ : \ \pi \vdash_{\mathsf{P}} F\}$. Again the same notation extends to quantified CNFs.

Note that for width in any calculus, whether the proof is tree-like or not is immaterial, since a proof can always be made tree-like by duplication without increasing the width. We therefore drop the T subscript when talking about proof width.

The third complexity measure for resolution calculi is *space*[1], first defined in [18]. Informally, it is the minimal number of clauses that must be kept simultaneously to refute a formula. We view a proof as a sequence of CNF formulas $F_0, F_1, \ldots, F_s$, where $F_0 = \emptyset$, $\Box \in F_s$, and each F_{i+1} is obtained from F_i by erasing some clause, downloading an axiom, or adding a clause derived by some P-rule from clauses in F_i. In the last case, one of the premises of the inference may also simultaneously be deleted. For such a proof σ, *CSpace*(σ) is the

[1] Also called clause space, to distinguish it from variable space or total space (see for example, [5]). We consider only clause space in this paper, and so we call it just space.

maximum number of clauses in any F_i, $i \in [s]$. The space to refute F, denoted $CSpace(\vdash_{\mathsf{P}} F)$, is the minimum $CSpace(\sigma)$ over all P-refutations σ for F. The same notions apply to QBFs, where $F_0, F_1, \ldots, F_s$ is a sequence of CNF formulas, all with the same quantifier prefix.

If we modify the inference step so that the clause(s) used to obtain the inference are erased in the same step, then any clause can be used at most once and we obtain a tree-like space-oriented P-proof. Correspondingly we can define $CSpace(\vdash_{\mathsf{P}_\mathsf{T}} F)$ as the minimum space used by any tree-like proof sequence refuting F.

3.2 Relations in Classical Resolution

We now state some of the main relations between size, width, and space for classical resolution. We start with the foundational size-width relations of Ben-Sasson and Wigderson [7].

▶ **Theorem 1** (Ben-Sasson, Wigderson [7]). *For all unsatisfiable CNFs F in n variables the following holds:* $S(\vdash_{\mathsf{Res}_T} F) \geq 2^{w\left(\vdash_{\mathsf{Res}} F\right) - w(F)}$ *and* $S(\vdash_{\mathsf{Res}} F) = \exp\left(\Omega\left(\frac{\left(w\left(\vdash_{\mathsf{Res}} F\right) - w(F)\right)^2}{n}\right)\right)$.

Space complexity was introduced in [18] and relations between space, size, and width are explored (cf. also [23, 14]), establishing the size-space relation for tree-like resolution:

▶ **Theorem 2** (Esteban, Torán [18]). *For all unsatisfiable CNFs F the following relation holds:* $S(\vdash_{\mathsf{Res}_T} F) \geq 2^{CSpace\left(\vdash_{\mathsf{Res}_T} F\right)} - 1$.

The fundamental relation between space and width for full resolution was obtained in [1]; a more direct proof was given recently in [19].

▶ **Theorem 3** (Atserias, Dalmau [1]). *For all unsatisfiable CNFs F the following relation holds:* $w(\vdash_{\mathsf{Res}} F) \leq CSpace(\vdash_{\mathsf{Res}} F) + w(F) - 1$.

3.3 Existential Width: What Is the Right Width Notion for QBF?

We wish to explore the possibility of a similar approach as in [7] to prove analogues of the classical results above for QBFs. The following simple example shows, however, that the relationships in Theorem 1 and Theorem 3 do not carry over for the system Q-Res.

▶ **Proposition 4.** *For the false QBFs* $\mathcal{F}_n = \forall u_1 \ldots u_n \exists e_0 \exists e_1 \ldots e_n . (e_0) \wedge \bigwedge_{i \in [n]} (\neg e_{i-1} \vee u_i \vee e_i) \wedge (\neg e_n)$ *we have* $S(\vdash_{\mathsf{Q\text{-}Res}_T} \mathcal{F}_n) = O(n)$ *and* $CSpace(\vdash_{\mathsf{Q\text{-}Res}_T} \mathcal{F}_n) = O(1)$*, but* $w(\vdash_{\mathsf{Q\text{-}Res}} \mathcal{F}_n) = \Omega(n)$.

As this example illustrates, it is easy to enforce that universal variables are accumulated in a clause, thus leading to large width. Hence the following question naturally arises: can we obtain size-width or space-width relations by using the tighter measure of only counting existential variables?

This aligns with the situation in the expansion systems ∀Exp+Res and IR-calc, where clauses contain only existential variables. In this respect, it is worth noting that the above example indeed does not demonstrate the failure of the size-width relationship in expansion-based calculi. For instance, in ∀Exp+Res, a tree-like refutation could download the existential variables of axioms annotated with $u_i/0$ for $i \in [n]$, and generate the empty clause in $O(n)$ steps with width just 2 at the leaves and 1 at the internal nodes.

Thus, to get a consistent and interesting width measure for QBF calculi, we consider the notion of *existential width* that just counts the number of existential literals. This approach is justified also for Q-Res as the calculus can only resolve on existential variables, and rules out the easy counterexamples above. Formally, we define the existential width of a clause C to be the number of existential literals in C, and denote it by $w_\exists(C)$. Using $w_\exists$ instead of w everywhere, we obtain the existential width of a formula $w_\exists(F)$, of a proof $w_\exists(\pi)$, and of refuting a false sentence $w_\exists(\vdash_{\mathsf{P}} \mathcal{F})$.

For the expansion systems ∀Exp+Res and IR-calc the notions of existential width and width coincide. (In particular, distinct annotations of the same existential variable are counted as distinct literals.) Hence we can drop the ∃ subscript in width of proofs in these systems. For the width of the sentence itself, there is still a difference between w and $w_\exists$.

4 Negative Results: Size-Width, Space-Width Relations Fail in Q-Res

In this section we show that in the Q-Res proof system, even replacing width by existential width, the relations to size or space as in classical resolution (Theorems 1 and 3) no longer hold for both tree-like and general proofs.

Firstly, we point out where the technique of [7] fails. A crucial ingredient of their proof is the following statement: if a clause A can be derived from $F|_{x=1}$ in width w, then the clause $A \vee \neg x$ can be derived from F in width $w + 1$ (possibly using a weakening rule at the end). We show that the statement no longer holds in Q-Res.

▶ **Proposition 5.** *There are false sentences ψ_n, with an existential literal b quantified at the innermost level, such that the sentence $\psi_n|_{b=1}$ is false and has a small existential-width proof, but ψ_n itself needs large existential width to refute in Q-Res.*

Proof. The sentence ψ_n is constructed by taking the conjunction of two sentences with distinct variables. The first sentence is a very simple one: $\exists a \forall u \exists b. (a \vee u \vee \neg b) \wedge (\neg a)$. It is a true sentence, but if b is set to 1, it becomes false. The second sentence is a false sentence of the form $\exists \vec{x}. G_n(\vec{x})$, where G_n is any unsatisfiable CNF formula over the $\vec{x}$ variables, such that G_n needs large width in classical resolution. One such example is the CNF formula described by Bonet and Galesi [16], that we denote as BG_n. BG_n is an unsatisfiable 3-CNF formula over $O(n^2)$ variables with $w(\vdash BG_n) = \Omega(n)$.

Now define ψ_n as $\exists \vec{x} \exists a \forall u \exists b. (a \vee u \vee \neg b) \wedge (\neg a) \wedge BG_n(\vec{x})$. Note that the clauses $(a \vee u \vee \neg b) \wedge (\neg a)$ contain a contradiction if and only if $b = 1$. Thus $\psi_n|_{b=1}$ can be refuted with existential width 1 using just these two clauses: a ∀-Red on $(a \vee u)$ yields a which can be resolved with $\neg a$. On the other hand, to refute ψ_n, the contradiction in BG_n must be exposed. Since all the variables involved are existential, Q-Res degenerates to classical resolution, requiring (existential) width $\Omega(n)$. ◀

The example in Proposition 5 can be made 'less degenerate' by interleaving more existential and universal variables disjoint from $\vec{x}$ and putting them in the first sentence. All we need is that b is quantified existentially at the end, the first sentence is true as a whole but false if $b = 1$, and this latter sentence can be refuted in Q-Res with small existential width.

We now show that it is not just the technique of [7] that fails for Q-Res. No other technique will work either, because the relation from Theorem 1 between size and existential width itself fails to hold. The same example shows that the relation from Theorem 3 between space and existential width also fails to hold.

We first give an example where the relation for tree-like proofs fails.

▶ **Theorem 6.** *There exist false QBFs CR'_n over $O(n^2)$ variables, such that $S(\vdash_{\textsf{Q-Res}_T} CR'_n) = n^{O(1)}$, $w_\exists(CR'_n) = 3$, $CSpace(\vdash_{\textsf{Q-Res}_T} CR'_n) = O(1)$, and $w_\exists(\vdash_{\textsf{Q-Res}_T} CR'_n) = \Omega(n)$.*

The formulas CR'_n are Tseitin transformations of a natural completion principle formula CR_n from [20]. The proof is similar, but slightly more involved than the proof for our next Theorem 7. Since tree-like space is at least as large as space, Theorem 6 also rules out the space-width relation for general dag-like Q-Res proofs.

However, Theorem 6 cannot be used to show that the size-existential-width relationship for general dag-like proofs fails in Q-Res, because CR'_n have $O(n^2)$ variables. We show via another example that the relation fails to hold in Q-Res as well. This example cannot be used for proving Theorem 6 because it is known to be hard for $\mathsf{Q\text{-}Res_T}$ [20].

▶ **Theorem 7.** *There is a family of false QBFs ϕ'_n in $O(n)$ variables such that $S(\vdash_{\mathsf{Q\text{-}Res}} \phi'_n) = n^{O(1)}$, $w_\exists(\phi'_n) = 3$, and $w_\exists(\vdash_{\mathsf{Q\text{-}Res}} \phi'_n) = \Omega(n)$.*

Proof. Consider the following formulas ϕ_n, introduced by Janota and Marques-Silva [20]: $\exists e_1 \forall u_1 \exists c_1 c_2 \ldots \exists e_n \forall u_n \exists c_{2n-1} c_{2n}$.

$$\bigwedge_{i \in [n]} \big((\neg e_i \vee c_{2i-1}) \wedge (\neg u_i \vee c_{2i-1}) \wedge (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i})\big) \wedge \bigvee_{i \in [2n]} \neg c_i.$$

We know from [20] that ϕ_n have polynomial-size proofs in Q-Res (but require exponential-size proofs in $\mathsf{Q\text{-}Res_T}$). However, we need a formula with constant initial width. To achieve this we consider quantified Tseitin transformations of ϕ_n, i.e. we introduce $2n+1$ new existential variables x_i at the innermost quantification level in ϕ_n, and replace the only large clause in ϕ_n by any CNF formula that preserves satisfiability. Let ϕ'_n denote the modified formula:

$$\phi'_n = \exists e_1 \forall u_1 \exists c_1 c_2 \ldots \exists e_n \forall u_n \exists c_{2n-1} c_{2n} \exists x_0 \ldots x_{2n}$$

$$\bigwedge_{i \in [n]} \big((\neg e_i \vee c_{2i-1}) \wedge (\neg u_i \vee c_{2i-1}) \wedge (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i})\big) \wedge \tag{1}$$

$$\neg x_0 \wedge \bigwedge_{i \in [2n]} (x_{i-1} \vee \neg c_i \vee \neg x_i) \wedge x_{2n}. \tag{2}$$

Note that $w_\exists(\phi'_n) = 3$. We refer to the clauses in (2) as x-clauses. It is clear that from the x-clauses, we can derive the large clause of ϕ_n in $2n+1$ resolution steps and get back ϕ_n. Thus $S(\vdash_{\mathsf{Q\text{-}Res}} \phi'_n) \leq S(\vdash_{\mathsf{Q\text{-}Res}} \phi_n) + 2n + 1 \in n^{O(1)}$.

We now show that ϕ'_n needs large existential width. Let π be a proof in Q-Res, $\pi \vdash_{\mathsf{Q\text{-}Res}} \phi'_n$. List the clauses of π in sequence, $\pi = \{D_0, D_1, \ldots, D_s = \Box\}$, where each clause in the sequence is either a clause from ϕ'_n, or is derived from clause(s) preceding it in the sequence using resolution or $\forall$-Red. There must be at least one universal reduction step in π, since all the initial clauses are necessary for refuting ϕ'_n, some of them contain universal variables, and the only way to remove a universal variable in Q-Res is by $\forall$-Red. Let i be the least index such that the clause D_i is obtained by $\forall$-Red on D_j for some $0 < i$. Since all x variables block all u variables, D_j and D_i cannot contain any x variables. We use this fact to show that $w_\exists(D_i) = \Omega(n)$. Our strategy is to associate some set with each clause in π in a specific way, and use the set size to bound existential width.

We associate the following sets with the literals of ϕ'_n and the clauses of π.

$$\begin{array}{lrcl}
 & \sigma(x_0) & = & \emptyset \\
\forall i \in [2n] & \sigma(x_i) & = & [i] = \{1, 2, \ldots, i\} \\
 & \sigma(\neg x_0) & = & [2n] \\
\forall i \in [2n] & \sigma(\neg x_i) & = & [2n] \setminus [i] = \{i+1, \ldots, 2n\} \\
\forall i \in [n] & \sigma(e_i) = \sigma(u_i) = \sigma(\neg c_{2i}) = \sigma(c_{2i-1}) & = & \{2i\} \\
\forall i \in [n] & \sigma(\neg e_i) = \sigma(\neg u_i) = \sigma(\neg c_{2i-1}) = \sigma(c_{2i}) & = & \{2i-1\} \\
\forall D \in \pi & \sigma(D) & = & \bigcup_{l \in D} \sigma(l).
\end{array}$$

Note that for any literal ℓ, $\sigma(\ell)$ and $\sigma(\neg\ell)$ are disjoint.

For $D \in \pi$, let π_D be the sub-DAG of π, rooted at D.

▶ **Claim 8.** *π_{D_i} contains at least one x-clause (axiom clause of type (2)).*

Proof. The parent D_j of node D_i contains a universal variable which is then removed through $\forall$-Red to get D_i. The universal variables appear only in clauses of type (1), but are blocked by the c-variables in every clause where they appear. Thus, before a reduction is permitted, a c-variable must be eliminated by resolution. Since all c-variables appear only positively in type (1) clauses, some x-clause must be used in the resolution. ◀

We show that all clauses in π_{D_i} that are descendants of some x-clause have large sets associated with them. In particular, we show:

▶ **Claim 9.** *Every clause D in π_{D_i} such that π_D contains an x-clause has $\sigma(D) = [2n]$.*

Deferring the proof briefly, we continue with our argument. From Claim 9 we conclude that $\sigma(D_i) = [2n]$. Recall that none of the x variables belongs to D_i. All other literals are associated with singleton sets, so D_i must contains at least $2n$ literals in order to be associated with the complete set $[2n]$. Since Q-Res proofs prohibit a variable and its negation in the same clause, at most n of the literals in D_i can be universal variables. Thus D_i has at least n existential literals, hence $w_\exists(D_i) = \Omega(n)$.

It remains to establish the claimed set size.

Proof of Claim 9. We proceed by induction on the depth of descendants of x-clauses in π_{D_i}. The base case is an x-clause itself and follows from the definition of σ.

For the inductive step, let D be obtained by resolving $(E \vee z)$ and $(F \vee \neg z)$. There are two cases to consider:

Case 1: Both $(E \vee z)$ and $(F \vee \neg z)$ are descendants of x-clauses (not necessarily the same x-clause). Then by induction, $\sigma(E \vee z) = \sigma(F \vee \neg z) = [2n]$. So $\sigma(E) \supseteq [2n] \setminus \sigma(z)$ and $\sigma(F) \supseteq [2n] \setminus \sigma(\neg z)$. Since $\sigma(z)$ and $\sigma(\neg z)$ are disjoint, $\sigma(E) \cup \sigma(F) = [2n]$. Thus $\sigma(D) = \sigma(E) \cup \sigma(F) = [2n]$ as claimed.

Case 2: Exactly one of $(E \vee z)$ and $(F \vee \neg z)$ is a descendant of an x-clause. Without loss of generality, let $F \vee \neg z$ be the descendant. Then $E \vee z$ is either a type-(1) clause or is derived solely from type-(1) clauses using resolution. However, observe that the only clauses derivable solely from type-(1) clauses via resolution, without creating tautologies as mandated in Q-Res, are of the form $(c_{2i-1} \vee c_{2i})$ for some i. It follows that z is not an x variable. Hence $\sigma(z)$ and $\sigma(\neg z)$ are distinct singleton sets. Further, z cannot be a u variable either, since resolution on universal variables is not permitted in Q-Res.

Now note that for any type-(1) clause C, $\sigma(C) = \{2i-1, 2i\}$ for the appropriate i. Similarly, $\sigma(c_{2i-1} \vee c_{2i}) = \{2i-1, 2i\}$. So if $E \vee z$ is one of these clauses, then $\sigma(E \vee z) = \sigma(z) \cup \sigma(\neg z)$ and $\sigma(E) = \sigma(\neg z)$. Further, as in Case 1, by induction we know that $\sigma(F \vee \neg z) = [2n]$ and $\sigma(F) \supseteq [2n] \setminus \sigma(\neg z)$. Hence, $\sigma(E \vee F) = [2n]$ as claimed. ◀

This completes the proof of the theorem. ◀

The above counterexamples are provided by formulas that require small size, but large existential width. We will now illustrate via another example that also *large size and large width* can occur. These examples are very natural formulas based on the parity function, which have recently been used in [11] to show exponential size lower bounds for Q-Res, and indeed a separation between Q-Res and ∀Exp+Res. We will later use these formulas in Section 5 to also show a separation for width between Q-Res and ∀Exp+Res.

Let $\mathrm{xor}(o_1, o_2, o)$ be the set of clauses expressing $o \equiv o_1 \oplus o_2$; that is, $\{\neg o_1 \vee \neg o_2 \vee \neg o,\ o_1 \vee o_2 \vee \neg o,\ \neg o_1 \vee o_2 \vee o,\ o_1 \vee \neg o_2 \vee o\}$. In [11], the sentence $\mathrm{QP{\scriptstyle ARITY}}_n$ is defined as follows:

$$\exists x_1, \ldots, x_n \,\forall z \,\exists t_2, \ldots, t_n.\ \mathrm{xor}(x_1, x_2, t_2) \cup \bigcup_{i=3}^{n} \mathrm{xor}(t_{i-1}, x_i, t_i) \cup \{z \vee t_n, \neg z \vee \neg t_n\}.$$

The x_i variables act as the input for the parity function, and the t_i variables are defined inductively to calculate $\mathrm{P{\scriptstyle ARITY}}(x_1, \ldots, x_i)$.

We now complement the exponential size lower bound from [11] by a width lower bound.

▶ **Theorem 10.** $w_\exists(\vdash_{\text{Q-Res}} \mathrm{QP{\scriptstyle ARITY}}_n) \geq n$.

Proof. In the formula $\mathrm{QP{\scriptstyle ARITY}}_n$, the contradiction occurs semantically because of the clauses $z \vee t_n$, $\neg z \vee \neg t_n$ asserting $z \neq t_n$ (along with the fact that the values of x variables uniquely determine the values of all t variables, in particular, t_n). Thus, at least one of these clauses must be used in any proof, necessitating a ∀-reduction. In Q-Res we cannot reduce z while any of the t variables are present; and due to the restrictions in Q-Res we cannot resolve any descendants of $z \vee t_n$ with any descendants of $\neg z \vee \neg t_n$ until there is at least one ∀-reduction.

Consider a smallest Q-Res proof, and assume without loss of generality that a first (lowest) ∀ reduction happens on the positive literal z. Therefore before this ∀-reduction step we have essentially a resolution proof π from $\Gamma = \mathrm{xor}(x_1, x_2, t_2) \cup \bigcup_{i=3}^{n} \mathrm{xor}(t_{i-1}, x_i, t_i) \cup \{t_n \vee z\}$. The clause D that occurs in π immediately before the ∀-reduction must only contain variables from $\{x_1, \ldots, x_n\}$ apart from the literal z, else the reduction is blocked.

We now use the following observation.

▶ **Claim 11.** *Suppose $x_1 \oplus \cdots \oplus x_n \vDash C$ for some clause C. Then C is either a tautology or C contains all variables $x_1, \ldots, x_n$.*

Any assignment to the x variables satisfying $x_1 \oplus \cdots \oplus x_n$ has a unique extension to z and the t variables satisfying all clauses of the formula $\mathrm{QP{\scriptstyle ARITY}}_n$. This extension necessarily has $t_n = x_1 \oplus \cdots \oplus x_n = 1$ and $z = 0$. Since it satisfies all axioms, by soundness of resolution, it also satisfies D.

This, along with Claim 11, implies that D is either a tautology or has all x variables. Since it cannot be a tautology (it appears in the proof, and besides, at the very least it has the variable z), it must have all x variables, and hence has existential width n. ◀

5 Simulations: Preserving Size, Width, and Space Across Calculi

After these strong negative results, ruling out size-width and space-width relations in Q-Res and Q-Res_T, we aim to determine whether any positive results hold in the expansion systems ∀Exp+Res and IR-calc. Before we can do this we need to relate the measures of size, width, and space across the three calculi Q-Res, ∀Exp+Res, IR-calc. Of course, such a comparison in terms of refined simulations is also interesting in its own as it determines the relative strength of the different proof systems. As size corresponds to running time, and space to memory consumption of QBF solvers, such a comparison yields interesting insights into the power of QBF solvers using CDCL vs. expansion techniques.

It is known that IR-calc p-simulates ∀Exp+Res and Q-Res [10], and that ∀Exp+Res p-simulates Q-Res_T [20]. We revisit these proofs, with special attention to the width parameter, and also obtain simulating proofs that are tree-like if the original proof is tree-like. The relationships we establish are stated in the following theorem:

▶ **Theorem 12.** *For all false QBFs $\mathcal{F}$, the following relations hold:*
1. $\frac{1}{2}S(\vdash_{\mathsf{IR_T\text{-}calc}} \mathcal{F}) \leq S\left(\vdash_{\forall\mathsf{Exp+Res_T}} \mathcal{F}\right) \leq S(\vdash_{\mathsf{IR_T\text{-}calc}} \mathcal{F}) \leq 3S(\vdash_{\mathsf{Q\text{-}Res_T}} \mathcal{F})$.
2. $w(\vdash_{\mathsf{IR\text{-}calc}} \mathcal{F}) = w(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F}) \leq w_\exists(\vdash_{\mathsf{Q\text{-}Res}} \mathcal{F})$.
3. $CSpace(\vdash_{\forall\mathsf{Exp+Res_T}} \mathcal{F}) = CSpace(\vdash_{\mathsf{IR_T\text{-}calc}} \mathcal{F}) \leq CSpace(\vdash_{\mathsf{Q\text{-}Res_T}} \mathcal{F})$.

These results follow from Proposition 13 and Lemmas 14, 15 below. Our first simulation of ∀Exp+Res by IR-calc only needs to complete partial annotations in axioms:

▶ **Proposition 13.** *Any proof in ∀Exp+Res of size S, width W, and space C can be efficiently converted into a proof in IR-calc of size at most $2S$, width W, and space C. If the proof in ∀Exp+Res is tree-like, so is the resulting IR-calc proof.*

▶ **Lemma 14.** *$\forall\mathsf{Exp+Res_T}$ p-simulates $\mathsf{IR_T}$-calc while preserving width, size, and space.*

Proof Sketch. The idea is to systematically transform an $\mathsf{IR_T}$-calc proof, proceeding downwards from the top where we have the empty clause, and modifying annotations as we go down, so that when all leaves have been modified the resulting proof is in fact an $\forall\mathsf{Exp+Res_T}$ proof. This crucially requires that we start with a tree-like proof; if the underlying graph is not a tree, we cannot always find a way of modifying the annotations that will work for all descendants. ◀

The simulation in Lemma 14 exhibits an interesting phenomenon: while it shows that the tree-like versions of ∀Exp+Res and IR-calc are p-equivalent, it was shown in [11] that in the dag-like versions, IR-calc is exponentially stronger than ∀Exp+Res. Thus ∀Exp+Res and IR-calc provide a rare example in proof complexity of two systems that coincide in the tree-like model, but are separated in the dag-like model.

▶ **Lemma 15.** *$\mathsf{IR_T}$-calc p-simulates $\mathsf{Q\text{-}Res_T}$ while preserving space and existential width exactly and size up to a factor of 3.*

Proof Sketch. We use the same simulation as given in [10]. This simulation was originally for dag-like proof systems, but here we check that it also works for tree-like systems and observe that space and existential width are preserved. ◀

As a by-product, these simulations enable us to give an easy and elementary proof of the simulation of $\mathsf{Q\text{-}Res_T}$ by ∀Exp+Res, shown in [20] via a more involved argument.

▶ **Corollary 16** (Janota, Marques-Silva [20])**.** *$\forall\mathsf{Exp+Res_T}$ p-simulates $\mathsf{Q\text{-}Res_T}$.*

Using again the width lower bound for $\textsc{QParity}_n$ (Theorem 10) we can show that item 2 of Theorem 12 cannot be improved, i.e. we obtain an optimal width separation between Q-Res and ∀Exp+Res.

▶ **Theorem 17.** *$w_\exists(\vdash_{\mathsf{Q\text{-}Res}} \textsc{QParity}_n) = \Omega(n)$, but $w(\vdash_{\forall\mathsf{Exp+Res}} \textsc{QParity}_n) = O(1)$.*

Proof. By Theorem 10, $\textsc{QParity}_n$ requires existential width n in Q-Res. To get the separation it remains to show $w(\vdash_{\forall\mathsf{Exp+Res}} \textsc{QParity}_n) = O(1)$. For this we use the following ∀Exp+Res proofs of $\textsc{QParity}_n$ from [11]: the formulas $\textsc{QParity}_n$ have exactly one universal variable z, which we expand in both polarities 0 and 1. This does not affect the x_i variables, but creates different copies $t_i^{z/0}$ and $t_i^{z/1}$ of the existential variables right of z. Using the clauses of $\mathrm{xor}(t_{i-1}, x_i, t_i)$, we can inductively derive clauses representing $t_i^{z/0} = t_i^{z/1}$. This lets us derive a contradiction using the clauses $t_n^{z/0}$ and $\neg t_n^{z/1}$.

Clearly, this proof only contains clauses of constant width, giving the result. ◀

6 Positive Results: Size, Width, and Space in Tree-like QBF Calculi

We are now in a position to show positive results on size-width and size-space relations for QBF resolution calculi. However, most of these results only apply to weak tree-like systems.

6.1 Relations in the Expansion Calculi ∀Exp+Res and IR-calc

We first observe that for ∀Exp+Res almost the full spectrum of relations from classical resolution remains valid.

▶ **Theorem 18.** *For all false QBFs $\mathcal{F}$, the following relations hold:*

1. $S\left(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F}\right) \geq 2^{w\left(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F}\right) - w_\exists(\mathcal{F})}$.
2. $S\left(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F}\right) \geq 2^{CSpace\left(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F}\right)} - 1$.
3. $CSpace\left(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F}\right) \geq CSpace(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F}) \geq w(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F}) - w_\exists(\mathcal{F}) + 1$.

Proof Sketch. Proofs in ∀Exp+Res first download the axioms, leading to clauses containing only annotated existential literals. After that only classical resolution steps are performed and Theorems 1, 2, and 3 can be applied. ◀

By the equivalence of $\forall\mathsf{Exp+Res}_\mathsf{T}$ and IR_T-calc with respect to all three measures size, width, and space (Theorem 12) we can transfer all results from Theorem 18 to IR_T-calc.

▶ **Theorem 19.** *For all false QBFs $\mathcal{F}$, the following relations hold:*

1. $S(\vdash_{\mathsf{IR}_T\text{-calc}} \mathcal{F}) \geq 2^{w\left(\vdash_{\mathsf{IR}\text{-calc}} \mathcal{F}\right) - w_\exists(\mathcal{F})}$.
2. $S(\vdash_{\mathsf{IR}_T\text{-calc}} \mathcal{F}) \geq 2^{CSpace\left(\vdash_{\mathsf{IR}_T\text{-calc}} \mathcal{F}\right)} - 1$.
3. $CSpace(\vdash_{\mathsf{IR}_T\text{-calc}} \mathcal{F}) \geq w(\vdash_{\mathsf{IR}\text{-calc}} \mathcal{F}) - w_\exists(\mathcal{F}) + 1$.

6.2 The Size-Space Relation in Tree-like Q-resolution

We finally return to Q-Res. Most relations were already ruled out in Section 4 for both Q-Res and $\mathsf{Q\text{-}Res}_\mathsf{T}$. The only relation that we can still show to hold is the classical size-space relation (Theorem 2), which we lift from Res_T to $\mathsf{Q\text{-}Res}_\mathsf{T}$.

In classical resolution, this relationship was obtained using pebbling games [18]. We observe that the same approach works for $\mathsf{Q\text{-}Res}_\mathsf{T}$ as well, giving the analogous relationship.

▶ **Theorem 20.** *For a false QBF sentence $\mathcal{F}$, $S(\vdash_{\mathsf{Q\text{-}Res}_T} \mathcal{F}) \geq 2^{CSpace\left(\vdash_{\mathsf{Q\text{-}Res}_T} \mathcal{F}\right)} - 1$.*

7 Conclusion

Our results show that the success story of width in resolution needs to be rethought when moving to QBF. Indeed, the question arises: is width a central parameter in QBF resolution? Is there another parameter that plays a similar role as classical width for understanding QBF resolution size and space?

Our findings almost completely uncover the picture for size, space, and width for the most basic and arguably most important QBF resolution systems Q-Res, ∀Exp+Res, and IR-calc. The most immediate open question arising from our investigation is whether size-width relations hold for general dag-like ∀Exp+Res or IR-calc proofs. The issue here is that in the classical size-width relation of [7] the number of variables enters the formula in a crucial way. For the instantiation calculi it is not clear what should qualify as the right count for this as different annotations of the same existential variable are formally treated as distinct variables (which is also clearly justified by the semantic meaning of expansions).

For further research it will also be interesting whether size-width or space-width relations apply to any of the stronger QBF resolution systems QU-Res [27], LD-Q-Res [2], or IRM-calc [10]. However, we conjecture that the negative picture also prevails for these systems.

Acknowledgements. This work was supported by the EU Marie Curie IRSES grant CORCON, grant no. 48138 from the John Templeton Foundation, EPSRC grant EP/L024233/1, and a Doctoral Training Grant from EPSRC (2nd author).

References

1 Albert Atserias and Víctor Dalmau. A combinatorial characterization of resolution width. *Journal of Computer and System Sciences*, 74(3):323–334, 2008.

2 Valeriy Balabanov and Jie-Hong R. Jiang. Unified QBF certification and its applications. *Form. Methods Syst. Des.*, 41(1):45–65, August 2012.

3 Valeriy Balabanov, Magdalena Widl, and Jie-Hong R. Jiang. QBF resolution systems and their proof complexities. In *Proc. Theory and Applications of Satisfiability Testing (SAT'14)*, pages 154–169, 2014.

4 Paul Beame, Christopher Beck, and Russell Impagliazzo. Time-space tradeoffs in resolution: superpolynomial lower bounds for superlinear space. In *Proc. ACM Symposium on Theory of Computing (STOC'12)*, pages 213–232, 2012.

5 Eli Ben-Sasson. Size space tradeoffs for resolution. In *Proc. Annual ACM Symposium on Theory of Computing (STOC'02)*, pages 457–464, 2002.

6 Eli Ben-Sasson and Jakob Nordström. Understanding space in proof complexity: Separations and trade-offs via substitutions. In *Proc. Innovations in Computer Science (ICS'11)*, pages 401–416, 2011.

7 Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow – resolution made simple. *Journal of the ACM*, 48(2):149–169, 2001.

8 Marco Benedetti and Hratch Mangassarian. QBF-based formal verification: Experience and perspectives. *JSAT*, 5(1-4):133–191, 2008.

9 Olaf Beyersdorff, Ilario Bonacina, and Leroy Chew. Lower bounds: from circuits to QBF proof systems. In *Proc. Innovations in Theoretical Computer Science (ITCS'16)*, 2016.

10 Olaf Beyersdorff, Leroy Chew, and Mikoláš Janota. On unification of QBF resolution-based calculi. In *Proc. Mathematical Foundations of Computer Science (MFCS'14)*, pages 81–93, 2014.

11 Olaf Beyersdorff, Leroy Chew, and Mikoláš Janota. Proof complexity of resolution-based QBF calculi. In *Proc. Symposium on Theoretical Aspects of Computer Science(STACS'15)*, pages 76–89, 2015.

12 Olaf Beyersdorff, Leroy Chew, Meena Mahajan, and Anil Shukla. Feasible interpolation for QBF resolution calculi. In *Proc. Automata, Languages, and Programming – 42nd International Colloquium (ICALP'15)*, pages 180–192, 2015.

13 Olaf Beyersdorff, Leroy Chew, and Karteek Sreenivasaiah. A game characterisation of tree-like Q-resolution size. In *Proc. International Conference on Language and Automata Theory and Applications(LATA'15)*, pages 486–498, 2015.

14 Olaf Beyersdorff and Oliver Kullmann. Unified characterisations of resolution hardness measures. In *Proc. Theory and Applications of Satisfiability Testing (SAT'14)*, pages 170–187, 2014.

15 Archie Blake. *Canonical expressions in boolean algebra.* PhD thesis, University of Chicago, 1937.

16 Maria Luisa Bonet and Nicola Galesi. A study of proof search algorithms for resolution and polynomial calculus. In *Proc. Annual Symposium on Foundations of Computer Science (FOCS'99)*, 1999.

17 Uwe Egly, Martin Kronegger, Florian Lonsing, and Andreas Pfandler. Conformant planning as a case study of incremental QBF solving. In *Proc. Artificial Intelligence and Symbolic Computation (AISC'14)*, pages 120–131, 2014.

18 Juan Luis Esteban and Jacobo Torán. Space bounds for resolution. *Information and Computation*, 171(1):84–97, 2001.

19 Yuval Filmus, Massimo Lauria, Mladen Miksa, Jakob Nordström, and Marc Vinyals. From small space to small width in resolution. In *International Symposium on Theoretical Aspects of Computer Science (STACS'14)*, pages 300–311, 2014.

20 Mikolás Janota and Joao Marques-Silva. Expansion-based QBF solving versus Q-resolution. *Theor. Comput. Sci.*, 577:25–42, 2015.

21 Hans Kleine Büning, Marek Karpinski, and Andreas Flögel. Resolution for quantified Boolean formulas. *Inf. Comput.*, 117(1):12–18, 1995.

22 Jan Krajíček. Interpolation theorems, lower bounds for proof systems and independence results for bounded arithmetic. *Journal of Symbolic Logic*, 62(2):457–486, 1997.

23 Oliver Kullmann. Investigating a general hierarchy of polynomially decidable classes of CNF's based on short tree-like resolution proofs. *Electronic Colloquium on Computational Complexity (ECCC)*, 99(41), 1999.

24 Jakob Nordström. Pebble games, proof complexity, and time-space trade-offs. *Logical Methods in Computer Science*, 9(3), 2013.

25 Jussi Rintanen. Asymptotically optimal encodings of conformant planning in QBF. In *AAAI*, pages 1045–1050. AAAI Press, 2007.

26 John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12:23–41, 1965.

27 Allen Van Gelder. Contributions to the theory of practical quantified Boolean formula solving. In *Proc. Principles and Practice of Constraint Programming (CP'12)*, pages 647–663, 2012.

28 Lintao Zhang and Sharad Malik. Conflict driven learning in a quantified boolean satisfiability solver. In *Proc. IEEE/ACM International Conference on Computer-aided Design (ICCAD'02)*, pages 442–449, 2002.