The Independence of Markov's Principle in Type Theory

Thierry Coquand[1] and Bassel Mannaa[2]

1 Department of Computer Science and Engineering, University of Gothenburg, Gothenburg, Sweden
thierry.coquand@cse.gu.se

2 Department of Computer Science and Engineering, University of Gothenburg, Gothenburg, Sweden
bassel.mannaa@cse.gu.se

Abstract

In this paper, we show that Markov's principle is not derivable in dependent type theory with natural numbers and one universe. One tentative way to prove this would be to remark that Markov's principle does not hold in a sheaf model of type theory over Cantor space, since Markov's principle does not hold for the generic point of this model. It is however not clear how to interpret the universe in a sheaf model [9, 17, 21]. Instead we design an extension of type theory, which intuitively extends type theory by the addition of a generic point of Cantor space. We then show the consistency of this extension by a normalization argument. Markov's principle does not hold in this extension, and it follows that it cannot be proved in type theory.

1998 ACM Subject Classification F.4.1 Mathematical Logic

Keywords and phrases Forcing, Dependent type theory, Markov's Principle, Cantor Space

Digital Object Identifier 10.4230/LIPIcs.FSCD.2016.17

1 Introduction

Markov's principle has a special status in constructive mathematics. One way to formulate this principle is that if it is impossible that a given algorithm does not terminate, then it does terminate. It is equivalent to the fact that if a set of natural number and its complement are both computably enumerable, then this set is decidable. This form is often used in recursivity theory. This principle was first formulated by Markov, who called it "Leningrad's principle", and founded a branch of constructive mathematics around this principle [14].

This principle is also equivalent to the fact that if a given real number is *not* equal to 0 then this number is *apart* from 0 (that is this number is $< -r$ or $> r$ for some rational number $r > 0$). On this form, it was explicitly *refuted* by Brouwer in intuitionistic mathematics, who gave an example of a real number (well defined intuitionistically) which is not equal to 0, but also not apart from 0. (The motivation of Brouwer for this example was to show the necessity of using *negation* in intuitionistic mathematics [4].) The idea of Brouwer can be represented formally using topological models [19].

In a neutral approach to mathematics, such as Bishop's [3], Markov's principle is simply left undecided. We also expect to be able to prove that Markov's principle is *not* provable in formal system in which we can express Bishop's mathematics. For instance, Kreisel [12] introduced *modified realizability* to show that Markov's principle is not derivable in the formal system HA^ω. Similarly, one would expect that Markov's principle is *not* derivable in

1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016).
Editors: Delia Kesner and Brigitte Pientka; Article No. 17; pp. 17:1–17:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Martin-Löf type theory [15], but, as far as we know, such a result has not been established yet. [1]

We say that a statement A is *independent* of some formal system if A cannot be derived in that system. A statement in the formal system of Martin-Löf type theory (MLTT) is represented by a closed type. A statement/type A is derivable if it is inhabited by some term t (written $\mathrm{MLTT} \vdash t : A$). This is the so-called propositions-as-types principle. Correspondingly we say that a statement A (represented as a type) is independent of MLTT if there is no term t such that $\mathrm{MLTT} \vdash t : A$.

The main result of this paper is to show that Markov's principle is independent of Martin-Löf type theory.[2]

The main idea for proving this independence is to follow Brouwer's argument. We want to extend type theory with a "generic" infinite sequence of 0 and 1 and establish that it is both absurd that this generic sequence is never 0, but also that we cannot show that it *has to* take the value 0. To add such a generic sequence is exactly like adding a *Cohen real* [5] in forcing extension of set theory. A natural attempt for doing this will be to consider a *topological model* of type theory (sheaf model over Cantor space), extending the work [19] to type theory. However, while it is well understood how to represent universes in *presheaf* model [9], it has turned out to be surprisingly difficult to represent universes in *sheaf* models, as we learnt from works of Chuangjie Xu and Martin Escardo [21] and works of Thomas Streicher [17]. Our approach is here instead a purely *syntactical* description of a forcing extension of type theory (refining previous work of [7]), which contains a formal symbol for the generic sequence and a proof that it is absurd that this generic sequence is never 0, together with a *normalization* theorem, from which we can deduce that we *cannot* prove that this generic sequence has to take the value 0. Since this formal system is an extension of type theory, the independence of Markov's principle follows.

As stated in [11], which describes an elegant generalization of this principle in type theory, Markov's principle is an important technical tool for proving termination of computations, and thus can play a crucial role if type theory is extended with general recursion as in [6].

This paper is organized as follows. We first describe the rules of the version of type theory we are considering. This version can be seen as a simplified version of type theory as represented in the system Agda [16], and in particular, contrary to the work [7], we allow η-conversion, and we express conversion as *judgment*. Markov's principle can be formulated in a natural way in this formal system. We describe then the forcing extension of type theory, where we add a Cohen real. For proving normalization, we follow Tait's computability method [18, 15], but we have to consider an extension of this with a computability *relation* in order to interpret the conversion judgment. This can be seen as a forcing extension of the technique used in [1]. Using this computability argument, it is then possible to show that we cannot show that the generic sequence has to take the value 0. We end by a refinement of this method, giving a consistent extension of type theory where the *negation* of Markov's principle is provable.

---

[1] The paper [10] presents a model of the calculus of constructions using the idea of modified realizability, and it seems possible to use also this technique to interpret the type theory we consider and prove in this way the independence of Markov's principle.

[2] Some authors define independence in the stronger sense "A statement is independent of a formal system if neither the statement nor its negation is provable in the system", e.g. [13]. We will establish the independence of Markov's principle in this stronger sense with the help of known results from the literature.

2 Type theory and forcing extension

A dependent type theory is given by: A syntax describing the set objects of discourse, forms of judgments, and rules of inference for deriving valid judgments.

The syntax of our type theory is given by the grammar:

$$\begin{aligned} t, u, A, B \coloneqq\ & x \mid \bot\mathsf{rec}\ (\lambda x.A) \mid \mathsf{unitrec}\ (\lambda x.A)\, t \mid \mathsf{boolrec}\ (\lambda x.A)\, t\, u \mid \mathsf{natrec}\ (\lambda x.A)\, t\, u \\ & \mid U \mid N \mid N_0 \mid N_1 \mid N_2 \mid 0 \mid 1 \mid \mathsf{S}\, t \\ & \mid \Pi(x\!:\!A)B \mid \lambda x.t \mid t\, u \mid \Sigma(x\!:\!A)B \mid (t, u) \mid t.1 \mid t.2 \end{aligned}$$

The terms N_0, N_1, N_2, and N will denote , respectively, the empty type, the unit type, the type of booleans, and the type of natural numbers. The term U will denote the universe, i.e. the type of small types. We use the notation $\bar{n}$ as a short hand for the term $\mathsf{S}^n\, 0$, where S is the successor constructor of natural numbers.

2.1 Type system

We describe a type theory with one universe à la Russell, natural numbers, functional extensionality and surjective pairing, hereafter referred to as MLTT.[3] The type theory has the following judgment forms: 1. $\Gamma \vdash$. 2. $\Gamma \vdash A$. 3. $\Gamma \vdash t : A$. 4. $\Gamma \vdash A = B$. 5. $\Gamma \vdash t = u : A$. The first expresses that Γ is a well-formed contexts, the second that A is a type in the context Γ, and the third that t is a term of type A in the context Γ. The fourth and fifth express type and term equality respectively. Below we outline the inference rules of this type theory. We use the notation $F \to G$ for $\Pi(x : F)G$ when G doesn't depend on F and $\neg A$ for $A \to N_0$.

Natural numbers:

$$\frac{\Gamma \vdash}{\Gamma \vdash N} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 0 : N} \qquad \textsc{nat-suc}\ \frac{\Gamma \vdash n : N}{\Gamma \vdash \mathsf{S}\, n : N}$$

$$\textsc{natrec-I}\ \frac{\Gamma, x : N \vdash F \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash g : \Pi(x : N)(F[x] \to F[\mathsf{S}\, x])}{\Gamma \vdash \mathsf{natrec}\ (\lambda x.F)\, a_0\, g : \Pi(x : N)F}$$

$$\textsc{natrec-0}\ \frac{\Gamma, x : N \vdash F \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash g : \Pi(x : N)(F[x] \to F[\mathsf{S}\, x])}{\Gamma \vdash \mathsf{natrec}\ (\lambda x.F)\, a_0\, g\, 0 = a_0 : F[0]}$$

$$\textsc{natrec-suc}\ \frac{\Gamma, x : N \vdash F \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash n : N \quad \Gamma \vdash g : \Pi(x : N)(F[x] \to F[\mathsf{S}\, x])}{\Gamma \vdash \mathsf{natrec}\ (\lambda x.F)\, a_0\, g\, (\mathsf{S}\, n) = g\, n\, (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g\, n) : F[\mathsf{S}\, n]}$$

$$\textsc{natrec-eq}\ \frac{\Gamma, x : N \vdash F = G \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash g : \Pi(x : N)(F[x] \to F[\mathsf{S}\, x])}{\Gamma \vdash \mathsf{natrec}\ (\lambda x.F)\, a_0\, g = \mathsf{natrec}\ (\lambda x.G)\, a_0\, g : \Pi(x : N)F}$$

Booleans:

$$\frac{\Gamma \vdash}{\Gamma \vdash N_2} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 0 : N_2} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 1 : N_2}$$

$$\textsc{boolrec-I}\ \frac{\Gamma, x : N_2 \vdash F \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash a_1 : F[1]}{\Gamma \vdash \mathsf{boolrec}\ (\lambda x.F)\, a_0\, a_1 : \Pi(x : N_2)F}$$

$$\textsc{boolrec-0}\ \frac{\Gamma, x : N_2 \vdash F \quad \Gamma \vdash a_0 : F[0] \quad \Gamma \vdash a_1 : F[1]}{\Gamma \vdash \mathsf{boolrec}\ (\lambda x.F)\, a_0\, a_1\, 0 = a_0 : F[0]}$$

[3] This is a type system similar to Martin-löf's [15] except that we have η-conversion and surjective pairing.

$$\text{BOOLREC-1}\ \frac{\Gamma, x\!:\!N_2 \vdash F \quad \Gamma \vdash a_0\!:\!F[0] \quad \Gamma \vdash a_1\!:\!F[1]}{\Gamma \vdash \mathsf{boolrec}\ (\lambda x.F)\ a_0\ a_1\ 1 = a_1\!:\!F[1]}$$

$$\text{BOOLREC-EQ}\ \frac{\Gamma, x\!:\!N_2 \vdash F = G \quad \Gamma \vdash a_0\!:\!F[0] \quad \Gamma \vdash a_1\!:\!F[1]}{\Gamma \vdash \mathsf{natrec}\ (\lambda x.F)\ a_0\ a_1 = \mathsf{natrec}\ (\lambda x.G)\ a_0\ a_1\!:\!\Pi(x\!:\!N_2)F}$$

Dependent functions:

$$\Pi\text{-I}\ \frac{\Gamma \vdash F \quad \Gamma, x\!:\!F \vdash G}{\Gamma \vdash \Pi(x\!:\!F)G} \qquad \Pi\text{-EQ}\ \frac{\Gamma \vdash F = H \quad \Gamma, x\!:\!F \vdash G = E}{\Gamma \vdash \Pi(x\!:\!F)G = \Pi(x\!:\!H)E}$$

$$\lambda\text{-I}\ \frac{\Gamma, x\!:\!F \vdash t\!:\!G}{\Gamma \vdash \lambda x.t\!:\!\Pi(x\!:\!F)G} \qquad \text{FUN-AP}\ \frac{\Gamma \vdash g\!:\!\Pi(x\!:\!F)G \quad \Gamma \vdash a\!:\!F}{\Gamma \vdash g\,a\!:\!G[a]} \qquad \beta\ \frac{\Gamma, x\!:\!F \vdash t\!:\!G \quad \Gamma \vdash a\!:\!F}{\Gamma \vdash (\lambda x.t)a = t[a]\!:\!G[a]}$$

$$\text{FUN}\ \frac{\Gamma \vdash g\!:\!\Pi(x\!:\!F)G \quad \Gamma \vdash u = v\!:\!F}{\Gamma \vdash g\,u = g\,v\!:\!G[u]} \qquad \text{FUN-EQ}\ \frac{\Gamma \vdash h = g\!:\!\Pi(x\!:\!F)G \quad \Gamma \vdash u\!:\!F}{\Gamma \vdash h\,u = g\,u\!:\!G[u]}$$

$$\text{FUN-EXT}\ \frac{\Gamma \vdash h\!:\!\Pi(x\!:\!F)G \quad \Gamma \vdash g\!:\!\Pi(x\!:\!F)G \quad \Gamma, x\!:\!F \vdash h\,x = g\,x\!:\!G[x]}{\Gamma \vdash h = g\!:\!\Pi(x\!:\!F)G}$$

Dependent product:

$$\Sigma\text{-I}\ \frac{\Gamma \vdash F \quad \Gamma, x\!:\!F \vdash G}{\Gamma \vdash \Sigma(x\!:\!F)G} \qquad \Sigma\text{-EQ}\ \frac{\Gamma \vdash F = H \quad \Gamma, x\!:\!F \vdash G = E}{\Gamma \vdash \Sigma(x\!:\!F)G = \Sigma(x\!:\!H)E}$$

$$\text{PR-I}\ \frac{\Gamma, x\!:\!F \vdash G \quad \Gamma \vdash a\!:\!F \quad \Gamma \vdash b\!:\!G[a]}{\Gamma \vdash (a,b)\!:\!\Sigma(x\!:\!F)G} \qquad \text{PR-E-1}\ \frac{\Gamma \vdash t\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash t.1\!:\!F} \qquad \text{PR-E-2}\ \frac{\Gamma \vdash t\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash t.2\!:\!G[t.1]}$$

$$\mathsf{pr}_1\ \frac{\Gamma \vdash (t,u)\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash (t,u).1 = t\!:\!F} \qquad \mathsf{pr}_2\ \frac{\Gamma \vdash (t,u)\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash (t,u).2 = u\!:\!G[t]}$$

$$\text{PR-EQ-1}\ \frac{\Gamma \vdash t = u\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash t.1 = u.1\!:\!F} \qquad \text{PR-EQ-1}\ \frac{\Gamma \vdash t = u\!:\!\Sigma(x\!:\!F)G}{\Gamma \vdash t.2 = u.2\!:\!G[t.1]}$$

$$\text{PR-EXT}\ \frac{\Gamma \vdash t\!:\!\Sigma(x\!:\!F)G \quad \Gamma \vdash u\!:\!\Sigma(x\!:\!F)G \quad \Gamma \vdash t.1 = u.1\!:\!F \quad \Gamma \vdash t.2 = u.2\!:\!G[t.1]}{\Gamma \vdash t = u\!:\!\Sigma(x\!:\!F)G}$$

Universe:

$$\frac{\Gamma \vdash}{\Gamma \vdash U} \qquad \frac{\Gamma \vdash F\!:\!U}{\Gamma \vdash F} \qquad \frac{\Gamma \vdash F = G\!:\!U}{\Gamma \vdash F = U} \qquad \frac{\Gamma \vdash}{\Gamma \vdash N\!:\!U} \qquad \frac{\Gamma \vdash}{\Gamma \vdash N_2\!:\!U}$$

$$\frac{\Gamma \vdash F : U \quad \Gamma, x\!:\!F \vdash G : U}{\Gamma \vdash \Pi(x\!:\!F)G : U} \qquad \frac{\Gamma \vdash F = H : U \quad \Gamma, x\!:\!F \vdash G = E : U}{\Gamma \vdash \Pi(x\!:\!F)G = \Pi(x\!:\!H)E\!:\!U}$$

$$\frac{\Gamma \vdash F : U \quad \Gamma, x\!:\!F \vdash G : U}{\Gamma \vdash \Sigma(x\!:\!F)G : U} \qquad \frac{\Gamma \vdash F = H : U \quad \Gamma, x\!:\!F \vdash G = E : U}{\Gamma \vdash \Sigma(x\!:\!F)G = \Sigma(x\!:\!H)E\!:\!U}$$

Congruence:

$$\frac{\Gamma \vdash t : F \quad \Gamma \vdash F = G}{\Gamma \vdash t : G} \quad \frac{\Gamma \vdash t = u : F \quad \Gamma \vdash F = G}{\Gamma \vdash t = u : G}$$

$$\frac{\Gamma \vdash F}{\Gamma \vdash F = F} \quad \frac{\Gamma \vdash F = G}{\Gamma \vdash G = F} \quad \frac{\Gamma \vdash F = G \quad \Gamma \vdash G = H}{\Gamma \vdash F = H}$$

$$\frac{\Gamma \vdash t : F}{\Gamma \vdash t = t : F} \quad \frac{\Gamma \vdash t = u : F}{\Gamma \vdash u = t : F} \quad \frac{\Gamma \vdash t = u : F \quad \Gamma \vdash u = v : F}{\Gamma \vdash t = v : F}$$

For brevity we omitted the rules for the types N_0 and N_1.

The following four rules are admissible in the this type system [1]:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash A} \quad \frac{\Gamma \vdash a = b : A}{\Gamma \vdash a : A} \quad \frac{\Gamma, x : F \vdash G \quad \Gamma \vdash a = b : F}{\Gamma \vdash G[a] = G[b]} \quad \frac{\Gamma, x : F \vdash t : G \quad \Gamma \vdash a = b : F}{\Gamma \vdash t[a] = t[b] : G[a]}$$

2.2 Markov's principle

Markov's principle can be represented in type theory by the type

$$\mathrm{MP} := \Pi(h : N \to N_2)[\neg\neg(\Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)) \to \Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)]$$

where $\mathsf{IsZero} : N_2 \to U$ is defined by $\mathsf{IsZero} := \lambda y.\mathsf{boolrec}\,(\lambda x.U)\,N_1\,N_0\,y$.

Note that $\mathsf{IsZero}\,(h\,n)$ is inhabited when $h\,n = 0$ and empty when $h\,n = 1$. Thus $\Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)$ is inhabited if there is n such that $h\,n = 0$.

The main result of this paper is the following:

▶ **Theorem 2.1.** *There is no term t such that* $\mathrm{MLTT} \vdash t : \mathrm{MP}$.

An *extension* of MLTT is given by introducing new objects, judgment forms and derivation rules. This means in particular that any judgment valid in MLTT is valid in the extension. A *consistent* extension is one in which the type N_0 is uninhabited.

To show Theorem 2.1 we will form a consistent extension of MLTT with a new consant f where $\vdash \mathsf{f} : N \to N_2$ and $\neg\neg(\Sigma(x : N)\,\mathsf{IsZero}\,(\mathsf{f}\,x)) \to \Sigma(x : N)\,\mathsf{IsZero}\,(\mathsf{f}\,x)$ is not derivable. Thus MP is not derivable in this extension and consequently not derivable in MLTT.

While this is sufficient to establish independence in the sense of non-derivability of MP. To establish the independence of MP in the stronger sense one also needs to show that $\neg$MP is not derivable in MLTT. This can achieved by reference to the work of Aczel [2] where it is shown that MLTT extended with $\vdash \mathsf{dne} : \Pi(A : U)(\neg\neg A \to A)$ is consistent. Since $h : N \to N_2, x : N \vdash \mathsf{IsZero}\,(h\,x) : U$ we have $h : N \to N_2 \vdash \Sigma(x : N)\,\mathsf{IsZero}\,(h\,x) : U$. Thus

$$h : N \to N_2 \vdash \mathsf{dne}\,(\Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)) : \neg\neg(\Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)) \to \Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)$$

By λ abstraction we have $\vdash \lambda h.\mathsf{dne}\,(\Sigma(x : N)\,\mathsf{IsZero}\,(h\,x)) : \mathrm{MP}$. We can then conclude that there is no term t such that $\mathrm{MLTT} \vdash t : \neg\mathrm{MP}$.

Finally, we will refine the result of Theorem 2.1 by building a consistent extension of MLTT where $\neg$MP is derivable.

2.3 Forcing extension

A *condition* p is a graph of a partial finite function from $\mathbb{N}$ to $\{0, 1\}$. We denote by $\langle\rangle$ the empty condition. We write $p(n) = b$ when $(n, b) \in p$. We say q *extends* p (written $q \leqslant p$) if p is a subset of q. A condition can be thought of as a compact open in Cantor space $2^{\mathbb{N}}$. Two conditions p and q are *compatible* if $p \cup q$ is a condition and we write pq for $p \cup q$, otherwise they are *incompatible*. If $n \notin \mathrm{dom}(p)$ we write $p(n,0)$ for $p \cup \{(n,0)\}$ and $p(n,1)$ for $p \cup \{(n,1)\}$. We define the notion of *partition* corresponding to the notion of finite covering of a compact open in Cantor space.

▶ **Definition 2.2** (Partition). We write $p \lhd p_1, \dots, p_n$ to say that $p_1, \dots, p_n$ is a partition of p and we define it as follows:

1. $p \lhd p$.
2. If $n \notin \mathrm{dom}(p)$ and $p(n,0) \lhd \dots, q_i, \dots$ and $p(n,1) \lhd \dots, r_j, \dots$ then $p \lhd \dots, q_i, \dots, r_j, \dots$.

Note that if $p \lhd p_1, \dots, p_n$ then p_i and p_j are incompatible whenever $i \neq j$. If moreover $q \leqslant p$ then $q \lhd \dots, qp_j, \dots$ where p_j is compatible with q.

We extend the given type theory by annotating the judgments with conditions, i.e. replacing each judgment $\Gamma \vdash J$ in the given type system with a judgment $\Gamma \vdash_p J$.

In addition we add the locality rule: $\textsc{loc}\dfrac{\Gamma \vdash_{p_1} J \quad \dots \quad \Gamma \vdash_{p_n} J}{\Gamma \vdash_p J} p \lhd p_1 \dots p_n$.

We add a term f for the generic point along with the introduction and conversion rules:

$$\mathsf{f}\text{-}\mathrm{I}\dfrac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{f} : N \to N_2} \qquad \mathsf{f}\text{-}\textsc{eval}\dfrac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{f}\,\overline{n} = p(n) : N_2} n \in \mathrm{dom}(p)\ .$$

We add a term w and the rule: $\textsc{w-term}\dfrac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{w} : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(\mathsf{f}\,x))}$.

Since w inhabits $\neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(\mathsf{f}\,x))$, our goal is then to show that no term inhabits $\Sigma(x:N)\,\mathsf{IsZero}(\mathsf{f}\,x)$.

It follows directly from the description of the forcing extension that:

▶ **Lemma 2.3.** *If $\Gamma \vdash J$ then $\Gamma \vdash_p J$ for all p. In particular, if $\vdash t : A$ then $\vdash_p t : A$ for all p.*

Note that if $q \leqslant p$ and $\Gamma \vdash_p J$ then $\Gamma \vdash_q J$ (monotonicity). A statement A (represented as a closed type) is derivable in this extension if $\vdash_{\langle\rangle} t : A$ for some t, which in turn implies $\vdash_p t : A$ for all p.

Similarly to [7] we can state a conservativity result for this extension. Let $\vdash g : N \to N_2$ and $\vdash v : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(g\,x))$ be two terms of standard type theory. We say that g is compatible with a condition p if g is such that $\vdash g\,\overline{n} = b : N_2$ whenever $(n,b) \in p$ and $\vdash g\,\overline{n} = 0 : N_2$ otherwise. We say that v is compatible with a condition p if g is compatible with p and v is given by $v \coloneqq \lambda x.x\,(\overline{n}_p, 0)$ where n_p is the smallest natural number such that $n_p \notin \mathrm{dom}(p)$. To see that v is well typed, note that by design $\Gamma \vdash g\,\overline{n}_p = 0 : N_2$ thus $\Gamma \vdash \mathsf{IsZero}\,(g\,\overline{n}_p) = N_1$ and $\Gamma \vdash (\overline{n}_p, 0) : \Sigma(x:N)\mathsf{IsZero}\,(g\,x)$. We have then $\Gamma, x : \neg(\Sigma(y:N)\,\mathsf{IsZero}\,(g\,y)) \vdash x\,(\overline{n}_p, 0) : N_0$ thus $\Gamma \vdash \lambda x.x\,(\overline{n}_p, 0) : \neg\neg(\Sigma(y:N)\,\mathsf{IsZero}\,(g\,y))$.

▶ **Lemma 2.4** (Conservativity). *Let $\vdash g : N \to N_2$ and $\vdash v : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(g\,x))$ be compatible with p. If $\Gamma \vdash_p J$ then $\Gamma[g/\mathsf{f}, v/\mathsf{w}] \vdash J[g/\mathsf{f}, v/\mathsf{w}]$, i.e. replacing f with g then w with v we obtain a valid judgment in standard type theory. In particular, if $\Gamma \vdash_{\langle\rangle} J$ where neither f nor w occur in Γ or J then $\Gamma \vdash J$ is a valid judgment in standard type theory.*

Proof. The proof is by induction on the type system and it is straightforward for all the standard rules. For (f-EVAL) we have $(\mathsf{f}\,\overline{n})[g/\mathsf{f}, v/\mathsf{w}] \coloneqq g\,\overline{n}$ and since g is compatible with p we have $\Gamma[g/\mathsf{f}, v/\mathsf{w}] \vdash g\,\overline{n} = p(n) : N_2$ whenever $n \in \mathrm{dom}(p)$. For (w-TERM) we have $(\mathsf{w} : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(\mathsf{f}\,x)))[g/\mathsf{f}, v/\mathsf{w}] \coloneqq (\mathsf{w} : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(g\,x)))[v/\mathsf{w}] \coloneqq v : \neg\neg(\Sigma(x:N)\,\mathsf{IsZero}\,(g\,x))$. For (LOC) the statement follows from the observation that when g is compatible with p and $p \lhd p_1, \dots, p_n$ then g is compatible with exactly one p_i for $1 \leqslant i \leqslant n$. ◀

3 A Semantics of the forcing extension

In this section we outline a semantics for the forcing extension given in the previous section. We will interpret the judgments of type theory by computability predicates and relations defined by reducibility to computable weak head normal forms.

3.1 Reduction rules

We extend the β, ι conversion with $\mathsf{f}\,\overline{n} \Rightarrow_p b$ whenever $(n, b) \in p$. In order to ease the presentation of the proofs and definitions we introduce *evaluation contexts* following [20].

$$\mathbb{E} ::= [\,] \mid \mathbb{E}\,u \mid \mathbb{E}.1 \mid \mathbb{E}.2 \mid \mathsf{S}\,\mathbb{E} \mid \mathsf{f}\,\mathbb{E}$$
$$\bot\mathsf{rec}\ (\lambda x.C)\,\mathbb{E} \mid \mathsf{unitrec}\ (\lambda x.C)\,a\,\mathbb{E} \mid \mathsf{boolrec}\ (\lambda x.C)\,a_0\,a_1\,\mathbb{E} \mid \mathsf{natrec}\ (\lambda x.C)\,c_z\,g\,\mathbb{E}$$

An expression $\mathbb{E}[e]$ is then the expression resulting from replacing the hole $[\,]$ by e. We reserve the symbols $\mathbb{E}$ and $\mathbb{C}$ for evaluation contexts. We have the following reduction rules:

$$\frac{}{\mathsf{unitrec}\ (\lambda x.C)\ c\ 0 \to c} \qquad \frac{}{\mathsf{boolrec}\ (\lambda x.C)\,c_0\,c_1\,0 \to c_0} \qquad \frac{}{\mathsf{boolrec}\ (\lambda x.C)\,c_0\,c_1\,1 \to c_1}$$

$$\frac{}{\mathsf{natrec}\ (\lambda x.C)\,c_z\,g\,0 \to c_z} \qquad \frac{}{\mathsf{natrec}\ (\lambda x.C)\,c_z\,g\,(\mathsf{S}\,\overline{k}) \to g\,\overline{k}\,(\mathsf{natrec}\ (\lambda x.C)\,c_z\,g\,\overline{k})}$$

$$\frac{}{(\lambda x.t)\,a \to t[a/x]} \qquad \frac{}{(u,v).1 \to u} \qquad \frac{}{(u,v).2 \to v}$$

$$\frac{e \to e'}{e \to_p e'} \qquad \textsc{f-red}\,\frac{k \in \mathrm{dom}(p)}{\mathsf{f}\,\overline{k} \to_p p(k)} \qquad \frac{e \to_p e'}{\mathbb{E}[e] \Rightarrow_p \mathbb{E}[e']}$$

Note that we reduce under S.

The relation $\Rightarrow$ is monotone, that is if $q \leqslant p$ and $t \Rightarrow_p u$ then $t \Rightarrow_q u$. We will also need to show that the reduction is local, i.e. if $p \lhd p_1, \dots, p_n$ and $t \Rightarrow_{p_i} u$ then $t \Rightarrow_p u$.

▶ **Lemma 3.1.** *If $m \notin \mathrm{dom}(p)$ and $t \to_{p(m,0)} u$ and $t \to_{p(m,1)} u$ then $t \to_p u$.*

Proof. By induction on the derivation of $t \to_{p(m,0)} u$. If $t \to_{p(m,0)} u$ is derived by (f-RED) then $t := \mathsf{f}\,\overline{k}$ and $u := p(m,0)(k)$ for some $k \in \mathrm{dom}(p(m,0))$. But since we also have a reduction $\mathsf{f}\,\overline{k} \to_{p(m,1)} u$, we have $p(m,1)(k) := u := p(m,0)(k)$ which could only be the case if $k \in \mathrm{dom}(p)$. Thus we have a reduction $\mathsf{f}\,\overline{k} \to_p u := p(k)$. Alternatively, we have a derivation $t \to u$, in which case we have $t \to_p u$ directly. ◀

▶ **Lemma 3.2.** *If $m \notin \mathrm{dom}(p)$ and $t \Rightarrow_{p(m,0)} u$ and $t \Rightarrow_{p(m,1)} u$ then $t \Rightarrow_p u$.*

Proof. From the reduction $t \Rightarrow_{p(k,0)} u$ we have $t := \mathbb{E}[e]$, $u := \mathbb{E}[e']$ and $e \to_{p(m,0)} e'$ for some context $\mathbb{E}$. But then we also have a reduction $\mathbb{E}[e] \Rightarrow_{p(m,1)} \mathbb{E}[e']$, thus $e \to_{p(m,1)} e'$. By Lemma 3.1, we have $e \to_p e'$ and thus $\mathbb{E}[e] \Rightarrow_p \mathbb{E}[e']$. ◀

▶ **Lemma 3.3.** *Let $q \leqslant p$. If $t \to_q u$ then either $t \to_p u$ or t has the form $\mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$.*

Proof. By induction on the derivation of $t \to_q u$. If the reduction $t \to_q u$ has the form $\mathsf{f}\,\overline{k} \to_q q(k)$ then either $k \notin \mathrm{dom}(p)$ and the statement follows or $k \in \mathrm{dom}(p)$ and we have $t \to_p u$. Alternatively, we have $t \to u$ and immediately $t \to_p u$. ◀

▶ **Lemma 3.4.** *Let $q \leqslant p$. If $t \Rightarrow_q u$ then either $t \Rightarrow_p u$ or t has the form $\mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$.*

Proof. If $t \Rightarrow_q u$ then $t \coloneqq \mathbb{E}[e]$, $u \coloneqq \mathbb{E}[e']$ and $e \rightarrow_q e'$ for some context $\mathbb{E}$. By Lemma 3.3 either $e \coloneqq \mathbb{C}[\mathsf{f}\,\overline{m}]$ for $m \notin \mathrm{dom}(p)$ and the statement follows or $e \rightarrow_p e'$ in which case we have $t \Rightarrow_p u$. ◀

▶ **Corollary 3.5.** *For any condition p and $m \notin \mathrm{dom}(p)$. Let $t \Rightarrow_{p(m,0)} u$ and $t \Rightarrow_{p(m,1)} v$. If $u \coloneqq v$ then $t \Rightarrow_p u$; otherwise, t has the form $\mathbb{E}[\mathsf{f}\,\overline{m}]$.*

Proof. Follows by Lemma 3.2 and Lemma 3.4. ◀

Next we define the relation $p \vdash t \Rightarrow u : A$ to mean $t \Rightarrow_p u$ and $\vdash_p t = u : A$ and we write $p \vdash A \Rightarrow B$ for $p \vdash A \Rightarrow B : U$. We note that it holds that if $p \vdash t \Rightarrow u : \Pi(x : F)G$ and $\vdash a : F$ then $p \vdash t\,a \Rightarrow u\,a : G[a]$ and if $p \vdash t \Rightarrow u : \Sigma(x : F)G$ then $p \vdash t.1 \Rightarrow u.1 : F$ and $p \vdash t.2 \Rightarrow u.2 : G[t.1]$. We define a closure for this relation as follows:

$$\frac{\vdash_p t : A}{p \vdash t \Rightarrow^* t : A} \qquad \frac{p \vdash t \Rightarrow u : A}{p \vdash t \Rightarrow^* u : A} \qquad \frac{p \vdash t \Rightarrow u : A \quad p \vdash u \Rightarrow^* v : A}{p \vdash t \Rightarrow^* v : A}$$

$$\frac{\vdash_p A}{p \vdash A \Rightarrow^* A} \qquad \frac{p \vdash A \Rightarrow B}{p \vdash A \Rightarrow^* B} \qquad \frac{p \vdash A \Rightarrow B \quad p \vdash B \Rightarrow^* C}{p \vdash A \Rightarrow^* C}$$

A term t is *in* p-whnf if whenever $t \Rightarrow_p u$ then $t \coloneqq u$. A whnf is canonical if it has the form 0,1,$\overline{n}$, $\lambda x.t$, f, w, $\bot\mathsf{rec}\;(\lambda x.C)$, $\mathsf{unitrec}\;(\lambda x.C)\,a$, $\mathsf{boolrec}\;(\lambda x.C)\,a_0\,a_1$, $\mathsf{natrec}\;(\lambda x.C)\,c_z\,g$, N_0,N_1,N_2,N,U,$\Pi(x : F)G$, or $\Sigma(x : F)G$. A p-whnf is proper if it is canonical or it is of the form $\mathbb{E}[\mathsf{f}\,\overline{k}]$ for $k \notin \mathrm{dom}(p)$.

We have the following corollaries to Lemma 3.2 and Corollary 3.5.

▶ **Corollary 3.6.** *Let $m \notin \mathrm{dom}(p)$. Let $p(m,0) \vdash t \Rightarrow_{p(m,0)} u : A$ and $p(m,1) \vdash t \Rightarrow_{p(m,1)} v : A$. If $u \coloneqq v$ then $p \vdash t \Rightarrow u : A$; otherwise t has the form $\mathbb{E}[\mathsf{f}\,\overline{m}]$.*

▶ **Corollary 3.7.** *If $p \vdash t \Rightarrow u : A$ and $q \leqslant p$ then $q \vdash t \Rightarrow u : A$. If $p \lhd p_1, \ldots, p_n$ and $p_i \vdash t \Rightarrow u : A$ for all i then $p \vdash t \Rightarrow u : A$.*

Proof. Let $q \leqslant p$. If $t \Rightarrow_p u$ we have $t \Rightarrow_q u$ and if $\vdash_p t = u : A$ then $\vdash_q t = u : A$. Thus $q \vdash t \Rightarrow u : A$ whenever $p \vdash t \Rightarrow u : A$. Let $p \lhd p_1, \ldots, p_n$. If for all i, $t \Rightarrow_{p_i} u : A$ then from Lemma 3.2, by induction on the partition, we have $t \Rightarrow_p u : A$. If $\vdash_{p_i} t = u : A$ for all i, then $\vdash_p t = u : A$. Thus we have $p \vdash t \Rightarrow u : A$ whenever $p_i \vdash t \Rightarrow u : A$ for all i. ◀

From the above we can show that closure $\Rightarrow^*$ is monotone, it is not however local.

For a closed term $\vdash_p t : A$, we say that t *has* a p-whnf if $p \vdash t \Rightarrow^* u : A$ and u is in p-whnf. If moreover u is canonical, respectively proper, we say that t has a canonical, respectively proper, p-whnf. Since the reduction relation is deterministic we have

▶ **Lemma 3.8.** *A term $\vdash_p t : A$ has at most one p-whnf.*

▶ **Corollary 3.9.** *Let $\vdash_p t : A$ and $m \notin \mathrm{dom}(p)$. If t has proper $p(m,0)$-whnf and a proper $p(m,1)$-whnf then t has a proper p-whnf.*

Proof. Let $p(m,0) \vdash t \Rightarrow^* u : A$ and $p(m,1) \vdash t \Rightarrow^* v : A$ with u in proper $p(m,0)$-whnf and v in proper $p(m,1)$-whnf. If $t \coloneqq u$ or $t \coloneqq v$ then t is already in proper p-whnf. Alternatively we have reductions $p(m,0) \vdash t \Rightarrow u_1 : A$ and $p(m,1) \vdash t \Rightarrow v_1 : A$. By Corollary 3.6 either t is in proper p-whnf or $u_1 \coloneqq v_1$ and $p \vdash t \Rightarrow u_1 : A$. It then follows by induction that u_1, and thus t, has a proper p-whnf. ◀

3.2 Computability predicate and relation

We define inductively a forcing relation $p \Vdash A$ to express that a type A is computable at p. Mutually by recursion we define relations $p \Vdash a : A$, $p \Vdash A = B$, and $p \Vdash a = b : A$. The definition fits the generalized mutual induction-recursion schema [8][4].

▶ **Definition 3.10** (Computibility predicate and relation).

$(\mathbf{F}_{\mathbf{N}_0})$ If $p \vdash A \Rightarrow^* N_0$ then $p \Vdash A$.

1. $p \Vdash t : A$ does not hold for all t.
2. $p \Vdash t = u : A$ does not hold for all t and u.
3. If $p \Vdash B$ then $p \Vdash A = B$ if
   - (i) $p \vdash B \Rightarrow^* N_0$.
   - (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

$(\mathbf{F}_{\mathbf{N}_1})$ If $p \vdash A \Rightarrow^* N_1$ then $p \Vdash A$.

1. $p \Vdash t : A$ if
   - (i) $p \vdash t \Rightarrow^* 0 : A$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \Vdash t : A$ and $p \Vdash u : A$ then $p \Vdash t = u : A$ if
   - (i) $p \vdash t \Rightarrow^* 0 : A$ and $p \vdash u \Rightarrow^* 0 : A$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
   - (iii) $p \vdash u \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \Vdash B$ then $p \Vdash A = B$ if
   - (i) $p \vdash B \Rightarrow^* N_1$.
   - (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

$(\mathbf{F}_{\mathbf{N}_2})$ If $p \vdash A \Rightarrow^* N_2$ then $p \Vdash A$.

1. $p \Vdash t : A$ if
   - (i) $p \vdash t \Rightarrow^* b : A$ for some $b \in \{0,1\}$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \Vdash t : A$ and $p \Vdash u : A$ then $p \Vdash t = u : A$ if
   - (i) $p \vdash t \Rightarrow^* b : A$ and $p \vdash u \Rightarrow^* b : A$ for some $b \in \{0,1\}$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
   - (iii) $p \vdash u \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \Vdash B$ then $p \Vdash A = B$ if
   - (i) $p \vdash B \Rightarrow^* N_2$.
   - (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

$(\mathbf{F}_{\mathbf{N}})$ If $p \vdash A \Rightarrow^* N$ then $p \Vdash A$.

1. $p \Vdash t : A$ if
   - (i) $p \vdash t \Rightarrow^* \overline{n} : A$ for some $n \in \mathbb{N}$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t : A$ for all $i \in \{0,1\}$.
3. If $p \Vdash t : A$ and $p \Vdash u : A$ then $p \Vdash t = u : A$ if
   - (i) $p \vdash t \Rightarrow^* \overline{n} : A$ and $p \vdash u \Rightarrow^* \overline{n} : A$ for some $n \in \mathbb{N}$.
   - (ii) $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
   - (iii) $p \vdash u \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash t = u : A$ for all $i \in \{0,1\}$.
4. If $p \Vdash B$ then $p \Vdash A = B$ if
   - (i) $p \vdash B \Rightarrow^* N$.
   - (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

[4] However, for the canonical proof below we actually need something weaker than an inductive-recursive definition (arbitrary fixed-point instead of *least* fixed-point), reflecting the fact that the universe is defined in an open way [15].

($\mathbf{F_\Pi}$) If $p \vdash A \Rightarrow^* \Pi(x:F)G$ then $p \Vdash A$ if $p \Vdash F$ and for all $q \leqslant p$, $q \Vdash G[a]$ whenever $q \Vdash a:F$ and $q \Vdash G[a] = G[b]$ whenever $q \Vdash a = b:F$.

1. If $\vdash_p f : A$ then $p \Vdash f : A$ if for all $q \leqslant p$, $q \Vdash f\,a : G[a]$ whenever $q \Vdash a : F$ and $q \Vdash f\,a = f\,b : G[a]$ whenever $q \Vdash a = b : F$.
2. If $p \Vdash f : A$ and $p \Vdash g : A$ then $p \Vdash f = g : A$ if for all $q \leqslant p$, $q \Vdash f\,a = g\,a : G[a]$ whenever $q \Vdash a:F$.
3. If $p \Vdash B$ then $p \Vdash A = B$ if
   (i) $\vdash_p A = B$ and $p \vdash B \Rightarrow^* \Pi(x : H)E$ and $p \Vdash F = H$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]$ whenever $q \Vdash a:F$.
   (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

($\mathbf{F_\Sigma}$) If $p \vdash A \Rightarrow^* \Sigma(x:F)G$ then $p \Vdash A$ if $p \Vdash F$ and for all $q \leqslant p$, $q \Vdash G[a]$ whenever $q \Vdash a:F$ and $q \Vdash G[a] = G[b]$ whenever $q \Vdash a = b:F$.

1. If $\vdash_p t:A$ then $p \Vdash t:A$ if $p \Vdash t.1:F$ and $p \Vdash t.2:G[t.1]$.
2. If $p \Vdash t:A$ and $p \Vdash u:A$ then $p \Vdash t = u:A$ if $p \Vdash t.1 = u.1:F$ and $p \Vdash t.2 = u.2:G[t.1]$.
3. If $p \Vdash B$ then $p \Vdash A = B$ if
   (i) $\vdash_p A = B$ and $p \vdash B \Rightarrow^* \Sigma(x : H)E$ and $p \Vdash F = H$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]$ whenever $q \Vdash a:F$.
   (ii) $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

($\mathbf{F_U}$) If $p \vdash A \Rightarrow^* U$ then $p \Vdash A$.

1. $p \Vdash C:A$ if
   (i) $p \vdash C \Rightarrow^* M:A$ for $M \in \{N_0, N_1, N_2, N\}$.
   (ii) $p \vdash C \Rightarrow^* \Pi(x:F)G:A$ and $p \Vdash F:A$ and for all $q \leqslant p$, $q \Vdash G[a]:A$ whenever $q \Vdash a:F$ and $q \Vdash G[a] = G[b]:A$ whenever $q \Vdash a = b:F$.
   (iii) $p \vdash C \Rightarrow^* \Sigma(x:F)G:A$ and $p \Vdash F:A$ and for all $q \leqslant p$, $q \Vdash G[a]:A$ whenever $q \Vdash a:F$ and $q \Vdash G[a] = G[b]:A$ whenever $q \Vdash a = b:F$.
   (iv) $p \vdash C \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]:A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash C:A$ for all $i \in \{0,1\}$.
5. If $p \Vdash C:A$ and $p \Vdash D:A$ then $p \Vdash C = D:A$ if
   (i) $p \vdash C \Rightarrow^* M:A$ and $D \Rightarrow^* M:A$ for $M \in \{N_0, N_1, N_2, N\}$.
   (ii) $p \vdash C \Rightarrow^* \Pi(x:F)G:A$ and $p \vdash D \Rightarrow^* \Pi(x:H)E:A$ and $p \Vdash F = H:A$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]:A$ whenever $q \Vdash a:F$.
   (iii) $p \vdash C \Rightarrow^* \Sigma(x:F)G:A$ and $p \vdash D \Rightarrow^* \Sigma(x:H)E:A$ and $p \Vdash F = H:A$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]:A$ whenever $q \Vdash a:F$.
   (iv) $p \vdash C \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash C = D : A$ for all $i \in \{0,1\}$.
   (v) $p \vdash D \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}] : A$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash C = D : A$ for all $i \in \{0,1\}$.
6. If $p \Vdash B$ then $p \Vdash A = B$ if $p \vdash B \Rightarrow^* U$.

($\mathbf{F_{Loc}}$) If $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for some $m \notin \mathrm{dom}(p)$ and $p(m,i) \Vdash A$ for all $i \in \{0,1\}$ then $p \Vdash A$.

1. If $p(m,i) \Vdash t:A$ for all $i \in \{0,1\}$ then $p \Vdash t:A$.
2. If $p \Vdash t:A$ and $p \Vdash u:A$ and $p(m,i) \Vdash t:A$ for all $i \in \{0,1\}$ then $p \Vdash t = u:A$.
3. If $p \Vdash B$ then $p \Vdash A = B$ if $p(m,i) \Vdash A = B$ for all $i \in \{0,1\}$.

We note from the definition that when $p \Vdash A = B$ then $p \Vdash A$ and $p \Vdash B$, when $p \Vdash a:A$ then $p \Vdash A$ and when $p \Vdash a = b:A$ then $p \Vdash a:A$ and $p \Vdash b:A$. We remark also if $p \vdash A \Rightarrow^* U$ then $A \coloneqq U$ since we have only one universe.

The clause ($\mathrm{F_{Loc}}$) gives semantics to *variable types*. For example, if $p \coloneqq \{(0,0)\}$ and $q \coloneqq \{(0,1)\}$ the type $R \coloneqq \mathsf{boolrec}\,(\lambda x.U)\,N_1\,N\,(\mathsf{f}\,0)$ has reductions $p \vdash R \Rightarrow^* N_1$ and $q \vdash R \Rightarrow^* N$. Thus $p \Vdash R$ and $q \Vdash R$ and since $\langle\rangle \lhd p, q$ we have $\langle\rangle \Vdash R$.

Immediately from Definition 3.10 we get:

▶ **Lemma 3.11.** *If* $p \Vdash A$ *then* $\vdash_p A$*. If* $p \Vdash a : A$ *then* $\vdash_p a : A$*.*

▶ **Lemma 3.12.** *If* $p \Vdash A$ *then there is a partition* $p \lhd p_1, \dots, p_n$ *where* A *has a canonical* p_i*-whnf for all* i*.*

Proof. The statement follows from the definition by induction on the derivation of $p \Vdash A$ ◀

▶ **Corollary 3.13.** *Let* $p \lhd p_1, \dots, p_n$*. If* $p_i \Vdash A$ *for all* i *then* A *has a proper* p*-whnf.*

Proof. Follows from Lemma 3.12 and Corollary 3.9 by induction on the partition. ◀

▶ **Lemma 3.14.** *If* $p \Vdash A$ *and* $q \leqslant p$ *then* $q \Vdash A$*.*

Proof. Let $p \Vdash A$ and $q \leqslant p$. By induction on the derivation of $p \Vdash A$:

($\mathrm{F_N}$) Since $p \vdash A \Rightarrow^* N$ and the reduction relation is monotone we have $q \vdash A \Rightarrow^* N$, thus $q \Vdash A$. The statement follows similarly for ($\mathrm{F_{N_0}}$), ($\mathrm{F_{N_1}}$), ($\mathrm{F_{N_2}}$) and ($\mathrm{F_U}$).

($\mathrm{F_\Pi}$) Let $p \vdash A \Rightarrow^* \Pi(x : F)G$. Since $p \Vdash F$, by induction $q \Vdash F$. Let $s \leqslant q$, we have then $s \leqslant p$. It then follows from $p \Vdash A$ that $s \Vdash G[a]$ whenever $s \Vdash a : F$ and $s \Vdash G[a] = G[b]$ whenever $s \Vdash a = b : F$. Thus $q \Vdash A$. The statement follows similarly for ($\mathrm{F_\Sigma}$).

($\mathrm{F_{Loc}}$) Let $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$. If $m \in \mathrm{dom}(q)$ then $q \leqslant p(m,0)$ or $q \leqslant p(m,1)$ and since $p(m,i) \Vdash A$, by induction $q \Vdash A$. Alternatively, $q \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$. But $q \lhd q(m,0), q(m,1)$ and $q(m,i) \leqslant p(m,i)$. By induction $q(m,i) \Vdash A$ for all $i \in \{0,1\}$ and thus $q \Vdash A$. ◀

▶ **Lemma 3.15.** *If* $p \Vdash t : A$ *and* $q \leqslant p$ *then* $q \Vdash t : A$*.*

Proof. Let $p \Vdash t : A$ and $q \leqslant p$. By induction on the derivation of $p \Vdash A$.

($\mathrm{F_N}$) Since $p \vdash A \Rightarrow^* N$ then $q \vdash A \Rightarrow^* N$. By induction on the derivation of $p \Vdash t : A$. If $p \vdash t \Rightarrow^* \bar{n} : A$ for $n \in \mathbb{N}$ then $q \vdash t \Rightarrow^* \bar{n} : A$, hence, $q \Vdash t : A$. Alternatively, $p \vdash t \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}] : A$ for some $k \notin \mathrm{dom}(p)$ and $p(k,b) \Vdash t : A$ for all $b \in \{0,1\}$. If $k \in \mathrm{dom}(q)$ then $q \leqslant p(k,1)$ or $q \leqslant p(k,0)$ and in either case, by induction, $q \Vdash t : A$. Otherwise, we have $q(k,b) \leqslant p(k,b)$ and by induction $q(k,b) \Vdash t : A$ for all b. By the definition $q \Vdash t : A$. The statement follows similarly for ($\mathrm{F_{N_0}}$), ($\mathrm{F_{N_1}}$), and ($\mathrm{F_{N_2}}$).

($\mathrm{F_U}$) We can show the statement by a proof similar to that of Lemma 3.14.

($\mathrm{F_\Pi}$) Let $p \vdash A \Rightarrow^* \Pi(x : F)G$. We have $q \vdash A \Rightarrow^* \Pi(x : F)G$. From $\vdash_p t : A$ we have $\vdash_q t : A$. Let $r \leqslant q$. If $r \Vdash a : F$ then since $r \leqslant p$ we have $r \Vdash t\,a : G[a]$. Similarly if $r \Vdash a = b : F$ then $r \Vdash t\,a = t\,b : G[a]$. Thus $q \Vdash t : A$.

($\mathrm{F_\Sigma}$) Let $p \vdash A \Rightarrow^* \Sigma(x : F)G$. We have $q \vdash A \Rightarrow^* \Sigma(x : F)G$. From $\vdash_p t : A$ we have $\vdash_q t : A$. Since $p \Vdash t : A$ we have $p \Vdash t.1 : F$ and $p \Vdash t.2 : G[t.1]$. By induction $q \Vdash t.1 : F$ and $q \Vdash t.2 : G[t.1]$, thus $q \Vdash t : A$.

($\mathrm{F_{Loc}}$) Let $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ for some $k \notin \mathrm{dom}(p)$. Since $p \Vdash t : A$ we have $p(k,b) \Vdash t : A$ for all $b \in \{0,1\}$. If $k \in \mathrm{dom}(q)$ then $q \leqslant p(k,0)$ or $q \leqslant p(k,1)$ and by induction $q \Vdash t : A$. Otherwise, $q \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ and since $q(k,b) \leqslant p(k,b)$, by induction, $q(k,b) \Vdash t : A$ for all b. By definition $q \Vdash t : A$. ◀

Using similar arguments we can also show the following two statements:

▶ **Lemma 3.16.** *Let* $p \Vdash A$*. If* $p \Vdash A = B$ *and* $q \leqslant p$ *then* $q \Vdash A = B$*.*

▶ **Lemma 3.17.** *Let* $p \Vdash A$*. If* $p \Vdash t = u : A$ *and* $q \leqslant p$ *then* $q \Vdash t = u : A$*.*

We collect the results of Lemmas 3.14, Lemma 3.15, Lemma 3.17, and Lemma 3.16 in the following corollary.

▶ **Corollary 3.18** (Monotonicity). *If $p \Vdash J$ and $q \leqslant p$ then $q \Vdash J$.*

We write $\Vdash J$ when $\langle\rangle \Vdash J$. By monotonicity $\Vdash J$ iff $p \Vdash J$ for all p.

▶ **Lemma 3.19.** *If $p(m,0) \Vdash A$ and $p(m,1) \Vdash A$ for some $m \notin \mathrm{dom}(p)$ then $p \Vdash A$.*

Proof. By Corollary 3.13, either A has a canonical p-whnf or $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ for $k \notin \mathrm{dom}(p)$.

- If $p \vdash A \Rightarrow^* M$ with $M \in \{N_0, N_1, N_2, N\}$ then we have immediately that $p \Vdash A$.
- If $p \vdash A \Rightarrow^* M$ with M of the form $\Pi(x : F)G$ or $\Sigma(x : F)G$ then $p(m,b) \vdash A \Rightarrow^* M$ for all $b \in \{0,1\}$. Since $p(m,b) \Vdash A$ we have $p(m,b) \Vdash F$ for all b and by induction $p \Vdash F$. Let $q \leqslant p$ and $q \Vdash a : F$. If $m \in \mathrm{dom}(q)$ then $q \leqslant p(m,b)$ for some $b \in \{0,1\}$. Assume, w.l.o.g, $q \leqslant p(m,0)$. Since $p(m,0) \Vdash A$ we have by the definition that $q \Vdash G[a]$. Alternatively, if $m \notin \mathrm{dom}(q)$ we have a partition $q \lhd q(m,0), q(m,1)$. By monotonicity $q(m,b) \Vdash a : F$, and since $q(m,b) \leqslant p(m,b)$, we have $q(m,b) \Vdash G[a]$ for all $b \in \{0,1\}$. By induction $q \Vdash G[a]$. Similarly we can show $q \Vdash G[a] = G[b]$ whenever $q \Vdash a = b : F$.
- Alternatively, $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ for some $k \notin \mathrm{dom}(p)$. If $k = m$ then by the definition $p \Vdash A$. Otherwise, $p(m,0) \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ and by the definition $p(m,0)(k,b) \Vdash A$ for all $b \in \{0,1\}$. Similarly, $p(m,1)(k,b) \Vdash A$ for all $b \in \{0,1\}$. But $p(k,b) \lhd p(m,0)(k,b), p(m,1)(k,b)$. By induction $p(k,b) \Vdash A$ for all $b \in \{0,1\}$ and thus $p \Vdash A$. ◀

Similarly we can show the following two statements:

▶ **Lemma 3.20.** *1. If $p(m,0) \Vdash t : A$ and $p(m,1) \Vdash t : A$ for some $m \notin \mathrm{dom}(p)$ then $p \Vdash t : A$.*
2. If $p(m,0) \Vdash t = u : A$ and $p(m,1) \Vdash t = u : A$ for some $m \notin \mathrm{dom}(p)$ then $p \Vdash t = u : A$.

▶ **Lemma 3.21.** *If $p(m,0) \Vdash A = B$ and $p(m,1) \Vdash A = B$ for some $m \notin \mathrm{dom}(p)$ then $p \Vdash A = B$.*

▶ **Corollary 3.22** (Local character). *If $p \lhd p_1, \dots, p_n$ and $p_i \Vdash J$ for all i then $p \Vdash J$.*

Proof. Follows from Lemma 3.19, Lemma 3.20, and Lemma 3.21 by induction. ◀

▶ **Lemma 3.23.** *Let $p \vdash A \Rightarrow^* M$ where $M \in \{N_1, N_2, N\}$. If $p \Vdash a : A$ then there is a partition $p \lhd p_1, \dots, p_n$ where a has a canonical p_i-whnf for all i. If $p \Vdash a = b : A$ then there is a partition $p \lhd q_1, \dots, q_m$ where a and b have the same canonical q_j-whnf for each j.*

Proof. Follows by induction from the definition. ◀

▶ **Lemma 3.24.** *Let $p \Vdash A = B$.*
1. *If $p \Vdash t : A$ then $p \Vdash t : B$ and if $p \Vdash u : B$ then $p \Vdash u : A$.*
2. *If $p \Vdash t = u : A$ then $p \Vdash t = u : B$ and if $p \Vdash v = w : B$ then $p \Vdash v = w : A$.*

Proof. By induction on the derivation of $p \Vdash A$.

($\mathrm{F_N}$) By induction on the derivation of $p \Vdash A = B$. (i) Let $p \vdash B \Rightarrow^* N$ then the statement follows directly. (ii) Let $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{m}]$ for $m \notin \mathrm{dom}(p)$ and $p(m,b) \Vdash A = B$ for all $b \in \{0,1\}$. Let $p \Vdash t : A$. By monotonicity $p(m,b) \Vdash t : A$ and by induction $p(m,b) \Vdash t : B$ for all b. By the definition $p \Vdash t : B$. Let $p \Vdash u : B$. By monotonicity $p(m,b) \Vdash u : B$ and $p(m,b) \Vdash A = B$. By induction $p(m,b) \Vdash u : A$ for all b. By local character $p \Vdash u : A$. Similarly we can show the second statement. The statement follows similarly for ($\mathrm{F_{N_1}}$) and ($\mathrm{F_{N_2}}$).

($\mathbf{F_\Pi}$) Let $p \vdash A \Rightarrow^* \Pi(x:F)G$. By induction on the derivation of $p \Vdash A = B$. (i) Let $\vdash_p A = B$ and $p \vdash B \Rightarrow^* \Pi(x:H)E$ and $p \Vdash F = H$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]$ whenever $q \Vdash a:F$. If $p \Vdash f:A$ then $\vdash_p f:A$, thus $\vdash_p f:B$. Let $q \leqslant p$ and $q \Vdash u:H$. By monotonicity $q \Vdash F = H$. By induction $q \Vdash u:F$, hence, $q \Vdash f\,u:G[u]$ and by induction $q \Vdash f\,u:E[u]$. Similarly, $q \Vdash f\,u = f\,v:E[u]$ whenever $q \Vdash u = v:H$. Thus $p \Vdash f:B$. Similarly, if $p \Vdash g:B$ we get $p \Vdash g:A$. (ii) Let $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}, \bar{k}]$ and $p(k,b) \Vdash A = B$ for all $b \in \{0,1\}$. If $p \Vdash f:A$ then by monotonicity $p(k,b) \Vdash f:A$ and by induction $p(k,b) \Vdash f:B$ for all b. By the definition $p \Vdash f:B$. If on the other hand $p \Vdash g:B$ then by definition $p(k,b) \Vdash g:B$ and by induction $p(k,b) \Vdash g:A$ for all b. By local character $p \Vdash g:A$. Similarly we can show the second statement.

($\mathbf{F_\Sigma}$) Let $p \vdash A \Rightarrow^* \Sigma(x:F)G$. By induction on the derivation of $p \Vdash A = B$. (i) Let $\vdash_p A = B$ and $p \vdash B \Rightarrow^* \Sigma(x:H)E$ and $p \Vdash F = H$ and for all $q \leqslant p$, $q \Vdash G[a] = E[a]$ whenever $q \Vdash a:F$. If $p \Vdash t:A$ then $\vdash_p t:A$, thus $\vdash_p t:B$. Since $p \Vdash t.1:F$, by induction $p \Vdash t.1:H$. Since $p \Vdash t.2:H[t.1]$, by induction $p \Vdash t.2:E[t.1]$. Thus $p \Vdash t:B$. Similarly if $p \Vdash u:B$ we have $p \Vdash u:A$. (ii) Let $p \vdash B \Rightarrow^* \mathbb{E}[\mathsf{f}, \bar{k}]$ and $p(k,b) \Vdash A = B$ for all $b \in \{0,1\}$. If $p \Vdash t:A$ then by monotonicity $p(k,b) \Vdash t:A$ and by induction $p(k,b) \Vdash t:B$ for all b. By the definition $p \Vdash f:B$. If on the other hand $p \Vdash g:B$ then by definition $p(k,b) \Vdash g:B$ and by induction $p(k,b) \Vdash g:A$ for all b. By local character $p \Vdash g:A$. Similarly we can show the second statement.

($\mathbf{F_U}$) Since $p \Vdash A = B$, we have $p \vdash B \Rightarrow^* U$ and the statements follow directly.

($\mathbf{F_{Loc}}$) Let $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]$ for some $k \notin \mathrm{dom}(p)$. Since $p \Vdash A = B$, we have $p(k,b) \Vdash A = B$ for all $b \in \{0,1\}$. If $p \Vdash t:A$ then $p(k,b) \Vdash t:A$ and by induction $p(k,b) \Vdash t:B$ for all b. By the definition $p \Vdash t:B$. If $p \Vdash u:B$ then $p(k,b) \Vdash u:B$ and by induction $p(k,b) \Vdash u:A$ for all b. By local character $p \Vdash u:A$. Similarly we can show the second statement. ◀

From the above results we can show that the relations $p \Vdash - = -$ and $p \Vdash - = -:A$ are equivalence relations. We omit the proof here.

▶ **Lemma 3.25.**
Reflexivity: *If $p \Vdash A$ then $p \Vdash A = A$ and if $p \Vdash t:A$ then $p \Vdash t = t:A$.*
Symmetry: *If $p \Vdash A = B$ then $p \Vdash B = A$ and if $p \Vdash t = u:A$ then $p \Vdash u = t:A$.*
Transitivity: *If $p \Vdash A = B$ and $p \Vdash B = C$ then $p \Vdash A = C$ and if $p \Vdash t = u:A$ and $p \Vdash u = v:A$ then $p \Vdash t = v:A$.*

4 Soundness

In this section we show that the type theory described in Section 2 is sound with respect to the semantics described in Section 3. That is, we aim to show that for any judgment J whenever $\vdash_p J$ then $p \Vdash J$.

▶ **Lemma 4.1.** *If $p \vdash A \Rightarrow^* B$ and $p \Vdash B$ then $p \Vdash A$ and $p \Vdash A = B$.*

Proof. Follows from the definition. ◀

▶ **Lemma 4.2.** *Let $p \Vdash A$. If $p \vdash t \Rightarrow u:A$ and $p \Vdash u:A$ then $p \Vdash t:A$ and $p \Vdash t = u:A$.*

Proof. Let $p \vdash t \Rightarrow u:A$ and $p \Vdash u:A$. By induction on the derivation of $p \Vdash A$.

($\mathbf{F_U}$) That is, $p \vdash A \Rightarrow^* U$. The statement follows similarly to Lemma 4.1.

($\mathbf{F_N}$) By induction on the derivation of $p \Vdash u:A$. If $p \vdash u \Rightarrow^* \bar{n}:N$ for some $n \in \mathbb{N}$ then $p \vdash t \Rightarrow^* \bar{n}:N$ and the statement follows by the definition. If $p \vdash u \Rightarrow^* \mathbb{E}[\mathsf{f}\,\bar{k}]:A$ for $k \notin \mathrm{dom}(p)$ and $p(k,b) \Vdash u:A$ for all $b \in \{0,1\}$ then since $p(k,b) \vdash t \Rightarrow u:A$, by induction,

$p(k,b) \Vdash t : A$ and $p(k,b) \Vdash t = u : A$. By the definition $p \Vdash t : A$ and $p \Vdash t = u : A$. The statement follows similarly for $(\mathrm{F}_{\mathrm{N}_1})$, $(\mathrm{F}_{\mathrm{N}_2})$.

($\mathbf{F}_\Pi$) Let $p \vdash A \Rightarrow^* \Pi(x:F)G$. Since $p \vdash t \Rightarrow u : A$ we have $\vdash_p t : A$. Let $q \leqslant p$ and $q \Vdash a : F$. We have $q \vdash t\,a \Rightarrow u\,a : G[a]$. By induction $q \Vdash t\,a : G[a]$ and $q \Vdash t\,a = u\,a : G[a]$. If $q \Vdash a = b : F$ we similarly get $q \Vdash t\,b : G[b]$ and $q \Vdash t\,b = u\,b : G[b]$. Since $q \Vdash G[a] = G[b]$, by Lemma 3.24, $q \Vdash t\,b = u\,b : G[a]$. But $q \Vdash u\,a = u\,b : G[a]$. By symmetry and transitivity $q \Vdash t\,a = t\,b : G[a]$. Thus $p \Vdash t : A$ and $p \Vdash t = u : A$.

($\mathbf{F}_\Sigma$) Let $p \vdash A \Rightarrow^* \Sigma(x:F)G$. From $p \vdash t \Rightarrow u : A$ we have $\vdash_p t : A$ and we have $p \vdash t.1 \Rightarrow u.1 : F$ and $p \vdash t.2 \Rightarrow u.2 : G[u.1]$. By induction $p \Vdash t.1 : F$ and $p \Vdash t.1 = u.1 : F$. By induction $p \Vdash t.2 : G[u.1]$ and $p \Vdash t.2 = u.2 : G[u.1]$. But since $p \Vdash A$ and we have shown $p \Vdash t.1 = u.1 : F$ we get $p \Vdash G[t.1] = G[u.1]$. By Lemma 3.24, $p \Vdash t.2 : G[t.1]$ and $p \Vdash t.2 = u.2 : G[t.1]$. Thus $p \Vdash t : A$ and $p \Vdash t = u : A$

($\mathbf{F}_{\mathbf{Loc}}$) Let $p \vdash A \Rightarrow^* \mathbb{E}[\mathsf{f}\,\overline{k}]$ for $k \notin \mathrm{dom}(p)$. Since $p \Vdash u : A$ we have $p(k,b) \Vdash u : A$ for all $b \in \{0,1\}$. But we have $p(k,b) \vdash t \Rightarrow u : A$. By induction $p(k,b) \Vdash t : A$ and $p(k,b) \Vdash t = u : A$. By the definition $p \Vdash t : A$ and $p \Vdash t = u : A$. ◀

▶ **Corollary 4.3.** *Let $p \vdash t \Rightarrow^* u : A$ and $p \Vdash A$. If $p \Vdash u : A$ then $p \Vdash t : A$ and $p \Vdash t = u : A$.*

▶ **Corollary 4.4.** $\Vdash \mathsf{f} : N \to N_2$.

Proof. It's direct to see that $\Vdash N \to N_2$. For an arbitrary condition p let $p \Vdash n : N$. By Lemma 3.23, we have a parition $p \lhd p_1, \dots, p_m$ where for each i, $p_i \vdash n \Rightarrow^* \overline{m}_i : N$ for some $m_i \in \mathbb{N}$. We have thus a reduction $p_i \vdash \mathsf{f}\,n \Rightarrow^* \mathsf{f}\,\overline{m}_i : N_2$. If $\overline{m}_i \in \mathrm{dom}(p_i)$ then $p_i \vdash \mathsf{f}\,n \Rightarrow^* \mathsf{f}\,\overline{m}_i \Rightarrow b_i : N_2$ for some $b_i \in \{0,1\}$ and by definition $p_i \Vdash \mathsf{f}\,n : N_2$. If for any j, $\overline{m}_j \notin \mathrm{dom}(p_j)$ then $p_j(m_j,0) \vdash \mathsf{f}\,n \Rightarrow^* \mathsf{f}\overline{m}_j \Rightarrow 0 : N_2$ and $p_j(m_j,1) \vdash \mathsf{f}\,n \Rightarrow^* \mathsf{f}\overline{m}_j \Rightarrow 1 : N_2$. Thus $p_j(m_j,0) \Vdash \mathsf{f}\,n : N_2$ and $p_j(m_j,1) \Vdash \mathsf{f}\,n : N_2$. By the definition $p_j \Vdash \mathsf{f}\,n : N_2$. We thus have that $p_i \Vdash \mathsf{f}\,n : N_2$ for all i and by local character $p \Vdash \mathsf{f}\,n : N_2$. Similarly we can show $p \Vdash \mathsf{f}\,n_1 = \mathsf{f}\,n_2 : N_2$ whenever $p \Vdash n_1 = n_2 : N$. Hence $\vdash \mathsf{f} : N \to N_2$. ◀

▶ **Lemma 4.5.** *If $\vdash_p t : \neg A$ and $p \Vdash A$ then $p \Vdash t : \neg A$ iff for all $q \leqslant p$ there is no term u such that $q \Vdash u : A$.*

Proof. Let $p \Vdash A$ and $\vdash_p t : \neg A$. We have directly that $p \Vdash \neg A$. Let $p \Vdash t : \neg A$. If $q \Vdash u : A$ for some $q \leqslant p$, then $q \Vdash t\,u : N_0$ which is impossible. Conversely, assume it is the case that for all $q \leqslant p$ there is no u for which $q \Vdash u : A$. Since $r \Vdash a : A$ and $r \Vdash a = b : A$ never hold for any $r \leqslant p$, the statement "$r \Vdash t\,a : N_0$ whenever $r \Vdash a : A$ and $r \Vdash t\,a = t\,b : N_0$ whenever $r \Vdash a = b : A$" holds trivially. ◀

▶ **Lemma 4.6.** $\Vdash \mathsf{w} : \neg\neg(\Sigma(x:N)\mathsf{IsZero}(\mathsf{f}\,x))$.

Proof. By Lemma 4.5 it is enough to show that for all q there is no term u for which $q \Vdash u : \neg(\Sigma(x:N)\mathsf{IsZero}(\mathsf{f}\,x))$. Assume $q \Vdash u : \neg(\Sigma(x:N)\mathsf{IsZero}(\mathsf{f}\,x))$ for some u. Let $m \notin \mathrm{dom}(q)$ we have then $q(m,0) \Vdash (\overline{m},0) : \Sigma(x:N)\mathsf{IsZero}(\mathsf{f}\,x)$ thus $q(m,0) \Vdash u\,(\overline{m},0) : N_0$ which is impossible. ◀

Let $\Gamma \coloneqq x_1 : A_1 \dots, x_n : A_n[x_1, \dots, x_{n-1}]$ and $\rho \coloneqq a_1, \dots, a_n$. We say $p \Vdash \rho : \Gamma$ if $p \Vdash a_1 : A, \dots, p \Vdash a_n : A_n[a_1, \dots, a_{n-1}]$. If moreover $\sigma \coloneqq b_1, \dots, b_n$ and $p \Vdash \sigma : \Gamma$, we say $p \Vdash \rho = \sigma : \Gamma$ if $p \Vdash a_1 = b_1 : A_1, \dots, p \Vdash a_n = b_n : A_n[a_1, \dots, a_{n-1}]$.

▶ **Lemma 4.7.** *Let $\Gamma \vdash_p$. For all $q \leqslant p$, if $q \Vdash \rho : \Gamma$, $q \Vdash \sigma : \Gamma$ and $q \Vdash \rho = \sigma : \Gamma$ then*

- *If $\Gamma \vdash_p A$ then $q \Vdash A\rho = A\sigma$ and if $\Gamma \vdash_p A = B$ then $q \Vdash A\rho = B\rho$.*
- *If $\Gamma \vdash_p a : A$ then $q \Vdash a\rho = a\sigma : A\rho$ and if $\Gamma \vdash_p a = b : A$ then $q \Vdash a\rho = b\rho : A\rho$*

Proof. The proof is by induction on the rules of the type system. We show that if the statement holds for the premise of the rule it holds for the conclusion. For economy of presentation we only present the proof for few selected rules. For the rest of the rules the proof follows in a similar fashion.

- For the elimination rules (β), (UNITREC-0), (BOOLREC-0), (BOOLREC-1), (NATREC-0), (NATREC-SUC), (pr_1), (pr_2) and (f-EVAL) the statement follows from Corollary 4.3.
- For the congruence rules the statement follows from Lemma 3.24, Lemma 3.25.
- The statement follows for (f-I) by Corollary 4.4, for (w-TERM) by Lemma 4.6, and for (LOC) by Lemma 3.22.
- (NAT-SUC) By induction $q \Vdash n\rho = n\sigma : N$. By Lemma 3.23 there is a partition $q \lhd q_1, \ldots, q_\ell$ where for each i, $q_i \vdash n\rho \Rightarrow^* \overline{m}_i : N$ and $q_i \vdash n\sigma \Rightarrow^* \overline{m}_i : N$ for some $m_i \in \mathbb{N}$. But then $q_i \vdash \mathsf{S}\, n\rho \Rightarrow^* \mathsf{S}\, \overline{m}_i : N$ and $q_i \vdash \mathsf{S}\, n\sigma \Rightarrow^* \mathsf{S}\, \overline{m}_i : N$ for all i. Thus $q_i \Vdash \mathsf{S}\, n\rho = \mathsf{S}\, n\sigma : N$ for all i and by local character $q \Vdash \mathsf{S}\, n\rho = \mathsf{S}\, n\sigma : N$.
- (Π-I) By induction $q \Vdash F\rho = F\sigma$. Let $r \leqslant q$. We have $r \Vdash (\rho, c) = (\rho, b) : (\Gamma, x : F)$ whenever $r \Vdash c = b : F$ and by induction $r \Vdash G\rho[c] = G\rho[b]$. We have then $q \Vdash \Pi(x : F\rho)G\rho$ and similarly $q \Vdash \Pi(x : F\sigma)G\sigma$. Whenever $r \Vdash a : F\rho$ then, by Lemma 3.24, $r \Vdash (\rho, a) = (\sigma, a) : (\Gamma, x : F)$ and by induction $r \Vdash G\rho[a] = G\sigma[a]$. Thus $q \Vdash \Pi(x : F\rho)G\rho = \Pi(x : F\sigma)G\sigma$.
- (λ-I) From $\Gamma, x : F \vdash_p t : G$ we have $\Gamma \vdash_p F$ and $\Gamma, x : F \vdash_p G$. Similarly to (Π-I) we can show $q \Vdash \Pi(x : F\rho)G\rho$, $q \Vdash \Pi(x : F\sigma)G\sigma$, and $q \Vdash \Pi(x : F\rho)G\rho = \Pi(x : F\sigma)G\sigma$. Let $r \leqslant q$ and $r \Vdash a : F\rho$. We have $r \Vdash (\rho, a) = (\sigma, a) : (\Gamma, x : F)$ and by induction $r \Vdash t\rho[a] = t\sigma[a] : G\rho[a]$. But $r \vdash (\lambda x.t\rho)\, a \Rightarrow t\rho[a] : G\rho[a]$ and $r \vdash (\lambda x.t\sigma)\, a \Rightarrow t\sigma[a] : G\sigma[a]$. By Lemma 4.2 one has $r \Vdash (\lambda x.t\rho)\, a = t\rho[a] : G\rho[a]$ and $r \Vdash (\lambda x.t\sigma)\, a = t\sigma[a] : G\sigma[a]$. Since by induction we have $r \Vdash G\rho[a] = G\sigma[a]$, by Lemma 3.24, $r \Vdash (\lambda x.t\sigma)\, a = t\sigma[a] : G\rho[a]$. By symmetry and transitivity $r \Vdash (\lambda x.t\rho)a = (\lambda x.t\sigma)a : G\rho[a]$. Similarly we can show $r \Vdash (\lambda x.t\rho)\, a = (\lambda x.t\rho)\, b : \Pi(x : F\rho)G\rho$ whenever $r \Vdash a = b : F\rho$ and $r \Vdash (\lambda x.t\sigma)\, a = (\lambda x.t\sigma)\, b : \Pi(x : F\sigma)G\sigma$ whenever $r \Vdash a = b : F\sigma$. Thus $q \Vdash (\lambda x.t\rho) = (\lambda x.t\sigma) : \Pi(x : F\rho)G\rho$
- ($\bot$REC-I-E) Follows trivially since $r \Vdash t : N_0$ never holds for any condition r.
- (NATREC-I) While we omit the proof here the basic idea is as follows: If for some $r \leqslant q$ we have $r \Vdash t : N$ then by Lemma 3.23, we have $r \lhd r_1, \ldots, r_n$ and for each i, $r_i \vdash t \Rightarrow^* \mathsf{S}^{k_i} 0$ for some $k_i \in \mathbb{N}$. By induction on k_i we can show $r_i \Vdash (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho\, t : F\rho[t]$ for all i. By local character we will then have $r \Vdash (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho\, t : F\rho[t]$. Similarly we can show $r \Vdash (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho\, t = (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho\, u : F\rho[t]$ whenever $r \Vdash t = u : N$. By the definition we will have $q \Vdash (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho : \Pi(x : N)F\rho$ and similarly we can show $q \Vdash (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\rho = (\mathsf{natrec}\ (\lambda x.F)\, a_0\, g)\sigma : \Pi(x : N)F\rho$. ◀

▶ **Theorem 4.8** (Fundamental Theorem). *If $\vdash_p J$ then $p \Vdash J$.*

5 Markov's principle

Now we have enough machinery to show the independence of MP from type theory. The idea is that if a judgment J is derivable in type theory (i.e. $\vdash J$) then it is derivable in the forcing extension (i.e. $\vdash_{\langle\rangle} J$) and by Theorem 4.8 it holds in the interpretation (i.e. $\Vdash J$). It thus suffices to show that there no t such that $\Vdash t : \mathrm{MP}$ to establish the independence of MP from type theory. First we recall the formulation of MP.

$$\mathrm{MP} \coloneqq \Pi(h : N \to N_2)[\neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\, (h\, x)) \to \Sigma(x : N)\, \mathsf{IsZero}\, (h\, x)]$$

where $\mathsf{IsZero} : N_2 \to U$ is given by $\mathsf{IsZero} \coloneqq \lambda y.\mathsf{boolrec}\ (\lambda x.U)\, N_1\, N_0\, y$.

▶ **Lemma 5.1.** *There is no term t such that* $\Vdash t : \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x)$.

Proof. Assume $\Vdash t : \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x)$ for some t. We then have $\Vdash t.1 : N$ and $\Vdash t.2 : \mathsf{IsZero}\,(\mathsf{f}\, t.1)$. By Lemma 3.23, one has a partition $\langle\rangle \lhd p_1, \ldots, p_n$ where for each i, $p_i \vdash t.1 \Rightarrow^* \overline{m}_i$ for some $\overline{m}_i \in \mathbb{N}$. Hence $p_i \vdash \mathsf{IsZero}\,(\mathsf{f}\, t.1) \Rightarrow^* \mathsf{IsZero}\,(\mathsf{f}\, \overline{m}_i)$ and by Lemma 4.1, $p_i \Vdash \mathsf{IsZero}\,(\mathsf{f}\, t.1) = \mathsf{IsZero}\,(\mathsf{f}\, \overline{m}_i)$. But, by definition, a partition of $\langle\rangle$ must contain a condition, say p_j, such that $p_j(k) = 1$ whenever $k \in \mathrm{dom}(p_j)$ (this holds vacuously for $\langle\rangle \lhd \langle\rangle$). Assume $m_j \in \mathrm{dom}(p_j)$, then $p_j \vdash \mathsf{IsZero}\,(\mathsf{f}\, t.1) \Rightarrow^* \mathsf{IsZero}\,(\mathsf{f}\, m_j) \Rightarrow^* N_0$. By monotonicity, from $\Vdash t.2 : \mathsf{IsZero}\,(\mathsf{f}\, t.1)$ we get $p_j \Vdash t.2 : \mathsf{IsZero}\,(\mathsf{f}\, t.1)$. But $p_j \vdash \mathsf{IsZero}\,(\mathsf{f}\, t.1) \Rightarrow^* N_0$ thus $p_j \Vdash \mathsf{IsZero}\,(\mathsf{f}\, t.1) = N_0$. Hence, by Lemma 3.24, $p_j \Vdash t.2 : N_0$ which is impossible, thus contradicting our assumption. If on the other hand $m_j \notin \mathrm{dom}(p_j)$ then since $p_j \lhd p_j(m_j, 0), p_j(m_j, 1)$ we can apply the above argument with $p_j(m_j, 1)$ instead of p_j. ◀

▶ **Lemma 5.2.** *There is no term t such that* $\Vdash t : \mathrm{MP}$.

Proof. Assume $\Vdash t : \mathrm{MP}$ for some t. From the definition, whenever $\Vdash g : N \to N_2$ we have $\Vdash t\, g : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\,(g\, x)) \to \Sigma(x : N)\, \mathsf{IsZero}\,(g\, x)$. Since by Corollary 4.4, $\Vdash \mathsf{f} : N \to N_2$ we have $\Vdash t\, \mathsf{f} : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x)) \to \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x)$. Since by Lemma 4.6, $\Vdash \mathsf{w} : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x))$ we have, $\Vdash (t\, \mathsf{f})\, \mathsf{w} : \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}\, x)$ which is impossible by Lemma 5.1. ◀

From Theorem 4.8, Lemma 5.2, and Lemma 2.3 we can then conclude:

▶ **Theorem 2.1.** *There is no term t such that* $\mathrm{MLTT} \vdash t : \mathrm{MP}$.

5.1 Many Cohen reals

We extend the type system in Section 2 further by adding a generic point f_q for each condition q. The introduction and conversion rules for f_q are given by:

$$\frac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{f}_q : N \to N_2} \quad \frac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{f}_q\, \overline{n} = 1}\, n \in \mathrm{dom}(q) \quad \frac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{f}_q\, \overline{n} = p(n)}\, n \notin \mathrm{dom}(q), n \in \mathrm{dom}(p)\,.$$

With the reduction rules: $\dfrac{n \in \mathrm{dom}(q)}{\mathsf{f}_q\, \overline{n} \to 1} \quad \dfrac{n \notin \mathrm{dom}(q), n \in \mathrm{dom}(p)}{\mathsf{f}_q\, \overline{n} \to_p p(n)}$.

We observe that the reduction relation is still monotone.

For each f_q we add a term $\dfrac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{w}_q : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}_q\, x))}$.

Finally we add a term mw witnessing the negation of MP $\dfrac{\Gamma \vdash_p}{\Gamma \vdash_p \mathsf{mw} : \neg\mathrm{MP}}$.

By analogy to Corollary 4.4 we have:

▶ **Lemma 5.3.** $\Vdash \mathsf{f}_q : N \to N_2$ *for all q.*

▶ **Lemma 5.4.** $\Vdash \mathsf{w}_q : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}_q\, x))$ *for all q.*

Proof. Assume $p \Vdash t : \neg(\Sigma(x : N) \mathsf{IsZero}\,(\mathsf{f}_q\, x))$ for some p and t. Let $m \notin \mathrm{dom}(q) \cup \mathrm{dom}(p)$, we have $p(\overline{m}, 0) \Vdash \mathsf{f}_q\, m = 0$. Thus $p(\overline{m}, 0) \Vdash (\overline{m}, 0) : \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}_q\, x)$ and $p(\overline{m}, 0) \Vdash t\, (\overline{m}, 0) : N_0$ which is impossible. ◀

▶ **Lemma 5.5.** *There is no term t for which* $q \Vdash t : \Sigma(x : N)\, \mathsf{IsZero}\,(\mathsf{f}_q\, x)$.

Proof. Assume $q \Vdash t : \Sigma(x : N)\, \mathsf{IsZero}\, (\mathsf{f}_q\, x)$ for some t. We then have $q \Vdash t.1 : N$ and $q \Vdash t.2 : \mathsf{IsZero}\, (\mathsf{f}_q\, t.1)$. By Lemma 3.23 one has a partition $q \lhd q_1, \ldots, q_n$ where for each i, $t.1 \Rightarrow^*_{q_i} \overline{m}_i$ for some $\overline{m}_i \in \mathbb{N}$. Hence $q_i \vdash \mathsf{IsZero}\, (\mathsf{f}_q\, t.1) \Rightarrow^* \mathsf{IsZero}\, (\mathsf{f}_q\, \overline{m}_i)$. But any partition of q contain a condition, say q_j, where $q_j(k) = 1$ whenever $k \notin \mathrm{dom}(q)$ and $k \in \mathrm{dom}(q_j)$. Assume $m_j \in \mathrm{dom}(q_j)$. If $m_j \in \mathrm{dom}(q)$ then $q_j \vdash \mathsf{f}_q\, m_j \Rightarrow 1 : N_2$ and if $m_j \notin \mathrm{dom}(q)$ then $q_j \vdash \mathsf{f}_q\, \overline{m}_j \Rightarrow q_j(k) \coloneqq 1 : N_2$. Thus $q_j \vdash \mathsf{IsZero}\, (\mathsf{f}_q\, t.1) \Rightarrow^* N_0$ and by Lemma 4.1, $q_j \Vdash \mathsf{IsZero}\, (\mathsf{f}\, t.1) = N_0$. From $\Vdash t.2 : \mathsf{IsZero}\, (\mathsf{f}\, t.1)$ by monotonicity and Lemma 3.24 we have $q_j \Vdash t.2 : N_0$ which is impossible. If on the other hand $m_j \notin \mathrm{dom}(q_j)$ then since $q_j \lhd q_j(m_j, 0), q_j(m_j, 1)$ we can apply the above argument with $q_j(m_j, 1)$ instead of q_j. ◀

▶ **Lemma 5.6.** $\Vdash \mathsf{mw} : \neg\mathrm{MP}$

Proof. Assume $p \Vdash t : \mathrm{MP}$ for some p and t. Thus whenever $q \leqslant p$ and $q \Vdash u : N \to N_2$ then $q \Vdash t\, u : \neg\neg(\Sigma(x : N)\, \mathsf{IsZero}\, (u\, x)) \to (\Sigma(x : N)\, \mathsf{IsZero}\, (u\, x))$. But we have $q \Vdash \mathsf{f}_q : N \to N_2$ by Lemma 5.3. Hence $q \Vdash t\, \mathsf{f}_q : \neg\neg(\Sigma(x : N) \mathsf{IsZero}\, (\mathsf{f}_q\, x)) \to (\Sigma(x : N) \mathsf{IsZero}\, (\mathsf{f}_q\, x))$. But $q \Vdash \mathsf{w}_q : \neg\neg(\Sigma(x : N) \mathsf{IsZero}\, (\mathsf{f}_q\, x))$ by Lemma 5.4. Thus $q \Vdash (t\, \mathsf{f}_q)\, \mathsf{w}_q : \Sigma(x : N)\, \mathsf{IsZero}\, (\mathsf{f}_q\, x)$ which is impossible by Lemma 5.5. ◀

We have then the following result.

▶ **Theorem 5.7.** *There is a consistent extension of* MLTT *where* ¬MP *is derivable.*

Acknowledgements. We thank Simon Huber, Thomas Streicher, Martin Escardo and Chuangjie Xu. We also thank the reviewers for providing valuable comments which helped improve the manuscript.

References

1. Andreas Abel and Gabriel Scherer. On irrelevance and algorithmic equality in predicative type theory. *Logical Methods in Computer Science*, 8(1):1–36, 2012. `doi:10.2168/LMCS-8(1:29)2012`.
2. Peter Aczel. On relating type theories and set theories. In Thorsten Altenkirch, Bernhard Reus, and Wolfgang Naraschewski, editors, *Types for Proofs and Programs: International Workshop, TYPES' 98 Kloster Irsee, Germany, March 27–31, 1998 Selected Papers*, pages 1–18, Berlin, Heidelberg, 1999. Springer Berlin Heidelberg.
3. Errett Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, 1967.
4. L. E. J. Brouwer. Essentially negative properties. In A. Heyting, editor, *Collected Works*, volume I, pages 478 – 479. North-Holland, 1975.
5. Paul J Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, 50(6):1143–1148, 12 1963.
6. Robert L. Constable and Scott Fraser Smith. Partial objects in constructive type theory. In *Proceedings of Second IEEE Symposium on Logic in Computer Science*, pages 183–193, 1987.
7. Thierry Coquand and Guilhem Jaber. A note on forcing and type theory. *Fundamenta Informaticae*, 100(1-4):43–52, 2010.
8. Peter Dybjer. A general formulation of simultaneous inductive-recursive definitions in type theory. *The Journal of Symbolic Logic*, 65(2):525–549, 2000.
9. Martin Hofmann and Thomas Streicher. Lifting grothendieck universes. unpublished note, publication year unknown. URL: `http://www.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf`.

10 J.M.E. Hyland and C.-H.L. Ong. Modified realizability toposes and strong normalization proofs (extended abstract). In *Typed Lambda Calculi and Applications, LNCS 664*, pages 179–194. Springer-Verlag, 1993.

11 Alexei Kopylov and Aleksey Nogin. Markov's principle for propositional type theory. In Laurent Fribourg, editor, *Computer Science Logic: 15th International Workshop, CSL 2001 10th Annual Conference of the EACSL Paris, France, September 10–13, 2001, Proceedings*, pages 570–584, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

12 Georg Kreisel. Interpretation of analysis by means of constructive functionals of finite types. In A. Heyting, editor, *Constructivity in Mathematics*, pages 101–128. Amsterdam, North-Holland Pub. Co., 1959.

13 Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, 1980.

14 Maurice Margenstern. L'école constructive de Markov. *Revue d'histoire des mathématiques*, 1(2):271–305, 1995.

15 Per Martin-Löf. An intuitionistic theory of types. reprinted in Twenty-five years of constructive type theory, Oxford University Press, 1998, 127–172, 1972.

16 Ulf Norell. *Towards a practical programming language based on dependent type theory.* PhD thesis, Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Göteborg, Sweden, September 2007.

17 Thomas Streicher. Universes in toposes. In Laura Crosilla and Peter Schuster, editors, *From Sets and Types to Topology and Analysis: Towards practicable foundations for constructive mathematics*, pages 78–90. Oxford University Press, 2005.

18 William W. Tait. Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic*, 32(2):198–212, 1967.

19 Dirk van Dalen. An interpretation of intuitionistic analysis. *Annals of Mathematical Logic*, 13(1):1 – 43, 1978.

20 Andrew K. Wright and Matthias Felleisen. A syntactic approach to type soundness. *Information and Computation*, 115(1):38 – 94, 1994.

21 Chuangjie Xu and Martín Escardó. Universes in sheaf models. University of Birmingham, 2016.