Linear-Time Recognition of Map Graphs with Outerplanar Witness[*]

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Abstract

Map graphs generalize planar graphs and were introduced by Chen, Grigni and Papadimitriou [STOC 1998, J.ACM 2002]. They showed that the problem of recognizing map graphs is in NP by proving the existence of a planar witness graph W. Shortly after, Thorup [FOCS 1998] published a polynomial-time recognition algorithm for map graphs. However, the run time of this algorithm is estimated to be $\Omega(n^{120})$ for n-vertex graphs, and a full description of its details remains unpublished.

We give a new and purely combinatorial algorithm that decides whether a graph G is a map graph having an outerplanar witness W. This is a step towards a first combinatorial recognition algorithm for general map graphs. The algorithm runs in time and space $O(n + m)$. In contrast to Thorup's approach, it computes the witness graph W in the affirmative case.

1 Introduction

Consider the adjacency graph of the states of the USA, where two states are adjacent if their borders intersect. Since Arizona, Colorado, New Mexico and Utah meet pairwise at a single common point, the adjacency graph will not be planar; however, it will be a *map graph*. In (much) more detail, a *map* of a graph $G = (V, E)$ is a function $\mathcal{M}$ that maps each vertex $v \in V$ to a disc homeomorph $\mathcal{M}(v)$ on the sphere (the states) such that, for any two distinct vertices $v, w \in V$, the interiors of $\mathcal{M}(v)$ and $\mathcal{M}(w)$ are disjoint, and v and w are adjacent in G if and only if the boundaries of $\mathcal{M}(v)$ and $\mathcal{M}(w)$ intersect. A graph G is a *map graph* if a map of G exists.

By definition, map graphs contain and exceed the class of planar graphs. They have applications in graph drawing, circuit board design and topological inference problems [4]. Chen, Grigni and Papadimitriou [2] characterized map graphs as the *half-squares* of sufficiently small planar bipartite graphs called *witnesses* (we give precise definitions for both terms in the next section). This result allows, similar to Kuratowski's Theorem for planar graphs, to use purely combinatorial arguments for an object that has been originally defined by

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topological properties. Since such witnesses can always be chosen small in size ($O(n)$ vertices for map graphs on n vertices), the recognition problem for map graphs is in NP. In 1998, Chen et al. therefore raised the question whether recognizing map graphs is in P.

This problem was resolved shortly after by Thorup [18], whose solution is based on a carefully designed topological treatment. However, a full version of the extended abstract [18] has, to the best of our knowledge, not yet appeared. The algorithm is complicated; its run time is not given explicitly, but estimated to be at least $\Omega(n^{120})$. Moreover, driven by topological arguments, the algorithm does not produce a witness if the graph is indeed a map graph, although a combinatorial description of this witness is at hand. In this view, an important question left open is whether there is a polynomial-time *certifying* algorithm in the sense of McConnell et al. [16], where a good candidate for a certificate would be the witness mentioned above.

Our Contribution.   We give a purely combinatorial recognition algorithm for map graphs that have an outerplanar witness (rather than a planar witness). Map graphs with an outerplanar witness are general enough that they can have unbounded treewidth; in particular, cliques of any size may belong to this class of graphs. Our algorithm runs in time and space $O(n + m)$ and is certifying. This is the first non-trivial step towards a combinatorial and efficient recognition algorithm for general map graphs. Although the restriction to outerplanar witnesses is somewhat specific compared to the general case of planar witnesses, we will show structural properties for certain classes beyond (e.g. for $K_{2,k}$-free witnesses, and for graphs with small separators), that might be important for solving the general case.

We remark that the main algorithmic task is to compute a witness W, or to decide that none exists. Creating a map from W is a simple task that can be accomplished in linear time [2]. The crucial part of computing a witness W is that we know only a *subset* of the vertices of W; we need to do non-trivial algorithmic work in order to compute the remaining vertices of W. This is the reason why recognizing graphs that are *half-squares* of planar graphs is more challenging than recognizing graphs that are squares of planar graphs [15].

Related Work.   By definition, planar graphs are an important subclass of map graphs, and planar graphs have been known since the 1970s to be recognizable in time $O(n)$ [12]. Nowadays, several other linear-time algorithms for planar graph recognition exist , and so it is natural to ask whether they can be generalized to the much wider class of map graphs. Let a *d-map* graph be a map graph that has a witness in which every intersection point has at most d neighbors (states). The planar graphs are exactly the map graphs for which at most three states meet at each single point; thus, by the well-known linear-time recognition algorithms for planar graphs, 3-map graphs can be recognized in $O(n)$ time.

An intricate cubic-time recognition algorithm for a subclass of 4-map graphs was given by Chen et al. [3]; here, the 4-map graphs are required to be *hole-free*, meaning that there is at most one connected region of the plane that is not covered by states or borders. However, even efficiently recognizing general 4-map graphs in polynomial time remains an open problem; Thorup's algorithm does not necessarily give an embedding minimizing the maximum degree of the intersection points, so it cannot be used to recognize 4-map graphs.

Another motivation for *d*-map graphs is the study of *1-planar* graphs, which are the graphs that can be embedded in the plane such that each edge crosses at most one other edge. Recognizing 1-planar graphs is NP-complete [10, 14]. Brandenburg [1] characterizes "fully triangulated" 1-planar graphs as hole-free 4-map graphs, which by Chen et al.'s algorithm [3] are hence efficiently recognizable. It would be interesting to know where exactly the recognition problem becomes NP-complete between these two graph classes.

Further interest stems from the parameterized complexity community: Generalizing earlier algorithms for problems on planar graphs, Demaine et al. [5] gave fixed-parameter algorithms for combinatorial optimization problems such as minimum dominating set in map graphs. Fomin et al. [9] gave PTAS's for optimization problems on map graphs; they later improved these to EPTAS's [8].

2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, and let $n := |V(G)|$ and $m := |E(G)|$. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G. For a subset $V' \subseteq V(G)$, let $G[V']$ denote the subgraph of G induced by V'. For a graph G, its *square* G^2 is the graph on vertex set $V(G)$ in which two vertices are adjacent if their distance in G is at most two.

Witnesses. A *witness* of a map graph $G = (V, E)$ is a bipartite planar graph $W = (V \uplus I, E_W)$ with $E_W \subseteq V \times I$ such that $W^2[V] = G$. The graph $W^2[V]$ is also called the *half-square* of W, as it is the square of W restricted to the side V of the bipartition. The vertices in I are called *intersection points*, the vertices in V *real vertices*. We say that a witness W is a *tree witness* if it is a tree; analogously, *outerplanar witnesses* are outerplanar and the usual witnesses, which are planar, are sometimes called *planar witnesses*.

▶ **Proposition 1** ([2]). *A graph G is a map graph if and only if it has a* witness. *If so, there is a witness with at most $3n - 6$ intersection points.*

A direct consequence of this result is that the recognition problem for map graphs is in NP. Let G be a map graph with witness $W = (V \uplus I, E_W)$. Throughout this paper, we assume, without loss of generality, that every intersection point in I has degree at least two.

Let G be a map graph with witness W and let $P = v_1, v_2, \ldots, v_k$ be a path in G. A path P_W in W *corresponds* to P if $P_W = v_1, x_1, v_2, x_2, \ldots, x_{k-1}, v_k$ such that x_i is an intersection point that is adjacent to v_i and v_{i+1}, for $i = 1, \ldots, k-1$. Observe that any path P in G has some corresponding path in W, and any induced path P in G has a corresponding induced path P' in W, as every chord of P' in the bipartite witness W would join an intersection point with a real vertex and therefore generate a chord of P. More generally, for a subgraph of G that is induced by some vertex subset $V' \subseteq V(G)$, we specify the corresponding part in a witness of G:

▶ **Definition 2.** Let G be a map graph with witness W and let $U \subseteq V(G)$. A vertex $w \in W$ is *touched* by U if either $w \in U$ or w is an intersection point with at least two neighbors in U. The *touched set* $T(U)$ of U is the set of all vertices in W touched by U. The *touched subgraph* of U is $W[T(U)]$.

By using half-squares, we can get back from an induced subgraph $W[U]$ of W for some $U \subseteq V(G) \cup I$ to the *original subgraph* $W^2[U \cap V(G)]$ in G. Clearly, $W[U]$ witnesses $W^2[U \cap V]$. We will often use the following observation.

▶ **Observation 3.** *For every $U \subseteq V$, $W[T(U)]$ is a witness of $G[U]$. Moreover, $G[U]$ is connected if and only if $W[T(U)]$ is connected.*

Outerplanar Graphs. The following characterizations of planar and outerplanar graphs in terms of forbidden minors are well-known.

▶ **Proposition 4** (Wagner [19])**.** *A graph is planar if and only if it neither contains a K_5-minor nor a $K_{3,3}$-minor.*

▶ **Proposition 5.** *A graph is outerplanar if and only if it neither contains a K_4-minor nor a $K_{2,3}$-minor.*

▶ **Proposition 6** (Sysło [17])**.** *A triangle-free graph is outerplanar if and only if it does not contain a $K_{2,3}$-minor.*

Connectivity and SPQR trees. A graph is *connected* if every two of its vertices are connected by a path; the maximal connected subgraphs of G are called *components* of G. A *separator S* of a graph G is a subset of V such that $G - S$ has more components than G. For an integer $c \in \mathbb{N}$, a connected graph is *c-connected* if it either has at most c vertices or removing any set of less than c vertices leaves a connected subgraph. A 2-connected resp. 3-connected graph is sometimes called *biconnected* resp. *triconnected*.

For a graph G, an *SPQR tree* [6, 7] is a tree T for which each node $x \in V(T)$ has an associated multigraph G_x, called *skeleton* of x, and one of the following four types:

- S-node: then G_x is a cycle on at least three vertices.
- P-node: then G_x is a multigraph with two vertices and at least three edges.
- Q-node: then G_x is a multigraph with two vertices and two parallel edges.
- R-node: then G_x is a 3-connected graph.

Each edge xy between two nodes of T is associated with two directed *virtual* edges, one in G_x and one in G_y. Each edge in G_x can be virtual for at most one edge of T. All edges of S-, P- and R-nodes are virtual for some edge of T, and we simply call them *virtual edges*. An edge that is not virtual for any edge of T is *real*. Only skeletons of Q-nodes contain real edges and every Q-node skeleton contains exactly one real edge.

An SPQR tree T represents a biconnected graph G_T, formed as follows. Whenever an edge $xy \in E(T)$ associates the virtual edge of G_x with the virtual edge of G_y, form a larger graph as the *2-clique-sum* of G_x and G_y: We identify the endpoints of the virtual edge of G_x with that of G_y, and then delete the resulting edge. Applying this step to each edge of T (in any order) produces the graph G_T.

We assume throughout that T is *minimal*, which implies that its S- and P-nodes are pairwise non-adjacent. Under this assumption, T is uniquely determined from G. The graphs G_x associated with the nodes of T are called the *triconnected components* of G.

While the above definition coincides with the classical definition of SPQR trees, it is often more convenient to omit the Q-nodes from the tree as they carry little information. To this end, we simply remove each Q-node and replace the corresponding virtual edge in the skeleton of the neighboring node by a real edge. In the following we will use this modified version of SPQR trees.

3 Reduction along Small Separators

Clearly, every separator S of W that contains only vertices of $V(G)$ and for which at least two components of $W - S$ contain vertices in $V(G)$ is also a separator in G, as no edge that is generated by the half-square can cross S.

▶ **Lemma 7.** *Let G be a map graph with witness W and let $S \subseteq V(G)$. Then $C \mapsto W[T(C)]$ is a bijection from the vertex sets C of the components of $G - S$ to the components of $W - S$ that contain a vertex of $V(G)$. In particular, every separator S of G is a separator of W and,*

conversely, every separator $S \subseteq V(G)$ of W such that each component of $W - S$ contains a vertex of V is a separator of G.

Proof. If S is not a separator of G or of W, the statement follows from Observation 3. Hence, assume that S separates both G and W. Let A and B be the vertex sets of two arbitrary components of $G - S$. By Observation 3, the touched subgraphs $W[T(A)]$ and $W[T(B)]$ are connected in $W - S$. Assume to the contrary that some vertices $a \in A$ and $b \in B$ are contained in the same component of $W - S$. Then $W - S$ contains a shortest path from a to b, whose original subgraph in G must be a path from a to b on the same real vertices, i.e., disjoint from S. This contradicts that A and B are different components of $G - S$; hence, the components of $G - S$ partition V in exactly the same way as the components of $W - S$. In order to show that every $W[T(C)]$ is a component of $W - S$, it remains to prove that no intersection point is contained in two touched subgraphs $W[T(A)]$ and $W[T(B)]$. However, in that case, A and B would be connected in $G - S$. ◄

▶ **Lemma 8.** *A map graph G has a planar (outerplanar, tree) witness if and only if all of its biconnected components have planar (outerplanar, tree) witnesses, respectively.*

Proof of Lemma 8. Assume G has a planar (outerplanar, tree) witness and let C be the vertex set of any biconnected component of G. Then $W[T(U)]$ is a planar (outerplanar, tree) witness for $G[C]$ by Observation 3, as trees, planar and outerplanar graphs are closed under taking induced connected subgraphs.

If, on the other hand, each biconnected component of G has a planar (outerplanar, tree) witness, we can identify these witnesses along the cutvertices of G, obtaining a planar (outerplanar, tree) witness of G. ◄

We will thus assume that G is biconnected throughout the paper. Lemma 8 can be generalized to separators of size two as follows (a similar generalization exists for separators of size three). Consider a separator $S = \{u, v\}$ of size two in a biconnected graph. An *S-bridge* is either the edge uv, or the graph that is obtained from a component C of $G - S$ by adding the edges of G that join C with S, as well as their endpoints.

▶ **Lemma 9.** *Let G be a biconnected map graph that is not triconnected, and let $S = \{u, v\}$ be a separator of G. If uv is an edge of G, let $G' = G[C \cup S]$ for some component C of $G - S$, otherwise let $G' = (V', E')$ be the graph obtained from G by contracting some S-bridge B of G to a single edge. Then G' is a map graph, and any witness of G contains some witness of G' as a minor.*

Proof. If uv is an edge of G, then G' is an induced subgraph of G, and is therefore a map graph. Hence, any witness of G' is an induced subgraph of some witness of G, and so in this case the statement of the lemma holds.

Now assume that G does not contain the edge S. Then there exists a shortest path P in B connecting u and v. Obtain the graph $G'' = G - (V(B) \setminus V(P))$. Then G'' is an induced subgraph of G and hence is a map graph. Hence, any witness of G'' is an induced subgraph of some witness of G. Now contract the path P (whose interior vertices have degree 2) to a single edge, and call the resulting graph G'. Then any witness of G'' contains a path P'' realizing the path P. The internal vertices of P'' all have degree 2, and so contracting this path P'' to a path of length 2 yields a corresponding witness for G'. Hence any witness of G contains as a minor a witness of G'', which in turn contains as a minor a witness of G'. ◄

Lemma 9 will allow us to reduce along separators of size two.

4   Map Graphs with a Tree Witness

We characterize the map graphs that admit a tree witness. The characterization implies immediately a linear-time recognition algorithm for such graphs.

▶ **Lemma 10.** *A biconnected map graph has a tree witness if and only if it is a clique.*

Proof. Clearly a clique has a tree witness (the star). Conversely, assume that G is a biconnected graph with tree witness W and assume that G is not a clique. Then W is not a star, and it hence has two adjacent non-leaf vertices u and v. Since intersection points are pairwise non-adjacent, one of them, without loss of generality v, is not an intersection point. Then v is a cutvertex in W and, by Lemma 7, a cutvertex in G. This contradicts the assumption that G is biconnected. ◀

Lemma 8 and Lemma 10 immediately imply the following characterization of map graphs with a tree witness.

▶ **Theorem 11.** *A map graph has a tree witness if and only if each of its biconnected components is a clique.*

▶ **Corollary 12.** *Map graphs with a tree witness can be recognized in $O(n + m)$ time.*

5   Map Graphs with an Outerplanar Witness

In this section we study the problem of recognizing map graphs with an outerplanar witness. Due to Lemma 8, we can assume that the input graph G is biconnected. As bipartite planar graphs are triangle-free, we know with Proposition 6 that G has an outerplanar witness if and only if G has a $K_{2,3}$-minor free witness. Thus, all of the following proofs work for recognizing map graphs admitting witnesses that are $K_{2,3}$-minor free.

The next result states that triconnected map graphs G have witnesses with a very simple structure. For $k \geq 2$, a set of paths $\Pi_1, \ldots, \Pi_k$ in a witness $W = (V(G) \cup I, E_W)$ of G is *internally $V(G)$-disjoint* if no two paths Π_i and Π_j share an internal vertex in $V(G)$.

▶ **Lemma 13.** *For $k \geq 3$, a k-connected map graph G has a $K_{2,k}$-minor free witness if and only if it is a clique.*

Proof. If G is a clique, it has a tree witness that is a star whose center is an intersection point, and this witness is $K_{2,k}$-minor free. If G is not a clique, two vertices, say $u, v \in V(G)$, are not adjacent; let W be a witness of G. Since G is k-connected, G contains k internally vertex-disjoint paths from u to v. Let $P_1, \ldots, P_k$ denote such internally vertex-disjoint uv-paths of minimum total length; in particular, each of the P_i is an induced path. Denote by u_i the neighbor of u in P_i for $i = 1, \ldots, k$; see Fig. 1a for an example for $k = 3$. Let Π_i denote a path in W corresponding to P_i for $i = 1, \ldots, k$. Clearly the Π_i are internally $V(G)$-disjoint and each of them is an induced path. For a path Π containing vertices a and b, let $\Pi[a, b]$ denote the subpath of Π from a to b. Let $A = \bigcup_{i=1}^{k} V(\Pi_i[u, u_i]) \setminus \{u_1, \ldots, u_k\}$ and $B = \bigcup_{i=1}^{k} V(\Pi_i[u_i, v]) \setminus \{u_1, \ldots, u_k\}$; see Fig. 1b. Note that A consists exactly of u and the neighbors of u on the paths Π_i.

We claim that
 (i) $W[A]$ and $W[B]$ are connected,
 (ii) $A \cap B = \emptyset$, and
 (iii) each vertex u_i has a neighbor in A and a neighbor in B.
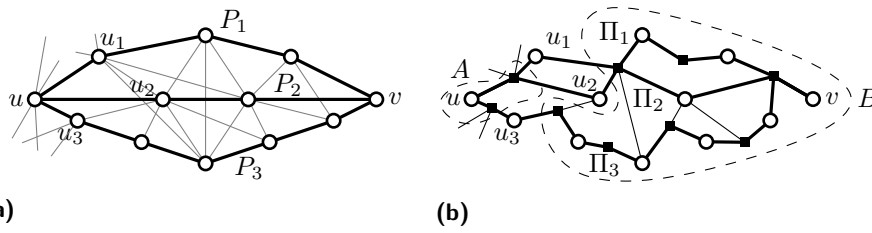
Figure 1 Illustration for the proof of Lemma 13 for $k = 3$. Vertices of G are empty disks, intersection points are small black squares.
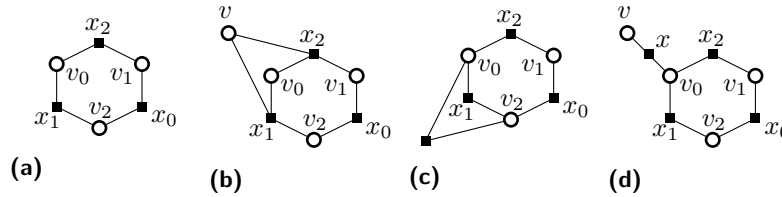
Figure 2 Illustration of the cases in the proof of Lemma 14.

Assume the claim holds and consider the graph $W' := W[A \cup B \cup \{u_1, \ldots, u_k\}]$. Since $W[A] \subseteq W'$ and $W[B'] \subseteq W'$ are connected (Claim (i)) and disjoint (Claim (ii)), we can contract these subgraphs into distinct vertices v_A and v_B, respectively. By Claim (iii), it follows that each of the u_i is adjacent to both v_A and v_B. Omitting a possible edge $v_A v_B$ yields a $K_{2,k}$-minor in W.

We now prove the claim. Statements (i) and (iii) follow immediately from the definitions of A and B via the paths Π_i. For (ii), assume that $A \cap B \neq \emptyset$ and let $x \in A \cap B$. Since the paths $\Pi_1, \ldots, \Pi_k$ are internally $V(G)$-disjoint and each of them is induced, x must be an intersection point. Since $x \in A$, x is adjacent to u. Since $x \in B$, it follows that x is adjacent to a vertex $w \in V(P_i) \cap B$ for some $i \in \{1, \ldots, k\}$, say without loss of generality $w \in V(P_1) \cap B$. Since x is adjacent to u and w, G contains the edge uw. Moreover, $w \neq u_1$, since $u_1 \notin B$. But then replacing the subpath from u to w in P_1, which contains u_1 in its interior, by the edge uw yields a shorter k-tuple of internally vertex-disjoint uv-paths in G. This contradicts the minimality of $P_1, \ldots, P_k$. ◀

In particular, a triconnected map graph with outerplanar witness, which is $K_{2,3}$-minor free, must be a clique. Hence, it suffices to investigate separators of size 2 in G.

In general map graphs, every two adjacent vertices have at least one neighboring intersection point in the witness, according to the definition of half-squares. Let G be a map graph with an outerplanar witness H. The intuition for the next lemma is that then every three vertices of a clique in G have a common neighboring intersection point in H. Note that this property is not true for arbitrary planar witnesses, as each of the pizza with crust, hamantash and riceball [2] contains three nations without any common intersection.

▶ **Lemma 14.** *Let G be a map graph with an outerplanar witness W and let v_0, v_1, v_2 be vertices of a clique of size at least 4. Then some intersection point in W is adjacent to v_0, v_1, v_2.*

Proof. Assume the contrary. Then there exist distinct intersection points x_0, x_1, x_2 such that x_i is adjacent to v_{i+1}, v_{i+2} but not to v_i, where indices are taken modulo 3; see Fig. 2a.

Now consider a vertex v of the clique that is distinct from the v_i. There is a path of length two from v to each of the v_i in W. If v is adjacent to two (or more) of the x_i, we

(a)   **(b)**
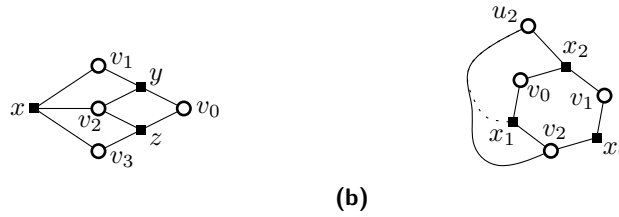
🟨 **Figure 3** Illustration of the proofs of Lemma 16(a) and Lemma 17(b).

immediately have a $K_{2,3}$-minor (Fig. 2b with branch vertices $\{x_1, x_2\}$); likewise, if there is an intersection point distinct from the x_i adjacent to two of the v_i ($\{v_0, v_2\}$ in Fig. 2c). It follows that v must reach one of the v_i, without loss of generality v_0, via an intersection point x distinct from the x_i, and x_i is not adjacent to v_1 and v_2 (Fig. 2d). But now, to reach v_1 and v_2, v either has to be adjacent to x_0, or it must use a new intersection point y adjacent to v_1 or v_2. In both cases, we obtain a $K_{2,3}$-minor. ◄

The proof of Lemma 14 shows that the bound "4" on the clique size is as small as possible.

▶ **Definition 15.** A clique C in a map graph G with witness W is *represented by an intersection point* if W contains at least one intersection point whose neighborhood is $V(C)$.

Cliques that are represented by exactly one intersection point are called "pizzas" by Chen et al. [2]. We now show that in outerplanar witnesses all cliques of size at least 4 must be represented in this way, thus significantly reducing the possible representations.

▶ **Lemma 16.** *Let G be a map graph with outerplanar witness W. Each maximal clique C of size at least 4 is represented by an intersection point.*

Proof. We first show that W contains an intersection point x that is adjacent to all vertices of C. This readily implies that x has no other neighbor, as any such neighbor would contradict the maximality of C. Let x be an intersection point in W with a maximum number of neighbors in C and assume to the contrary that C contains a vertex v_0 that is not adjacent to x. By Lemma 14, x has at least three neighbors v_1, v_2, v_3 in C. Let $V' = \{v_0, v_1, v_2, v_3\}$.

If there were a single intersection point $y \neq x$ adjacent to $\{v_1, v_2, v_3\}$, this would result in a $K_{2,3}$ with branch vertices x and y. This implies that any intersection point different from x can be adjacent to at most three of the vertices in V' (omitting at least one of $\{v_1, v_2, v_3\}$). On the other hand, by Lemma 14, for any such subset a corresponding intersection point exists. Thus, there exist intersection points $y \neq x$ and $z \neq x$ with $N(y) \cap V' = \{v_0, v_1, v_2\}$ and $N(z) \cap V' = \{v_0, v_2, v_3\}$; see Fig. 3a. Contracting the two edges yv_0 and zv_0 yields a $K_{2,3}$-minor in W. This contradicts outerplanarity.

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According to the definition of witnesses, maximal cliques of size 2 must also be represented by intersection points. Thus, the only cliques for which the representation is unclear are maximal cliques of size 3. In the following we show that cliques that cannot be represented by an intersection point in an outerplanar witness induce a special structure. Namely, any two of its vertices form a separator, and we further describe the way in which these separators decompose the graph.

▶ **Lemma 17.** *Let G be a map graph with outerplanar witness and let W be an outerplanar witness of G that maximizes the number of maximal cliques of size 3 that are represented by*
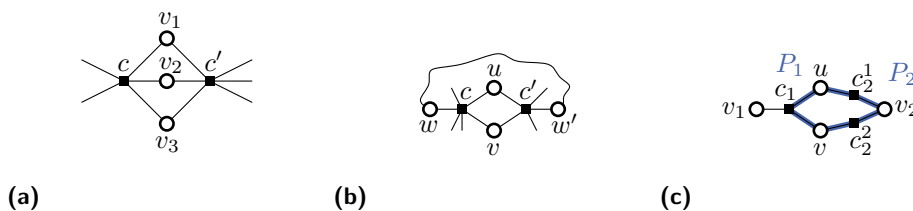
Figure 4 Illustration for the proof of Lemma 18. The wiggly line in (b) indicates an arbitrary path from w to w' avoiding u and v. The paths P_1 and P_2 are drawn bold (and blue) in (c).

intersection points. Let v_0 and v_1 be any two vertices of a maximal clique $C = \{v_0, v_1, v_2\}$ that is not represented by an intersection point. Then $\{v_0, v_1\}$ is a separator in G that separates v_2 from every other maximal clique of G containing v_0 and v_1.

Proof. Since C is not represented by an intersection point, there exist witness points x_0, x_1, x_2 such that x_i is adjacent to v_{i+1} and v_{i+2} but not to v_i (indices modulo 3). The cycle containing the x_i and v_i does not contain any vertex inside because of outerplanarity of W; in addition, any interior edge would contradict that C is not represented by an intersection point. Thus the x_i and v_i form the boundary of a face of W; see Fig. 3b.

If some x_i has degree 2, we can add the edge $x_i v_i$, which would result in an intersection point for the clique, contradicting the maximality of W. It follows that each x_i is adjacent to some $u_i \in V(G) - C$. We claim that $\{v_0, v_1\}$ separates u_2 from v_2 in W. Otherwise, there exists a path from u_2 to v_2 in $W - v_0 - v_1$. Together with the cycle formed by the v_i and x_i, this yields a $K_{2,3}$-minor, contradicting the outerplanarity. Thus, $\{v_0, v_1\}$ is also a separator in G that separates u_2 and v_2. ◀

▶ **Lemma 18.** *For a map graph G with outerplanar witness, the following statements hold.*
(i) *Any two maximal cliques of G share at most two vertices.*
(ii) *Any two vertices that are shared by two maximal cliques of G form a separator of G separating these cliques.*
(iii) *Any two vertices are shared by at most two maximal cliques.*

Proof. For (i), assume that C, C' are maximal cliques sharing vertices v_1, v_2, v_3. Since C, C' are distinct, they have size at least 4. By Lemma 16, they are represented by distinct intersection points c, c'; see Fig. 4a. Then v_1, v_2, v_3, c, c' induce a $K_{2,3}$; a contradiction.

For (ii), let $\{u, v\}$ be two vertices that are shared by two maximal cliques C and C'. Consider an outerplanar witness W of G that maximizes the number of cliques of size 3 that are represented by intersection points. If C and C' are realized by intersection points c and c', respectively, consider w and w' in $C \setminus C'$ and $C' \setminus C$, respectively. A path between these two vertices avoiding u and v yields a $K_{2,3}$-minor with branch vertices c and c', contradicting outerplanarity; see Fig. 4b. Hence, assume that one of the cliques is not realized as an intersection point. With Lemma 16, this clique has at most three vertices and the statement follows from Lemma 17.

For (iii), assume that two vertices u and v are shared by at least three maximal cliques C_0, C_1 and C_2. Note that each C_i has size at least three, as otherwise the C_i would not be distinct. According to (ii), every C_i contains a vertex v_i that is not in $C_{i+1} \cup C_{i+2}$ (indices taken modulo 3). If C_i is represented by an intersection point c_i, then let P_i denote the path $u c_i v$ in W (path P_1 in Fig. 4c). If C_i is not represented by an

intersection point, then C_i has size 3, and W contains a path uc_i^1, v_i, c_i^2, v, where c_i^1 and c_i^2 are intersection points. We define P_i to be this path (path P_2 in Fig. 4c). Paths P_i and P_j for $i \neq j$ are internally disjoint by the definition of the v_i, and so three internally disjoint paths from u to v in W yield a $K_{2,3}$-minor. This contradicts the outerplanarity of W. ◄

▶ **Lemma 19.** *Let G be a biconnected map graph with an outerplanar witness W. Every separator $S = \{u, v\}$ of G of size two separates exactly two components.*

Proof. Assume to the contrary that $G - S$ contains at least three components C_i, $1 \leq i \leq 3$. By Lemma 7, the touched subgraphs $W[T(V(C_i))]$ are different components of $W - S$. For every i, there is a path P_i in G from u to v that contains a vertex of C_i as inner vertex, since S is minimal in G. In W, each P_i corresponds to a path from u to v that contains a vertex of $W[T(V(C_i))]$ as inner vertex (this may be either an intersection point or a real vertex). Since each P_i has length at least two, W contains a $K_{2,3}$-minor with branch vertices u, v. ◄

5.1 Structural Properties of Map Graphs with Outerplanar Witness

To obtain an efficient recognition algorithm for map graphs with outerplanar witness, two things remain to be done. First, we need to better understand the structure of those maximal cliques for which the representation in the witness is not already decided by the previous results. Second, we need to find a way to quickly enumerate all the relevant cliques in order to decide upon their representation in the witness.

As we have seen, the maximal cliques for which the representation cannot be an intersection point induce separating pairs in the input graph. This, together with the fact that certainly all cliques of size at least 4 belong to a single triconnected component of the input graph motivates the study of the triconnected components of the input graph. Essentially, we show:

1. Every maximal clique of size at least three shows up as a triconnected component of G.
2. A description of the maximal cliques of size three that cannot be represented by an intersection point.

The first item allows us to quickly compute all maximal cliques by exploiting the SPQR tree, which can be computed in linear time [11], rather then by a maximal clique enumeration algorithm, which might be much slower. The second item is used to determine the correct intersection points for all maximal cliques.

The following corollary follows immediately from applying Lemma 9 along the recursive definition of SPQR trees. Afterwards we derive further structural results on the triconnected components of a map graphs with outerplanar witnesses.

▶ **Corollary 20.** *Let G be a biconnected map graph with an outerplanar witness. Then each skeleton of the SPQR tree of G is a map graph with an outerplanar witness.*

▶ **Lemma 21.** *Let G be a biconnected map graph with an outerplanar witness. Then the SPQR tree of G satisfies the following properties:*
 (i) *Every P-node skeleton consists of three parallel edges of which one is a real edge.*
(ii) *Every R-node skeleton is a clique.*

Proof. For (i), observe that, according to Lemma 19, there are exactly two components in $G - S$ for every separator $S = \{u, v\}$ of G of size two. Thus, every parallel P-node in the SPQR tree has at most three parallel edges (and at least three by definition of SPQR trees): two virtual ones and one edge from G.

For (ii), observe that the skeleton of an R-node is a triconnected graph. According to Corollary 20, this skeleton is a map graph with an outerplanar witness. Applying Lemma 13 with $k = 3$ implies that the skeleton is a clique. ◄

In fact, not only is every R-node skeleton a clique, but any such clique is a subgraph of G.

▶ **Lemma 22.** *Let G be a map graph with outerplanar witness. Then every R-node skeleton is a maximal clique that is a subgraph of G.*

Proof of Lemma 22. Consider an R-node skeleton S, which is a clique by Lemma 21(ii), and let uv be an edge of S that is not in G (thus, a virtual edge). Let G' be the graph that is obtained from G by contracting the subgraph corresponding to each remaining virtual edge into a single edge. Let G'' be the subgraph obtained from G' by removing all vertices in the subgraph that corresponds to the virtual edge uv, except for u, v and a shortest path between them. The graph G' is a map graph with outerplanar witness by Lemma 9, and G'' is a map graph with outerplanar witness by Observation 3, as G'' is an induced subgraph of G'. Thus, G'' contains the two cliques with vertex sets $V_1 = V(S) - \{u\}$ and $V_2 = V(S) - \{v\}$, which are maximal, as uv is not in G''. But then $|V_1 \cap V_2| \geq 2$, since $|V(S)| \geq 4$ and $|V_1 \cap V_2| \leq 2$ by Lemma 18(i). Hence $|V_1 \cap V_2| = 2$, $|S| = 4$ and thus, G'' is the graph obtained from K_4 by replacing an edge with a path of length at least two. This contradicts that, according to Lemma 18(ii), $V_1 \cap V_2$ is a separator of G''. ◄

▶ **Lemma 23.** *Let G be a biconnected map graph with outerplanar witness and let $C = \{u, v, w\}$ be a maximal clique in G. Then there is an S-node skeleton with vertex set $\{u, v, w\}$.*

Proof. By definition, the induced subgraph $G[C]$ is triconnected, hence there is a skeleton of a node q in the SPQR tree of G that contains all three vertices of C. However, q cannot be a P-node (as it contains only two vertices) and it cannot be an R-node, whose skeletons are well-known to contain at least four vertices. Hence, it must be an S-node. The skeleton of q cannot contain any other vertex than those in C, as it then would not be a cycle. ◄

It follows from Lemma 22 and Lemma 23 that we find all maximal cliques by considering the skeletons of the SPQR tree. In particular, this allows us to enumerate all maximal cliques in linear time. Recall that by Lemma 16 each maximal clique of size at least 4, which are precisely the cliques corresponding to R-nodes, must be represented by an intersection point. It remains to understand which maximal cliques of size three may not be represented by an intersection point.

▶ **Lemma 24.** *Let G be a biconnected map graph with outerplanar witness, let $C = \{u, v, w\}$ be a maximal clique in G, and let W be a witness that maximizes the number of cliques of size 3 represented by an intersection point. Then C is not represented by an intersection point in W if and only if the skeleton of the S-node corresponding to C has 3 virtual edges.*

Proof. Let S be the skeleton of the S-node on vertex set $\{u, v, w\}$, which exists by Lemma 23. Suppose, for the sake of contradiction, that C is not represented by an intersection point but one of the edges of S, say uv, is not virtual. Then the edges of C are represented by three distinct intersection points x_1, x_2, x_3, which form a cycle K together with the vertices of $\{u, v, w\}$ in W (cf. Fig. 2a). Assume, without loss of generality, that x_1 is adjacent to u and v. Since the edge uv is not virtual, the separator $\{u, v\}$ has exactly two split components of which one is an edge. As the other split-component is the one containing w, it follows that C is the only maximal clique that contains uv. Hence, x_1 has degree two in W.

We claim that K bounds a face of W in any outerplanar embedding of W. To justify the claim, observe that there cannot be a vertex embedded inside K due to W being outerplanar, and an edge embedded inside K would contradict the assumption that C is not represented by an intersection point. Thus, the claim holds, and the interior of K is empty.

We can then insert the edge $x_1 w$ to W, resulting in an outerplanar witness of G where C is represented by an intersection point. This, however, contradicts the maximality of W.

For proving sufficiency, assume to the contrary that C is represented by an intersection point c, but all edges of S are virtual. Then, the witness W contains for each virtual edge ab a path from a to b that avoids c and all inner vertices from each subgraph corresponding to a virtual edge different from ab. Thus, we obtain three internally disjoint paths connecting u to v, v to w and w to u, respectively. These three paths are all vertex-disjoint from $\{c\}$, and thus gives a K_4-minor in W with branching vertices $\{c, u, v, w\}$. This, however, contradicts the outerplanarity of W.   ◀

5.2   Recognition Algorithm

Based on our structural observations, we give a linear-time algorithm for recognizing map graphs that admit an outerplanar witness, Algorithm 1.

Algorithm 1 Linear-Time Recognition Algorithm for Map Graphs with Outerplanar Witness

Input: A graph G.
Output: An outerplanar witness W of G if G is a map graph, "no" otherwise.
 1: Create a candidate bipartite graph W; let V be one side of the bipartition of W.
 2: **for** each biconnected component H_i of G **do**
 3:     Compute an SPQR tree T_i of H_i.
 4:     **for** each R-node R of T_i **do**
 5:        **if** the skeleton of R is not a clique of non-virtual edges **return** "no"
 6:        Add an intersection point p_R with neighborhood $V(R)$ to W.
 7:     **for** each S-node S of T_i whose skeleton is a clique of size 3 with some real edge **do**
 8:        Add an intersection point p_S with neighborhood $V(S)$ to W.
 9: **for** each edge $e = uv$ in G that is not yet represented by an intersection point **do**
10:     Add an intersection point p_{uv} of degree 2 with neighborhood $\{u, v\}$ to W.
11: Test outerplanarity of W: **if** "yes", **return** W, **else return** "no".

The algorithm takes as input an arbitrary graph G. First, it decomposes G into its biconnected components $H_1, \ldots, H_t$. We know that G is a map graph with outerplanar witness if and only if each H_i is a map graph with outerplanar witness (see Lemma 8).

We seek to construct a bipartite witness candidate W of G as follows. Let the vertices of G be one side of the bipartition of W. For each H_i, compute the decomposition into its triconnected components, i.e., its SPQR tree T_i in linear time [11, 13]. For each R-node of T_i, check whether it is a clique of non-virtual edges. If not, then reject the graph H_i (and hence G) as not being a map graph with outerplanar witness. Otherwise, add an intersection point to W that represents that clique.

For each S-node of T_i that forms a clique of size 3 and that has a non-virtual edge, add an intersection point to W representing the clique.

Finally, for each edge of G that is not yet represented by one of the previously constructed intersection points, add a separate intersection point of degree 2 to W representing exactly this edge. Let W be the resulting candidate witness graph. Test whether W is outerplanar.

If W is outerplanar, then output the outerplanar witness W, otherwise reject the input graph G as not being a map graph with outerplanar witness.

▶ **Theorem 25.** *Map graphs with outerplanar witness can be recognized in $O(n + m)$ time.*

Proof. It is not hard to see that the above algorithm can be implemented to run in $O(n+m)$ time. In the following we prove the correctness.

First, assume that the algorithm outputs a witness W in the end. We show that W is a witness of the input graph G. Note that all intersection points we create represent either cliques of various sizes of G, or they only represent a single edge (if they are added in the last step). Thus, $W^2[V(G)] \subseteq G$. On the other hand, the last step ensures that $W^2[V(G)] \supseteq G$, and thus we have $W^2[V(G)] = G$, which shows that indeed G is a map graph with outerplanar witness W.

Conversely, assume that the algorithm rejects G, although G is a map graph with outerplanar witness. Let W^* be an outerplanar witness that minimizes the number of cliques of size 3 that are not represented by an intersection point.

The are only two steps in the algorithm where G may be rejected. First, when an R-node skeleton is not a clique that is subgraph of G. But in this case, G is not a map graph with outerplanar witness by Lemma 22. Second, G may be rejected when W is found not to be outerplanar. In this case, we will give an isomorphism from W to an induced subgraph of W^* whose restriction to V is the identity. This contradicts that W^* is outerplanar.

It suffices to give the mapping for intersection points only, as every witness contains an identical set V of real vertices. Let $w \in W$ be an intersection point of degree at least 4. Then $N(w)$ is a clique of size at least 4 in G. The intersection point w was added due to an R-node clique of that size, and hence $N(w)$ is a maximal clique of size at least 4. By Lemma 16, any outerplanar witness contains an intersection point representing that clique; we thus find an image for w in W^*. Note that a second vertex with the same neighborhood is not created; this would imply the existence of a second R-node skeleton with the same vertex set, which is impossible since any two skeletons share at most two vertices.

Let $w \in W$ be an intersection point of degree 3. Then $N(w)$ is a maximal clique of size 3, and w was created due to an S-node skeleton that contained a non-virtual edge. By Lemma 24, $N(w)$ is represented by an intersection point in W^* as well, and we thus find an image of w in W^*. Again, any intersection point mapped to the same image would imply the existence of a second S-node skeleton with the same vertex set, which is not possible.

Finally, let $w \in W$ be an intersection point of degree 2, which must have been added in the last step, and let $N(w) = \{u, v\}$. Clearly, W^* must contain an intersection point x adjacent to u and v. If x has degree at least 4, then $N(x)$ is a clique of size at least 4, which must show up as an R-node of the SPQR tree. But then the edge uv was already represented in W by an intersection point corresponding to x and the algorithm would not have added w. Similarly, if x has degree 3, then $N(x)$ is a maximal 3-clique. Since it is represented by the intersection point x, it follows that the corresponding S-node contains a virtual edge by Lemma 24. But then, again, the algorithm would have added an intersection point to W that corresponds to x. Thus, the degree of x must be 2, and we can choose it as an image for w. Since all degree-2 intersection points inserted in the last step represent distinct edges of G, no two degree-2 intersection points of W are mapped to the same intersection point of W^*. Hence, we have found an isomorphism from W to a subgraph of W^*.   ◀

The correctness proof shows that the algorithm computes a smallest (with respect to subgraph inclusion) outerplanar witness, and that this witness is unique up to isomorphism.

6 Discussion

We gave an $O(n+m)$ time and space recognition algorithm for map graphs with an outerplanar witness. The algorithm is certifying. This result is a first step towards improving Thorup's recognition algorithm for map graphs with planar witness that requires time about $\Omega(n^{120})$.

For map graphs with outerplanar witness, Lemma 18 shows that any two maximal cliques of G intersect in at most two vertices. However, this property does not generalize, as for arbitrary map graphs any $k \geq 2$ maximal cliques can have intersections of unbounded size: namely, one can show that for every pair $k, \ell \in \mathbb{N}$ there exist map graphs $G_{k,\ell}$ with exactly k maximal cliques such that these cliques intersect in exactly ℓ vertices.

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