Algorithmic Complexity for the Realization of an Effective Subshift By a Sofic*

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Abstract

Realization of d-dimensional effective subshifts as projective sub-actions of $d + d'$-dimensional sofic subshifts for $d' \geq 1$ is now well known [6, 4, 2]. In this paper we are interested in qualitative aspects of this realization. We introduce a new topological conjugacy invariant for effective subshifts, the speed of convergence, in view to exhibit algorithmic properties of these subshifts in contrast to the usual framework that focuses on undecidable properties.

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Introduction

A d-dimensional subshift is a set of colorings of $\mathbb{Z}^d$ by a finite set of colors in which a set of forbidden patterns never appear. The simplest class, called subshifts of finite type, corresponds at finite sets of forbidden patterns. In dimension 2, they are equivalent to the usual notion of tilings introduced by Wang [16]. Applying a block map on a subshift of finite type, one obtains a sofic subshift which can be characterized, in dimension 1, by a set of forbidden patterns accepted by a finite automaton [17].

For multidimensional subshifts, we can consider their stability according to another dynamical operation: projective subaction which consists of restricting the configurations of a subshift to a sublattice of $\mathbb{Z}^d$. The smallest class stable under this operation which contains the class of sofic shifts is the set of effective subshifts defined by a set of forbidden patterns enumerated by a Turing machine. A consequence of the main result of [6] states that every d-dimensional effective subshift can be obtained via projective subaction of a $d+2$-dimensional sofic. This result was improved in [4, 2] to hold for $d+1$-dimensional sofics.

These three classes evoked are stable by conjugacy and underline links between dynamical characterization and computability property of their set of forbidden patterns. Other classes are exhibited in [1], using forbidden patterns recursively enumerated by Turing machine with oracle. In this article, we introduce new conjugacy invariant classes which subdivide the class of effective subshift based on the speed of convergence of the realization via projective subaction. In contrast to the usual framework that focuses on undecidable properties and their position relatively to some hierarchies [7, 15, 9, 10], the approach proposed here emphasizes the algorithmic properties of subshifts using time and space complexity.

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In [14], the authors characterize one-dimensional sofic subshifts obtained by a projective subaction of subshift of finite type. It appears a difference between certain types of sofic subshifts, according to whether their realization can be stable or unstable that is to say if a bounded strip around the central one is necessary to obtain the desired sofic subshift or whether there is no bounds which guarantee to the central row to be in the subshift. This approach is inspired by the notion of stable and unstable limit-set for cellular automata [12].

In this article, we would like to go beyond the dichotomy stable vs unstable realization and try to quantify this notion. We introduce the notion of speed of convergence of the realization of an effective subshift by projective subaction of a sofic. This is defined as the function which, for a given integer k, returns the width of the strip necessary to obtain the language of the effective subshift up to a word of size k in the central rows.

Given an effective subshift, we study the set of speeds of convergence which realizes it as projective subaction. Modulo an equivalence relation this set is invariant under conjugacy (Sections 1.3). In Section 2 we compare the general constructions of realization of an effective subshift given in [6, 2] and we propose a quicker construction if the effective subshift has a periodic point. Moreover we show that when the dimension of the sofic increase the convergence is quicker. These results give upper bounds for realization by sofic, but is also possible to obtain lower bounds (see Section 3). In Section 4 we present some examples of different classes which exhibit the optimality of the different previous results.

1 Definitions and first properties

1.1 Classes of subshifts

Subshifts. Let $\mathcal{A}$ be a finite alphabet, a *configuration* x is an element of $\mathcal{A}^{\mathbb{Z}^d}$. Let $\mathbb{U}$ be a finite subset of $\mathbb{Z}^d$, denote $x_{\mathbb{U}}$ the *restriction* of x to $\mathbb{U}$. A *d-dimensional pattern* of support $\mathbb{U}$ is an element $p \in \mathcal{A}^{\mathbb{U}}$. Denote by $\mathcal{A}^*$ the set of d-dimensional patterns and $p \in \mathcal{A}^{\mathbb{U}}$ *appears* in a configuration x, denoted by $p \sqsubset x$, if there exists $\mathbf{i} \in \mathbb{Z}^d$ such that $p = x_{\mathbf{i}+\mathbb{U}}$.

For the product topology, $\mathcal{A}^{\mathbb{Z}^d}$ is a compact metric space on which $\mathbb{Z}^d$ acts by translation via the shift map σ defined for all $\mathbf{i} \in \mathbb{Z}^d$ by $\sigma^{\mathbf{i}}(x)_{\mathbf{j}} = x_{\mathbf{i}+\mathbf{j}}$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{j} \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-dynamical system $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ is called the *fullshift* and a *subshift* is a σ-invariant closed subset of $\mathcal{A}^{\mathbb{Z}^d}$. Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift, define $\mathcal{L}(\mathbf{T}) = \{p \in \mathcal{A}^* : \exists x \in \mathbf{T} \text{ such that } p \sqsubset x\}$ the *language of* $\mathbf{T}$ and $\mathcal{L}_n(\mathbf{T}) = \{p \in \mathcal{A}^{[0,n-1]^d} : p \in \mathcal{L}(\mathbf{T})\}$ the *square language of size n.*

Finite type condition. Let F be a set of patterns, define the *subshift of forbidden patterns* F by $\mathbf{T}_F = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \in F, p \not\sqsubset x\right\}$. Every subshift can be defined in this way and this allows to define classes of subshifts according to the complexity of F. Let $\mathbf{T}$ be a subshift,

- $\mathbf{T}$ is a *subshift of finite type* if there exists a finite set of patterns F such that $\mathbf{T} = \mathbf{T}_F$;
- $\mathbf{T}$ is an *effective subshift* if there exists a recursively enumerable set of patterns F (that is to say enumerated by a Turing machine) such that $\mathbf{T} = \mathbf{T}_F$.

Factor. A *block map* is a continuous function $\pi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ such that $\pi \circ \sigma^{\mathbf{i}} = \sigma^{\mathbf{i}} \circ \pi$ for all $\mathbf{i} \in \mathbb{Z}^d$. Equivalently, there exists a local function $\overline{\pi} : \mathcal{A}^{\mathbb{U}} \longrightarrow \mathcal{B}$ where $\mathbb{U} \subset \mathbb{Z}^d$ is a finite set called *neighborhood* such that $\pi(x)_{\mathbf{i}} = \overline{\pi}(x_{\mathbf{i}+\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{i} \in \mathbb{Z}^d$.

Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $\pi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ a block map, then $\pi(\mathbf{T}) \subset \mathcal{B}^{\mathbb{Z}^d}$ is a subshift called *factor subshift* of $\mathbf{T}$ by π which is called the *factor map.* A subshift $\mathbf{T}$ is called *sofic* if there exists a subshift of finite type $\mathbf{T}_F$ and a factor map π such that $\mathbf{T} = \pi(\mathbf{T}_F)$. The factor map π can be considered letter to letter.

Two subshifts $\mathbf{T}$ and $\mathbf{T}'$ are *conjugate* if there exists a bijective factor map $\psi : \mathbf{T} \longrightarrow \mathbf{T}'$. The different classes of subshifts defined here (finite type, sofic and effective subshifts) are stable under conjugacy.

Projective subactions. Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $d' < d$, the *projective subdynamics* of $\mathbf{T}$ of dimension d' is the subshift $\mathbf{SA}_{d'}(\mathbf{T})$ where $\mathbf{SA}_{d'} : \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{A}^{\mathbb{Z}^{d'}}$ is defined by $\mathbf{SA}_{d'}(x) = x_{\mathbb{Z}^{d'}\times\{\mathbf{0}\}}$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$.

▶ **Theorem 1.** *[6, 2, 4] Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}^d}$ be an effective subshift, then the $d+1$-dimensional subshift $\widetilde{\Sigma} = \{x \in \mathcal{A}^{\mathbb{Z}^{d+1}} : \exists y \in \Sigma$ such that $x_{\mathbb{Z}^d\times\{i\}} = y$ for all $i \in \mathbb{Z}\}$ is sofic.*

In particular there exists a subshift of finite type $\mathbf{T} \subset \mathcal{B}^{\mathbb{Z}^{d+1}}$ and a factor map $\pi : \mathcal{B}^{\mathbb{Z}^{d+1}} \to \mathcal{A}^{\mathbb{Z}^{d+1}}$, which can be considered letter to letter, such that $\mathbf{SA}_d(\pi(\mathbf{T})) = \Sigma$.

1.2 Speed of convergence

Approximation row. Let F be a finite set of d-dimensional forbidden patterns on $\mathcal{B}$ and $d' < d$. Define $\mathbb{B}_n = \{k_{d'+1}\mathbf{e_{d'+1}}, \dots, k_d\mathbf{e_d} : (k_{d'+1}, \dots, k_d) \in [-n, n]^{d-d'}\}$ where $\mathbf{e_1}, \dots, \mathbf{e_d}$ are the canonical vectors of $\mathbb{Z}^d$ and denote $\mathrm{Proj}_{\mathbf{i}} : \mathcal{B}^{\mathbb{B}_n} \longrightarrow \mathcal{B}$ the projection according to the coordinates $\mathbf{i} \in \mathbb{B}_n$.

One considers the n-*approximation row of* $\mathbf{T}_F \subset (\mathcal{B}^{\mathbb{B}_n})^{\mathbb{Z}^{d'}}$ the d'-dimensional subshift of finite type defined by the finite condition where no patterns of F appears in the row of width n. Formally, it is defined by:

$$\mathbf{T}_F^{n,d\to d'} = \left\{x \in \left(\mathcal{B}^{\mathbb{B}_n}\right)^{\mathbb{Z}^{d'}} : \forall p \in F, p \not\sqsubset (\mathrm{Proj}_{\mathbf{j}}(x_{\mathbf{i}}))_{(\mathbf{i},\mathbf{j})\in\mathbb{Z}^{d'}\times\mathbb{B}_n}\right\}.$$

Let $\pi : \mathcal{B}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ be a factor map. One has $\mathbf{SA}_{d'}(\pi(\mathbf{T}_F)) = \bigcap_{n\in\mathbb{N}} \mathbf{SA}_{d'}\left(\pi\left(\mathbf{T}_F^{n,d\to d'}\right)\right)$ where for n sufficiently large $\mathbf{SA}_{d'}\left(\pi\left(\mathbf{T}_F^{n,d\to d'}\right)\right)$ denote the central row of $\pi\left(\mathbf{T}_F^{n,d\to d'}\right)$.

Speed of convergence. By definition of $\mathbf{T}_F^{n,d\to d'}$, if $u \in \mathcal{L}(\mathbf{SA}_{d'}(\pi(\mathbf{T}_F)))$, then $u \in \mathcal{L}\left(\mathbf{SA}_{d'}\left(\pi\left(\mathbf{T}_F^{n,d\to d'}\right)\right)\right)$. We want to quantify the reciprocal, that is to say given a k, find the smallest n such that $u \notin \mathcal{L}_k(\mathbf{SA}_{d'}(\pi(\mathbf{T}_F))) \Longrightarrow u \notin \mathcal{L}_k\left(\mathbf{SA}_{d'}\left(\pi(\mathbf{T}_F^{n,d\to d'})\right)\right)$. This allows to quantify when a word is forbidden by the local rules F in the approximation row. The *speed of convergence as sofic* of the cover $\mathbf{T}_F$ with the factor π is the following function:

$$\begin{array}{rccl}\varphi_{F,\pi,d\to d'} : & \mathbb{N} & \longrightarrow & \mathbb{N}\\ & k & \longmapsto & \min\{n \in \mathbb{N} : u \notin \mathcal{L}_k(\mathbf{SA}_{d'}(\pi(\mathbf{T}_F)))\\ & & & \qquad\Longrightarrow u \notin \mathcal{L}_k\left(\mathbf{SA}_{d'}\left(\pi(\mathbf{T}_F^{n,d\to d'})\right)\right)\}.\end{array}$$

$\varphi_{F,\pi,d\to d'}(k)$ corresponds to the minimum size of the row to detect a forbidden pattern in the effective subshift realized as projective subaction of $\pi(\mathbf{T}_F)$.

▶ **Example 2.** Consider the following set of 2-dimensional forbidden patterns

$$F = \left\{\begin{array}{ccc} \begin{array}{cc} & \bar{\alpha}\\ \bar{\beta} & \bar{a}\end{array}, & \begin{array}{cc} & \bar{\beta}\\ \bar{a} & \bar{a}\end{array}, & \bar{a}\,\bar{\$}\,, \\ & & \\ \begin{array}{cc} \bar{\alpha} & \\ \bar{b} & \bar{\gamma}\end{array}, & \begin{array}{cc} \bar{\gamma} & \\ \bar{b} & \bar{b}\end{array}, & \bar{\$}\,\bar{b} \end{array} \quad \text{such that} \quad \begin{array}{l} \bar{\alpha} \in \{\bar{a}, \bar{b}\},\\ \bar{\beta} \in \{\bar{\$}, \bar{b}\},\\ \bar{\gamma} \in \{\bar{\$}, \bar{a}\}. \end{array}\right\}.$$

$\mathbf{SA}_1(\mathbf{T}_F)$ is the subshift where $\{\$a^nb^m\$, \$a^nb^ma, ba^nb^m\$, ba^nb^ma : m \neq n\}$ are the forbidden patterns. The idea is that in a configuration of $\mathbf{T}_F$, if a line contains $\$a^nb^m\$$ with

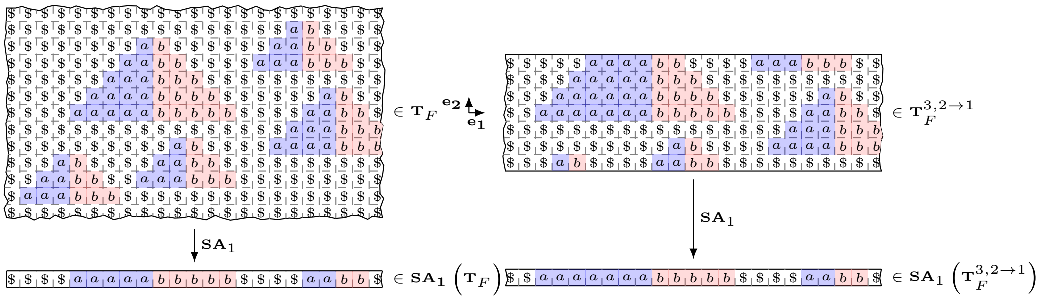


Figure 1 Application of $\mathbf{SA}_1$ under a configuration of $\mathbf{T}_F$ and $\mathbf{T}_F^{3,2\to 1}$.

$n \neq m$ then the next line in the direction $\mathbf{e_2}$ contains $\$a^{n-1}b^{m-1}\$$ and recursively. Thus the pattern $a\$$ or $\$b$ appear and the configuration considered is excluded (see Figure 1).

In $\mathbf{T}_F^{n,2\to 1}$ there is only n lines to detect a forbidden pattern so $\mathbf{SA_1}\left(\mathbf{T}_F^{n,2\to 1}\right)$ is the subshift where the forbidden patterns are $\{\$a^pb^m\$, \$a^pb^ma, ba^pb^m\$, ba^pb^ma : m \neq p \text{ and } \max(p,m) \leq n\}$. We deduce that $\varphi_{F,\mathrm{Id},2\to 1}(n) = \lfloor \frac{n}{2} \rfloor$. In Section 4 we will see that it is possible to obtain $\mathbf{SA}_1(\mathbf{T}_F)$ thanks to another sofic but with a better speed.

1.3 Subshift (φ, d)-realizable by sofic

A speed of convergence is in $\mathcal{F}$, the set of non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. Denote

$$\mathcal{F}_{\Sigma,d\to d'}^{Sofic} = \left\{\varphi \in \mathcal{F} : \exists F \underset{\text{finite}}{\subset} \mathcal{B}^* \text{ and } \pi : \mathcal{B} \to \mathcal{A} \text{ with } \varphi = \varphi_{F,\pi,d\to d'} \text{ and } \mathbf{SA}_{d'}(\pi(\mathbf{T}_F)) = \Sigma\right\}.$$

By Theorem 1, $\mathcal{F}_{\Sigma,d\to d'}^{Sofic} \neq \emptyset$ if and only if Σ is effective. Using the fact that a sofic subshift can superpose different layers and delete them with the factor, it is easy to verify that $\mathcal{F}_{\Sigma,d\to d'}^{Sofic}$ is stable by min, max, multiplication by an integer, division by an integer, addition and multiplication. Moreover $\mathcal{F}_{\Sigma,d\to d'}^{Sofic} \subset \mathcal{F}_{\Sigma,d+1\to d'}^{Sofic}$.

Invariance of (φ, d)-realizable subshift under conjugacy. We need to introduce a preorder relation on $\mathcal{F}$. We say that $\varphi \prec \varphi'$ if there exists $r, M \in \mathbb{N}$ such that $\varphi(k) \leq M\varphi'(k+r)$ for all $k \in \mathbb{N}$. We say that $\varphi \sim \varphi'$ if $\varphi \prec \varphi'$ and $\varphi' \prec \varphi$. Multiplication by M comes from the fact that a given speed can be improved by division by an integer and addition by r allows stability by conjugacy.

A d'-dimensional subshift Σ is *(φ, d)-realizable by projective subaction of sofic* if there exist a finite set of d-dimensional forbidden patterns F and a factor π such that $\mathbf{SA}_{d'}(\pi(\mathbf{T}_F)) = \Sigma$ and $\varphi_{F,\pi,d\to d'} \prec \varphi$. The subshift Σ is *sharp (φ, d)-realizable* if moreover $\varphi \prec \varphi'$ for all $\varphi' \in \mathcal{F}_{\Sigma,d\to d'}^{Sofic}$.

▶ **Proposition 3.** *Let Σ and Σ' be two conjugated d'-dimensional subshfits. The subshift Σ is (φ, d)-realizable by projective subaction of a sofic if and only if it is the same for Σ'.*

Proof. Let $\psi : \Sigma \longrightarrow \Sigma'$ be the conjugation map of neighborhood $\mathbb{U} = [-r, r]^{d'}$. The local function can be extended in a function $\psi : \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{B}^{\mathbb{Z}^d}$ of neighborhood $\mathbb{U} = [-r, r]^{d'} \times \{\mathbf{0}\}$.

Let $\mathbf{T}_F$ be a subshift of finite type and π be a factor such that $\mathbf{SA}_{d'}(\pi(\mathbf{T}_F)) = \Sigma$, one has $\mathbf{SA}_{d'}(\psi \circ \pi(\mathbf{T}_F)) = \Sigma'$. Let $u \in \mathcal{B}^{[0,k-1]^{d'}}$ and $\varphi = \varphi_{F,\pi,d\to d'}$, one has

$$\begin{aligned} u \notin \mathcal{L}(\Sigma') &\implies \left\{v \in \mathcal{A}^{[-r,k+r-1]^{d'}} : \psi(v) = u\right\} \not\subset \mathcal{L}(\Sigma) \\ &\implies \left\{v \in \mathcal{A}^{[-r,k+r-1]^{d'}} : \psi(v) = u\right\} \not\subset \mathcal{L}\left(\mathbf{SA}_{d'}\left(\pi\left(\mathbf{T}_F^{\varphi(k+2r),d\to d'}\right)\right)\right) \\ &\implies u \notin \mathcal{L}\left(\mathbf{SA}_{d'}\left(\psi \circ \pi\left(\mathbf{T}_F^{\varphi(k+2r),d\to d'}\right)\right)\right) \end{aligned}$$

Thus $\varphi_{F,\psi\circ\pi,d\to d'}(k) \leq \varphi_{F,\pi,d\to d'}(k+2r)$, the reciprocal is obtained using ψ^{-1}. ◀

2 Speed of convergence in general constructions

2.1 Notion of Turing machines

A *k-tapes Turing machine* $\mathcal{M} = (k, Q, \Gamma, \#, q_0, \delta, Q_F)$ is defined by:

- Γ a finite alphabet, with a blank symbol $\# \in \Gamma$. Initially, k infinite memory tapes represented as an element of $(\Gamma^k)^{\mathbb{Z}}$, are filled with $\#$, except for a finite prefix on the first tape (the input), and a computing head is located on the first letter of the tape;
- Q the finite set of states of the head and $q_0 \in Q$ is the initial state;
- $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{\leftarrow, \cdot, \to\}^k$ the transition function. Given the state of the head and the letter associated, it reads on the tape, depending on its position, the head can change state, replace the letter and move by one cell at most.
- $Q_F \subset Q$ the set of final states, when a final state is reached, the computation stops and the output is the value currently written on the tape.

Turing machines are a very robust model of computation, there exist several variants in the literature which are equivalent from a decidability point of view. Nevertheless these modifications on the definition are not without effects on the time and space complexities (time unit is one application of the transition function, space unit is one cell of the tape). To detect forbidden patterns in the projective subaction, one of the fundamental construction is the use of laical rules to encode Turing machine computations. In this article we choose to use the basic version of $\mathcal{M}$ but the reader should have in mind that it is possible to improve time and space complexities, using by instance these non-exhaustive acceleration techniques:

- **Compare-Copy:** compare or copy instantaneously a word between two markers between two tapes;
- **Transfer head:** transfer instantaneously the head to another cell of the tape marked by a special symbol;
- **Fill:** fill instantaneously a part of a tape with a periodic pattern.

Let F be a recursively enumerable set of forbidden patterns, then the complementary of $\mathcal{L}(\mathbf{T}_F)$ in $\mathcal{A}^*$, denoted $\mathcal{L}(\mathbf{T}_F)^c$ is also recursively enumerable. Consider $\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}$ be a Turing machine which enumerates $\mathcal{L}(\mathbf{T}_F)^c$, denote

- $\mathrm{Dtime}_{\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}}(k)$ the maximal time needed by the Turing machine $\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}$ to know if a pattern of size k is not in the language of $\mathbf{T}_F$;
- $\mathrm{Dspace}_{\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}}(k)$ the maximal space needed by the Turing machine $\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}$ to know if a pattern of size k is not in the language of $\mathbf{T}_F$ (only the space necessary for the computation is taken in consideration and the input is considered in an auxiliary tape).

▶ Remark. $\mathrm{Dtime}_{\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}}$ and $\mathrm{Dspace}_{\mathcal{M}_{\mathcal{L}(\mathbf{T}_F)^c}}$ are not computable if $\mathcal{L}(\mathbf{T}_F)^c$ is not recursive.

Let F be a set of patterns and $\mathcal{M}_F$ a Turing machine, called *enumerative Turing machine of* F, with the following behavior: it starts on the empty tape and successively writes the patterns of F on its tape. A set of finite patterns F *forbids the pattern* w if $w \notin \mathcal{L}(\mathbf{T}_F)$. Let $\mathcal{M}_F$ be an enumerative Turing machine of F, denote $\mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}(k)$ (resp. $\mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k)$) the smallest time (resp. the smallest space) taken by the Turing machine $\mathcal{M}_F$ such that the subset $F_{\mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}(k)}$ of F (resp. $F_{\mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k)} \subset F$) generated at this time (resp. at this space) forbid all the words of $\mathcal{L}_k(\mathbf{T}_F)^c$.

2.2 Speed of convergence for previous constructions

In this section, we give some elements to determine the speed of convergence given by the construction of [6] and [2]. The idea is to "program" a d-dimensional subshift of finite type, denoted $\mathbf{T}_{\mathtt{Final}}$ whose projective subaction is a given effective subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ where $d = 3$ in [6] and $d = 2$ in [2]. In the two constructions, $\mathbf{T}_{\mathtt{Final}}$ is constituted by three layers:

- the first one is $\mathcal{A}^{\mathbb{Z}^d}$ and contains different copies of the same configuration $y \in \mathcal{A}^{\mathbb{Z}}$ superposed on additional directions, the additional finite type conditions check if $y \in \Sigma$;
- the second is $\mathbf{T}_{\mathtt{Grid}} \subset \mathcal{A}_{\mathtt{Grid}}^{\mathbb{Z}^d}$ and constructs a grid which allows to implement well initialized Turing machine in all configurations with different sizes for time and space;
- the third is $\mathbf{T}_{\mathcal{M}} \subset \mathcal{A}_{\mathcal{M}}^{\mathbb{Z}^d}$ and checks if no forbidden pattern appears: the purpose is to implement a Turing machine $\mathcal{M}_F$ which enumerates forbidden patterns which define Σ and an additional procedure $\mathcal{M}_{\mathtt{Search}}$ which checks if the patterns produced appear in the configuration of the first layer (if it is the case, the Turing machine enters in a special state which is forbidden by $\mathbf{T}_{\mathtt{Final}}$).

Thus $x \in \mathbf{T}_{\mathtt{Final}} \subset \mathcal{A}^{\mathbb{Z}^d} \times \mathbf{T}_{\mathtt{Grid}} \times \mathbf{T}_{\mathcal{M}}$ if and only if there exists $y \in \Sigma$ such that $y = \pi(x)_{\mathbf{i}+\mathbb{Z}\mathbf{e_1}}$ for all $\mathbf{i} \in \langle \mathbf{e_2}, \dots, \mathbf{e_d} \rangle_{\mathbb{Z}}$ where π is the factor on the first layer which deletes computation states. In particular $\Sigma = \mathbf{SA}_1(\pi(\mathbf{T}_{\mathtt{Final}}))$ but moreover Σ is conjugate to a sub-action of $\pi(\mathbf{T}_{\mathtt{Final}})$. This result is stronger that just realization by projective subaction and allows to construct local rules for exotic tilings [3, 5].

In the two articles, $\mathbf{T}_{\mathrm{Grid}}$ is defined by a substitution. Mozes' result [13] gives local rules which force a cell to be in a super tile of order n well formed without considering the whole configuration. To determine $\varphi_{F_{\mathtt{Final}},\pi,d\to 1}$, it is sufficient to analyze the size in $\mathbf{T}_{\mathtt{Grid}}$ necessary for that $\mathcal{M}_F$ enumerates patterns of size k and all zones are checked by the additional procedure $\mathcal{M}_{\mathtt{Search}}$. This depends of $\mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}(k)$ and $\mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k)$.

Speed of convergence in the construction of [6]. As it is described in Section 4 of [6], $\mathbf{T}_{\mathtt{Grid}}$ gives a rectangular partition of $\mathbb{Z}^3$ generated by $\widehat{W}_3 \times \widehat{W}_5$ where $\widehat{W}_3$ and $\widehat{W}_5$ are obtained by a substitution. Thus for $s, t \in \mathbb{N}$ there exists $\mathbb{M} \subset \mathbb{Z}$ such that for all $i \in \mathbb{M}$, the slice $\{i\} \times \mathbb{Z}^2$ is partitioned into rectangles of size $3^s \times 5^t$ which delimits computation zones. Moreover $\mathbb{M}$ does not have gap bigger than $3^s 5^t$. To copy the initial configuration onto the first layer, we need an approximation row of width $O(3^s 5^t)$ to detect a forbidden word enumerated in space less than 3^s and in time less than 5^t. One deduces that

$$(k \longmapsto \varphi_{F_{\mathtt{Final}},\pi,3\to 1}) \sim \left(k \longmapsto \mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k)\mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}(k)\right).$$

Speed of convergence in the construction of [2]. As it is described in Section 2, Fact 2.4, of [2], $\mathbf{T}_{\mathtt{Grid}}$ defines fractured zone of computation to implement the Turing machine of size $2^n \times 2^{2^n}$, the first coordinate according to $\mathbf{e_1}$ corresponds to the space and the second according to $\mathbf{e_2}$ corresponds to the time. By the substitution rules and the clock rules, this fractured zone of computation is included in a pattern of $\mathbf{T}_{\mathtt{Grid}}$ of size $4^n \times (2^{n+2^n})$

and every row $\mathbf{T}_{\texttt{Final}}^{2^{n+2^n},2\to 1}$ contains such computation zone every 4^n cells. Since the time to cheek if a forbidden pattern of size k appears in the responsibility zone ($n^2 2^n$ steep in direction $\mathbf{e_2}$ by Fact 3.4 of [2]) is negligible according to the time given to the Turing machine to compute forbidden patterns (2^{n+2^n} steep in direction $\mathbf{e_2}$), one deduces that $(k \mapsto \varphi_{F_{\texttt{Final}},\pi,2\to 1}(k)) \sim \left(k \mapsto 2^{n(k)+2^{n(k)}}\right)$ where $n(k) = \min\{n : \mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k) < 2^n\}$. So

$$(k \longmapsto \varphi_{F_{\texttt{Final}},\pi,2\to 1}) \sim \left(k \longmapsto \mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k) 2^{\mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}(k)}\right).$$

2.3 A more efficient construction

In the particular case where Σ is an effective subshift with a periodic configuration, the construction can be highly simplified and the speed of convergence is improved. In a few words, the same type of construction with different layers is built, however the computation checks if no forbidden patterns appear only in one line, the other lines are mapped into the periodic configuration by the factor map. Thus the computation zones do not need to be fractionated and simplified layer $\mathbf{T}_{\texttt{Grid}}$ allows a computation in real time.

▶ **Theorem 4.** *Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}^d}$ be an effective subshift of dimension d with a periodic point $({}^\infty w^\infty \in \Sigma)$ defined by a set F of forbidden patterns enumerated by a Turing machine $\mathcal{M}_F$. Then there exists a subshift of finite type $\mathbf{T}_{F_{Final}}$ of dimension $d+1$ and a factor map π such that $\mathbf{SA}_d\left(\pi(\mathbf{T}_{F_{Final}})\right) = \Sigma$ and $\varphi_{F_{Final},\pi,d+1\to d} \sim \mathit{Dtime}^{enu}_{\mathcal{M}_F}$.*

Proof. Assume that $\mathcal{M}_F$ enumerates patterns of F on the first tape separated by the symbol \$ and that the tapes of $\mathcal{M}$ are onesided. The different layers of $\mathbf{T}_{F_{\texttt{Final}}}$ are:

- **Layer 1:** The first layer is $\mathbf{T}_{\texttt{Line}} \subset \left((\mathcal{A}\times\{\square, \square\})\cup\{\blacksquare\}\right)^{\mathbb{Z}^2}$ the subshift of finite type such that for $x \in \mathbf{T}_{\texttt{Line}}$ there is at most one $i \in \mathbb{Z}$ such that $\mathbf{SA}_1\left(\sigma^{i\mathbf{e_2}}(x)\right) = {}^\infty \blacksquare^\infty$ and for all $j \neq i$ one has $\mathbf{SA}_1\left(\sigma^{j\mathbf{e_2}}(x)\right) \in \{\sigma^k({}^\infty w^\infty) : k \in \mathbb{Z}\} \times \{\square, \square\}^{\mathbb{Z}}$.
- **Layer 2:** The second layer is the subshift $\mathbf{T}_{\texttt{Config}} = \{x \in \mathcal{A}^{\mathbb{Z}^2} : \sigma^{\mathbf{e_1}-\mathbf{e_2}}(x) = x\}$, the configuration is shifted in view to scan two adjacent areas (and their frontier) during the comparison.
- **Layer 3:** The third layer is the subshift of finite type $\mathbf{T}_{\texttt{Grid}} \subset \{\square, \boxed{*}, \square, \boxed{*}\}^{\mathbb{Z}^2}$ such that on each line of $x \in \mathbf{T}_{\texttt{Grid}}$, the two colors alternates and this alternation is repeated above until it crosses a line which contains the symbol $*$. In this case the transitions red/blue become monochromatic and the transitions blue/red force the alternation. Thus the sequences of monochromatic colors become larger. We remark that if a line contains the periodic configuration ${}^\infty(\square\square)^\infty$, then all lines below contain this periodic configuration and above, if we have crossed n times a line with the symbol $*$, we obtain a line with the periodic configuration ${}^\infty(\square^{2^n}\square^{2^n})^\infty$ (see Figure 2).
- **Layer 4:** Denote $\mathcal{A}_{\mathcal{M}} = ((Q\times\Gamma)\cup\Gamma)^k$ where k is the number of tapes, the fourth layer is a subshift of finite type $\mathbf{T}_{\mathcal{M}} \subset \mathcal{A}_{\mathcal{M}}^{\mathbb{Z}^2}$ where the local rules are given by the transition rules δ of $\mathcal{M}_F$.
- **Layer 5:** The fifth layer is the full-shift $\mathbf{T}_{\texttt{Compar}} = \{\square\ \square\}^{\mathbb{Z}^2}$.

To obtain the subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathbf{T}_{\texttt{Line}} \times \mathbf{T}_{\texttt{Config}} \times \mathbf{T}_{\texttt{Grid}} \times \mathbf{T}_{\mathcal{M}} \times \mathbf{T}_{\texttt{Compar}}$ we add a finite set of forbidden patterns $F_{\texttt{SynchroLine}} \cup F_{\texttt{Init}} \cup F_{\texttt{Extend}} \cup F_{\texttt{Compar}}$ which codes the interaction between the different layers. These local rules are:

- **Rules $F_{\texttt{SynchroLine}}$:** These rules imply that if a line ${}^\infty\blacksquare^\infty$ appears in the layer $\mathbf{T}_{\texttt{Line}}$ of a configuration, then it is synchronized with a periodic point ${}^\infty\square\square^\infty$ or ${}^\infty\boxed{*}\boxed{*}^\infty$ in the layer $\mathbf{T}_{\texttt{Grid}}$.

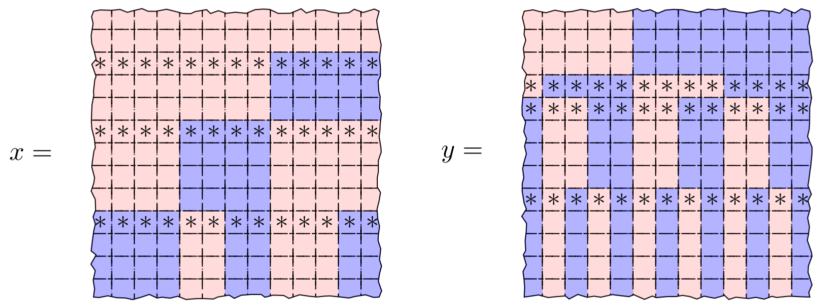


Figure 2 x and y are two examples of configurations of $\mathbf{T}_{\texttt{Grid}}$ and y contains $^\infty(\ )^\infty$.

- **Rules $F_{\texttt{Init}}$:** They imply that the initialization state appears in the layer $\mathbf{T}_{\mathcal{M}}$ on each cell in correspondence to the line $^\infty \blacksquare ^\infty$ in the layer $\mathbf{T}_{\texttt{Line}}$.
- **Rules $F_{\texttt{Extend}}$:** They imply that if a computation needs more space, the symbol $*$ appears in the layer $\mathbf{T}_{\texttt{Grid}}$ (thus the computation zones is doubled) and the tape in the layer $\mathbf{T}_{\mathcal{M}}$ corresponding to the old red zone is erased (to have only one computation by computation zone). Thus the space allowed by a Turing machine is doubled if the head was in a blue zone.
- **Rules $F_{\texttt{Compar}}$:** They imply that if a forbidden pattern appears in the enumeration obtained in $\mathbf{T}_{\mathcal{M}}$ then it is compared with the corresponding pattern which appears in $\mathbf{T}_{\texttt{Config}}$. If the two patterns coincide then the configuration is forbidden in $\mathbf{T}_{\texttt{Final}}$.

Define the factor map $\pi_{\texttt{Final}} : \mathbf{T}_{\texttt{Final}} \to \mathcal{A}^{\mathbb{Z}^2}$ such that for $x \in \mathbf{T}_{\texttt{Final}}$ and $\mathbf{i} \in \mathbb{Z}^2$, $\pi(x)_{\mathbf{i}}$ is the cell of the layer $\mathbf{T}_{\texttt{Config}}$ if we are in the line $^\infty \blacksquare ^\infty$ in $\mathbf{T}_{\texttt{Config}}$ and the cell corresponding to the periodic orbit of $\mathbf{T}_{\texttt{Line}}$ if not.

For $x \in \Sigma$ it is easy to construct $y \in \mathbf{T}_{\texttt{Final}}$ such that $\mathbf{SA}_1(\pi_{\texttt{Final}}(y)) = x$. Reciprocally, consider $y \in \mathbf{T}_{\texttt{Final}}$. If $\pi_{\texttt{Line}}(y)_{(0,0)} \neq \blacksquare$ then $\mathbf{SA}_1(\pi_{\texttt{Final}}(y)) = ^\infty w^\infty \in \Sigma$. If $\pi_{\texttt{Line}}(y)_{(0,0)} = \blacksquare$, we consider u a sub-pattern of $x = \mathbf{SA}_1(\pi_{\texttt{Final}}(y))$ of size n. Assume that $u \notin \mathcal{L}(\Sigma)$, so there exists a word $w \sqsubset x$ enumerated by $\mathcal{M}$ in time $t_F(n) = \mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}}(n)$ and space $s_F(n) = \mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}}(n)$ such that $w \sqsubset u$. By construction of $\mathbf{T}_{\texttt{Final}}$, one has $\mathbf{SA}_1\left(\pi_{\texttt{Grid}}(\sigma^{t_F(n)}(y))\right) = ^\infty (\ ^{2^k}\ ^{2^k})^\infty$ where $k = \min\{k' : s_F(n) < 2^{k'}\}$. Since the configuration is shifted on $\mathbf{T}_{\texttt{Config}}$ and compared instantaneously thanks to $\mathbf{T}_{\texttt{Compar}}$, we conclude there exists k' such that $t_F(n) \leq k' \leq t_F(n) + 2^{1+min\{k:s_F(n)<2^k\}}$ where the word w is detected in the line $y_{\mathbb{Z},k'}$. By the condition $F_{\texttt{Compar}}$ this is impossible.

Thus $\mathbf{SA}_{\mathbf{e}_1\mathbb{Z}}\left(\pi_{\texttt{Final}}(\mathbf{T}_{\texttt{Final}})\right) = \Sigma$ and $\varphi_{F,\pi,2\to1}(k) = t_F(k) + 2^{1+\min\{n:s_F(k)<2^n\}}$ for all $k \in \mathbb{N}$. In particular $\varphi_{F,\pi,2\to1} \sim \max(\mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}, \mathrm{Dspace}^{\mathrm{enu}}_{\mathcal{M}_F}) = \mathrm{Dtime}^{\mathrm{enu}}_{\mathcal{M}_F}$. ◀

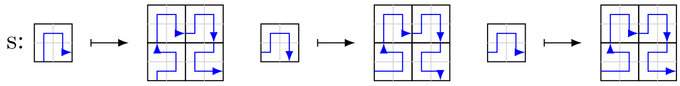
2.4 Increase the dimension to increase the speed

Generally, properties studied on subshifts of finite type exhibit a gap between dimension one and dimension two. The most famous is the undecidability of the domino problem in dimension $d \geq 2$. In this section we exhibit a gap which appears in an algorithmic point of view.

▶ **Theorem 5.** *Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift which is $(\varphi, d+d')$-realizable by projective subaction of a sofic then it is $(\varphi^{\frac{d'}{d''}}, d+d'')$-realizable by projective subaction of a sofic for $d'' \geq d'$.*

Proof. Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be an effective subshift. Consider $\mathbf{T}_F \subset \mathcal{B}^{\mathbb{Z}^2}$ a subshift of finite type such that $\mathbf{SA}_1(\pi(\mathbf{T}_F)) = \Sigma$. Denote $\varphi = \varphi_{F,\pi,2\to1}$. One constructs $\mathbf{T}_{F'} \subset \mathcal{B}'^{\mathbb{Z}^3}$ a subshift of finite type and $\pi' : \mathcal{B}' \to \mathcal{A}$ a factor map such that $\mathbf{SA}_1(\pi'(\mathbf{T}_{F'})) = \Sigma$ and $\varphi_{F',\pi',3\to 1} \in \Theta(\sqrt{\varphi})$. This prove the Theorem for $d = 1$, $d' = 1$ and $d'' = 2$.

Construction of a tangled grid. Consider the alphabet $\mathcal{C}$ formed by  ,  , , their rotations and their symmetrized about to the axis, thus $\mathrm{card}(\mathcal{C}) = 3 \times 4 \times 2 = 24$ and define the following substitution on $\mathcal{C}$ (modulo rotations and symmetries):



By iterating substitution s on a letter $a \in \mathcal{C}$, we construct for every $n \in \mathbb{N}$ the pattern $s^n(a)$ called the *super-tile of order n and type a*. The substitutive subshift defined by

$$\mathbf{T}_s = \left\{ x \in \mathcal{C}^{\mathbb{Z}^2} : u \sqsubset x \text{ if there exists } n \in \mathbb{N} \text{ and } a \in \mathcal{C} \text{ which verifies } u \sqsubset s^n(a) \right\},$$

is sofic according to Mozes' result [13]. Thus there exists a finite set of forbidden patterns F_s and a factor map $\pi_s : \mathcal{C}_s \to \mathcal{C}$ such that $\pi_s(\mathbf{T}_{F_s}) = \mathbf{T}_s$. In the Mozes' construction the local rules F_s force every super tile of order n to be assembled in a super tile of order $n+1$. Thus if $p \in \mathcal{C}_s^{[-k,k]^2}$ does not contain patterns of F_s, then the center letter $p_{\mathbf{0}}$ is in a super-tile of order n such that $2^n \leq k < 2^{n+1}$. In this super tile, the arrows form a connected tangled segment of size 2^{n^2}.

Construction of a three-dimensional sofic subshift which realizes Σ. Consider the subshift of finite type $\mathbf{T}_{F'} \subset \mathcal{B}'^{\mathbb{Z}^3}$ where $\mathcal{B}' = \mathcal{B} \times \mathcal{C}_s$ such that

- for all $i \in \mathbb{Z}$ the $\mathbb{Z}^2$-configuration $\pi_{\mathcal{C}_s}(x)_{i\mathbf{e_1}+\mathbb{Z}^2}$ is an element of $\mathbf{T}_s$;
- the 2-dimensional forbidden patterns of F are transfered in 3-dimensional forbidden patterns where the second coordinate is wrapped following the tangled grid (see Figure 3).

Let π' be the application of π following the tangled grid, we obtain $\mathbf{SA}_1(\pi'(\mathbf{T}_{F'})) = \Sigma$.

$\pi'(\mathbf{T}_{F'})$ has the expected speed of convergence. By definition of the speed of convergence, for any $u \in \mathcal{A}^k$, if $u \notin \Sigma$ then $u \notin \mathcal{L}\left(\pi\left(\mathbf{SA}_1\left(\mathbf{T}_F^{\varphi(k),2\to 1}\right)\right)\right)$.

Let $z \in \mathbf{T}_{F'}^{2\left\lceil\sqrt{\varphi(k)}\right\rceil,3\to 1}$. The condition F_s verified on $z_{\{0\}\times\left[-2\left\lceil\sqrt{\varphi(k)}\right\rceil,2\left\lceil\sqrt{\varphi(k)}\right\rceil\right]^2}$ implies that $z_{\mathbf{0}}$ is included in a super-tile of order $n = \left\lfloor \log_2\left(\left\lceil\sqrt{\varphi(k)}\right\rceil\right)\right\rfloor$. One deduces that $z_{\mathbf{0}}$ is in the center of a segment constituted following the arrows of $\mathcal{C}$ of amplitude $\left(2^{\left\lfloor \log_2\left(\left\lceil\sqrt{\varphi(k)}\right\rceil\right)\right\rfloor}\right)^2$. Like the local transitions F are transfered, there exists $y \in \mathbf{SA}_1\left(\mathbf{T}_F^{\varphi(k),2\to 1}\right)$ which correspond to z in the wrapped zone. Thus $u \notin \mathcal{L}\left(\mathbf{T}_{F'}^{2\left\lceil\sqrt{\varphi(k)}\right\rceil,3\to 1}\right)$ that is to say $\varphi_{F',\pi',3\to 1} \prec \sqrt{\varphi}$. In the same way the reverse holds and so $\varphi_{F',\pi',3\to 1} \succ \sqrt{\varphi}$. ◀

▶ Remark. Exemples of Section 4 show that this theorem is optimal.

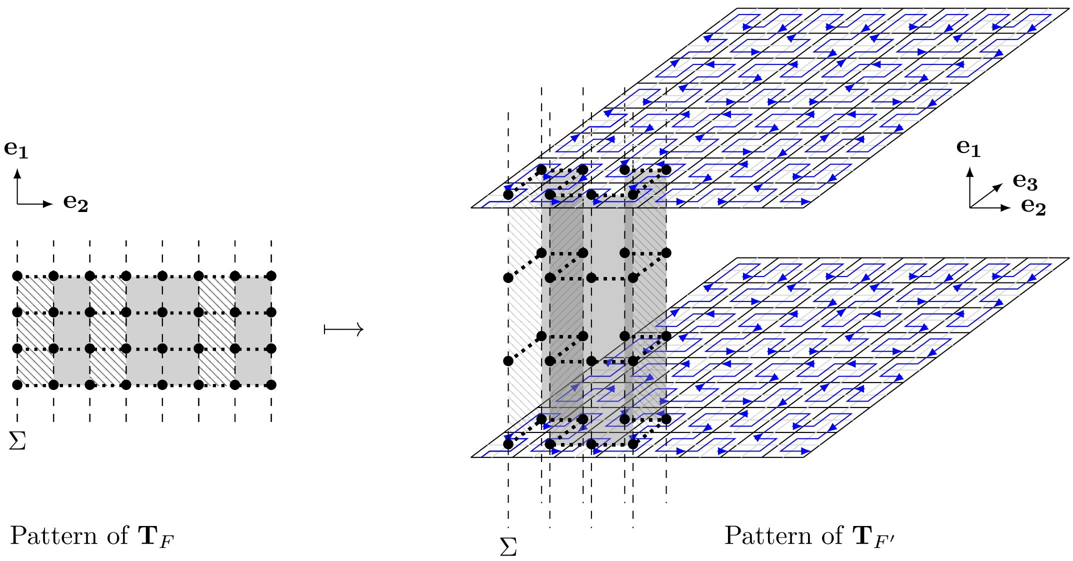


Figure 3 Pattern of $\mathbf{T}_F$ wrapped following the tangled grid in a pattern of $\mathbf{T}_{F'}$. The subshift Σ is obtained taking factor π or π' and projective subaction following $\mathbf{e_1}$.

3 Lower bounds for the speed of convergence of a subshift

3.1 Combinatorial lower bounds

Let Σ be a one dimensional subshift and let $u \in \mathcal{A}^*$, the *follower set of word of size* k of u is $\mathrm{Fol}^k_\Sigma(u) = \{v \in \mathcal{L}_k(\Sigma) : uv \in \mathcal{L}(\Sigma)\}$. If $u \notin \mathcal{L}(\Sigma)$ then $\mathrm{Fol}^k_\Sigma(u) = \emptyset$. Moreover one has $\mathrm{card}(\{\mathrm{Fol}^{k_2}_\Sigma(u) : u \in \mathcal{A}^{k_1}\}) \leq \mathrm{card}(\mathcal{A})^{k_1}$.

▶ **Theorem 6.** *Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be an one dimensional effective subshift and $\varphi \in \mathcal{F}^{Sofic}_{\Sigma,d}$ with $d \geq 2$. Then there exists a constant M such that for all $k_1, k_2 \in \mathbb{N}$ one has:*

$$M\varphi_{F,\pi,d\to 1}(k_1+k_2) \geq \left(\log(card(\{Fol^{k_2}_\Sigma(u) : u \in \mathcal{A}^{k_1}\}))\right)^{\frac{1}{d-1}}.$$

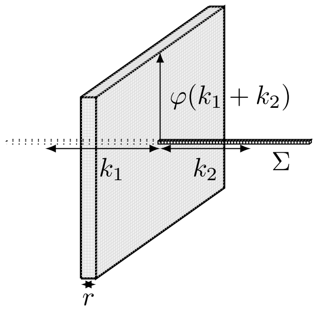
Proof. Assume that $\Sigma = \mathbf{SA}_1(\pi(\mathbf{T}_F))$ and $\varphi = \varphi_{F,\pi,d\to 1}$. For $u \in \mathcal{L}_{k_1}(\Sigma)$, one has

$$\mathrm{Fol}^{k_2}_\Sigma(u) = \Big\{\mathbf{SA}_1(\pi(x))_{[0,k_2-1]} \in \mathcal{A}^{k_2} : \\ x \in \mathbf{T}_F^{\varphi(k_1+k_2),d\to 1} \text{ such that } \mathbf{SA}_1(\pi(x))_{[-k_1,-1]} = u\Big\}.$$

Let r such that the support of every pattern of F is included in $[0,r-1]^d$. For $x \in \mathbf{T}_F^{\varphi(k_1+k_2),d\to 1} \subset \mathcal{B}^{\mathbb{Z}^d}$ such that $\mathbf{SA}_1(\pi(x))_{[-k_1,-1]} = u \in \mathcal{A}^{k_1}$, the knowledge of $x_{[-r,-1]\times[-\varphi(k_1+k_2),\varphi(k_1+k_2)]^{d-1}}$ is sufficient to determine which set of $\{\mathrm{Fol}^{k_2}_\Sigma(u') : u' \in \mathcal{A}^{k_1}\}$ is allowed to complete $u \in \mathcal{A}^{k_1}$ by a word $v \in \mathcal{A}^{k_2}$ such that

$$uv \in \mathcal{L}_{k_1+k_2}(\mathbf{SA}_1\left(\pi(\mathbf{T}_F^{\varphi(k_1+k_2),d\to 1}\right)) = \mathcal{L}_{k_1+k_2}(\Sigma).$$

Thus $\mathrm{card}(\{\mathrm{Fol}^{k_2}_\Sigma(u) : u \in \mathcal{A}^{k_1}\}) \leq \mathcal{B}^{r(2\varphi(k_1+k_2)+1)^{d-1}}$.



◀

3.2 Computational lower bounds

▶ **Theorem 7.** *Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be an one dimensional effective subshift and $\varphi \in \mathcal{F}^{Sofic}_{\Sigma,d}$. There exists a Turing machine $\mathcal{M}$ whose the domaine is $\mathcal{L}(\Sigma)^c$ such that*
- $\max(\log, (\varphi_{F,\pi,d\to 1})^{d-1}) \succ \log \circ Dtime_{\mathcal{M}}$;
- $(\varphi_{F,\pi,d\to 1})^{d-1} \succ Dspace_{\mathcal{M}}$.

Since $\mathcal{L}(\Sigma)^c$ is not necessarily recursive, $\mathrm{Dtime}_{\mathcal{M}}$ and $\mathrm{Dspace}_{\mathcal{M}}$ are not necessarily computable.

Proof. Let F be a finite set of forbidden patterns of maximal size r such that $\Sigma = \mathbf{SA}_1(\pi(\mathbf{T}_F))$ and $\varphi = \varphi_{F,\pi,d\to 1}$. Denote $\mathbb{B}_n = \{k_2\mathbf{e_2}+\dots+k_d\mathbf{e_d} : (k_2,\dots,k_d) \in [-n,n]^{d-1}\}$ and $\mathbf{T}^m = \mathbf{T}_F^{m,d\to 1}$. One has $\mathcal{L}_k\left(\mathbf{SA}_1\left(\pi(\mathbf{T}^{\varphi(k)})\right)\right) = \mathcal{L}_k(\Sigma)$ and $\mathbf{T}^{\varphi(k)} \subset \left(\mathcal{B}^{\mathbb{B}_{\varphi(k)}}\right)^{\mathbb{Z}}$ is a one-dimensional subshift of finite type of order r. This subshift can be represented by a graph where the vertices are $\left(\mathcal{B}^{\mathbb{B}_{\varphi(k)}}\right)^r \cap \mathcal{L}(\mathbf{T}^{\varphi(k)})$ and there is an edge from u to v if the two words coincide except for the extremal letters (see [11]). Thus this graph has at most $\mathrm{card}(\mathcal{B})^{r(2\varphi(k))^{d-1}}$ vertices and can be viewed as an automaton which accepts words of $\mathcal{L}(\pi(\mathbf{T}^{\varphi(k)}))$, this takes a linear time in the size of the graph.

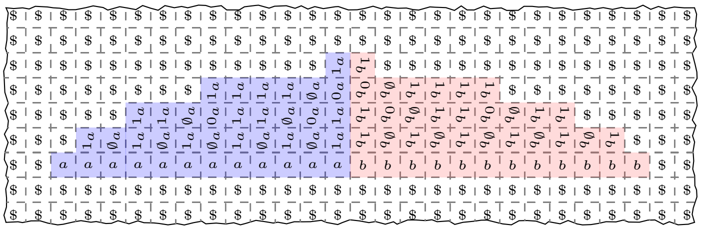
To determine if $u \notin \mathcal{L}(\Sigma)$, it is sufficient that $u \notin \mathcal{L}(\mathbf{SA}_1(\pi(\mathbf{T}^m)))$ for some $m \in \mathbb{N}$. We implement an algorithm which explores the graph generated by $\mathbf{T}^m$ for each $m \in \mathbb{N}$ and search if u is accepted with the corresponding automaton. One knows if $u \in \mathcal{L}(\mathbf{SA}_1(\pi(\mathbf{T}^m)))$ in time $O(k\,\mathrm{card}(\mathcal{A})^{r(2m)^{d-1}})$. This algorithm halts on $u \notin \mathcal{L}(\Sigma)$ in time

$$\mathrm{Dtime}_{\mathcal{M}}(k) \leq M\,k \sum_{m=1}^{\varphi(k)} \mathrm{card}(\mathcal{A})^{r(2m)^{d-1}} \leq M\,k\,\varphi(k)\mathrm{card}(\mathcal{A})^{r(2\varphi(k))^{d-1}}$$

Since $\varphi(k) \leq \varphi^{d-1}(k)$, it follows that $\max(\log, \varphi^{d-1}) \succ \log \circ \mathrm{Dtime}_{\mathcal{M}}$.We deduce the first point of the theorem.

To prove the second point, the naive procedure to find a configuration of $\mathbf{SA}_1(\pi(\mathbf{T}^m))$ which contains u in the center is to start from an element of $\left(\mathcal{B}^{\mathbb{B}_n}\right)^r \cap \mathcal{L}(\mathbf{T}^m)$ and complete it respecting the condition F until it finds again a one-sided periodic orbit. To be sure to explore all the orbits it is possible to order them lexicographically. Thus, the algorithm just needs to know the last orbit checked, this needs $r(2m)^{d-1}$ space to know if $u \in \mathcal{L}(\mathbf{SA}_1(\pi(\mathbf{T}^m)))$. If $u \notin \mathcal{L}(\Sigma)$, the algorithm halts when it explores $\left(\mathcal{B}^{\mathbb{B}_n}\right)^r \cap \mathcal{L}(\mathbf{T}^{\varphi(k)})$. So there exists $M > 0$ such that $M(\varphi(k))^{d-1} \geq \mathrm{Dspace}_{\mathcal{M}}(k)$. We recall that the word u is written on an annex tape which is only used for the reading and which is not counted in $\mathrm{Dspace}_{\mathcal{M}}$. ◀

▶ Remark. These theorems do not generalize to dimension 2: Theorem 6 uses a characterization of one dimensional sofic subshifts with follower sets and Theorem 7 is blocked by the undecidability of emptiness of two-dimensional subshifts of finite type.



A configuration of $\mathbf{T}_{F_{\log}}$

Figure 4 A configuration of $\mathbf{T}_{F_{\log}}$.

4 Some classes of speed of convergence and perspectives

In this section we give the sharp realization of some one-dimensional subshifts.

▶ **Sofic subshift.** A subshift is constant-realizable by sofic if and only if it is sofic (see [14]).

▶ **Gap under constant-realizable.** If a subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is $(\varphi, 2)$-realizable by sofic with $\varphi \in o(\log(\log(n)))$ then this subshift is sofic. Indeed, by Theorem 7, $\mathcal{L}(\Sigma)^c$ can be recognized in space $o(\log(\log(n)))$, thus $\mathcal{L}(\Sigma)^c$ is rational (see [8]), that is to say Σ is sofic.

Let $\mathcal{L} \subset \mathcal{A}^*$ be a language and $\$ \notin \mathcal{A}$. Define the subshift $\mathbf{T}(\mathcal{L}) = \mathbf{T}_{F_{\mathcal{L}}} \subset \mathcal{A}'^{\mathbb{Z}}$ where $\mathcal{A}' = \mathcal{A} \cup \{\$\}$ and $F = \{\$u\$: u \notin \mathcal{L}\}$. If $\mathcal{L}$ is effective then $\mathbf{T}(\mathcal{L})$ is an effective subshift.

▶ **log-realizable.** Consider $\mathcal{L}_= = \{a^n b^n : n \in \mathbb{N}\}$. The subshift $\mathbf{T}(\mathcal{L}_=) \subset \{a, b, \$\}^{\mathbb{Z}}$ is sharp $((\log)^{\frac{1}{d-1}}, d)$-realizable by sofic for $d \geq 2$. Theorem 6 gives the lower bound since

$$\operatorname{card}(\{\operatorname{Fol}^n_\Sigma(u) : u \in \mathcal{A}^n\}) \geq \operatorname{card}(\{\operatorname{Fol}^k_\Sigma(\$a^k) : k \in [0, n-1]\}) = n.$$

For $d = 2$, the upper bound is obtained considering the subshift of finite type $\mathbf{T}_{F_{\log}} \subset \{a, b, \$, 0_a, 1_a, \emptyset_a, 0_b, 1_b, \emptyset_b\}^{\mathbb{Z}^2}$ where $F_{\log}$ are the forbidden patterns of shape $\mathbb{U} =$ which do not appear in the configuration represented in Figure 4. The factor π maps \$ on \$, $\{0_a, 1_a, \emptyset_a\}$ on a and $\{0_b, 1_b, \emptyset_b\}$ on b. The idea is to implement counters which grow when going from \$'s region and compare them at the frontier. The upper bound for $d \geq 3$ is obtained using Theorem 5.

In the same way the subshift $\mathbf{T}(\mathcal{L}_{\text{square}})$ defined with the langage $\mathcal{L}_{\text{square}} = \{a^n b^{n^2} : n \in \mathbb{N}\}$ is sharp $((\log)^{\frac{1}{d-1}}, d)$-realizable by sofic.

▶ **Linear-realizable.** For $u \in \{0,1\}^*$, define $\overline{u}$ the miror of u. Consider $\mathcal{L}_{\text{palin}} = \{u\overline{u} : u \in \{0,1\}^*\}$, the subshift $\mathbf{T}(\mathcal{L}_{\text{palin}}) \subset \{0, 1, \$\}^{\mathbb{Z}}$ is sharp $((\mathrm{Id})^{\frac{1}{d-1}}, d)$-realizable by projective subaction of sofic for $d \geq 2$ where $\mathrm{Id} : k \mapsto k$. Theorem 6 gives the lower bound.

For $d = 2$, the upper bound is obtained considering the subshift $\mathbf{T}_{F_{\text{lin}}} \subset \{\$, 0_l, 1_l, 0_r, 1_r\}^{\mathbb{Z}^2}$ where F_{lin} are the patterns of shape $\mathbb{U} =$ or which do not appear in the configuration represented in Figure 5. The factor π maps \$ on \$, $\{0_l, 0_r\}$ on 0 and $\{1_l, 1_r\}$ on 1. The principle is to compare vertically the two words of $\{0,1\}^*$. The upper bound for $d \geq 3$ is obtained using Theorem 5.

▶ **$\mathrm{Dspace}_{\mathcal{M}}$ realizable.** Let $\mathcal{L}$ be a computable language in space $\mathrm{Dspace}_{\mathcal{M}}$ and $\# \notin \mathcal{L}$. Consider $\mathcal{L}' = \{u\#^{\mathrm{Dtime}_{\mathcal{M}}(|u|)}\}$, then $\mathbf{T}(\mathcal{L}')$ is sharp $(\mathrm{Dspace}_{\mathcal{M}}, 2)$-realizable (the time of the Turing machine is coded following $\mathbf{e_1}$ in the sofic which realizes $\mathbf{T}(\mathcal{L}')$).

▶ **Substitutive subshift.** Let s be a one-dimensional substitution, $\mathbf{T}_s \cup \{^\infty a^\infty\}$ is sharp $(\log, 2)$-realisable. The lower bound is given by Theorem 6 and the upper bound follows from the sofic subshift where elements of $\mathbf{T}_s$ are in at most in one row. If it appears, this row is de-substituted in the next row in direction $\mathbf{e_2}$.

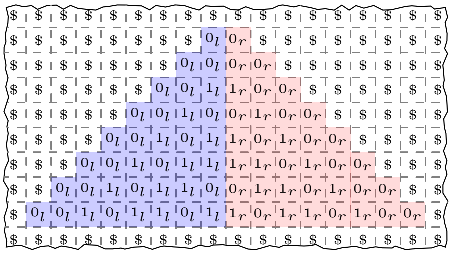


Figure 5 A configuration of $\mathbf{T}_{F_{\text{lin}}}$.

▶ **No-computable realization.** Consider the recursively enumerable set $F = \{01^n0 : n \text{ such that the Turing machine of number } n \text{ halts}\}$. Then $\varphi \in \mathcal{F}_{\Sigma,2}^{\mathcal{S}ofic}$ is larger than any recursive function, otherwise it is possible to decide if the Turing machine of number n halts.

▶ **Perspectives.** This article highlights the importance of algorithmic properties and optimality in the realization of effective subshifts by sofic. Particularly, the last section exhibits the existence of different subclasses of effective subshift but does not present systematic study: characterization of classes of subshifts with the same speed of convergence, links between dynamical properties and speed of convergence, sharp realization for effective subshift without periodic point (as the substitutive subshift $\mathbf{T}_s$)...

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