Price of Competition and Dueling Games[*]

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Abstract

We study competition in a general framework introduced by Immorlica, Kalai, Lucier, Moitra, Postlewaite, and Tennenholtz and answer their main open question. Immorlica et al. considered classic optimization problems in terms of competition and introduced a general class of games called dueling games. They model this competition as a zero-sum game, where two players are competing for a user's satisfaction. In their main and most natural game, the ranking duel, a user requests a webpage by submitting a query and players output an ordering over all possible webpages based on the submitted query. The user tends to choose the ordering which displays her requested webpage in a higher rank. The goal of both players is to maximize the probability that her ordering beats that of her opponent and gets the user's attention. Immorlica et al. show this game directs both players to provide suboptimal search results. However, they leave the following as their main open question: "does competition between algorithms improve or degrade expected performance?" (see the introduction for more quotes) In this paper, we resolve this question for the ranking duel and a more general class of dueling games.

More precisely, we study the quality of orderings in a competition between two players. This game is a zero-sum game, and thus any Nash equilibrium of the game can be described by minimax strategies. Let the value of the user for an ordering be a function of the position of her requested item in the corresponding ordering, and the social welfare for an ordering be the expected value of the corresponding ordering for the user. We propose the price of competition which is the ratio of the social welfare for the worst minimax strategy to the social welfare obtained by a social planner. Finding the price of competition is another approach to obtain structural results of Nash equilibria. We use this criterion for analyzing the quality of orderings in the ranking duel. Although Immorlica et al. show that the competition leads to suboptimal strategies, we prove the quality of minimax results is surprisingly close to that of the optimum solution. In particular, via a novel factor-revealing LP for computing price of anarchy, we prove if the value of the user for an ordering is a linear function of its position, then the price of competition is at least 0.612 and bounded above by 0.833. Moreover we consider the cost minimization version of the problem. We prove, the social cost of the worst minimax strategy is at most 3 times the optimal social cost.

Last but not least, we go beyond linear valuation functions and capture the main challenge for bounding the price of competition for any arbitrary valuation function. We present a principle which states that the lower bound for the price of competition for all 0-1 valuation functions is the same as the lower bound for the price of competition for all possible valuation functions. It is

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worth mentioning that this principle not only works for the ranking duel but also for all dueling games. This principle says, in any dueling game, the most challenging part of bounding the price of competition is finding a lower bound for 0-1 valuation functions. We leverage this principle to show that the price of competition is at least 0.25 for the generalized ranking duel.

1   Introduction

The conventional wisdom is that competition among suppliers will increase social welfare by providing consumers with competitive prices, high-quality products, and a wide range of options. A classic example is the Bertrand competition [10] where suppliers compete in price to incentivize consumers to buy from them and as a result the market price decreases to the point that it matches the marginal cost of production. Indeed there are many theoretical and empirical studies for supporting this belief in the economic literature (see, e.g., [29, 25, 24, 2]). However while in many markets the competition steers businesses to optimize their solutions for consumers, there are competitive markets in which businesses do not offer the best option to consumers. An interesting example for describing this situation is a dueling game, namely, a zero-sum game where two players compete to attract users. Immorlica, Kalai, Lucier, Moitra, Postlewaite, and Tennenholtz [19] showed surprisingly if players are aimed to beat their opponents in a dueling game, they may offer users suboptimal results. However, they raised this question regarding the efficiency of the competition as the authors write, *"Perhaps more importantly, one could ask about performance loss inherent when players choose their algorithms competitively instead of using the (single-player) optimal algorithm. In other words, what is the price of anarchy[1] of a given duel? ... Our main open question is (open question 1): does competition between algorithms improve or degrade expected performance?"* As we describe below, we study this open question for a set of dueling games and in particular for the ranking duel which is an appropriate representative of dueling games due to Immorlica et al. [19].

Dueling games.   A *dueling game* $\mathcal{G}$ is a zero-sum game where two players compete for the attention of a user [2]. In a dueling game both players try to beat the other player and offer a better option with a higher value to the user. In particular, while the user's request is unknown to both players and they only have access to probability distribution p, the goal for each player is to maximize the probability that her offer is better than her opponent's offer. This framework falls within a general and natural class of ranking or social context games [7, 11], where each player plays a base game separately and then ultimate payoffs are determined by both their own outcomes and the outcomes of others. Immorlica et al. argue that this class of games models a variety of scenarios of competitions between algorithm designers, such as, competition between search engines (who must choose how to rank search

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[1] Indeed Immorlica et al. [19] use the term of the *price of anarchy* in their aforementioned open question for the same concept of the *price of competition* in this paper.
[2] One can see the user as a population of users with the same behavior.

results), or competition between hiring managers (who must choose from a pool of candidates in the style of the secretary problem).

To be more precise a dueling game is defined by 4-tuple $\mathcal{G} = (\Omega, p, S, v)$, where Ω is the set of all possible requests from the user, p is a probability distribution over set Ω i.e., p_ω is the probability of requesting $\omega \in \Omega$ by the user, S is the set of all possible pure strategies for both players, and $v_\omega(s)$ is the value of pure strategy $s \in S$ for the user upon request $\omega \in \Omega$. Note that v is usually considered to be the valuation of the players, but in this paper valuation function v denotes the value for the user. While a mixed strategy is a probability distribution over all possible pure strategies in S, we write the value of mixed strategy $\mathbf{x}$ as $v_\omega(\mathbf{x}) = E_{s \sim \mathbf{x}}[v_\omega(s)]$.

A social planner is often interested in choosing a strategy which maximizes the social welfare, even though it may be a bad strategy in the competition between players. This means the social welfare maximizer strategy may not appear in any Nash equilibrium of the game, and thus the competition between players results in a suboptimal outcome for the users. Knowing the fact that Nash equilibria of a dueling game can be formed by suboptimal strategies, the following question seems to be an important question to ask regarding the inefficiency of this competition:

> What is the social welfare of any Nash equilibrium in a dueling game in comparison to the social welfare of the optimal strategy?

Price of competition. As aforementioned while in so many cases the competition motivates businesses to optimize their solutions for consumers, there are competitive markets and in particular dueling games of Immorlica et al. [19], in which businesses do not offer the best option to consumers. We define *price of competition* in this paper to capture this phenomenon.

First we note that since dueling games are two-player zero-sum games, Nash equilibria of these games are characterized by minimax strategies. Therefore, one can measure the inefficiency of any Nash equilibrium by comparing the welfare of any minimax strategy, in a game of competition between two players, with the welfare achieved by a social welfare maximizer. We are now ready to define the following criterion for measuring the quality of minimax strategies in a dueling game.

▶ **Definition 1. Price of competition (PoC)** is the ratio between the social welfare of the worst minimax strategy and the social welfare of the best possible strategy.

The proposed concept of the price of competition has the same spirit as the concept of the price of anarchy, and both concepts try to measure the inefficiency of Nash equilibria quantitatively. The price of anarchy, introduced by the seminal work of Koutsoupias and Papadimitriou [23], is a well-known concept in game theory that measures the ratio of the social welfare of the worst Nash equilibrium to the optimal social welfare. Although these two concepts are defined to capture properties of Nash equilibria, they are meaningfully different. In the price of anarchy, the social welfare is defined as the expected utility of all players in an equilibrium "outcome"[3] which is always zero for any zero-sum game. However, in the price of competition, the social welfare is the expected utility of the user (which is not a player) in a minimax "strategy". In fact, the price of competition is aimed to analyze the impact of the competition between players on an external user.

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[3] Which is essentially the same as the sum of utilities of all players.

Since the price of competition captures the inefficiency of minimax strategies in two-player zero-sum games and all Nash equilibria of any two-player zero-sum game can be described by the set of minimax strategies, we believe the price of competition sheds new light on the structural analysis of Nash equilibria in two-player zero-sum games. Indeed as Alon, Demaine, Hajiaghayi, and Leighton [5] mention understanding the structure of Nash equilibiria, and not just the price of anarchy, is very important in general and thus our work is exactly toward this direction.

Due to the space constraints, all the missing proofs are provided in the appendices.

1.1   Our results

Ranking duel: To define the ranking duel more precisely, consider a ranking duel with two players. When a user submits a query to a player, she is basically searching a webpage which is unknown to the player. The player only has a prior knowledge about the requested webpage, i.e., for each webpage the probability that this webpage is requested by the user is known. The strategy of each player is an ordering for displaying webpages. When the requested webpage is realized, the player which puts this webpage in a higher rank gets the user attention, and thus wins the competition. The goal of each player is to maximize the probability of winning the competition. In this situation, a social planner who wants to minimize the expected rank of the requested webpage lists webpages in a decreasing order of their probabilities. However, this strategy may lose the competition to another strategy.[4]

We first investigate the quality of minimax strategies and prove that surprisingly the social welfare of any minimax strategy is not far from that of the optimal solution; it is 0.612 of the optimal solution for the linear valuation functions and 0.25 of the optimal for any arbitrary valuation function.

▶ **Theorem 1.** *Consider an instance of the raking duel. If the valuation function is a non-negative linear function of the rank, the price of competition is at least* 0.612 *for* $|\Omega| \geq 10$, *and at most* 0.833.

Our proof needs a careful understanding of properties for minimax strategies and has three main steps. First, we prove nice structural properties of minimax strategies. This step is the main step toward bounding the price of competition and gives an insight into properties of the polytope of minimax strategies. For example for every two webpages ω_1 and ω_2 with $p_{\omega_1} > p_{\omega_2}$, we prove there is a lower bound on the probability that any minimax strategy ranks webpage ω_1 before webpage ω_2. In the next step, we leverage these properties to write a factor-revealing mathematical program for bounding PoC. At last, we find a linear program where the set of its feasible solutions is a superset of the set of feasible solutions of the former mathematical program. We find the optimal solution of this linear program to formally prove the theorem for $|\Omega| \geq 10$. Moreover, we write a computer program to find the optimal solution of the corresponding linear program and show the price of competition is at least 0.637 for $|\Omega| \geq 100$ (which is slightly better the case that $\Omega \geq 10$). To the best of our knowledge, we are the first to use factor-revealing techniques to bound the inefficiency of equilibria.

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[4] For example consider a situation when the user submits a query and she is interested in webpages w_1, w_2, and w_3 with probabilities 0.35, 0.33 and 0.32 respectively. In this situation the social planner ranks webpage w_i at position i, for $i = 1, 2, 3$. However, if a player plays based on this strategy, her opponent puts webpages w_2, w_3, and w_1 at positions 1, 2, and 3 respectively, and thus wins the competition when the user requests webpages w_2 or w_3. This means the social planner strategy loses the competition with probability 0.65.

Afterwards, we consider the cost minimization version of the ranking duel and prove a constant upper bound for the social cost of the game using the same technique of Theorem 1. Note that the only difference between the cost minimization and welfare maximization of a dueling game is that function v is a cost function rather than a valuation function, and once a webpage is searched the winner of the cost minimization game is the player who provides a solution with a lower cost. Moreover, we define the PoC_{cost} of the ranking duel as the ratio between the minimax strategy with the highest cost and the strategy with the least cost. In the following theorem we show that $\text{PoC}_{cost} \leq 3$.

▶ **Theorem 2.** *For a ranking duel with a linear cost function, we have $PoC_{cost} \leq 3$.*

It is worth mentioning that the structural properties of minimax strategies do not depend on the valuation function, and thus the polytope of minimax strategies remains unchanged for every valuation function which is a decreasing (increasing) function of rank in the welfare maximization (cost minimization) variant of the game. Therefore, we leverage structural properties of the polytope of minimax strategies, which is presented in Theorem 1, for proving Theorem 2 and in general one can apply our techniques for characterizing the polytope of minimax strategies for an arbitrary valuation function. Nevertheless, writing the factor-revealing mathematical program totally depends on the linearity of the valuation function.

General valuation functions: There are situations where the value of the user is not a linear function of rank. For example, consider a user that only cares about the top search results and will be satisfied if and only if her requested webpage is ranked higher than a certain threshold. We investigate the efficiency of minimax strategies for any non-negative non-linear valuation function. Moreover, we go beyond the ranking duel and consider other dueling games, in the pioneering work of [19]. While bounding the social welfare for arbitrary valuation functions and general dueling games seems to be challenging, we present a general principle to capture the main challenge of this problem. The proposed principle has the same spirit as the classic 0-1 principle in the sorting network which states: "a sorting network will sort any given input if and only if it sorts any given 0-1 input [13]." The following principle has the same message and shows if one can bound the social welfare for any 0-1 valuation function, the same bound holds for any arbitrary valuation function. This means the main challenge for bounding the social welfare is to bound it for 0-1 valuation functions. The main idea for proving Theorem 3 is to decompose any valuation function into 0-1 valuation functions.

▶ **Theorem 3.** *0-1 Principle: Consider a dueling game. If the price of competition is greater than α when the social welfare is defined based on any 0-1 valuation function, then it is greater than α when the social welfare is defined based on any valuation function.*

One can leverage this principle to analyze the efficiency of competition in any dueling game. For example, we show that the price of competition in the ranking duel is at least 0.25 for an arbitrary valuation function.

▶ **Theorem 4.** *The price of competition is at least* 0.25 *for the ranking duel, when the social welfare is defined based on an arbitrary valuation function.*

In the proof of Theorem 4, based on the 0-1 principle, we first consider the problem with pseudo-valuation functions in which the value of each position is either 0 or 1. We consider $\mathbf{x}^*$ as the minimax strategy with the least social welfare and construct a response strategy $\mathbf{x}'_i$ for the second player for every $1 \leq i \leq n$ as follows:

  ⬛ Draw a permutation randomly based on strategy $\mathbf{x}^*$. If the value of the position of the i-th webpage is 1 then play that permutation. Otherwise, swap the position of the i-th webpage with one of the positions with value 1 at random and play the new permutation.

Next, we use the fact that minimax strategy $\mathbf{x}^*$ does not lose to strategy $\mathbf{x}'_i$ for proving a set of inequalities which later on helps us to bound the price of competition. Finally, we use the 0-1 principle to show that this lower bound holds for all possible valuation functions.

This principle also helps us to provide upper bounds on the price of competition when one considers a general valuation function. For example we show that the PoC of the following game introduced by Immorlica et al. [19] cannot be bounded by any constant value:

Binary search duel: The binary search duel is a dueling game where each player chooses a binary search tree over the set of all possible requests Ω. When the user's request $\omega \in \Omega$ is realized, the value for each strategy is defined based on the depth of request ω in the corresponding binary search tree.

▶ **Theorem 5.** *The price of competition is $\mathcal{O}(\frac{1}{|\Omega|})$ for the binary search duel, when the social welfare is defined based on an arbitrary valuation function.*

In order to construct bad instances for these duels, we design a valuation function which is 1 for low depths, 0 for high depths, and a small positive value ϵ in between. We show the price of competition is less than any given number $\beta > 0$ for the binary search duel by constructing an instance of the binary search duel with $|\Omega| = \Theta(\frac{1}{\beta})$.

1.2 Related work

Immorlica et al. [19] are the first who considered the concept of dueling games. They present dueling games in the context of dueling algorithms, where two competitive algorithms try to maximize the probability of outperforming their opponent for an unknown stochastic input. While we employ the same model in this paper, our goal completely differs from that of Immorlica et al. [19]. Immorlica et al. [19] present polynomial-time algorithm for finding a minimax strategy of a dueling game when the polytope of minimax strategies can be represented by a polynomial number of linear constraints. Knowing the fact that the polytope of minimax strategies of any ranking duel has polynomially many facets, they propose a polynomial-time algorithm for finding a minimax strategy of ranking duels. This method was later generalized by [3] to solve the Colonel Blotto game. Immorlica et al. [19] leave the problem of analyzing the social welfare of competitive algorithms as their main open question. In this paper, we do not deal with the computational complexity of finding minimax strategies, but we focus on answering the posted open question and analyze the social welfare of minimax strategies for a given duel.

As we are interested in quantifying the inefficiency of Nash equilibria, our proposed concept of the price of competition has the same flavor as the concept of the price of anarchy [23, 26]. The price of anarchy is commonly used for quantifying the inefficiency of a system which is constructed by selfish agents. For example, it has been used to analyze the inefficiency of Nash equilibria in congestion games [27, 12], network creation games [16, 14, 4, 5], and selfish scheduling games [6, 20]. (See, e.g. [26] for more examples).

Kempe and Lucier [21] recently study the impact of competition on the social welfare in a competitive sponsored search market. In their model, which is a departure from the model of Immorlica et al. [19], search engines again compete to obtain more users. A user's request is defined by a set S of webpages which is unknown to search engines, and the user is satisfied if and only if at least one of webpages in S is ranked in a better position than a given threshold t. The strategy of each search engine is an ordering over all possible webpages.

At last, the user chooses a search engine based on a selection rule which is a function of probability of being satisfied by each search engine. Kempe and Lucier [21] prove that if search engines extract utility from satisfied users or the search engine selection rule is convex, then the social welfare of the game is at least half of the optimum social welfare. Moreover, they show if the utility of search engines is driven from all customers and the search engine selection rule is concave, then the social welfare of the game is bounded away from that of the optimum solution by a factor of $\Omega(n)$, where n is the number of all possible webpages. We would like to note that our model is a general model for studying all dueling games which is exactly the same as the model of Immorlica et al. [19], and is significantly different from that of Kempe and Lucier [21].

There is a line of research that study a competition between advertisers in sponsored search auctions [1, 9, 17, 15, 22]. These works analyze the revenue of a single search engine in various settings regarding users' behavior and the business model of advertisers. However, in ranking duel we investigate a competition between players who provide orderings rather than advertisers.

There is a rich literature in economics that explains product differentiation in competitive markets. While producing similar products is supported by classical models such as the Hotelling model [18], Aspremont, Gabszewicz, and Thisse [8] argue that competitive producers may improve their revenue by producing different products. See, e.g., [29, 25, 24] for details on this literature. The same phenomenon can be seen in the sponsored search market, e.g., Telang, Rajan, and Mukhopadhyay [28] show low-quality search engines may extract revenue from the sponsored search market.

2   Model

2.1   Dueling games

In dueling game $\mathcal{G}$ both players try to beat the other player and offer a better value in the competition. Assume players A and B play pure strategies s_A and s_B respectively, and event ω has occurred. In this situation, player A wins the competition if and only if $v_\omega(s_A) > v_\omega(s_B)$, and thus the utility of player A given event ω can be written as follows:

$$u_\omega^A(s_A, s_B) = \begin{cases} +1 & \text{if } v_\omega(s_A) > v_\omega(s_B) \\ 0 & \text{if } v_\omega(s_A) = v_\omega(s_B) \\ -1 & \text{if } v_\omega(s_A) < v_\omega(s_B) \end{cases}$$

Now consider a situation where players A and B play mixed strategies $\mathbf{x}$ and $\mathbf{y}$ respectively and event ω has occurred. The utility of player A is the probability that player A wins the competition minus the probability that player B wins the competition and can be defined as follows:

$$u_\omega^A(\mathbf{x}, \mathbf{y}) = Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}}[v_\omega(s_A) > v_\omega(s_B)] - Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}}[v_\omega(s_A) < v_\omega(s_B)]$$

Finally the overall utility of player A is $u^A(\mathbf{x}, \mathbf{y}) = \sum_\omega p_\omega u_\omega^A(\mathbf{x}, \mathbf{y})$. Since dueling game $\mathcal{G}$ is a zero-sum game the utility of player B is the negation of the utility of player A for each ω, i.e., $u_\omega^B(\mathbf{x}, \mathbf{y}) = -u_\omega^A(\mathbf{x}, \mathbf{y})$ and thus $u^B(\mathbf{x}, \mathbf{y}) = -u^A(\mathbf{x}, \mathbf{y})$.

▶ **Definition 2** (Minimax strategy). Strategy $\mathbf{x}$ of player A is minimax if $\mathbf{x} \in \text{argmax}_{\mathbf{x}'}\{\min_{\mathbf{y}}\{u^A(\mathbf{x}', \mathbf{y})\}\}$. Similarly, Strategy $\mathbf{y}$ of player B is minimax if $\mathbf{y} \in \text{argmax}_{\mathbf{y}'}\{\min_{\mathbf{x}}\{u^B(\mathbf{x}, \mathbf{y}')\}\}$.

Based on the definition of dueling games and the fact that the set of all possible pure strategies for both players is S, we can conclude the outcome of both players in any Nash equilibrium is 0 and moreover the set of minimax strategies of both players coincide. We define the set of minimax strategies by $\mathcal{M}$.

▶ **Definition 3** (Social welfare). Consider dueling game $\mathcal{G} = (\Omega, p, S, v)$. The social welfare of pure strategy s is the expected value of this strategy over all possible events and can be written as $\mathrm{SW}(s) = \sum_\omega p_\omega v_\omega(s)$. The social welfare of mixed strategy $\mathbf{x}$ is $\mathrm{SW}(\mathbf{x}) = E_{s \sim \mathbf{x}}[\mathrm{SW}(s)]$.

In this paper, we are interested to study the social welfare of the game in equilibria. Note that the customer locks into one of the players in long term. On the other hand, both players only try to offer the customer a better option than the other one, and thus play a minimax strategy in the competition. These cause inefficiency in the game. Here we define a new criterion to measure this inefficiency in the game.

▶ **Definition 4** (Price of competition). The price of competition is the ratio of the worst minimax strategy to the optimal solution which is:

$$\frac{\min_{\mathbf{x} \in \mathcal{M}} \mathrm{SW}(\mathbf{x})}{\max_{\mathbf{x}} \mathrm{SW}(\mathbf{x})} = \frac{\min_{\mathbf{x} \in \mathcal{M}} \mathrm{SW}(\mathbf{x})}{\max_{s \in S} \mathrm{SW}(s)}.$$

Similar to the welfare maximization model, we consider the cost minimization model in which players try to beat the opponent by offering a lower cost to the user. In particular we have a cost function c, such that $c_\omega(s)$ denotes the cost of strategy s and event ω. Hence, the utility of player A would be defined as

$$u_\omega^A(\mathbf{x}, \mathbf{y}) = Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}}[c_\omega(s_A) < c_\omega(s_B)] - Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}}[c_\omega(s_A) > c_\omega(s_B)].$$

Similarly we define the social cost $\mathrm{SC}(s) = \sum_\omega p_\omega c_\omega(s)$ for a pure strategy s and $\mathrm{SC}(\mathbf{x}) = E_{s \sim \mathbf{x}}[\mathrm{SC}(s)]$ for a mixed strategy $\mathbf{x}$. Finally the price of competition in cost minimization version is defined as

$$\frac{\max_{\mathbf{x} \in \mathcal{M}} \mathrm{SC}(\mathbf{x})}{\min_{\mathbf{x}} \mathrm{SC}(\mathbf{x})} = \frac{\max_{\mathbf{x} \in \mathcal{M}} \mathrm{SC}(\mathbf{x})}{\min_{s \in S} \mathrm{SC}(s)}.$$

2.2 Ranking duel

Ranking duel is a dueling game where $\Omega = \{1, \cdots, n\}$ is the set of n webpages which can be requested by a user. In this game, the set of pure strategies S is equal to the set of all possible permutations over Ω, i.e., each player outputs an ordering of webpages for the user. We denote each pure strategy of the ranking duel by π (instead of s) where $\pi(\omega)$ is the rank of webpage ω. The valuation function v of a raking duel can be defined based on function $f : \{1, \cdots, n\} \to R^+ \cup \{0\}$ as $v_\omega(\pi) = f(\pi(\omega))$. Consider mixed strategy $\mathbf{x}$ where $\mathbf{x}_\pi$ is the probability that strategy $\mathbf{x}$ outputs permutation π. The *social welfare* of strategy $\mathbf{x}$ can be defined as:

$$\mathrm{SW}(\mathbf{x}) = \sum_\omega \sum_\pi p_\omega \mathbf{x}_\pi f(\pi(\omega)). \tag{1}$$

3 Price of competition in the linear ranking duel

3.1 Welfare maximization ranking duel

In this section we give bounds for the PoC in the ranking duel when the valuation function is non-negative and linear, in other words $f(i) = c(n - i) + d$, where $c, d \geq 0$.

First we formulate the social welfare of strategy $\mathbf{x}$ and the optimal social welfare. Without loss of generality in this section we assume $p_1 \geq p_2 \geq \ldots \geq p_n$. Let $Pr_{\pi \sim \mathbf{x}}[\pi(a) = i]$ denote the probability that in a randomly drawn permutation π from strategy $\mathbf{x}$, the rank of webpage a is i. Similarly let $Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]$ denote the probability that in a randomly drawn permutation π from strategy $\mathbf{x}$, webpage a comes before webpage b.

▶ **Proposition 5.** *In a ranking duel with valuation function f and n webpages, the social welfare of a strategy $\mathbf{x}$ is $SW_f(\mathbf{x}) = \sum_{a=1}^{n} \sum_{i=1}^{n} p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i]f(i)$.*

Let OPT be the strategy with the maximum social welfare. Hence SW(OPT) is formulated as follows.

▶ **Proposition 6.** *In a ranking duel with valuation function f and n webpages, the optimal social welfare is $SW_f(OPT) = \sum_{a=1}^{n} p_a f(a)$.*

Lemma 7 shows that for any minimax strategy $\mathbf{x}$ and any linear function $f(i) = c(n-i)+d$ with $c, d \geq 0$, the PoC is no less than the case in which $f(i) = n - i$.

▶ **Lemma 7.** *For valuation functions $f(i) = n - i$, $f'(i) = c(n - i) + d$ with $c, d \geq 0$, and any strategy $\mathbf{x}$, $\frac{SW_f(\mathbf{x})}{SW_f(OPT)} \leq \frac{SW_{f'}(\mathbf{x})}{SW_{f'}(OPT)}$.*

Thus any lower bound for the PoC with $f(i) = n - i$, is also a lower bound for the PoC with any other linear valuation function. Therefore, from now on we assume $f(i) = n - i$, and use SW($\mathbf{x}$) and SW(OPT) instead of $SW_f(\mathbf{x})$ and $SW_f(OPT)$, respectively. Hence SW(OPT) $= \sum_{a=1}^{n} p_a(n - a)$. Now we try to compute SW($\mathbf{x}$) from a different perspective.

▶ **Proposition 8.** *In a ranking duel with n webpages, the social welfare of strategy $\mathbf{x}$ is*

$$SW(\mathbf{x}) = \sum_{a=1}^{n} \sum_{b=a+1}^{n} p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)].$$

Intuitively by Proposition 8 we can compute the social welfare of a strategy by comparing the ranks of every pairs of webpages. Therefore we define $h_{ab}(\mathbf{x})$ to be the amount that the pair of webpages a and b adds to the social welfare in strategy $\mathbf{x}$, i.e. $h_{ab}(\mathbf{x}) = p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$. Thus we can rewrite Proposition 8 as SW($\mathbf{x}$) $= \sum_{a=1}^{n} \sum_{b=a+1}^{n} h_{ab}(\mathbf{x})$. Hence for every strategy $\mathbf{x}$,

$$\frac{\text{SW}(\mathbf{x})}{\text{SW}(\text{OPT})} = \frac{\sum_{a=1}^{n} \sum_{b=a+1}^{n} h_{ab}(\mathbf{x})}{\sum_{a=1}^{n} p_a(n - a)}.$$

In Lemma 9 we provide our main tool for bounding the price of competition in the linear ranking duel.

▶ **Lemma 9.** *Given a strategy $\mathbf{x}$, if there exist an integer k such that $2 \leq k \leq n$ and for all k different indices $i_1 < i_2 < \ldots < i_k$,*

$$\frac{\sum_{a=1}^{k} \sum_{b=a+1}^{k} h_{i_a i_b}(\mathbf{x})}{\sum_{a=1}^{k} p_{i_a}(k - a)} \geq \alpha,$$

then $\frac{SW(\mathbf{x})}{SW(OPT)} \geq \alpha$.

Now our goal is to provide a lower bound for α when $\mathbf{x}$ is a minimax strategy. In order to do that, first we provide some structural properties of the minimax strategies. Leveraging

these properties we write a mathematical program with k variables p_a and $\binom{k}{2}$ variables h_{ab}. Finally, we provide a factor-revealing linear program to obtain a close lower bound for α in the corresponding mathematical program.

In Lemmas 10, 11, 13, and Proposition 12 we provide the structural properties of the minimax strategies.

▶ **Lemma 10.** *Let $\mathbf{x}$ be a minimax strategy and a and b be two webpages such that $p_a \geq p_b$. Let π_{ba} be any permutation in the support of $\mathbf{x}$ in which b precedes a. Let $i < j$ be the respective position of a and b in π_{ba}, then strategy $\mathbf{x}$ must satisfy,*

$$Pr_{\pi \sim \mathbf{x}}[i < \pi(b) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(b) < j] \geq \frac{p_a}{p_b}(Pr_{\pi \sim \mathbf{x}}[i < \pi(a) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(a) < j]).$$

Intuitively Lemma 10 shows that if $p_a \geq p_b$ and there is a permutation in which b comes before a, then the probability that $\mathbf{x}$ ranks b in interval $[i, j]$ (counting the non-endpoint elements twice) is greater than the probability that $\mathbf{x}$ ranks a in this interval by a factor of $\frac{p_a}{p_b}$. Otherwise, by swapping the rank of a and b we can achieve a strategy that beats $\mathbf{x}$.

▶ **Lemma 11.** *Let $\mathbf{x}$ be a minimax strategy and $\mathbf{x}_\pi$ be the probability that strategy $\mathbf{x}$ plays permutation π. For every pair of webpages a and b with $p_a \geq p_b$, we have*

$$Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \geq (\frac{p_a}{2p_b} - 1)Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]. \tag{2}$$

Briefly, in the proof of Lemma 11 we propose an algorithm to find a set of permutations Π in $\mathbf{x}$, such that 1) for each $\pi \in \Pi$, b comes before a, 2) for each permutation π' in $\mathbf{x}$ in which b comes before a, there is a permutation $\pi \in \Pi$, such that $\pi(b) \leq \pi'(a) \leq \pi(a)$, and 3) the interval of the ranks of b and a are distinct, i.e. for two permutations $\pi, \pi' \in \Pi, [\pi(b), \pi(a)] \cap [\pi'(b), \pi'(a)] = \emptyset$. We apply the inequality in Lemma 10 for all permutations in Π to achieve Lemma 11.

In Proposition 12 and Lemma 13 we provide lower bounds for $h_{ab}(\mathbf{x})$ when $\mathbf{x}$ is a minimax strategy. Hence we can use these lower bounds in the proposed mathematical program to achieve a lower bound for the PoC.

▶ **Proposition 12.** *For minimax strategy $\mathbf{x}$ and webpages a and b such that $p_a \geq p_b$, $h_{ab}(\mathbf{x}) \geq p_b$.*

▶ **Lemma 13.** *For minimax strategy $\mathbf{x}$ and webpages a and b such that $p_a \geq p_b$, $h_{ab}(\mathbf{x}) \geq p_a - 2p_b + \frac{2p_b^2}{p_a}$.*

Leveraging the properties of the minimax strategies we write MP 3. In MP 3, Constraints 5 and 6 force p_a's to satisfy the probability constraints. Using Proposition 12, Constraint 7 forces h_{ab} to be no less than p_b and due to Lemma 13, Constraint 7 forces h_{ab} to be no less than $p_a - 2p_b + \frac{2p_b^2}{p_a}$. By Lemma 9, α in Constraint 4 gives a lower bound for the PoC.
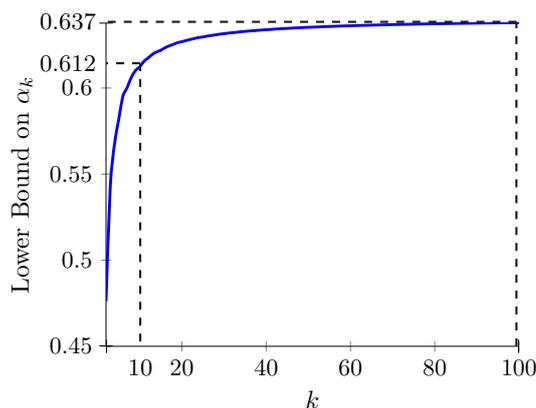
$$\text{minimize } \alpha \tag{3}$$

$$\text{subject to } \alpha = \frac{\sum_{a=1}^{k} \sum_{b=a+1}^{k} h_{ab}}{\sum_{a=1}^{k} p_a(k-a)} \tag{4}$$

$$p_a \geq 0 \qquad \forall 1 \leq a \leq k \tag{5}$$

$$\sum_{1 \leq a \leq k} p_a \leq 1 \tag{6}$$

$$h_{ab} \geq p_b \qquad \forall 1 \leq a < b \leq k \tag{7}$$

$$h_{ab} \geq p_a - 2p_b + \frac{2p_b^2}{p_a} \quad \forall 1 \leq a < b \leq k \tag{8}$$

Figure 1 Lower bound on the solution of MP 3. While we formally prove $\alpha_{10} \geq 0.612$, this figure shows lower bounds on α_k for $2 \leq k \leq 100$ found by a computer program. Note that, by Lemma 14 the PoC of the ranking duel with linear valuation function is at least α_k for all $n \geq k$.

For each k, let α_k be the optimal value of the objective function in MP 3.

▶ **Lemma 14.** *α_k is a lower bound for the PoC of the linear ranking duel where $n \geq k$.*

In Theorem 15 we formally prove $\alpha_{10} \geq 0.612$, which results in PoC ≥ 0.612 for any ranking duel with $n \geq 10$ webpages. Moreover, we write a computer program to find α_k for $2 \leq k \leq 100$ (see Figure 1).

▶ **Theorem 15.** *For a linear ranking duel with $n \geq 10$ webpages, PoC ≥ 0.612.*

The proof idea is as follows. We design a linear program from MP 3. Constraints 4 and 8 in MP 3 are not linear, thus we first try to replace Constraint 8 by three linear constraints. Afterwards, intuitively we scale the probabilities such that $\sum_{a=1}^{k} p_a(k-a)$ equals 1, hence we can have a linear constraint instead of Constraint 4. Then we show each feasible solution for MP 3 is also a feasible solution for the achieved linear program. Finally by finding a feasible solution for the dual of the corresponding LP, we provide a lower bound for α in the primal LP, which is a lower bound for the optimal solution of MP 3 as well.

4   General framework

In this section we present a general framework for analyzing the price of competition in dueling games. Proving lower bounds for the price of competition in dueling games highly depends on the valuation functions and it becomes more challenging when the valuation functions are complex. However, the behavior of minimax strategies only depends on the comparison of the valuation functions rather than actual values. We leverage this fact to provide Theorem 16 which enables us to prove bounds for the price of competition without concerning the complexities of the valuation functions. We refer to this theorem as the *0-1 principle*.

Let (Ω, p, S, v) be a dueling game and α be a non-negative real number. We define the trigger function $v_\omega^\alpha(s)$ for a pure strategy s in the following way:

$$v_\omega^\alpha(s) = \begin{cases} 1 & \text{if } v_\omega(s) \geq \alpha \\ 0 & \text{if } v_\omega(s) < \alpha \end{cases}$$

Moreover, we define the pseudo-welfare function $\text{SW}^\alpha(s)$ as the summation of the values of the trigger functions when a player is playing strategy s with respect to α, $\text{SW}^\alpha(s) = \sum_{\omega \in \Omega} p_\omega v_\omega^\alpha(s)$. Furthermore, the pseudo-welfare function for a mixed strategy $\mathbf{x}$ is defined as $\text{SW}^\alpha(\mathbf{x}) = E_{s \sim \mathbf{x}}[\text{SW}^\alpha(s)]$. Let PoC^α be the pseudo-welfare of the minimax strategy with the least social welfare over SW(OPT) which can be formulated by $\text{PoC}^\alpha = \frac{\text{SW}^\alpha(\mathbf{x}^*)}{\text{SW}^\alpha(\text{OPT})}$, where $\mathbf{x}^*$ is the minimax strategy with the least social welfare and OPT is the strategy with highest social welfare. Note that optimal and minimax strategies are determined regardless of the pseudo-welfare function. For simplicity, we consider $\text{PoC}^\alpha = 1$ when $\text{SW}^\alpha(\text{OPT}) = 0$. In the following we show that the PoC of every dueling game is bounded by $\min_{\alpha \geq 0}\{\text{PoC}^\alpha\}$.

▶ **Theorem 16** (0-1 principle). *For every dueling game we have $PoC \geq \min_{\alpha \geq 0}\{PoC^\alpha\}$.*

In the following subsections we show how we can apply the 0-1 principle to dueling games in order to present lower bounds for the PoC. In Subsection 4.1 we show that the PoC of the ranking duel is at least $\frac{1}{4}$ regardless of the valuation function. Note that, for every α, one could design a valuation function in such a way that $|v_\omega^\alpha(\mathbf{x}) - v_\omega(\mathbf{x})| \leq \epsilon$ while the optimal and minimax strategies remain the same. Therefore, we have the lowest PoC when the range of the valuation function is $[0, \epsilon] \cup [1, 1 + \epsilon]$.

4.1   Ranking duel with general valuation function

Recall that in the ranking duel each position of the permutation has a valuation $f(i)$, each pure strategy of the players is a permutation of webpages $\pi = \langle \pi^{-1}(1), \pi^{-1}(2), \pi^{-1}(3), \ldots, \pi^{-1}(n) \rangle$, and $\Omega = \{1, 2, \ldots, n\}$ is the set of elements of uncertainty. For a webpage $\omega \in \Omega$, $v_\omega(\pi) = f(\pi(\omega))$, where $\pi(\omega)$ is the rank of ω in π. In the following, we use the 0-1 principle to show that the PoC of the ranking duel with an arbitrary valuation function is at least $\frac{1}{4}$.

▶ **Theorem 17.** *The PoC of the ranking duel is at least $\frac{1}{4}$.*

4.2   Binary search duel with general valuation function

In this subsection we study the binary search duel and show that the PoC of this game can be $\Omega(\frac{1}{n})$. In this game $\Omega = \{1, 2, \ldots, n\}$ and each pure strategy of the players is a binary tree such that its in-order traversal visits the elements of Ω in the sorted order. Moreover, $v_\omega(s)$ is determined by $f(d_s(\omega))$ where $d_s(\omega)$ denotes the depth of element ω in the binary search tree corresponding to s and $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a decreasing function.

▶ **Theorem 18.** *For every $\beta > 0$ there is an instance of the binary search duel with $|\Omega| = \theta(\frac{1}{\beta})$ and $PoC \leq \beta$.*

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