The Strongly Stable Roommates Problem

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Abstract

An instance of the strongly stable roommates problem with incomplete lists and ties (SRTI) is an undirected non-bipartite graph $G = (V, E)$, with an adjacency list being a linearly ordered list of ties, which are vertices equally good for a given vertex. Ties are disjoint and may contain one vertex. A matching M is a set of vertex-disjoint edges. An edge $\{x, y\} \in E \setminus M$ is a *blocking edge* for M if x is either unmatched or strictly prefers y to its current partner in M, and y is either unmatched or strictly prefers x to its current partner in M or is indifferent between them. A matching is *strongly stable* if there is no blocking edge with respect to it. We present an $O(nm)$ time algorithm for computing a strongly stable matching, where we denote $n = |V|$ and $m = |E|$. The best previously known solution had running time $O(m^2)$ [16]. We also give a characterisation of the set of all strongly stable matchings. We show that there exists a partial order with $O(m)$ elements representing the set of all strongly stable matchings, and we give an $O(nm)$ algorithm for constructing such a representation. Our algorithms are based on a simple reduction to the bipartite version of the problem.

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1 Introduction

An instance of the STABLE ROOMMATES PROBLEM WITH TIES AND INCOMPLETE LISTS (SRTI) involves a non-bipartite graph $G = (V, E)$, where an adjacency list of each vertex is a linearly ordered list of ties, which are subsets of vertices equally good for a given vertex. Ties are disjoint and may contain one vertex. Thus if vertices b_1 and b_2 are neighbours of a in G then one of the following holds:

- a strictly prefers b_1 to b_2, which we denote by $b_1 \succ_a b_2$
- b_1 and b_2 are tied on the preference list of a, which we denote by $b_1 =_a b_2$
- a strictly prefers b_2 to b_1, which we denote by $b_1 \prec_a b_2$

If vertex a strictly prefers b_1 to b_2 or is indifferent between them, then we say that a *weakly prefers* b_1 to b_2 and we denote it by $b_1 \succeq_a b_2$. A matching M is a set of edges, no two of which share an endpoint. Let $e = (v, w)$ be an edge contained in a matching M. Then we say that vertices v and w are matched in M and that v is the partner of w in M, which we also denote as $v = M(w)$. If a vertex v has no edge of M incident to it, then we say that v is *free* or *unmatched* in M. An edge $(x, y) \in E \setminus M$ is a *blocking edge* for M if x is either unmatched or strictly prefers y to its current partner in M, and y is either unmatched or weakly prefers x to its current partner. A matching is *strongly stable* if there is no edge blocking it. The goal is to determine a strongly stable matching of a given instance or to

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report that no such matching exists. The STABLE MARRIAGE PROBLEM WITH TIES AND INCOMPLETE LISTS (SMTI) is a version of the problem, such that the underlying graph G is bipartite.

Motivation. As the problem name suggests, applications of SRTI arise in the context of assigning students to dormitories [13], [14]. In the problem we try to assign students to share two-person rooms. An instance of SRTI arises in a natural way based on students preferences over one another. The notion of strong stability allows us to prevent the following scenarios. Suppose that we assign a student a to share a room with a student $M(a)$ and a student b to share a room with a student $M(b)$. Assume also that a and b accept each other as a potential rommmates and that a prefers b to $M(a)$ but b is indifferent between a and $M(b)$. Then to improve their situation student a may try to bribe student b, in order to convince b to accept them. Since b is indifferent between a and $M(b)$, they may be happy to share a room with a, denying our assignment.

Previous results. Several algorithms for computing a strongly stable matching in bipartite instances of SMTI have already been given. Let us denote $n = |V|$, $m = |E|$. Irving [5] gave an $O(n^4)$ algorithm for computing strongly stable matchings for instances of SMTI in which the graph is complete. In [9] Manlove extended the algorithm to general SMTI instances, obtaining $O(m^2)$ runtime. In [7] Kavitha, Mehlhorn, Michail and Paluch gave an $O(nm)$ algorithm for the problem. Several structural results related to the problem in instances of SMTI have been given. In [10] the set of strongly stable matchings has been shown to form a distributive lattice (defined in Preliminaries). Recently, in [8] Ghosal, Kunysz and Paluch characterised the set of strongly stable matchings. They described an $O(nm)$ algorithm for constructing a partial order with $O(m)$ elements representing the set of solutions to the problem.

Contrary to the bipartite version of the problem, its non-bipartite generalisation has not received much attention in the literature. The problem of computing a strongly stable matching in non-bipartite instances of SRTI was first solved by Scott [16]. He obtained an $O(m^2)$ algorithm for the problem (The algorithm contained some flaws that can be removed using results from this paper). To the best of our knowledge no structural results related to the problem have been published so far.

Our results. Scott [16] and Manlove [11] asked whether it was possible to use techniques from [7] in order to speed up Scott's algorithm for computing a single strongly stable matching in instances of SRTI from $O(m^2)$ time to $O(nm)$ time. We describe an $O(nm)$ time algorithm for the problem, however we would like to remark that our algorithm is not an extension of Scott's algorithm. Our approach is based on a simple reduction to the bipartite version of the problem. Let $\mathcal{I}$ be an instance of SRTI, and $G = (V, E)$ be an underlying graph. We define an auxiliary instance $\mathcal{I}'$ of SMTI along with its underlying graph $G' = (A \cup B, E')$ as follows. We make two copies $v^p \in A$ and $v^r \in B$ of each vertex $v \in V$. For each edge $\{v, w\} \in E$ we add (v^p, w^r) and (w^p, v^r) to E'. Preference lists in $\mathcal{I}'$ are inherited from preference lists in $\mathcal{I}$. Most of the strongly stable matchings in G' correspond to certain cycles in G, however a deep understanding of the structure of the bipartite instance $\mathcal{I}'$ allows us to filter out matchings which do not correspond to strongly stable matchings in G. This approach allows us not only to obtain a faster algorithm for computing a single strongly stable matching, but also to characterise the set of all strongly stable matchings in instances of SRTI. Our characterisation is based on the construction of a certain partial order with

$O(m)$ elements which allows us to represent all the strongly stable matchings. No such characterisation has been known so far in this setting, however we would like to remark that our construction resembles the one given by Gusfield and Irving [3] for instances of SRI. The presented characterisation can be used to solve a number of problems connected with strongly stable matchings such as enumeration of strongly stable matchings, the minimum regret matching problem and the problem of computing all strongly stable pairs. We would like to point out that we do not address these problems in the paper. The main advantage of our approach is its simplicity. Due to the complicated nature of the problem, it would require a lot of effort to extend Scott's algorithm in order to achieve an $O(nm)$ running time for finding a single strongly stable matching and construct a representation of all the strongly stable matchings. Our algorithms completely avoid the need for low level technical details. The reduction to the bipartite version of the problem allows us also to construct an alternative version of Irving's algorithm [4] for computing stable matchings in instances of SRI. We remark that this has already been observed by Dean and Munshi in [1], where they also use the bipartite formulation of the problem to obtain their results.

Related work. Depending on the way we define a blocking edge in an instance of SRTI we can get two other versions of the stable matching problem. In the weakly stable matching problem an edge $e = (x, y)$ is blocking if by getting matched to each other both x and y would become better off. In the super stable matching problem an edge e is said to be blocking if neither x nor y would become worse off. A matching is respectively *weakly stable* or *super stable* if no blocking edge exists with respect to it.

Super stable matchings in instances of SRTI were investigated by Irving and Manlove [6]. They gave an $O(m)$ time algorithm for finding a super stable matching or reporting that no such matching exists. The algorithm is an extension of Irving's algorithms for finding stable matching in the SRI setting [4] and for finding super stable matching in instances of SMTI [5]. Using a polynomial time reduction from SRTI under super stability to 2-SAT, Fleiner, Irving and Manlove [2] deduced a number of structural results involving super stable matchings. These structural results allowed authors to give algorithms for computing all super stable pairs, enumeration of super stable matchings and finding a minimum regret super stable matching.

In contrast to strongly stable matchings and super stable matchings, the problem of determining whether a weakly stable matching exists in a non-bipartite graph was proven to be NP-complete by Ronn [15]. He proved that NP-completeness holds even if each preference list is either strictly ordered or contains a tie of length 2 at the head.

2 Preliminaries

We start with some additional notation. Let $\mathcal{I}$ be an instance of either SMTI or SRTI. Denote the set of all strongly stable matchings in $\mathcal{I}$ by $\mathcal{M}(\mathcal{I})$. Denote by $V(\mathcal{I})$, $E(\mathcal{I})$ the set of vertices and edges respectively of the underlying graph of $\mathcal{I}$. We say that an instance $\mathcal{I}$ is solvable if there is a strongly stable matching in the underlying graph. We define the rank of w in v's preference list, denoted by $rank(v, w)$, to be 1 plus the number of ties which are preferred to w by v. If $\mathcal{I}$ is a an instance of SMTI, then the underlying bipartite graph is of the form $G = (A \cup B, E)$. As is customary we call the vertices of A and B respectively men and women.

Below we give an overview of known structural results related to strongly stable matchings in bipartite graphs.

▶ **Theorem 1** ([7]). *There is an $\mathcal{O}(nm)$ algorithm to determine a man-optimal strongly stable matching of the given instance or report that no such matching exists.*

We say that a matching is *man-optimal* if every man gets the best partner among all his possible partners in any strongly stable matching. It can be proven that such a matching always exists if a given instance is solvable.

▶ **Theorem 2** (Rural Hospitals Theorem [9]). *In a given instance of* SMTI*, the same vertices are matched in all strongly stable matchings.*

We define an equivalence relation $\sim$ on $\mathcal{M}(\mathcal{I})$ as follows.

▶ **Definition 3.** For two strongly stable matchings M and N, $M \sim N$ if and only if each man m is indifferent between $M(m)$ and $N(m)$. Denote by $[M]$ the equivalence class containing M and denote by $\mathcal{X}$ the set of equivalence classes of $\mathcal{M}(\mathcal{I})$ under $\sim$.

Note that if there are no ties in the instance i.e. $\mathcal{I}$ is an instance of SMI, then each equivalence class of $\sim$ contains exactly one matching. It turns out that if ties are present in the instance, then an equivalence class can contain exponentially many matchings. To see that consider any bipartite graph G which admits a perfect matching. We construct an instance $\mathcal{J}$ of SMTI from G such that the preference list of every vertex is a single tie. Note that perfect matchings in G are strongly stable in $\mathcal{J}$ and all perfect matchings belong to the same equivalence class. If G admits exponentially many perfect matchings, then there are exponentially many strongly stable matchings in $\mathcal{J}$ as well.

Strongly stable matchings belonging to the same equivalence class can be easily characterised (more details in [10]). Thus we focus on structural results related to the set of equivalence classes of $\sim$.

For two strongly stable matchings M and N we say that M dominates N and write $N \preceq M$ if each man m weakly prefers $M(m)$ to $N(m)$. If M dominates N and there exists a man m who prefers $M(m)$ to $N(m)$ then we say that M strictly dominates N and we call N a successor of M.

Next we define a partial order $\preceq^*$ on $\mathcal{X}$.

▶ **Definition 4.** For any two equivalence classes $[M]$ and $[N]$, $[M] \preceq^* [N]$ if and only if $M \preceq N$.

Let M and N be two strongly stable matchings. Consider the symmetric difference $M \oplus N$. By Theorem 2 this set contains only alternating cycles. These cycles display an interesting property captured in:

▶ **Lemma 5** ([10]). *Let M and N be two strongly stable matchings. Consider any alternating cycle C of $M \oplus N$. Let $(m_0, w_0, m_1, w_1, ..., m_{k-1}, w_{k-1})$ be the sequence of vertices of C where m_i are men and w_i are women. Then there are only three possibilities:*
- $(\forall m_i) w_i =_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i =_{w_i} m_{i-1}$
- $(\forall m_i) w_i \prec_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i \succ_{w_i} m_{i-1}$
- $(\forall m_i) w_i \succ_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i \prec_{w_i} m_{i-1}$

Subscripts are taken modulo k.

Below we introduce two operations transforming pairs of strongly stable matchings into other strongly stable matchings.

▶ **Definition 6.** Let M and N be two strongly stable matchings. Consider any man m and his partners $M(m)$ and $N(m)$.

By $M \wedge N$ we denote the matching such that:

- if $M(m) \succeq_m N(m)$ then $(m, M(m)) \in M \wedge N$
- if $M(m) \prec_m N(m)$ then $(m, N(m)) \in M \wedge N$

Similarly by $M \vee N$ we denote the matching such that:

- if $M(m) \succ_m N(m)$ then $(m, N(m)) \in M \vee N$
- if $M(m) \preceq_m N(m)$ then $(m, M(m)) \in M \vee N$

From [10] it follows that both $M \vee N$ and $M \wedge N$ are strongly stable matchings, and $M, N \succeq M \vee N$ and $M, N \preceq M \wedge N$.

We extend operations $\vee$ and $\wedge$ to the set $\mathcal{X}$ of equivalence classes. Let $[M], [N] \in \mathcal{X}$. Denote $[M] \vee [N] = [M \vee N]$, $[M] \wedge [N] = [M \wedge N]$.

A *lattice* is a partially ordered set in which every two elements a, b have a unique infimum (denoted $a \vee b$) and a unique supremum (denoted $a \wedge b$). A lattice L with operations join $\vee$ and meet $\wedge$ is *distributive* if for any three elements x, y, z of L the following holds: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

▶ **Theorem 7** ([10]). *The partial order $(\mathcal{X}, \preceq^*)$ with operations meet $\vee$ and join $\wedge$ defined above forms a distributive lattice.*

It is easy to give an example such that $\mathcal{X}$ is of exponential size. It turns out that it is possible to build a representation of the lattice which is of polynomial size. In order to describe its construction a few more definitions are needed.

▶ **Definition 8.** Let M and N be two strongly stable matchings such that $N \prec M$. We say that N is a **strict successor** of M if and only if there is no strongly stable matching M' such that $N \prec M' \prec M$.

Let M_0 be a man-optimal strongly stable matching, and let M_z be a woman optimal strongly stable matching. We call a sequence $(M_0, M_1, \ldots, M_z)$ such that $M_0 \succ M_1 \succ \ldots \succ M_z$ and M_{i+1} is a strict successor of M_i, a *maximal sequence of strongly stable matchings.*

▶ **Theorem 9** ([8]). *There is an $\mathcal{O}(nm)$ time algorithm to compute a maximal sequence of strongly stable matchings.*

This algorithm works as follows. We first compute a man-optimal matching M_0 in $\mathcal{O}(nm)$ time using the standard algorithm [7], then given a matching M_i we find a strict successor M_{i+1} or determine that M_i is a woman-optimal matching (more details in [8]). We iterate over i until we reach a woman-optimal matching. Using amortized analysis it can be proven that the algorithm runs in $\mathcal{O}(nm)$ time. The algorithm can be easily modified so that it starts with an arbitrary strongly stable matching instead of a man-optimal one.

▶ **Corollary 10.** *Let M_0 be a strongly stable matching. There is an $\mathcal{O}(nm)$ algorithm to compute a sequence of strongly stable matchings $(M_0, M_1, \ldots, M_k)$ such that M_k is a woman-optimal matching, and M_{i+1} is a strict successor of M_i for each i.*

An important property of this algorithm is that in successive matchings each vertex either stays matched to the same partner or gets a partner of a different rank.

▶ **Corollary 11.** *Let $\mathcal{S} = (M_0, M_1, \ldots, M_z)$ be any sequence of strongly stable matchings produced by the algorithm of Corollary 10. Then for each $v \in V(\mathcal{I})$ and $i < z$ we have either $M_i(v) = M_{i+1}(v)$ or $rank(v, M_i(v)) \neq rank(v, M_{i+1}(v))$.*

Let M and N be two strongly stable matchings such that N is a strict successor of M. Recall that $M \oplus N$ is a set of alternating cycles. Consider some matchings $M' \in [M]$, $N' \in [N]$. Depending on the choice of matchings M' and N' it might happen that $M \oplus N \neq M' \oplus N'$. Note that from the definition of $\sim$ it follows that for every vertex v we have $rank(v, M(v)) - rank(v, N(v)) = rank(v, M'(v)) - rank(v, N'(v))$. In other words when we transform a matching from $[M]$ into some matching from $[N]$ the change of v's rank does not depend on the choice of matchings from equivalence classes. This observation motivates the following definition.

▶ **Definition 12.** Let M and N be two strongly stable matchings such that N is a strict successor of M. For any vertex v denote $r_v = rank(v, M(v))$ and $r'_v = rank(v, N(v))$. We say that a set of triples $\rho([M],[N]) = \{(v, r_v, r'_v) : v \in V(\mathcal{I}), r_v \neq r'_v\}$ is a **rotation** transforming $[M]$ into $[N]$.

Let ρ be a rotation and M, N be two strongly stable matchings such that N is a strict successor of M. We say that the set of alternating cycles $M \oplus N$ *realizes* a rotation ρ if $\rho = \rho([M],[N])$. As noted above there are potentially many sets of cycles realizing a given rotation. A rotation ρ is *exposed* in $[M]$ if $\rho = \rho([M],[N])$ for some N which is a strict successor of M. We say that such a rotation *transforms* $[M]$ into $[N]$. Note that a given rotation may be exposed in many equivalence classes.

▶ **Theorem 13** ([8]). *Let $\mathcal{S} = (M_0, M_1, \ldots, M_z)$ be a maximal sequence of strongly stable matchings. For $i \in \{0, 1, \ldots, z-1\}$ denote $\rho_i = \rho([M_i],[M_{i+1}])$. Then the set $D(\mathcal{I}) = \{\rho_0, \rho_1, \ldots, \rho_{z-1}\}$ does not depend on the choice of $\mathcal{S}$, and $\rho_i \neq \rho_j$ for $i \neq j$.*

Note that given a maximal sequence of strongly stable matchings $(M_0, M_1, \ldots, M_z)$ we can easily compute rotations $(\rho_0, \rho_1, \ldots, \rho_{z-1})$ where $\rho_i = \rho([M_i],[M_{i+1}])$. Moreover the set $C_{\rho_i} = M_i \oplus M_{i+1}$ realizes ρ_i for each i. It is important to note that depending on the choice of the maximal sequence $\mathcal{S}$ alternating cycles in C_{ρ_i} may differ.

Let $v \in V(\mathcal{I})$, and let ρ be a rotation. If $(v, f, s) \in \rho$, then we say that ρ *moves* v from rank f to rank s. If a particular maximal sequence of strongly stable matchings $\mathcal{S}$ is given and $\rho = \rho([M_i],[M_{i+1}])$ for some i then we say that a rotation ρ moves v from $M_i(v)$ to $M_{i+1}(v)$.

Note that in an instance of SMI for each rotation there is exactly one set of alternating cycles realizing this rotation. This follows easily from the definition of a rotation and the fact that each equivalence class consists of exactly one matching. It can be proven that in this setting a set of cycles realizing a given rotation always consists of one cycle. Thus in an instance of SMI a rotation can be viewed as a single cycle. In the more general SMTI setting it may happen that a set of cycles realizing a given rotation consists of more than one cycle (Figure 1).

▶ **Definition 14.** Let $D(\mathcal{I})$ be the set of all rotations in a given instance $\mathcal{I}$. We define the order $\prec$ on elements of $D(\mathcal{I})$ as follows. We say that a rotation ρ precedes rotation ρ' and write $\rho \prec \rho'$ if and only if for every maximal sequence $\mathcal{S} = (M_0, M_1, \ldots, M_z)$ of strongly stable matchings we have $\rho = \rho([M_i],[M_{i+1}])$ and $\rho' = \rho([M_j],[M_{j+1}])$ for some i, j such that $i < j$.

Let Z be a subset of $D(\mathcal{I})$. We say that Z is a *closed set* if there is no $\rho \in D(\mathcal{I}) \setminus Z$ such that $\rho \prec \rho'$ for some $\rho' \in Z$. It turns out that each closed set corresponds to an equivalence class of $\sim$. Moreover given such a set we can efficiently find an equivalence class corresponding to it. We briefly explain how to do it.

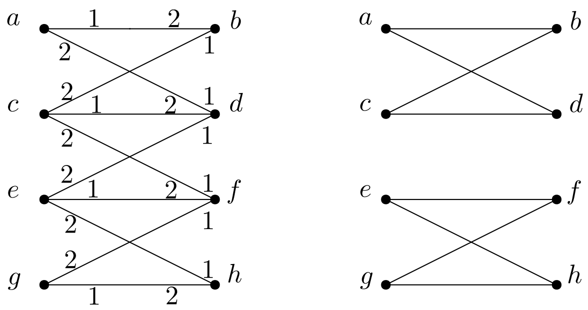

Figure 1 An example instance of SMTI is presented on the left hand side. Natural numbers denote ranks of the corresponding edges. This instance admits two strongly stable matchings $M_1 = \{(a,b),(c,d),(e,f),(g,h)\}$ and $M_2 = \{(a,d),(c,b),(e,h),(g,f)\}$. Cycles realizing the only rotation are presented on the right hand side.

First assume that we are given some particular maximal sequence $\mathcal{S} = (M_0, M_1, \ldots, M_z)$ of strongly stable matchings, the set of rotations $D(\mathcal{I})$, and for each rotation $\rho_i = \rho([M_i],[M_{i+1}])$ a set of cycles $C_{\rho_i} = M_i \oplus M_{i+1}$ realizing it.

▶ **Definition 15.** Let $Z = \{\rho_{a_0}, \rho_{a_1}, \ldots, \rho_{a_{k-1}}\}$ be a closed set. We order its elements so that there are no i, j such that $i < j$ and $\rho_{a_i} \succ \rho_{a_j}$. Let $N_0 = M_0$, $N_{i+1} = N_i \oplus C_{\rho_{a_i}}$. We denote $f_{\mathcal{S}}(Z) = N_k$.

Note that the sequence $\{N_i\}$ depends on the ordering of elements of Z, however its last element $f_{\mathcal{S}}(Z) = M_0 \oplus C_{\rho_{a_0}} \oplus C_{\rho_{a_1}} \oplus \ldots \oplus C_{\rho_{a_{k-1}}}$ is the same regardless of the ordering.

Intuition behind this definition is as follows. We start with an equivalence class $[N_0]$ ($N_0 = M_0$), and some ordering of elements of Z. First we apply a rotation ρ_{a_0} and get a matching $N_1 = N_0 \oplus C_{\rho_{a_0}}$ belonging to $[N_1]$. Then we apply ρ_{a_1} to N_1 and get $N_2 = N_1 \oplus C_{\rho_{a_1}}$ belonging to $[N_2]$. We continue this process until we apply all the rotations. In the end we get a matching $f_{\mathcal{S}}(Z) = N_k$, and a class $[N_k]$. Note that depending on the ordering of Z, sequences $\{N_i\}$ may go through different equivalence classes, however all possible orderings result in the same matching N_k, and an equivalence class $[N_k]$.

The next lemma says that every equivalence class can be obtained in this way from some closed set of rotations.

▶ **Lemma 16.** *For each equivalence class $[M]$ there is a closed set X such that $f_{\mathcal{S}}(X) \in [M]$. Let Z_1 and Z_2 be closed sets. Then $Z_1 \neq Z_2$ implies that $[f_{\mathcal{S}}(Z_1)] \neq [f_{\mathcal{S}}(Z_2)]$.*

For each closed set Z we define $g_{\mathcal{S}}(Z) = [f_{\mathcal{S}}(Z)]$. It can be proven that $g_{\mathcal{S}}$ does not depend on the choice of $\mathcal{S}$ and that $g_{\mathcal{S}}$ is a bijection between closed sets of $D(\mathcal{I}, \prec)$ and the set $\mathcal{X}$. The above discussion is summarized in the following theorem.

▶ **Theorem 17** ([8]). *There is a one-to-one correspondence between the set $\mathcal{X}$ of equivalence classes of $\sim$ and the closed sets of $(D(\mathcal{I}), \prec)$.*

It is important to note that given the function $f_{\mathcal{S}}$ we can get one strongly stable matching from each equivalence class and that depending on the choice of $\mathcal{S}$ these matchings may differ. In other words if $\mathcal{S} \neq \mathcal{S}'$ then it may happen that $f_{\mathcal{S}}(Z) \neq f_{\mathcal{S}'}(Z)$ for some Z, however regardless of the choice of $\mathcal{S}$ and $\mathcal{S}'$ we have $[f_{\mathcal{S}}(Z)] = [f_{\mathcal{S}'}(Z)]$. We emphasize this fact as our algorithm for the non-bipartite version of the problem is based on it.

Note that from Definition 14 alone it is non-trivial how to efficiently construct the relation $\prec$ on $D(\mathcal{I})$. Construction of an explicit representation of the relation $\prec$ would take $\Omega(m^2)$

time, because $D(\mathcal{I})$ might have $\Omega(m)$ elements. It can be proven that we can efficiently construct a sparse subgraph of $(D(\mathcal{I}), \prec)$ such that the closed sets of these two posets are identical.

▶ **Theorem 18** ([8]). *There is a graph $G' = (D(\mathcal{I}), E')$ such that $|E'| = \mathcal{O}(m)$, and the closed sets in G' are exactly the same as the closed sets in the poset $(D(\mathcal{I}), \prec)$. Such a graph can be constructed in $\mathcal{O}(nm)$ time.*

3 The Strongly Stable Roommates Problem

Let $\mathcal{I}$ be an instance of SRTI and let $G = (V, E)$ be the underlying graph. We define an auxiliary instance $\mathcal{I}'$ of SMTI and its underlying bipartite graph $H = (A \cup B, E')$. We make two copies of each vertex $v \in V(G)$, $v^p \in A$ – a *proposing node* and $v^r \in B$ – a *responding node*. For each edge $\{v, w\} \in E$ we add two edges (v^p, w^r) and (w^p, v^r) to E'. Each node in H inherits its preference list from the original instance i.e. for each edge $\{v, w\} \in E$ we have $rank(v^p, w^r) = rank(v, w)$ and $rank(w^r, v^p) = rank(w, v)$. Following the notation from [3] we denote edges of non-bipartite graphs as $\{x, y\}$ rather than (x, y) to emphasise the fact that pairs are unordered.

In the next few lemmas we show that we can derive some useful properties of the structure of strongly stable matchings in $\mathcal{I}$ from the structure of $\mathcal{I}'$. Throughout this section we assume that $\mathcal{I}$ is an instance of SRTI and that $\mathcal{I}'$ is defined as above.

▶ **Definition 19.** Let M be a matching in $\mathcal{I}'$. We say that M is a **symmetric matching** if for each edge $(v, w) \in E$ we have $(v^p, w^r) \in M \iff (w^p, v^r) \in M$.

▶ **Lemma 20.** *There is a one-to-one correspondence between strongly stable matchings in $\mathcal{I}$ and symmetric strongly stable matchings in $\mathcal{I}'$.*

Proof. Let M be a strongly stable matching in $\mathcal{I}$. Let M' be a symmetric matching such that for each $\{v, w\} \in M$ we add (v^p, w^r) and (w^p, v^r) to M'. We can easily see that M' is strongly stable. If there was some edge (x^p, y^r) blocking M' then M would be blocked by $\{x, y\}$. Similarly we can see that for each symmetric strongly stable matching M in $\mathcal{I}'$ there is a corresponding strongly stable matching M' in $\mathcal{I}$. For each $(v^r, w^p) \in M$ we add $\{v, w\}$ to M'. Stability of M' is simple to establish. It is obvious that it is a one-to-one correspondence. ◀

Let M be a strongly stable matching M in $\mathcal{I}$. We denote its symmetric counterpart in $\mathcal{I}'$ by $s(M)$. One can easily see that the equivalence class $[s(M)]$ might contain some matchings which are not symmetric. Moreover, some classes may not contain any symmetric matchings. For instance let M be a matching in $\mathcal{I}'$ such that for some vertex $v \in V(\mathcal{I})$ we have $rank(v^p, M(v^p)) \neq rank(v^r, M(v^r))$. It is clear that none of the matchings from $[M]$ can be symmetric. This observation leads to the following definition.

▶ **Definition 21.** Let M be a strongly stable matching in $\mathcal{I}'$. We say that the class $[M]$ is **symmetric** if for each $v \in V(\mathcal{I})$ we have $rank(v^p, M(v^p)) = rank(v^r, M(v^r))$.

It is obvious that if M is a symmetric matching then $[M]$ is also symmetric. Note that if a matching belongs to a symmetric equivalence class it does not imply that it is actually a symmetric matching. Do all the symmetric equivalence classes contain at least one symmetric matching? It turns out that the answer to this question is negative. A counterexample is presented in Figure 2. It turns out that the following theorem holds:

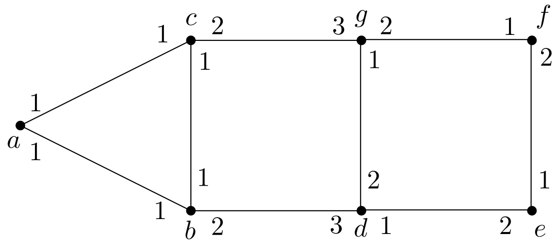

Figure 2 An example of an unsolvable instance $\mathcal{I}$ of SRTI. There are four strongly stable matchings in the corresponding auxiliary instance $\mathcal{I}'$ of SMTI:

- $M_1 = \{(a^p, b^r), (b^p, c^r), (c^p, a^r), (d^p, e^r), (e^p, f^r), (f^p, g^r), (g^p, d^r)\}$
- $M_2 = \{(a^p, b^r), (b^p, c^r), (c^p, a^r), (d^p, e^r), (e^p, d^r), (f^p, g^r), (g^p, f^r)\}$
- $M_3 = \{(a^p, b^r), (b^p, c^r), (c^p, a^r), (d^p, g^r), (g^p, d^r), (e^p, f^r), (f^p, e^r)\}$
- $M_4 = \{(a^p, b^r), (b^p, c^r), (c^p, a^r), (d^p, g^r), (g^p, f^r), (f^p, e^r), (e^p, d^r)\}$

Matchings M_2 and M_3 belong to symmetric equivalence classes but neither of them is symmetric.

▶ **Theorem 22.** *An instance $\mathcal{I}$ is solvable if and only if the following hold:*

- *at least one symmetric equivalence class exists in $\mathcal{I}'$*
- *each symmetric equivalence class in $\mathcal{I}'$ contains at least one symmetric strongly stable matching*

The above theorem implies that if we find a symmetric equivalence class which does not contain any symmetric matching, then the instance is unsolvable. Moreover if we find a strongly stable matching in a given instance $\mathcal{I}$ then every single symmetric equivalence class of $\mathcal{I}'$ contains at least one symmetric matching. Note that the above theorem is somewhat trivial in the case of strict preferences. In an instance of SMI each equivalence class contains exactly one matching. There are no ties, thus M is symmetric if and only if $[M]$ is symmetric.

In order to prove Theorem 22 we are going to make use of the function $f_{\mathcal{S}}$ (Definition 15). It turns out that if an instance $\mathcal{I}$ is solvable, then we can pick a maximal sequence $\mathcal{S}$ such that if a closed set Z corresponds to a symmetric equivalence class, then $f_{\mathcal{S}}(Z)$ is a symmetric matching. Below we give a candidate for such a sequence.

Denote $q(v^r) = v^p$ and $q(v^p) = v^r$ for $v \in V(\mathcal{I})$. For a given matching M in $\mathcal{I}'$ we denote $S(M) = \{(q(w), q(v)) : (v, w) \in M\}$.

▶ **Definition 23.** We say that a maximal sequence of strongly stable matchings $\mathcal{S} = (M_0, M_1, \dots M_{2k})$ is **symmetric** if M_k is a symmetric matching and $S(M_i) = M_{2k-i}$ for each i.

It turns out that if a given instance of $\mathcal{I}'$ admits a symmetric strongly stable matching, then there is a symmetric maximal sequence of strongly stable matchings containing this matching.

▶ **Lemma 24.** *Let M_k be a woman-optimal matching in $\mathcal{I}'$, M_0 be a symmetric strongly stable matching, and let $(M_0, M_1, \dots, M_k)$ be a sequence of strongly stable matchings such that M_{i+1} is a strict successor of M_i. Then the sequence:*

$$\mathcal{Q} = (S(M_k), S(M_{k-1}), \dots, S(M_1), M_0, M_1, \dots, M_k)$$

is a symmetric maximal sequence of strongly stable matchings.

Proof. It suffices to prove that for each i the matching $S(M_i)$ is strongly stable, $S(M_k)$ is man-optimal, and that $S(M_{i-1})$ is a strict successor of $S(M_i)$. Strong stability of $S(M_i)$

follows easily from strong stability of M_i. If there was an edge (v^p, w^r) blocking $S(M_i)$ then (w^p, v^r) would block M_i. It can be easily proven that for any two strongly stable matchings N, N' we have $N \prec N' \iff S(N') \prec S(N)$. If there was a matching N such that $S(M_k) \prec N$, then $S(N) \prec M_k$ would hold, thus M_k would not be woman-optimal. Similarly if there was a strongly stable matching M' such that $S(M_i) \succ M' \succ S(M_{i-1})$, then there would be $M_{i-1} \succ S(M') \succ M_i$. This would contradict the assumption that M_i is a strict successor of M_{i-1}. Thus for each i the matching $S(M_{i-1})$ is a strict successor of $S(M_i)$. Hence $\mathcal{Q}$ is a symmetric maximal sequence of strongly stable matchings. ◀

Corollary 10 along with the proof of Lemma 24 imply that given a symmetric strongly stable matching M in $\mathcal{I}'$ we can compute a symmetric maximal sequence of strongly stable matchings.

▶ **Corollary 25.** *Assume that we are given a symmetric strongly stable matching M in $\mathcal{I}'$. Then there exists a symmetric maximal sequence of strongly stable matchings $(M_0, M_1, \ldots, M_{2k})$ such that $M_k = M$. We can compute such a sequence in $\mathcal{O}(nm)$ time.*

Before we prove Theorem 22 we need to investigate the structure of the set of rotations $D(\mathcal{I}')$. Let M be a symmetric strongly stable matching. Consider a symmetric sequence of strongly stable matchings $\mathcal{S} = (M_0, M_1, \ldots, M_{2k})$ such that $M_k = M$. Let $\rho_i = \rho([M_i], [M_{i+1}])$ for each i. Theorem 13 implies that $D(\mathcal{I}') = \{\rho_0, \rho_1, \ldots, \rho_{2k-1}\}$. From Theorem 17 we know that the set $Z = \{\rho_0, \rho_1, \ldots, \rho_{k-1}\}$ is closed and it corresponds to the equivalence class $[M]$. It turns out that the set of remaining rotations $D(\mathcal{I}') \setminus Z$ has very similar structure to Z. In order to see this we consider matchings M_i, M_{i+1} and their symmetric counterparts $M_{2k-i} = S(M_i)$, $M_{2k-i-1} = S(M_{i+1})$. Let $v \in V(\mathcal{I})$, $w_1^r = M_i(v^p)$, and $w_2^r = M_{i+1}(v^p)$. Observe that rotation ρ_i moves v^p from w_1^r to w_2^r. We can easily see that ρ_{2k-i-1} moves v^r from w_2^p to w_1^p. This observation motivates the following definition.

For a given rotation ρ in $\mathcal{I}'$ we denote $\overline{\rho} = \{(q(v), s, f) : (v, f, s) \in \rho\}$. Similarly for a given cycle C in $\mathcal{I}'$ we denote $\overline{C} = \{(q(w), q(v)) : (v, w) \in C\}$. We say that $\overline{\rho}$ is the *rotation dual* to ρ. Analogously for a given cycle C we say that $\overline{C}$ is the *cycle dual* to C. Note that the rotation dual to $\overline{\rho}$ is equal to ρ. Similarly the cycle dual to $\overline{C}$ is equal to C. We say that the set of all rotations $D(\mathcal{I}')$ is *symmetric* if for each $\rho \in D(\mathcal{I}')$ both $\overline{\rho} \in D(\mathcal{I}')$ and $\rho \neq \overline{\rho}$ hold.

▶ **Theorem 26.** *If $\mathcal{I}$ is solvable, then the set of rotations $D(\mathcal{I}')$ is symmetric.*

Proof. Let $\mathcal{S} = (M_0, M_1, \ldots, M_{2k})$ be a symmetric maximal sequence of strongly stable matchings. Such a sequence exists from Corollary 25. Denote $\rho_i = \rho([M_i], [M_{i+1}])$ for each i. From Theorem 13 we know that $D(\mathcal{I}') = \{\rho_0, \rho_1, \ldots, \rho_{2k-1}\}$, and that $\rho_i \neq \rho_j$ for $i \neq j$. One can easily check that ρ_i is dual to ρ_{2k-i-1} for each i. Thus $D(\mathcal{I}')$ is symmetric. ◀

From the above theorem we know that if $D(\mathcal{I}')$ is not symmetric, then for sure $\mathcal{I}$ does not admit any strongly stable matching. From Theorems 9 and 13 we know how to compute the set $D(\mathcal{I}')$ in $\mathcal{O}(nm)$ time. We can easily check if this set is symmetric.

▶ **Corollary 27.** *There is an $\mathcal{O}(nm)$ algorithm to determine if $D(\mathcal{I}')$ is symmetric.*

It turns out that if the set $D(\mathcal{I}')$ is symmetric then we can characterise the set of symmetric equivalence classes of $\mathcal{I}'$. We say that a set $Z \subseteq D(\mathcal{I}')$ is *complete* if for each rotation $\rho \in D(\mathcal{I}')$ either $\rho \in Z$ or $\overline{\rho} \in Z$. Similarly to Lemma 16, for a given maximal sequence $\mathcal{S}$ we consider the function $f_{\mathcal{S}}$ (Definition 15) however this time we restrict its domain to the set of complete closed sets.

▶ **Lemma 28.** *Let $D(\mathcal{I}')$ be symmetric, $\mathcal{S} = (M_0, M_1, \ldots, M_z)$ be any maximal sequence of strongly stable matchings. For each set $Z \subseteq D(\mathcal{I}')$ which is complete and closed $f_{\mathcal{S}}(Z)$ belongs to a symmetric equivalence class. Moreover for each symmetric equivalence class $[M]$ there is a closed and complete set Z such that $f_{\mathcal{S}}(Z) \in [M]$.*

Proof. Let Z be a complete and closed set of rotations. We prove that $f_{\mathcal{S}}(Z)$ belongs to a symmetric equivalence class. First let us order rotations of $Z = \{\rho_0, \rho_1, \ldots, \rho_k\}$, so that if $i < j$, then $\rho_i \not\succ \rho_j$. Then we denote $N_0 = M_0$, $N_{i+1} = N_i \oplus C_{\rho_i}$ for $0 \leq i < k$. From the definition of rotation it follows that N_{i+1} is a strict successor of N_i. Consider an arbitrary vertex $v \in V(\mathcal{I})$. In order to prove that $f_{\mathcal{S}}(Z)$ belongs to a symmetric equivalence class we need to show that $rank(v^p, N_k(v^p)) = rank(v^r, N_k(v^r))$. For each i the rotation ρ_i moves v^p from $N_i(v^p)$ to $N_{i+1}(v^p)$, thus Z moves v^p from $N_0(v^p)$ to $N_k(v^p)$. From the definition of dual rotation we know that $\overline{\rho_i}$ moves v^r from $q(N_{i+1}(v^p))$ to $q(N_i(v^p))$, hence $D(\mathcal{I}') \setminus Z$ moves v^r from $q(N_k(v^p))$ to $q(N_0(v^p))$ From the completeness of Z we obtain that Z moves v^r from $N_0(v^r)$ to $q(N_k(v^p))$. Thus $rank(v^r, N_k(v^r)) = rank(v^r, q(N_k(v^p)) = rank(v^p, N_k(v^p))$, hence $f_{\mathcal{S}}(Z)$ belongs to a symmetric equivalence class.

Let $[M]$ be a symmetric equivalence class. We prove that there is a closed and complete set Z such that $f_{\mathcal{S}}(Z) \in [M]$. From Lemma 24 there exists a symmetric maximal sequence of strongly stable matchings $(M_0, M_1, \ldots, M_{2k})$ such that $M = M_k$. Analogously to the proof of Theorem 26 we denote $\rho_i = \rho([M_i], [M_{i+1}])$ for each i. From Theorem 13 we know that $D(\mathcal{I}') = \{\rho_0, \rho_1, \ldots, \rho_{2k-1}\}$. Consider the set $Z = \{\rho_0, \rho_1, \ldots, \rho_{k-1}\}$. This set is obviously closed. One can easily check that ρ_i is dual to ρ_{2k-i-1}, thus Z is complete. From Theorem 17 we have $f_{\mathcal{S}}(Z) \in [M]$. ◀

Analogously to Theorem 17 we consider a function $g_{\mathcal{S}}(Z) = [f_{\mathcal{S}}(Z)]$ and obtain the following theorem.

▶ **Theorem 29.** *There is a one-to-one correspondence between the symmetric equivalence classes and the complete closed subsets of rotations in $(D(\mathcal{I}'), \prec)$.*

It may happen that the set of rotations is symmetric but none of the symmetric classes contains a symmetric strongly stable matching (Figure 2). If an instance $\mathcal{I}$ is solvable, then we can pick a maximal sequence $\mathcal{S}$ of strongly stable matchings, such that all values of the function $f_{\mathcal{S}}$ are symmetric strongly stable matchings. Note that this implies Theorem 22 and allows us to characterise strongly stable matchings in a non-bipartite instance $\mathcal{I}$. Below we prove that it suffices to take a sequence obtained from Corollary 25 as $\mathcal{S}$.

▶ **Theorem 30.** *Let $\mathcal{Q} = (M_0, M_1, \ldots, M_{2k})$ be a symmetric maximal sequence of strongly stable matchings obtained as in Corollary 25. For each complete and closed set Z of rotations $f_{\mathcal{Q}}(Z)$ is a symmetric matching.*

Proof. Let Z be a closed and complete set of rotations. Denote $M = f_{\mathcal{Q}}(Z)$. Let $v \in V(\mathcal{I})$ be any vertex, and let $M(v^p) = w^r$. Our goal is to prove that $M(v^r) = w^p$. From Lemma 28 we know that M belongs to a symmetric equivalence class, thus $rank(v^p, M(v^p)) = rank(v^r, M(v^r))$. Denote this rank by r. From the definition of $f_{\mathcal{Q}}$ if an edge belongs to M then it must belong to one of the matchings M_i. Recall that since $\mathcal{Q}$ is a sequence obtained from Corollary 25 we have that for each i, and a vertex x either $M_i(x) = M_{i+1}(x)$ or $rank(x, M_i(x)) \neq rank(x, M_{i+1}(x))$ (Corollary 11). This observation, and the fact that $M_0 \succ M_1 \succ \ldots \succ M_{2k}$ imply that there exists exactly one vertex of rank r matched to v^p amongst all vertices matched to v^p in matchings of $\mathcal{Q}$. Let us denote this vertex by w^r, and assume that $M_j(v^p) = w^r$ for some j. Similarly it can be proven that there exists exactly

Algorithm 1 for computing a symmetric equivalence class

let $\mathcal{I}'$ be an auxiliary symmetric instance of SMTI
if $\mathcal{I}'$ is unsolvable **or** $D(\mathcal{I}')$ is not symmetric **then**
    return no
build a graph G' representing the poset of rotations (Theorem 18)
$Z \leftarrow \emptyset$
for $\rho : \rho \in D(\mathcal{I}')$ **do**
    $indeg(\rho) \leftarrow 0$
    $marked(\rho) \leftarrow 0$
for $\rho : \rho \in D(\mathcal{I}')$ **do**
    for $\rho' : (\rho, \rho') \in G'$ **do**
        $indeg(\rho') \leftarrow indeg(\rho') + 1$
$Q \leftarrow \{\rho \in D(\mathcal{I}') : indeg(\rho) = 0\}$
while $Q \neq \emptyset$ **do**
    $\rho \leftarrow$ any element from Q
    $Q \leftarrow Q \setminus \rho$
    if $marked(\rho) = 0$ **then**
        $Z \leftarrow Z \cup \{\rho\}$
        $marked(\overline{\rho}) \leftarrow 1$
        for $\rho' : (\rho, \rho') \in G'$ **do**
            $indeg(\rho') \leftarrow indeg(\rho') - 1$
            if $indeg(\rho') = 0$ **then**
                $Q \leftarrow Q \cup \{\rho'\}$
return a matching corresponding to Z

one vertex of rank r matched to v^r amongst all vertices matched to v^r in matchings of $\mathcal{Q}$. The fact that $\mathcal{Q}$ is symmetric implies that $M_{2k-j}(v^r) = w^p$, so v^r must be matched to w^p in M. This implies that M is a symmetric matching. ◀

Assuming that we are given one symmetric matching in $\mathcal{I}'$ we can efficiently construct $f_{\mathcal{Q}}$ as shown in Corollary 25. It remains to show how to find some symmetric equivalence class and check if there exists a symmetric matching belonging to this class. If such a matching does not exist then Theorem 22 implies that $\mathcal{I}$ is unsolvable.

Algorithm 1 for determining a symmetric equivalence class works as follows. We first build the symmetric instance $\mathcal{I}'$. If this instance is unsolvable or $D(\mathcal{I}')$ is not symmetric then $\mathcal{I}$ is unsolvable from Theorem 26. From now on we assume that $D(\mathcal{I}')$ is symmetric. We prove that in this case we can find a symmetric equivalence class in $\mathcal{I}'$. The intuition behind the algorithm is very simple. We first construct a set Z which is closed and complete. Initially we set $Z = \emptyset$. During each step of the algorithm we add to Z some rotations ρ such that there does not exist any rotation $\rho' \notin Z$ such that $\rho' \prec \rho$. It is enough to make sure that Z is closed at all times of the execution. In order to make sure that the resulting set is complete, when we add a rotation ρ to Z we mark $\overline{\rho}$, and never add any marked rotations to Z. Once the set Z is constructed we use Theorem 29 to obtain a symmetric equivalence class corresponding to it. Before we prove the correctness of the algorithm we need the following technical lemma.

▶ **Lemma 31.** *Let $\rho, \rho' \in D(\mathcal{I}')$ be two rotations such that $\rho \prec \rho'$. Then $\overline{\rho'} \prec \overline{\rho}$ holds.*

Proof. Assume that $\overline{\rho'} \prec \overline{\rho}$ does not hold. From the definition of $\prec$ we know that there exists a maximal sequence of strongly stable matchings $\mathcal{S} = (M_0, M_1, \ldots, M_{2k})$ such that $\overline{\rho'} = \rho_j$, $\overline{\rho} = \rho_l$ for some $l < j$, where we denote $\rho_i = \rho([M_i], [M_{i+1}])$ for each i. One can easily see that $\mathcal{S}' = (S(M_{2k}), S(M_{2k-1}), \ldots, S(M_0))$ is also a maximal sequence of strongly stable matchings. It can be proven that $\overline{\rho_i} = \rho([S(M_{i+1})], [S(M_i)])$ for each i. Thus $\rho' = \overline{\rho_j}$ and $\rho = \overline{\rho_l}$ – a contradiction with the definition of $\prec$. ◀

Correctness of the algorithm is proven in the following theorem.

▶ **Theorem 32.** *Let $\mathcal{I}$ be an instance of* SRTI*, and let $\mathcal{I}'$ be the auxiliary instance of* SMTI*. Algorithm 1 determines a matching belonging to a symmetric equivalence class of $\mathcal{I}'$ or reports that such a matching does not exist. The runtime of the algorithm is bounded by $\mathcal{O}(nm)$.*

Proof. It is clear from Theorem 26 that if the algorithm returns *no* in line 4, then $\mathcal{I}$ is unsolvable.

We first prove that Z is closed. Closed sets in the poset $(D(\mathcal{I}'), \prec)$, and in G' are identical, hence it suffices to prove that Z is closed in G'. At the start of the execution we have $Z = \emptyset$, so Z is closed. A rotation can be added to Z only if all its immediate predecessors in G' have already been added to Z. It implies that when a rotation is added to Z this set remains closed, hence Z is closed during the entire execution of the algorithm.

We prove that Z is complete when we exit the *while* loop (lines $14 - 23$). Assume to the contrary that Z is not complete. Then there is some rotation ρ such that $\rho, \overline{\rho} \notin Z$. From the pseudocode we can see that neither ρ nor $\overline{\rho}$ is marked. Since $\rho \notin Z$ there exists some rotation ρ' such that $(\rho', \rho) \in G'$ and $\rho' \notin Z$, otherwise at some point ρ would have been added to Q in the line 23, and either ρ or $\overline{\rho}$ would be added to Z. If ρ' is marked then $\overline{\rho'} \in Z$. From Lemma 31 we know that $\rho' \prec \rho$ implies that $\overline{\rho} \prec \overline{\rho'}$ – a contradiction with the fact that Z is closed. Hence ρ' cannot be marked. The rotation $\overline{\rho'}$ is also unmarked, because $\rho' \notin Z$. We can do the same reasoning for ρ' and $\overline{\rho'}$ and get another rotation $\rho'' \notin Z$, such that $\overline{\rho''} \notin Z$ and neither ρ'' nor $\overline{\rho''}$ is marked. We continue this process building a sequence of rotations $\rho \succ \rho' \succ \rho'' \succ \ldots$, and we eventually get a contradiction because our poset is finite. Thus Z is complete, and it corresponds to a symmetric equivalence class.

Let us estimate the complexity of the algorithm. From Corollary 27 we know that computations in lines $3 - 4$ take $\mathcal{O}(nm)$ time. Then we build a graph G' in time $\mathcal{O}(nm)$ (Theorem 18). Number of operations performed in lines $14 - 23$ is proportional to the number of edges of G', which is bounded by $\mathcal{O}(nm)$. Thus the algorithm runs in $\mathcal{O}(nm)$ time overall. ◀

The last step of the algorithm is to show how to determine if a symmetric matching exists in a given symmetric equivalence class.

▶ **Theorem 33.** *Let M be a strongly stable matching belonging to a symmetric equivalence class. We can determine in $\mathcal{O}(\sqrt{n}m)$ time if there is a symmetric strongly stable matching belonging to $[M]$.*

Proof. We will construct a subgraph $G' = (V', E')$ of G (recall that G is the underlying graph of $\mathcal{I}$) such that perfect matchings in G' correspond to symmetric matchings in $[M]$. Let us consider a matching M. For each $v \in V(\mathcal{I})$ such that v^p and v^r are matched in M we add v to V'. We also add to E' each edge (v, w) such that $rank(v, M(v)) = rank(v, w)$ and $rank(w, v) = rank(w, M(w))$. Similarly to the proof of Lemma 20 it can be shown that each symmetric matching in $[M]$ corresponds to a perfect matching in G'. It remains to compute a maximum matching in the non-bipartite graph G' [12]. ◀

The algorithm for computing a single strongly stable matching follows easily from the above discussion. Given an instance $\mathcal{I}$ of SRTI, we first use Algorithm 1 in order to compute a strongly stable matching M belonging to a symmetric equivalence class in an auxiliary instance $\mathcal{I}'$. Then from Theorem 33 we determine a symmetric matching belonging to $[M]$, and output its counterpart in $\mathcal{I}$. Thus we obtain the following theorem:

▶ **Theorem 34.** *There is an $O(nm)$ time algorithm to determine a single strongly stable matching in an instance of* SRTI *or to report that no such matching exists.*

In the Introduction we mentioned that Scott's $O(m^2)$ algorithm contained some flaws. We explain the problem with his algorithm using our terminology. Scott's algorithm can correctly determine whether a symmetric equivalence class exists in an auxiliary instance $\mathcal{I}'$. He claims that a symmetric strongly stable matching can always be found in a given symmetric equivalence class (Lemma 3.3.4 in [16]). This is a false statement as we can see in Figure 2. The algorithm can be repaired using for instance Theorem 33, however an analogue of Theorem 22 is needed to prove its correctness.

Given a single strongly stable matching we can easily construct a representation of the set of all strongly stable matchings. Using Corollary 25 we compute a symmetric maximal sequence of strongly stable matchings and then construct the poset of rotations as in Theorem 17. Closed and complete subsets of rotations correspond to strongly stable matchings from Theorem 29 hence the following holds:

▶ **Theorem 35.** *There is an $O(nm)$ time algorithm to construct a poset $(D(\mathcal{I}'), \prec)$, such that closed and complete subsets of $D(\mathcal{I}')$ correspond to symmetric equivalence classes of an auxiliary instance $\mathcal{I}'$.*

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