Stable States of Perturbed Markov Chains*

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Abstract

Given an infinitesimal perturbation of a discrete-time finite Markov chain, we seek the states that are stable despite the perturbation, *i.e.* the states whose weights in the stationary distributions can be bounded away from 0 as the noise fades away. Chemists, economists, and computer scientists have been studying irreducible perturbations built with monomial maps. Under these assumptions, Young proved the existence of and computed the stable states in cubic time. We fully drop these assumptions, generalize Young's technique, and show that stability is decidable as long as $f \in O(g)$ is. Furthermore, if the perturbation maps (and their multiplications) satisfy $f \in O(g)$ or $g \in O(f)$, we prove the existence of and compute the stable states and the metastable dynamics at all time scales where some states vanish. Conversely, if the big-O assumption does not hold, we build a perturbation with these maps and no stable state. Our algorithm also runs in cubic time despite the weak assumptions and the additional work. Proving its correctness relies on new or rephrased results in Markov chain theory, and on algebraic abstractions thereof.

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1 Introduction

Motivated by the dynamics of chemical reactions, Eyring [4] and Kramers [12] studied how infinitesimal perturbations of a Markov chain affect its stationary distributions. It has been further investigated by, *e.g.*, probability theorists, economists, and computer scientists. In fields of application such as learning and game theory, it is sometimes unnecessary to describe the exact values of the limit stationary distributions: it suffices to know whether these values are zero or not. Thus, the *stochastically stable states* ([5], [10], [17]) were defined in several contexts as the states with positive probability in the limit. We rephrase a definition below.

▶ **Definition 1** (Markov chain and stability). A finite discrete-time Markov chain is a function $m : S \times S \to [0, 1]$ such that $\sum_{y \in S} m(x, y) = 1$ for all x in finite state space S. A stationary distribution is a probability distribution over the states that is invariant under one step of the MC. Let I be a subset of positive real numbers with 0 as a limit point for the usual topology[1]. A perturbation is a function $p : I \times S \times S \to [0, 1]$ such that p_ϵ is a discrete-time MC for all $\epsilon \in I$. If p_ϵ is irreducible for all $\epsilon \in I$, then p is said to be an irreducible perturbation.

A state $x \in S$ is stochastically stable if there is a family of corresponding stationary distributions $(\mu_\epsilon)_{\epsilon \in I}$ s.t. $\liminf_{\epsilon \to 0} \mu_\epsilon(x) > 0$, *i.e.* $1 \in O(\mu(x))$. It is stochastically fully vanishing if $\limsup_{\epsilon \to 0} \mu_\epsilon(x) = 0$ for all $(\mu_\epsilon)_{\epsilon \in I}$. Non-stable states are called vanishing.

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[1] This implies that I is infinite. $]0, 1]$ and $\{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ are typical I.

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Let us motivate Definition 1: we want to find out which states of a real-world system are the most likely to occur a lot in the long run. The system behaves like a MC that we are unable to know exactly, but we know what it is likely to look like, *i.e.* we know a family of MC indexed by ϵ where it occurs for sure, most likely for a very small ϵ. This alone is far too weak to decide state likeliness in the long run, but assuming a good asymptotic behavior of the family for small ϵ implies the existence of a likely state. This is our main result below.

▶ **Theorem 2.** *Consider a perturbation with state space S such that $f \in O(g)$ or $g \in O(f)$ for all f and g in the multiplicative closure of the transition probability functions $\epsilon \mapsto p_\epsilon(x, y)$ with $x \neq y$. Then the perturbation has stable states. Furthermore, any oracle deciding $f \in O(g)$ in constant time allows us to decide stability in $O(|S|^3)$.*

The finiteness of the state space implies that all perturbations have non-fully vanishing states.

1.1 Related works and comparisons

In 1990 Foster and Young [5] defined the stochastically stable states of a general (continuous) evolutionary process, as an alternative to the evolutionary stable strategies [16]. Stochastically stable states were soon adapted in [10] for 2×2 evolutionary games. Then Young [17, Theorem 4] proved a "finite version of results obtained by Freidlin and Wentzel" in [6] and characterized the stochastically stable states if the perturbation satisfies the following assumptions: 1) the p_ϵ are aperiodic and irreducible; 2) the p_ϵ converge to some p_0 when ϵ approaches zero; 3) every transition probability is a function of ϵ that is equivalent to $c \cdot \epsilon^\alpha$ for some non-negative real numbers c and α. The main tool in Young's proof was proved in [11] and is the special case for irreducible chains of the Markov chain tree theorem (see [13] or [6]). Young's characterization involves minimum directed spanning trees, which can be computed in $O(n^2)$ [7] for graphs with n vertices. Since there are at most n roots for directed spanning trees in a graph with n vertices, Young can compute the stable states in $O(n^3)$.

In 2000 Ellison [3] studied the stable states *via* the alternative notion of the radius of a basin of attraction, and wrote that the major drawback of his work compared to Young's is that it is "not universally applicable". In 2005, Greenwald and Wicks [9] designed an algorithm expressing the exact values of the limit stationary distribution of a perturbation, which, as a byproduct, also computes the stable states. Like [17] they consider perturbations that are related to the functions $\epsilon \mapsto \epsilon^\alpha$, but they only require that the functions converge exponentially fast. Also, instead of requiring that the P^ϵ be irreducible for $\epsilon > 0$, they only require that they have exactly one essential class. They do not analyze the complexity of their algorithm, though. We improve upon [17], [3], and [9] in several ways.

1. The perturbations in the literature relate to the maps $\epsilon \mapsto \epsilon^\alpha$. Their specific form and their continuity, especially at 0, are used in the existing proofs. Theorem 2 relaxes this assumption. Continuity, even at 0, is irrelevant, which allows for aggressive, *i.e.*, non-continuous "perturbations". We show that our assumption is (almost) unavoidable.
2. The perturbations in the literature are irreducible (or almost in [9]). It is general enough for perturbations using the maps $\epsilon \mapsto \epsilon^\alpha$, since it suffices to process each sink (aka bottom) irreducible component independently, and gather the results. This trick does not work for general perturbation maps, but Theorem 2 still does not assume irreducibility.
3. The perturbation is abstracted into a weighted graph and shrunk by combining recursively a shortest-path algorithm (w.r.t. some tropical-like semiring) and a strongly-connected-component algorithm. Using tropical-like algebra to abstract over Markov chains has already been done before, *e.g.* in [8], but not to solve the stable state problem.

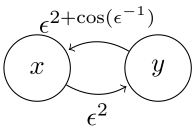

Figure 1

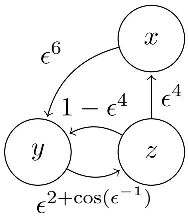

Figure 2

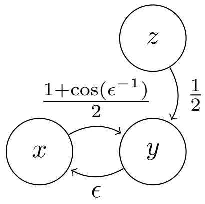

Figure 3

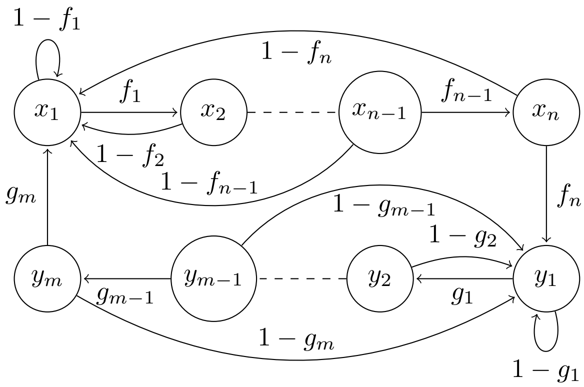

Figure 4

4. We compute the stable states in $O(n^3)$, the best known complexity as in [17], and the computation itself is a summary of the asymptotic behavior of the perturbation: it says at which time scales the vanishing states vanish, and the intermediate graph at each recursive stage of the algorithm accounts for the metastable dynamics at this time scale.

Section 1.2 analyses which assumptions are relevant for the existence of stable states; Section 2 proves the existential part of Theorem 2, *i.e.* it develops the probabilistic machinery to prove the existence of stable states; hinging on this, Section 3 proves the algorithmic part of Theorem 2, *i.e.* it abstracts the relevant objects using a new algebraic structure, presents the algorithm, and proves its correctness and complexity; Section 4 discusses two important special cases and an induction proof principle related to the termination of our algorithm. (Standard notations and proofs can be found in [1].)

1.2 Towards general assumptions

Even continuous perturbations that converge when ϵ approaches 0 may fail to have stable states. For instance let $S := \{x, y\}$ and for all $\epsilon \in]0,1]$ let $p_\epsilon(x,y) := \epsilon^2$ and $p_\epsilon(y,x) := \epsilon^{2+\cos(\epsilon^{-1})}$ as in Figure 1, where the self-loops are omitted. In the unique stationary distribution x has a weight $\mu_\epsilon(x) = (1+\epsilon^{-\cos(\epsilon^{-1})})^{-1}$. Since $\mu_{(2n\pi)^{-1}}(x) = \frac{2n\pi}{1+2n\pi} \to_{n\to\infty} 1$ and $\mu_{(2(n+1)\pi)^{-1}}(x) = \frac{1}{1+2(n+1)\pi} \to_{n\to\infty} 0$, neither x nor y is stable.

As mentioned above, the usual perturbations relate to the maps $\epsilon \mapsto \epsilon^\alpha$ with $\alpha \geq 0$, which rules out Figure 1 and implies the existence of stable states [17]. Here, however, we want to assume as little as possible about the perturbations, while still guaranteeing the existence of stable states. Towards it let us rephrase the big O notation as a binary relation. (Its useful and well-known algebraic properties are mentioned in [1].)

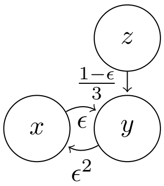

Figure 5

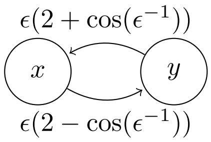

Figure 6

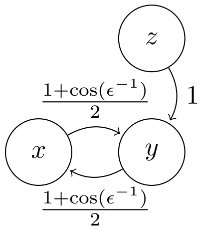

Figure 7

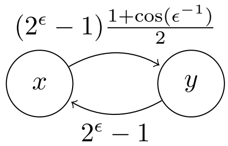

Figure 8

▶ **Definition 3** (Order). For $f, g : I \to [0,1]$, let us write $f \precsim g$ if there exist positive b and ϵ such that $f(\epsilon') \leq b \cdot g(\epsilon')$ for all $\epsilon' < \epsilon$; let $f \cong g$ stand for $f \precsim g \wedge g \precsim f$.

Requiring that every two transition probability maps f and g occurring in the perturbation satisfy $f \precsim g$ or $g \precsim f$ rules out the example from Figure 1, but not the one from Figure 2. There $\mu_\epsilon(z) \leq \mu_\epsilon(x) = \frac{\epsilon^{\cos(\epsilon^{-1})}}{1+\epsilon^{\cos(\epsilon^{-1})}(1+\epsilon^2)}$ and $\mu_\epsilon(y) = \frac{1}{1+\epsilon^{\cos(\epsilon^{-1})}(1+\epsilon^2)}$. So $\mu_\epsilon(z) \to_{\epsilon\to 0} 0$ and $\mu_{2n\pi}(y) \to_{n\to\infty} 0$ and $\mu_{2(n+1)\pi}(x) \to_{n\to\infty} 0$, no state is stable. Informally, z is not stable because it gives everything but receives at most ϵ; neither x nor y is stable since their interaction resembles Figure 1 due to ϵ^6 and $\epsilon^4 \cdot \epsilon^{2+\cos(\epsilon^{-1})}$. This remark is turned into a general Observation 4 below, which will motivate the "unavoidable" Assumption 5.

▶ **Observation 4.** *For $1 \leq i \leq n$ and $1 \leq j \leq m$ let $f_i, g_j : I \to [0,1]$ be s.t. $\prod_i f_i$ and $\prod_j g_j$ are not $\precsim$-comparable. Then there is a perturbation without stable states that is built only with the $f_1, \ldots, f_n, g_1, \ldots, g_m$ and the $1 - f_1, \ldots, 1 - f_n, 1 - g_1, \ldots, 1 - g_m$. See Figure 4.*

▶ **Assumption 5.** *The multiplicative closure of the maps $\epsilon \mapsto p_\epsilon(x, y)$ with $x \neq y$ is totally preordered by $\precsim$.*

E.g, the maps $\epsilon \mapsto c \cdot \epsilon^\alpha$ with $c > 0$ and $\alpha \in \mathbb{R}$ satisfy Assumption 5. We can afford such a weak Assumption 5 because we are not interested in the exact weights of some putative limit stationary distribution, but only whether the weights are bounded away from zero.

Let us show the significance of Assumption 5, which is satisfied in Figures 5 to 8: Young's result shows that y is the unique stable state of the perturbation in Figure 5, but it cannot say anything about Figures 6 to 8: Figure 6 is not regular, *i.e.*, $\frac{2+\cos(\epsilon^{-1})}{2-\cos(\epsilon^{-1})}$ does not converge, and neither do the weights $\mu_\epsilon(x)$ and $\mu_\epsilon(y)$, but it is possible to show that both limits inferior are $1/4$ nonetheless, so both x and y are stable; the transition probabilities in Figure 7 do not converge, and $\frac{1+\cos(\epsilon^{-1})}{2}$ and $1 - \frac{1+\cos(\epsilon^{-1})}{2}$ are not even comparable, but it is easy to see that $\mu_\epsilon(x) = \mu_\epsilon(y) = \frac{1}{2}$; and in Figure 8 the only stable state is x since its weight oscillates between $\frac{1}{2}$ and 1. Note that Assumption 5 rules out Figure 1 to 4 without stable states.

2 Existence of stable states

This section presents three transformations that simplify perturbations while retaining the relevant information about stability. Two of them are defined *via* the dynamics of the original perturbation. Their relevance relies on the close relation between the stationary distributions and the dynamics of MCs. Lemma 6 below pinpoints this relation, where $\mathbb{P}^x(\tau_y^+ < \tau_x^+)$ is the probability that starting from x the MC hits y before returning to x.

▶ **Lemma 6.** *A distribution μ of a finite Markov chain is stationary iff its support involves only essential states and for all states x and y we have $\mu(x)\mathbb{P}^x(\tau_y^+ < \tau_x^+) = \mu(y)\mathbb{P}^y(\tau_x^+ < \tau_y^+)$.*

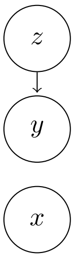

Figure 9

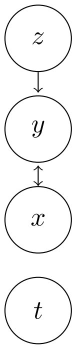

Figure 10

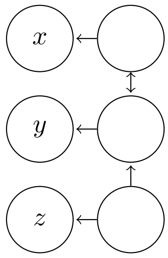

Figure 11

Lemma 6 can already help us find the stable states of small examples such as in Figures 1 and 6. In Figure 1 it says that $\mu_\epsilon(x)\epsilon^2 = \mu_\epsilon(y)\epsilon^{2+\cos(\epsilon^{-1})}$ so we find $\liminf \mu_\epsilon(x) = \liminf \mu_\epsilon(y) = 0$ without calculating the stationary distributions. In Figure 6 it says that $\mu_\epsilon(x)(2-\cos(\epsilon^{-1})) = \mu_\epsilon(y)(2+\cos(\epsilon^{-1}))$, so $\mu_\epsilon(x) \leq 3\mu_\epsilon(y)$ and $\frac{1}{4} \leq \mu_\epsilon(y)$, and likewise for x.

The dynamics, *i.e.*, terms like $\mathbb{P}^x(\tau_y^+ < \tau_x^+)$ are usually hard to compute, and so will be the two transformations that are defined *via* the dynamics, but Lemma 7 below shows that approximating them is safe as far as the stable states are concerned. *E.g.*, the coefficients in Figure 6 (19) can safely be replaced with ϵ (1), and Figure 13 with Figure 14. Lemma 7, where p, μ, *etc.* depend on ϵ, will simplify the computation of the stable states dramatically.

▶ **Lemma 7.** *Let p and $\tilde{p}$ be perturbations with the same state space, s.t. $x \neq y \Rightarrow p(x,y) \cong \tilde{p}(x,y)$. For all stationary distribution maps μ for p, there exists $\tilde{\mu}$ for $\tilde{p}$ such that $\mu \cong \tilde{\mu}$.*

2.1 Essential graph

The *essential graph* of a perturbation captures the non-infinitesimal flow between different states at the normal time scale. It is a very coarse yet useful description of the perturbation.

▶ **Definition 8** (Essential graph)**.** Given a perturbation with state space S, the essential graph is a binary relation over S and possesses the arc (x,y) if $x \neq y$ and $p(z,t) \precsim p(x,y)$ for all $z,t \in S$. The essential classes are the sink SCCs of the graph. The other SCCs are the transient classes. A state in an essential class is essential, the others are transient.

The essential classes will be named $E_1, \ldots, E_k$. Observation 9 below implies that the essential graph is made of the arcs (x,y) such that $x \neq y$ and $p(x,y) \cong 1$, as expected.

▶ **Observation 9.** *Let p be a perturbation. There exist positive c and ϵ_0 such that for all $\epsilon < \epsilon_0$, for all simple paths γ in the essential graph, $c < p_\epsilon(\gamma)$.*

For example, the perturbations (with $I =]0,1]$) that are described in Figures 2, 3, 5, and 7 all have Figure 9 as essential graph, and $\{x\}$ and $\{y\}$ as essential class. Figure 10 (11) is the essential graph of Figure 12 (15), and $\{x,y\}$ and $\{t\}$ (x, y, z) are its essential classes. Note that the essential states of a perturbation and of the related MCs are not the same: in Figure 12, for all $\epsilon \in]0,1]$ all the states are essential for the related MCs. However, if $p_\epsilon = m$ for some m and all ϵ, the essential graph of p is the underlying graph of m. Thus the essential graphs generalize the underlying graphs like the perturbations generalize the MCs.

The essential graph alone cannot tell which states are stable: *e.g.*, swapping ϵ and ϵ^2 in Figure 5 yields the same graph, but then by Lemma 6 the only stable state is x instead of y.

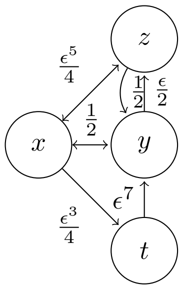

Figure 12

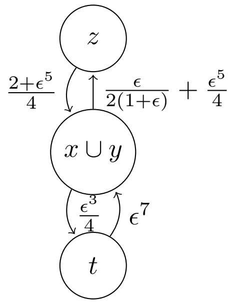

Figure 13

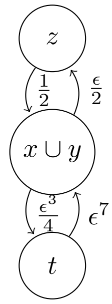

Figure 14

Yet, the graph makes the following case disjunction possible, along which we will either say that all states are stable, or perform one of the transformations from the next subsections.

1. Either the graph is empty (*i.e.* totally disconnected) and the perturbation is trivial, or
2. it is empty and the perturbation is non-trivial, or
3. it is non-empty and has a non-singleton essential class, or
4. it is non-empty and has only singleton essential classes.

Observation 9 motivates the convenient Assumption 10 below. Note that it is just made wlog, *i.e.*, up to focusing on a smaller neighborhood of 0 inside I, whereas Assumption 5 above is a key condition that will appear explicitly in our final result.

▶ **Assumption 10.** *There is $c > 0$ s.t. $p(\gamma) > c$ for all simple paths γ in the essential graph.*

2.2 Essential collapse

The essential collapse, defined below, amounts to merging one essential class of a perturbation into one meta-state and letting this state represent faithfully the whole class in terms of dynamics between the whole class and each of the outside states. ($\mathbb{P}^x(X_{\tau^+_{S\setminus E\cup\{x\}}} = y)$ is the probability that starting from x, the first state hit in $S \setminus E \cup \{x\}$ is y.)

▶ **Definition 11** (Essential collapse of a perturbation)**.** Let p be a perturbation on state space S. Let x be a state in an essential class E, and let $\tilde{S} := (S \setminus E) \cup \{\cup E\}$. The essential collapse $\kappa(p, x) : I \times \tilde{S} \times \tilde{S} \to [0, 1]$ of p around x is defined below.

$$\begin{aligned}
\kappa(p,x)(y,z) &:= p(y,z) && \text{for all } y,z \in S \setminus E\\
\kappa(p,x)(\cup E,\cup E) &:= \mathbb{P}^x(X_{\tau^+_{S\setminus E\cup\{x\}}} = x) &&\\
\kappa(p,x)(\cup E,y) &:= \mathbb{P}^x(X_{\tau^+_{S\setminus E\cup\{x\}}} = y) && \text{for all } y \in S \setminus E\\
\kappa(p,x)(y,\cup E) &:= \sum_{z\in E} p(y,z) && \text{for all } y \in S \setminus E
\end{aligned}$$

▶ **Observation 12.** *$\kappa(p, x)$ is again a perturbation, κ preserves irreducibility, and if $\{x\}$ is an essential class, $\kappa(p, x) = p$.*

For example, collapsing around x or y in Figure 6 has no effect. Figure 12 has essential classes $\{x, y\}$ and $\{t\}$. Figure 13 displays its essential collapse around x. It was calculated by noticing that $\mathbb{P}^x(X_{\tau^+_{\{x,z,t\}}} = t) = \frac{\epsilon^3}{4}$, and $\mathbb{P}^x(X_{\tau^+_{\{x,z,t\}}} = x) = \frac{1}{2} - \frac{\epsilon^3}{4} - \frac{\epsilon^5}{4} + \frac{1}{2} \cdot \mathbb{P}^y(X_{\tau^+_{\{x,z,t\}}} = x)$, and $\mathbb{P}^y(X_{\tau^+_{\{x,z,t\}}} = x) = \frac{1}{2} + \frac{1-\epsilon}{2} \cdot \mathbb{P}^y(X_{\tau^+_{\{x,z,t\}}} = x)$.

Proposition 16 will show that it suffices to compute the stable states of Figure 13 to compute those of Figure 12, and by Lemma 7 it suffices to compute those of the simpler

Figure 14. However, computing the exact values $\mathbb{P}^x(X_{\tau^+_{S\setminus E\cup\{x\}}} = y)$ can be difficult even on simple examples like above. Fortunately, Lemma 13 shows that they are $\cong$-equivalent to maxima that are easy to compute. *E.g.*, using Lemma 13 to approximate the essential collapse of Figure 12 around x yields Figure 14 directly, without Figure 13 as an intermediate.

▶ **Lemma 13.** *Let p be a perturbation with state space S satisfy Assumption 5, and let $\tilde{p}$ be the essential collapse $\kappa(p,x)$ of p around x in some essential class E. For all $y \in S \setminus E$, we have $\tilde{p}(\cup E, y) \cong \max_{z\in E} p(z,y)$ and $\tilde{p}(y, \cup E) \cong \max_{z\in E} p(y,z)$.*

Note that by Lemma 13, only the essential class is relevant during the essential collapse up to $\cong$, the exact state is irrelevant. Lemma 13 is also a tool used to prove, *e.g.*, Proposition 14 which shows that the essential graph may contain useful information about stability.

▶ **Proposition 14.** *Let a perturbation p with state space S satisfy Assumption 5, let μ be a corresponding stationary distribution map.*

1. *If y is a transient state, $\liminf_{\epsilon\to 0} \mu_\epsilon(y) = 0$.*
2. *If two states x and y belong to the same essential or transient class, $\mu(x) \cong \mu(y)$.*

Proposition 14.1 says that the transient states are vanishing, *e.g.* the nameless states in Figure 11. Proposition 14.2 says that two states in the same class are either both stable or both vanishing, *e.g.* $\{x\}$ and $\{y\}$ in Figure 10.

The essential collapse is useful since it preserves (non-)stability, as stated in Proposition 16. Its proof invokes Lemma 15 below, which shows that the essential collapse preserves the dynamics up to $\cong$, and Lemma 6, which relates the dynamics and the stationary distributions.

▶ **Lemma 15.** *Given a perturbation p with state space S, let $\tilde{p}$ be the essential collapse of p around x in some essential class E, and let $\tilde{x} := \cup E$. The following holds for all $y \in S \setminus E$.*
$\mathbb{P}^y(\tau_x < \tau_y) \cong \tilde{\mathbb{P}}^y(\tau_{\tilde{x}} < \tau_y) \quad \wedge \quad \mathbb{P}^x(\tau_y < \tau_x) \cong \tilde{\mathbb{P}}^{\tilde{x}}(\tau_y < \tau_{\tilde{x}})$

▶ **Proposition 16.** *Let a perturbation p with state space S satisfy Assumption 5, and let x be in an essential class E.*

1. *Let $\tilde{p}$ be the chain after the essential collapse of p around x. Let μ ($\tilde{\mu}$) be a stationary distribution map of p ($\tilde{p}$). There exists a stationary distribution map $\tilde{\mu}$ for $\tilde{p}$ (μ for p) such that $\tilde{\mu}(\cup E) \cong \mu(x)$ and $\tilde{\mu}(y) \cong \mu(y)$ for all $y \in S \setminus E$.*
2. *A state $y \in S$ is stable for p iff either $y \in E$ and $\cup E$ is stable for $\kappa(p,x)$, or $y \notin E$ and y is stable for $\kappa(p,x)$.*

By definition, collapsing an essential class preserves the structure of the perturbation outside of the class, so Proposition 16 implies that the collapse commutes up to $\cong$. Especially, the order in which the collapses are performed is irrelevant when computing the stable states.

2.3 Transient deletion

If all the essential classes of a perturbation are singletons, Observation 12 says that the essential collapse is useless. If in addition the essential graph has arcs, there are transient states, and Definition 17 below deletes them to shrink the perturbation further.

▶ **Definition 17** (Transient deletion). Let a perturbation p with state space S, transient states T, and singleton essential classes, satisfy Assumption 5. The function $\delta(p)$ over $S\setminus T$ is derived from p by transient deletion: for all distinct $x, y \in S \setminus T$ let $\delta(p)(x,y) := \mathbb{P}^x(X_{\tau^+_{S\setminus T}} = y)$

▶ **Observation 18.** *$\delta(p)$ is again a perturbation, δ preserves irreducibility, and if all states are essential, $\delta(p) = p$.*

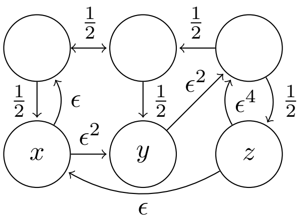

Figure 15

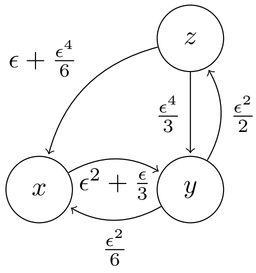

Figure 16

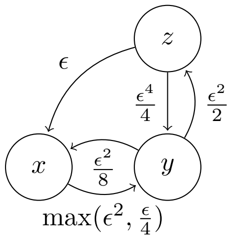

Figure 17

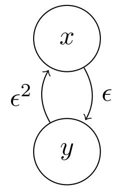

Figure 18

For example, in Figure 5 the essential classes are $\{x\}$ and $\{y\}$, z is transient, and the transient deletion yields Figure 18. Also, in Figure 15, the essential classes are $\{x\}$, $\{y\}$, and $\{z\}$, the transient states are nameless, and the transient deletion yields Figure 16. The transient deletion is useful thanks to Proposition 19 below.

▶ **Proposition 19.** *If a perturbation p satisfy Assumption 5 and has singleton essential classes, p and $\delta(p)$ have the same stable states.*

Like the essential collapse, the transient deletion is defined *via* the dynamics and is hard to compute. Like Lemma 13 did for the collapse, Lemma 20 approximates the transient deletion by an expression that is easy to compute. *E.g.*, Figure 15 yields Figure 17 without computing Figure 16. Note that $\max(\epsilon^2, \frac{\epsilon}{4})$ in Figure 17 may be simplified into ϵ by Lemma 7.

▶ **Lemma 20.** *If p with states S, transient states T, satisfies Assumption 5 and has singleton essential classes, then $\mathbb{P}^x(X_{\tau^+_{S\setminus T}} = y) \cong \max\{p(\gamma) : \gamma \in \Gamma_T(x,y)\}$ for all $x, y \in S \setminus T$.*

2.4 Outgoing scaling and existence of stable states

If the essential graph has no arc, the collapse and the deletion are useless to compute the stable states. This section says how to transform a non-trivial perturbation with empty (*i.e.* totally disconnected) essential graph into a perturbation with the same stable states but a non-empty essential graph, so that collapse or deletion may be applied. Intuitively, it is done by speeding up time until a first non-infinitesimal flow is observable between different states.

Towards it, the *ordered division* is defined in Definition 21. It allows us to divide a function by a function with zeros by returning a default value in the zero case. It is named ordered because we will "divide" f by g only if $f \precsim g$, so that only 0 may be "divided" by 0. Then Observation 22 further justifies the terminology.

▶ **Definition 21** (Ordered division)**.** For $f, g : I \to [0,1]$ and $n > 1$ let us define $(f \div_n g) : I \to [0,1]$ by $(f \div_n g)(x) := \frac{f(x)}{g(x)}$ if $0 < g(x)$ and otherwise $(f \div_n g)(x) := \frac{1}{n}$.

▶ **Observation 22.** *$(f \div_n g) \cdot g = f$ for all n and $f, g : I \to [0,1]$ such that $f \precsim g$.*

▶ **Definition 23** (Outgoing scaling)**.** Let a perturbation p with state space S satisfy Assumption 5, let $m := |S| \cdot \max\{p(z,t) \mid z, t \in S \wedge z \neq t\}$, and let us define the following.

- $\sigma(p)(x,y) := p(x,y) \div_{|S|} m$ for all $x \neq y$
- $\sigma(p)(x,x) := (p(x,x) + m - 1) \div_{|S|} m$.

For example Figure 6 satisfies Assumption 5 and its essential graph is empty. Scaling it yields Figure 19, which satisfies Assumption 5 and whose essential graph has two arcs. Note that collapsing around x or y in Figure 6 has no effect, but in Figure 19 it yields a one-state perturbation. Proposition 24 below states how well the outgoing scaling behaves.

▶ **Proposition 24.** *1. If a perturbation p satisfies Assumption 5, so does $\sigma(p)$, and the essential graph of $\sigma(p)$ is non-empty.*
2. A state is stable for p iff it is stable for $\sigma(p)$.

The outgoing scaling divides the weights of the proper arcs by m, as if time were sped up by m^{-1}. The self-loops thus lose their meaning, but Proposition 24 proves it harmless. Note that the self-loops are also ignored in Assumption 5, Lemma 7, and Definition 8.

Let us describe a recursive computation of the stable states: if the perturbation is constant identity, all its states are stable; else, if the essential graph is empty, apply the outgoing scaling; else, apply one collapse or the transient deletion. This procedure is correct by Propositions 24.2, 16.2, and 19, hence Theorem 25 below (the existential part of Theorem 2).

▶ **Theorem 25.** *Let p be a perturbation such that $f \precsim g$ or $g \precsim f$ for all f and g in the multiplicative closure of the $p(x, y)$ with $x \neq y$. Then p has stable states.*

3 Abstract and quick algorithm

Following the procedure described before Theorem 25 but using the approximation Lemmas 13 and 20 instead of the precise collapse and deletion computes the stable states in $O(n^4)$, where n is the number of states. To improve the speed to $O(n^3)$ we split the deletion into two stages, depending on the lengths of the relevant paths. To improve it further by a multiplicative factor we merge the collapses into these two stages. It would have been cumbersome to define the collapse-deletion merge directly *via* the dynamics in Section 2, and to prove its correctness *via* probabilistic techniques, hence the usefulness of the rather atomic collapse and deletion in the first part of our work. Ensuring that they are safely performed up to $\cong$ is a straightforward sanity check, by Lemma 7, but handling the collapse-deletion merge requires particular attention. Also, the proof for the scaling involves a new algebraic structure accommodating the ordered division: we call it an order-division semiring $(F, 0, 1, \cdot, \leq, \div)$, where F is the quotient of the transition-probability maps by $\cong$, 0 is the zero function, $\div$ is the abstraction of the ordered division, *etc.*, and $[\chi]$ and $[\sigma]$ are the abstractions of the collapse-deletion merge and of the scaling, respectively. All this is well-defined thanks to Assumption 5. Definitions and proofs can be found in Section 3.1 in [1].

Based on these abstractions, this section presents the algorithm (computing the stable states and more), its correctness, and its complexity in $O(n^3)$. Algorithm 1 mainly consists in applying recursively the function $[\chi] \circ [\sigma]$ until a totally disconnected graph is produced. It does not explicitly refer to perturbations since this notion was abstracted on purpose. Instead, the algorithm manipulates digraphs with arcs labeled in an ordered-division semiring, in which inequality, multiplication and ordered division are implicitly assumed to be computable.

One call to the `FindHubRec` corresponds to $[\chi] \circ [\sigma]$, *i.e.* Lines 7 and 9 to $[\sigma]$, and Lines 10 till 18 to $[\chi]$. Before calling `FindHubRec`, Lines 2 and 3 produce an isomorphic copy of the input that is easier to handle when making unions and keeping track of the remaining vertices. Note that Line 9 does not update the $P(x, x)$: it would be useless indeed, since the self-loops are irrelevant by Observation 34 in [1]. Line 10 computes the essential graph up to self-loops, and Line 11 computes the essential classes by a modified Tarjan's algorithm. The computation of $[\chi](P)(\cup E_i, \cup E_j) := \max_{\leq}\{P(\gamma) : \gamma \in \Gamma_T(E_i, E_j)\}$ is performed in two stages: the first stage at Line 12 considers only paths of length one; the second stage at Line 18 considers the paths with a vertex in T. This case disjunction reduces the size of the graph on which the shortest path algorithm from Line 16 is run (and thus the complexity of `FindHub`). Note that it is called with laws $\cdot$ and max instead of $+$ and min. Moreover,

Algorithm 1: FindHub

1 **Function** `FindHub` **is**
  input : (S, P), where $P : S \times S \to F$
  `//` $(F, 0, 1, \cdot, \leq, \div)$ `is an ordered-division semiring.`
  output : a subset of S
2 $\hat{S} \leftarrow \{\{s\} | s \in S\}$; `// For bookkeeping.`
3 **for** $x, y \in S$ **do** $\hat{P}(\{x\}, \{y\}) \leftarrow P(x, y)$; `// For bookkeeping.`
4 **return** `FindHubRec(`$\hat{S}$,$\hat{P}$`)`;
5 **end**
6 **Function** `FindHubRec` **is**
  input : (S, P), where S is a set of sets and $P : S \times S \to F$
  output : a subset of S
7 $M \leftarrow \max\{P(x, y) \,|\, (x, y) \in S \times S \wedge x \neq y\}$;
8 **if** $M = 0$ **then return** $\cup S$; `// Recursion base case`
9 **for** $x, y \in S$ *and* $x \neq y$ **do** $P(x, y) \leftarrow P(x, y) \div M$; `// Outgoing scaling.`
10 $A \leftarrow \{(x, y) \in S \times S \,|\, P(x, y) = 1\}$; `//` A `is a digraph.`
11 $(E_1, \dots, E_k) \leftarrow$`TarjanSinkSCC(`$S$,$A$`)`; `// Returns the sink SCCs of` A.
  `// Maximal labels of direct arcs, below.`
12 **for** $1 \leq i, j \leq k$ **do** $P'(\cup E_i, \cup E_j) \leftarrow \max\{P(x, y) \,|\, (x, y) \in E_i \times E_j\}$;
  `// Maximal labels of all relevant paths, in the remainder.`
13 $T \leftarrow S \setminus (E_1 \cup \dots \cup E_k)$;
14 $P_T \leftarrow P$; `// Initialisation.`
15 **for** $(x, y) \in (S \setminus T) \times S$ **do** $P_T(x, y) \leftarrow 0$; `// Drops arcs not starting in` T.
16 **for** $y \in T$ **do** $P_T(y, \_) \leftarrow$`Dijkstra(`S,P_T,y, $\cdot$, $\max$`)`;
  `//` $P_T(y, \_)$ `is the "distance" function from` $y \in T$`, using` $\cdot$ `and` $\max$.
17 **for** $1 \leq i, j \leq k$ *and* $i \neq j$ *and* $(x_i, x_j, y) \in E_i \times E_j \times T$ **do**
18   $P'(\cup E_i, \cup E_j) \leftarrow \max(P'(\cup E_i, \cup E_j), P(x_i, y) \cdot P_T(y, x_j))$;
19 **end for**
20 **return** `FindHubRec(`$\{\cup E_1, \dots, \cup E_k\}, P'$`)`
21 **end**

since our weights are at most 1 we can use [14] or [2] (which assume non-negative weights) to implement Line 16. Proposition 26 below shows that our algorithm is fast, and our main algorithmic result follows, which is the algorithmic part of Theorem 2.

▶ **Proposition 26.** *The algorithm* `FindHub` *terminates within* $O(n^3)$ *computation steps, where* n *is the number of vertices of the input graph.*

▶ **Theorem 27.** *Let a perturbation* p *satisfy Assumption 5. A state is stochastically stable iff it belongs to* `FindHub`$(S, [p])$. *Provided that inequality, multiplication, and ordered division between equivalence classes of perturbation maps can be computed in constant time, stability can be decided in* $O(n^3)$*, where* n *is the number of states.*

One achievement of our algorithm is that it processes all weighted digraphs (*i.e.* abstractions of perturbations) uniformly: neither irreducibility nor any kind of connectedness is required. *E.g.* in Figures 20 to 25, the four-state perturbation is the disjoint union of two

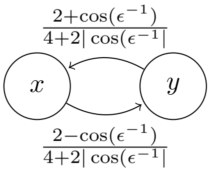

Figure 19

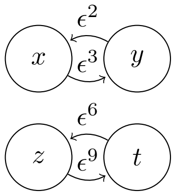

Figure 20 Initial perturbation.

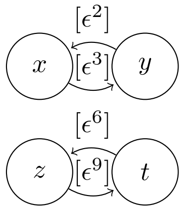

Figure 21 Abstraction.

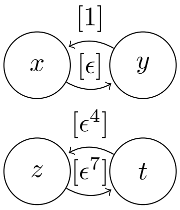

Figure 22 Outgoing scaling.

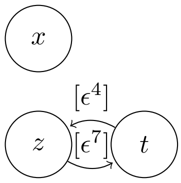

Figure 23 Transient deletion.

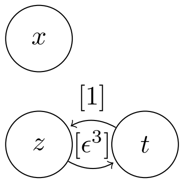

Figure 24 Outgoing scaling.

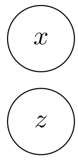

Figure 25 Transient deletion.

smaller perturbations. As expected the stable states of the union are the union of the stable states, *i.e.* $\{x, z\}$, but whereas the outgoing scaling applied to the bottom of Figure 21 (the perturbation restricted to $\{z, t\}$) would yield the bottom of Figure 24 directly by division by $[\epsilon^6]$, two rounds of ougtoing scaling lead to this stage when processing the four-state perturbations.

4 Discussion

This section studies two special cases of our setting: first, how assumptions that are stronger than Assumption 5 make not only some proofs easier but also one result stronger; second, how far Young's technique can be generalized. Then we notice that the termination of our algorithm defines an induction proof principle, which is used to show that the algorithm computes a well-known object when fed a strongly connected graph. Eventually, we discuss how to give the so-far-informal notion of time scale a formal flavor.

Stronger assumption

We consider Assumption 28, a stronger version of Assumption 5. It yields Proposition 29, a stronger version of Proposition 14.1. (The proofs are similar but the new one is simpler.)

▶ **Assumption 28.** *If $x \neq y$ and $p(x, y)$ is non-zero, it is positive; and $f \cong g$ or $f \in o(g)$ or $g \in o(f)$ for all f and g in the multiplicative closure of the $\epsilon \mapsto p_\epsilon(x, y)$ with $x \neq y$.*

▶ **Proposition 29.** *Let a perturbation p with state space S satisfy Assumption 28, and let μ be a stationary distribution map for p. If y is a transient state, $\lim_{\epsilon\to 0} \mu_\epsilon(y) = 0$.*

Under Assumption 5 some states may be neither stable nor fully vanishing: y in Figure 8 and x in Figure 1 where the bottom ϵ^2 is replaced with ϵ. Assumption 28 rules out such cases.

▶ **Corollary 30.** *States of perturbations under Assumption 28 are stable or fully vanishing.*

Generalization of Young's technique

Our proof of existence of and computation of the stable states of a perturbation are very different from Young's [17] who uses a finite version of the Markov chain tree theorem. Here we investigate how far Young's technique can be generalized. This will suggest that we were right to change approaches, but it will also yield a decidability result in Proposition 34.

Lemma 31 generalizes [17, Lemma 1]. Both proofs use the Markov chain tree theorem, but they are significantly different nonetheless. Let p be a perturbation with state space S. As in [17] or [8], for all $x \in S$ let $\mathcal{T}_x$ be the set of the spanning trees of (the complete graph of) $S \times S$ that are directed towards x. For all $x \in S$ let $\beta^x_\epsilon := \max_{T \in \mathcal{T}_x} \prod_{(z,t)\in T} p_\epsilon(z,t)$.

▶ **Lemma 31.** *A state x of irreducible p with state space S is stable iff $\beta^y \precsim \beta^x$ for all $y \in S$.*

Assumption 5 and Lemma 31 together yield Observation 32, a generalization of existing results about existence of stable states, such as [17, Theorem 4]. The underlying algorithm runs in time $O(n^3)$ where n is the number of states, just like Young's.

▶ **Observation 32.** *Let a perturbation p on state space S satisfy Assumption 5. If for all $x \neq y$ the map $p(x,y)$ is either identically zero or strictly positive, p has stable states.*

The stable states of a perturbation are computable even without the positivity assumption from Observation 32, but their existence is no longer guaranteed by the same proof. In this way, Observation 33 is like the existential part of Theorem 2, but with a bad complexity.

▶ **Observation 33.** *Let F be a set of perturbation maps of type $I \to [0,1]$ for some I. Let us assume that F is closed under multiplication by elements in F and by characteristic functions of decidable subsets of I, that $\precsim$ is decidable on $F \times F$, and that the supports of the functions in F are uniformly decidable. If $f \precsim g$ or $g \precsim f$ for all $f, g \in F$, stability is decidable in $O(n^5)$ for the perturbations p such that $x \neq y \Rightarrow p(x,y) \in F$.*

The assumption $f \precsim g$ or $g \precsim f$ for all $f, g \in F$ from Observation 33 relates to Assumption 5. Proposition 34 drops it while preserving decidability of stability, but at the cost of an exponential blow-up since the supports of the maps are no longer ordered by inclusion.

▶ **Proposition 34.** *Let F be a set of maps of type $I \to [0,1]$ for some I. Let us assume that F is closed under multiplication by elements in F and by characteristic functions of decidable subsets of I, that $\precsim$ is decidable on $F \times F$, and that the supports of the functions in F are uniformly decidable. Then stability is decidable for the p such that $x \neq y \Rightarrow p(x,y) \in F$.*

What does Algorithm 1 compute?

Applying sequentially the scaling, collapse, and deletion terminates, so it amounts to an *induction proof principle* for finite graphs with arcs labeled in an ordered-division semiring. Observation 35 is proved along this principle. It can also be proved by an indirect argument using Lemma 31 and Theorem 27, but the inductive proof is simple and from scratch.

▶ **Observation 35.** *Let $(F, 0, 1, \cdot, \leq, \div)$ be an ordered-division semiring, and let $P : S \times S \to F$ correspond to a strongly connected digraph, where an arc is absent iff its weight is 0. Then* `FindHub`(S, P) *returns the roots of the maximum directed spanning trees.*

Note that finding the roots from Observation 35 is also doable in $O(n^3)$ by computing the maximum spanning trees rooted at each vertex, by [7] which uses the notion of *heap*, whereas `FindHub` uses a less advanced algorithm.

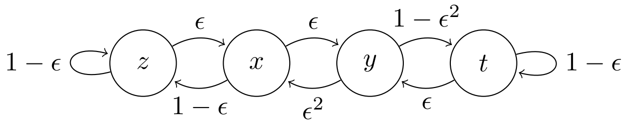

Figure 26 Vanishing time scale.

Vanishing time scales

Computing `FindHub` induces an order in which the states are found to be vanishing. Under the stronger Assumption 28, a notion of *vanishing time scale* may be defined, with the flavor of non-standard analysis [15]. Let $(\mathcal{T}, \cdot)$ be a group of functions $I \to]0, +\infty[$ such that $f \cong g$ or $f \in o(g)$ or $g \in o(f)$ for all f and g in $\mathcal{T}$. The elements of $[\mathcal{T}]$ are called the time scales. Let p over states S satisfy Assumption 28 and let $x \in S$ be deleted at the d-th recursive call of $\texttt{FindHub}(S, [p])$. Let $M_1, \dots, M_d$ be the maxima (*i.e.* M) from Line 7 in Algorithm 1 at the 1st,...,d-th recursive calls, respectively. We say that x vanishes at time scale $\prod_{1 \le i \le d} M_i^{-1}$.

Figure 26 suggests that a similar account of vanishing times scales, even just a qualitative one, would be much more difficult to obtain by invoking the Markov chain tree theorem as in [17]. The only stable state is t; the state z vanishes at time scale $[\epsilon]^{-2}$; and x and y vanish at the same time scale [1] although the maximum spanning trees rooteed at x and y have different weights: ϵ^4 and ϵ^3, respectively.

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