Connected Reversible Mealy Automata of Prime Size Cannot Generate Infinite Burnside Groups *

Thibault Godin[1] and Ines Klimann[2]

1 **Univ. Paris Diderot, Sorbonne Paris Cité, IRIF, UMR 8243 CNRS, F-75013 Paris, France** godin@liafa.univ-paris-diderot.fr

2 **Univ. Paris Diderot, Sorbonne Paris Cité, IRIF, UMR 8243 CNRS, F-75013 Paris, France** klimann@liafa.univ-paris-diderot.fr

Abstract

The simplest example of an infinite Burnside group arises in the class of automaton groups. However there is no known example of such a group generated by a reversible Mealy automaton. It has been proved that, for a connected automaton of size at most 3, or when the automaton is not bireversible, the generated group cannot be Burnside infinite. In this paper, we extend these results to automata with bigger stateset, proving that, if a connected reversible automaton has a prime number of states, it cannot generate an infinite Burnside group.

1998 ACM Subject Classification F.4.3 Formal Languages, F.1.1 Models of Computation, G.2.m Miscellaneous

Keywords and phrases Burnside problem, Automaton groups, Reversibility, Orbit trees

Digital Object Identifier 10.4230/LIPIcs.MFCS.2016.44

1 Mealy automata and the General Burnside problem

The Burnside problem is a famous, long-standing question in group theory. In 1902, Burnside asked if a *finitely generated* group whose all elements have finite order –henceforth called a *Burnside group*– is necessarily finite [3].

The question stayed open until Golod and Shafarevitch exhibit in 1964 an infinite group satisfying Burnside's conditions [8, 9], hence solving the general Burnside problem. In the early 60's, Glushkov suggested using automata to attack the Burnside problem [6]. Later, Aleshin [2] in 1972 and then Grigorchuk [10] in 1980 gave simple examples of automata generating infinite Burnside groups. Over the years, automaton groups have been successfully used to solve several other group theoretical problems and conjectures such as Atiyah, Day, Gromov or Milnor problems; the underlying automaton structure can indeed be used to better understand the generated group.

It is remarkable that every known examples of infinite Burnside automaton groups are generated by non reversible Mealy automata, that is, Mealy automata where the input letters do not all act like permutations on the stateset. We conjecture that it is in fact impossible for a reversible Mealy automaton to generate an infinite Burnside group. Our past work with several co-authors has already given some partial results to this end. In [7] it is proven that non bireversible Mealy automata cannot generate Burnside groups. For the whole class of reversible automata, it has been proved in [12] that 2-state reversible Mealy automata cannot generate infinite Burnside groups. This result has later been extended in [13] to 3-state

* This work was partially supported by the French *Agence Nationale pour la Recherche*, through the Project **MealyM** ANR-JS02-012-01.

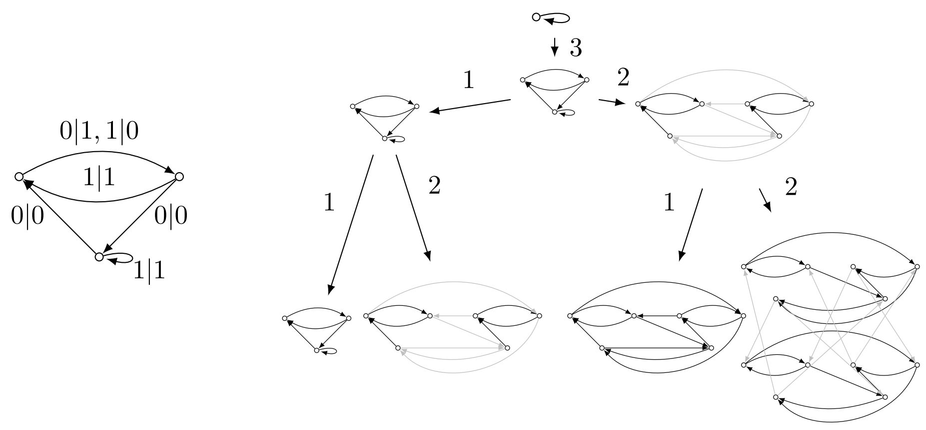

Figure 1 The Bellaterra automaton $\mathcal{B}$ and four levels of the orbit tree $\mathfrak{t}(\mathcal{B})$.

connected reversible automata. In this paper we generalize these results to any connected revertible automaton with a prime number of states:

▶ **Theorem.** *A connected invertible-reversible Mealy automaton of prime size cannot generate an infinite Burnside group.*

Our proof is inspired by the former work in the 3-state case of the second author with Picantin and Savchuk [13]. However the extension from 3 to any prime p required the introduction of a new machinery. This constitutes the main part of our paper, see Section 5.

The paper is organized as follows. In Section 2 we set up notations and recall useful facts on Mealy automata, automaton groups, and rooted trees. Then in Section 3 we link some characteristics of an automaton group to the connected components of the powers of the generating automaton. In Section 4 we introduce a tool developed in [13], the labeled orbit tree, that is used in Section 5 to define our main tool, the *jungle tree*. In this former section we also present some constructions and properties connected to this jungle tree. At last, in Section 6, we gather our information and prove our main result.

2 Basic notions

2.1 Groups generated by Mealy automata

We first recall the formal definition of an automaton. A *(finite, deterministic, and complete) automaton* is a triple $\big(Q, \Sigma, \delta = (\delta_i \colon Q \to Q)_{i\in\Sigma}\big)$, where the *stateset* Q and the *alphabet* Σ are non-empty finite sets, and the δ_i are functions.

A *Mealy automaton* is a quadruple $\mathcal{A} = (Q, \Sigma, \delta, \rho)$, where both (Q, Σ, δ) and (Σ, Q, ρ) are automata. In other terms, it is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet. Its *size* $\#\mathcal{A}$ is the cardinality of its stateset.

The graphical representation of a Mealy automaton is standard, see Figure 1 left.

A Mealy automaton $(Q, \Sigma, \delta, \rho)$ is *invertible* if the functions ρ_x are permutations of Σ and *reversible* if the functions δ_i are permutations of Q.

In a Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}(\mathcal{A}) = (\Sigma, Q, \rho, \delta)$. Obviously, a Mealy automaton is reversible if and only if its dual is invertible.

An invertible Mealy automaton is *bireversible* if it is reversible (i.e. the input letters of the transitions act like permutations on the stateset) and the output letters of the transitions act like permutations on the stateset.

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton. Each state $x \in Q$ defines a mapping from Σ^* into itself recursively defined by:

$$\forall i \in \Sigma,\ \forall \mathbf{s} \in \Sigma^*, \qquad \rho_x(i\mathbf{s}) = \rho_x(i)\rho_{\delta_i(x)}(\mathbf{s})\ .$$

The mapping ρ_x for each $x \in Q$ is length-preserving and prefix-preserving: it is the function *induced* by x. For $\mathbf{x} = x_1 \cdots x_n \in Q^n$ with $n > 0$, set $\rho_\mathbf{x} \colon \Sigma^* \to \Sigma^*, \rho_\mathbf{x} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}$.

Denote dually by $\delta_i \colon Q^* \to Q^*, i \in \Sigma$, the functions induced by the states of $\mathfrak{d}(\mathcal{A})$. For $\mathbf{s} = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, set $\delta_\mathbf{s} \colon Q^* \to Q^*,\ \delta_\mathbf{s} = \delta_{s_n} \circ \cdots \circ \delta_{s_1}$.

The semigroup of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$ is called the *semigroup generated by* $\mathcal{A}$ and is denoted by $\langle \mathcal{A} \rangle_+$. When $\mathcal{A}$ is invertible, the functions induced by its states are permutations on words of the same length and thus we may consider the group of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$. This group is called the *group generated by* $\mathcal{A}$ and is denoted by $\langle \mathcal{A} \rangle$.

2.2 Terminology on trees

Throughout this paper, we will use different sorts of labeled trees. Here we set up some common terminology for all of them.

All our trees are rooted, *i.e.* with a selected vertex called the *root*. We visualize the trees traditionally as growing down from the root. A *path* is a (possibly infinite) sequence of adjacent edges without backtracking from top to bottom. A path is *initial* if it starts at the root of the tree. A *branch* is an infinite initial path. The lead-off vertex of a non-empty path $\mathbf{e}$ is denoted by $\top(\mathbf{e})$ and its terminal vertex by $\bot(\mathbf{e})$ whenever the path is finite.

The *level of a vertex* is its distance to the root and the *level of an edge* or *a path* is the level of its initial vertex.

If the edges of a rooted tree are labeled by elements of some finite set, the *label* of a (possibly infinite) path is the ordered sequence of labels of its edges.

Extending the notions of children, parents and descendent to the edges, we will say that an edge f is the *child* of an edge e if $\bot(e) = \top(f)$ (*parent* being the converse notion, and *descendent* the transitive closure).

All along this article we will follow walks on some trees. A *walk* is just a path in a tree, which is build gradually. In particular if $\mathbf{e}$ is a finite path (or can identify one), to say that it can be followed by f in some tree means that $\mathbf{e}f$ is (or identifies) also a path in that tree.

3 Connected components of the powers of an automaton

In this section we detail the basic properties of the connected components of the powers of a reversible Mealy automaton, as it has been done in [13]. The link between these components is central in our construction.

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a reversible Mealy automaton.

By reversibility, all the connected components of its underlying graph are strongly connected.

Consider the powers of $\mathcal{A}$: for $n > 0$, its *n-th power* $\mathcal{A}^n$ is the Mealy automaton

$$\mathcal{A}^n = \big(\ Q^n, \Sigma, (\delta_i \colon Q^n \to Q^n)_{i\in\Sigma}, (\rho_\mathbf{x} \colon \Sigma \to \Sigma)_{\mathbf{x}\in Q^n}\ \big)\ .$$

By convention, $\mathcal{A}^0$ is the trivial automaton on the alphabet Σ.

As $\mathcal{A}$ is reversible, so are its powers and the connected components of $\mathcal{A}^n$ coincide with the orbits of the action of $\langle \mathfrak{d}(\mathcal{A}) \rangle$ on Q^n.

Since $\mathcal{A}$ is reversible, there is a very particular connection between the connected components of $\mathcal{A}^n$ and those of $\mathcal{A}^{n+1}$ as highlighted in [12]. More precisely, take a connected component $\mathcal{C}$ of some $\mathcal{A}^n$, and let $\mathbf{u} \in Q^n$ (also written $|\mathbf{u}| = n$) be a state of $\mathcal{C}$. Take also $x \in Q$ a state of $\mathcal{A}$, and let $\mathcal{D}$ be the connected component of $\mathcal{A}^{n+1}$ containing the state $\mathbf{u}x$. Then, for any state $\mathbf{v}$ of $\mathcal{C}$, there exists a state of $\mathcal{D}$ prefixed with $\mathbf{v}$:

$$\exists \mathbf{s} \in \Sigma^* \mid \delta_{\mathbf{s}}(\mathbf{u}) = \mathbf{v} \quad \text{and so} \quad \delta_{\mathbf{s}}(\mathbf{u}x) = \mathbf{v}\delta_{\rho_{\mathbf{u}}(\mathbf{s})}(x) \ .$$

Furthermore, if $\mathbf{u}y$ is a state of $\mathcal{D}$, for some state $y \in Q$ different from x, then $\delta_{\mathbf{s}}(\mathbf{u}x)$ and $\delta_{\mathbf{s}}(\mathbf{u}y)$ are two different states of $\mathcal{D}$ prefixed with $\mathbf{v}$, because of the reversibility of $\mathcal{A}^{n+1}$: the transition function $\delta_{\rho_{\mathbf{u}}(\mathbf{s})}$ is a permutation. Hence $\mathcal{D}$ can be seen as consisting of several copies of $\mathcal{C}$ and $\#\mathcal{C}$ divides $\#\mathcal{D}$. They have the same size if and only if, for each state $\mathbf{u}$ of $\mathcal{C}$ and any different states $x, y \in Q$, $\mathbf{u}x$ and $\mathbf{u}y$ cannot simultaneously lie in $\mathcal{D}$.

The connected components of the powers of a Mealy automaton and the finiteness of the generated group or of a monogenic subgroup are closely related, as shown in the following propositions (obtained independently in [13, 4]).

▶ **Proposition 1.** *A reversible Mealy automaton generates a finite group if and only if the connected components of its powers have bounded size.*

▶ **Proposition 2.** *Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be an invertible-reversible Mealy automaton and let $\mathbf{u} \in Q^+$ be a non-empty word. The following conditions are equivalent:*

(i) *$\rho_{\mathbf{u}}$ has finite order,*

(ii) *the sizes of the connected components of $(\mathbf{u}^n)_{n\in\mathbb{N}}$ are bounded,*

(iii) *there exists a word $\mathbf{v}$ such that the sizes of the connected components of $(\mathbf{v}\mathbf{u}^n)_{n\in\mathbb{N}}$ are bounded,*

(iv) *for any word $\mathbf{v}$, the sizes of the connected components of $(\mathbf{v}\mathbf{u}^n)_{n\in\mathbb{N}}$ are bounded.*

4 The Labeled Orbit Tree and the Order Problem

Most of the notions of this section have been introduced in [13]. We refer the reader to this reference for the proofs of the results in this section.

We build a tree capturing the links between the connected components of consecutive powers of a reversible Mealy automaton. See an example in Figure 1. As recalled at the end of this section, the existence of elements of infinite order in the semigroup generated by an invertible-reversible automaton is closely related to some path property of this tree.

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a reversible Mealy automaton. Consider the tree with vertices the connected components of the powers of $\mathcal{A}$, and the incidence relation built by adding an element of Q: for any $n \geq 0$, the connected component of $\mathbf{u} \in Q^n$ is linked to the connected component(s) of $\mathbf{u}x$, for any $x \in Q$. This tree is called the *orbit tree* of $\mathfrak{d}(\mathcal{A})$ [5, 11]. It can be seen as the quotient of the tree Q^* under the action of the group $\langle \mathfrak{d}(\mathcal{A}) \rangle$.

We label any edge $\mathcal{C} \to \mathcal{D}$ of the orbit tree by the ratio $\frac{\#\mathcal{D}}{\#\mathcal{C}}$, which is always an integer (less than or equal to $\#\mathcal{A}$) by the reversibility of $\mathcal{A}$. We call this labeled tree the *labeled orbit tree* of $\mathfrak{d}(\mathcal{A})$ [13]. We denote by $\mathfrak{t}(\mathcal{A})$ the labeled orbit tree of $\mathfrak{d}(\mathcal{A})$. Note that for each vertex of $\mathfrak{t}(\mathcal{A})$, the sum of the labels of all edges going down from this vertex always equals to $\#Q$, the size of $\mathcal{A}$.

Each vertex of $\mathfrak{t}(\mathcal{A})$ is labeled by a connected automaton with stateset in Q^n, where n is the level of this vertex in the tree. By a minor abuse, we can consider that each vertex is labeled by a finite language in Q^n, or even by a word in Q^n.

Let $\mathbf{u}$ be a (possibly infinite) word over Q. The *path of* $\mathbf{u}$ in the orbit tree $\mathfrak{t}(\mathcal{A})$ is the unique initial path going from the root through the connected components of the prefixes of $\mathbf{u}$; $\mathbf{u}$ can be called *a representative* of this initial path (we can say equivalently that this path is *represented* by $\mathbf{u}$ or that the word $\mathbf{u}$ *represents* the path).

▶ **Definition 3.** Let $\mathcal{A}$ be a reversible Mealy automaton and $\mathfrak{s}$ be a subtree of $\mathfrak{t}(\mathcal{A})$. An $\mathfrak{s}$*-word* is a word in $Q^* \cup Q^\infty$ representing an initial path of $\mathfrak{s}$. A *cyclic* $\mathfrak{s}$*-word* is a word in Q^* whose all powers are $\mathfrak{s}$-words (equivalently, it is an $\mathfrak{s}$-word viewed as a cyclic word).

The structure of an orbit tree is not arbitrary and it is possible to identify some similarities inside this tree.

▶ **Definition 4.** Let e and f be two edges in the orbit tree $\mathfrak{t}(\mathcal{A})$. We say that e *is liftable to* f if each word of $\bot(e)$ admits some word of $\bot(f)$ as a suffix.

Consider $\mathbf{u}$ in $\top(e)$ and its suffix $\mathbf{v}$ in $\top(f)$: any state $x \in Q$ such that $\mathbf{u}x \in \bot(e)$ satisfies $\mathbf{v}x \in \bot(f)$. Informally, "e liftable to f" means that what can happen after $\top(e)$ by following e can also happen after $\top(f)$ by following f. This condition is equivalent to a weaker one:

▶ **Lemma 5.** *Let $\mathcal{A}$ be a reversible Mealy automaton, and let e and f be two edges in the orbit tree $\mathfrak{t}(\mathcal{A})$. If there exists a word of $\bot(e)$ which admits a word of $\bot(f)$ as suffix, then e is liftable to f.*

Obviously if e is liftable to f, then f is closer to the root of the orbit tree. The fact that an edge is liftable to another one reflects a deeper relation stated below. The following lemma is one of the key observations.

▶ **Lemma 6.** *Let e and f be two edges in the orbit tree $\mathfrak{t}(\mathcal{A})$. If e is liftable to f, then the label of e is less than or equal to the label of f.*

The notions of children of an edge and of being liftable to it are not linked, but it is interesting to consider their intersection.

▶ **Definition 7.** Let e and f be two edges in an orbit tree: e is a *legitimate child* of f if f is its parent and e is liftable to f.

The notion of liftability can be generalized to paths:

▶ **Definition 8.** Let $\mathbf{e} = (e_i)_{i\in I}$ and $\mathbf{f} = (f_i)_{i\in I}$ be two paths of the same (possibly infinite) length in the orbit tree $\mathfrak{t}(\mathcal{A})$. The path $\mathbf{e}$ *is liftable to* the path $\mathbf{f}$ if, for any $i \in I$, the edge e_i is liftable to the edge f_i.

▶ **Definition 9.** Let $\mathcal{A}$ be a bireversible Mealy automaton and $\mathfrak{s}$ be a (possibly infinite) path or subtree of $\mathfrak{t}(\mathcal{A})$. For $k > 0$, $\mathfrak{s}$ is k*-self-liftable* whenever any path in $\mathfrak{s}$ starting at level $i + k$ is liftable to a path in $\mathfrak{s}$ starting at level i, for any $i \geq 0$. A path or a subtree is *self-liftable* if it is k-self-liftable for some $k > 0$.

The path represented by x^ω, for some state x, is an example of an infinite initial 1-self-liftable path.

▶ **Lemma 10.** *Let* $\mathbf{e}$ *be a non-empty finite initial* 1*-self-liftable path of some orbit tree* $\mathfrak{t}(\mathcal{A})$*, with last edge* e*. The edge* e *has at least one legitimate child. The sum of the labels of the legitimate children of* e *is equal to the label of* e*.*

Proof. Denote by k the label of e. Let $\mathbf{u}$ be some state in $\top(e)$ and x some state of $\mathcal{A}$ such that $\mathbf{u}x$ is a state of $\bot(e)$ —this is possible by the definition of an orbit tree. We decompose $\mathbf{u}$ in its first letter and some suffix: $\mathbf{u} = z\mathbf{v}$. As $\mathbf{e}$ is a 1-self-liftable path and $z\mathbf{v}x$ is a state in $\bot(\mathbf{e}) = \bot(e)$, we know that $\mathbf{v}x$ is a state in $\top(e)$. Hence by the construction of a label orbit tree, there exist exactly k states $(y_i)_{1\leq i\leq k}$ such that $(\mathbf{v}xy_i)_i$ are states of $\bot(e)$. So the connected components of the $(z\mathbf{v}xy_i)_i$ label legitimate children of e. Clearly e cannot have another legitimate child. ◀

We recall here a characterization of the existence of elements of infinite order in the semigroup generated by a reversible Mealy automaton $\mathcal{A}$ in terms of path properties of the associated orbit tree $\mathfrak{t}(\mathcal{A})$ [13].

▶ **Definition 11.** Any branch labeled by a word not suffixed by 1^ω is called *active*.

▶ **Theorem 12.** *[13] The semigroup generated by an invertible-reversible automaton* $\mathcal{A}$ *admits elements of infinite order if and only if the orbit tree* $\mathfrak{t}(\mathcal{A})$ *admits an active self-liftable branch.*

5 Jungle Trees

Our main result being known for non bireversible automata [7], we focus on the bireversible case. All the tools introduced in this section are new. They are used to get rid of the particularity of the stateset of size 3 in [13].

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a connected bireversible Mealy automaton with no active self-liftable branch. From Theorem 12, all the elements of the semigroup $\langle\mathcal{A}\rangle_+$ have finite order. In this section we introduce the tools to prove that such an automaton of prime size generates a finite group (Theorem 32).

5.1 Jungle trees and stems

We focus on some particular subtrees of $\mathfrak{t}(\mathcal{A})$:

▶ **Definition 13.** Let $\mathbf{e}$ be a finite initial 1-self-liftable path such that:
- $\bot(\mathbf{e})$ has at least two legitimate children;
- every legitimate child of $\bot(\mathbf{e})$ has label 1.

The *jungle tree* $\mathfrak{j}(\mathbf{e})$ of $\mathbf{e}$ is the subtree of $\mathfrak{t}(\mathcal{A})$ build as follows:
- it contains the path $\mathbf{e}$ — its *trunk*;
- it contains the regular tree rooted by $\bot(\mathbf{e})$, and formed by all the edges which are descendant of $\bot(\mathbf{e})$ and liftable to the lowest (*i.e.* the last) edge of $\mathbf{e}$.

The *arity* of this jungle tree is the number of legitimate children of $\bot(\mathbf{e})$. Since every legitimate child has label 1, it is also the label of the last edge of $\mathbf{e}$.

Words in $\bot(\mathbf{e})$ are called *stems*. They have all the same length which is the length of the trunk of $\mathfrak{j}(\mathbf{e})$.

A tree is a *jungle tree* if it is the jungle tree of some finite initial 1-self-liftable path. An example of such a tree is depicted in Figure 3.

Graphically, a jungle tree starts with a linear part whose labels decrease (its trunk) and eventually ends as a regular tree with all labels 1. Any jungle tree is 1-self-liftable.

Note that: (i) there exists at least one jungle tree, from Lemma 10 and the hypothesis that $\mathcal{A}$ has no active self-liftable branch; (ii) there are finitely many jungle trees.

From now on, $\mathfrak{j}$ **denotes a jungle tree of** $\mathcal{A}$**, whose trunk has length** n**.**

As shown below, any cyclic $\mathfrak{j}$-words has finite order.

▶ **Remark 14.** *If* $\mathbf{uv}$ *is a* $\mathfrak{j}$*-word, with* $|\mathbf{v}| \geq n$*, what can follow* $\mathbf{uv}$ *in* $\mathfrak{j}$ *is independent from* $\mathbf{u}$*. In particular, if* $\mathbf{vw}$ *is also a* $\mathfrak{j}$*-word, then so is* $\mathbf{uvw}$*.*

The existence of cyclic $\mathfrak{j}$-words is ensured by the simple fact that any $\mathfrak{j}$-word of length $n \times (1 + \#Q^n)$ admits at least two identical factors of length n, and hence has a cyclic $\mathfrak{j}$-word as a factor by Remark 14.

▶ **Proposition 15.** *Every cyclic* $\mathfrak{j}$*-word induces an action of finite order, bounded by a uniform constant depending on* $\mathfrak{j}$*.*

Proof. Let $\mathbf{u}$ be a cyclic $\mathfrak{j}$-word, then, for any integer $k > n$, $\mathbf{u}^k$ is a $\mathfrak{j}$-word. By the definition of a jungle tree, the label of the path of $\mathbf{u}^\omega$ is ultimately 1 and, by Proposition 1, the action induced by $\mathbf{u}$ has finite order, bounded by a constant which depends on the connected component at the end of the trunk of $\mathfrak{j}$. ◀

Because of the self-liftability of $\mathfrak{j}$, any factor of a $\mathfrak{j}$-word is itself a $\mathfrak{j}$-word. Hence any factor of length n of a $\mathfrak{j}$-word is a stem. And by the construction of $\mathfrak{j}$, the end of its trunk has only one vertex whose label is hence a connected component, and all the stems are states of that same connected component.

▶ **Definition 16.** Let $\mathfrak{j}$ be a jungle tree of trunk of length n. A *liana covering up* $\mathfrak{j}$ is a language of $\mathfrak{j}$-words, of the form $\mathbf{w}L_\mathbf{w}$, where $\mathbf{w} \in Q^n$ is a stem, and $L_\mathbf{w} \subseteq Q^* \cup Q^\infty$ is a prefix-preserving language which, seen as a tree, is regular of the same arity than $\mathfrak{j}$.

Each vertex of $\mathfrak{j}$ has exactly one representative in $\mathbf{w}L_\mathbf{w}$. For each stem $\mathbf{w}$ there is exactly one suitable $L_\mathbf{w}$.

▶ **Remark 17.** *Let* $\mathbf{w}L_\mathbf{w}$ *be a liana covering up a jungle tree* $\mathfrak{j}$ *and* $\mathbf{uv}$ *be a finite* $\mathfrak{j}$*-word such that* $|\mathbf{v}| = n$*: if* $L_\mathbf{v}$ *is the greatest language such that* $\mathbf{uv}L_\mathbf{v} \subseteq \mathbf{w}L_\mathbf{w}$*, then* $\mathbf{v}L_\mathbf{v}$ *is also a liana covering up* $\mathfrak{j}$*.*

In what follows, we try to better understand the stucture of jungle trees and lianas. Let $S = \mathbf{s}L_\mathbf{s}$ be a liana covering up $\mathfrak{j}$ ($\mathbf{s} \in Q^n$). Our goal is to prove the following result:

▶ **Theorem 18.** *Let* $\mathbf{u}$ *be a factor in* S*. Then* $\mathbf{u}$ *has the following property:*

If $\mathbf{uv} \in Q^*$ *is a factor in* S*, then* $\mathbf{u}$ *exists further in* S*.* (**Ubiquity**)

More formally: if $\mathbf{tuv} \in S$*, there exists* $\mathbf{w} \in Q^*$ *such that* $\mathbf{tuvwu} \in S$*.*

The graphical sense of this theorem is that if you are walking on a $\mathfrak{j}$-word and you have already seen some factor, you can find eventually this same factor.

Proof. First, remember that if $\mathbf{u}$ is a stem (*i.e.* $\mathbf{u}$ is a factor in S of length n), what can follow $\mathbf{u}$ (in S) does not depend either of the choice of the liana (as long as you are in a liana covering up the same jungle tree), or of the location of $\mathbf{u}$ in this liana. Hence it is sufficient to prove Theorem 18 for $|\mathbf{u}| = n$.

We start by proving that there is at least one stem $\mathbf{u}_0$ with Property (**Ubiquity**). To obtain this word, we travel along $S = \mathbf{s}L_\mathbf{s}$ in the following way, starting with $\mathbf{u}_0 = \mathbf{s}$:

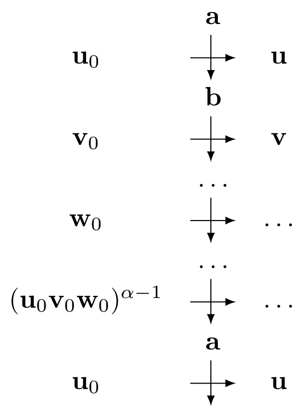

Figure 2 Extension of Property (**Ubiquity**) from $\mathbf{u}_0$ to $\mathbf{u}$.

- if $\mathbf{u}_0$ answers to the question, our journey is over;
- otherwise, at the end of $\mathbf{u}_0$ we follow some finite path such that $\mathbf{u}_0$ does not exist anymore after this path; then we replace $\mathbf{u}_0$ by the next word of length n in S, and back to the previous step.

Since S is infinite but has a finite arity and a finite number of factors of length n, the previous algorithm ends returning a stem $\mathbf{u}_0$ satisfying Property (**Ubiquity**). By Remark 17, the jungle tree $\mathfrak{j}$ is covered up by a liana of the form $\mathbf{u}_0 L_{\mathbf{u}_0}$.

The extension of Property (**Ubiquity**) to other words is illustrated by Figure 2. Let $\mathbf{uv}$ be a factor in S, with $|\mathbf{u}| = n$. In particular $\mathbf{u}$ is a stem, hence $\mathbf{u}_0$ and $\mathbf{u}$ are states of the same connected component, and there exists a path in this component from $\mathbf{u}_0$ to $\mathbf{u}$, say by the action of some $\mathbf{a} \in \Sigma^*$. The automaton being reversible, $\mathbf{v}$ is the image of some $\mathbf{v}_0 \in Q^{|\mathbf{v}|}$ by $\mathbf{b} = \delta_{\mathbf{u}_0}(\mathbf{a})$ (see Figure 2).

Now, we know that on the left part of Figure 2 we can find eventually $\mathbf{u}_0$, after some $\mathbf{w}_0$ (because $\mathbf{u}_0$ has Property (**Ubiquity**)). And by the invertibility of the automaton, there exists some power α of $\mathbf{u}_0\mathbf{v}_0\mathbf{w}_0$ which stabilizes $\mathbf{a}$ (see Figure 2).

Hence $\mathbf{u}$ can be seen again eventually. Furthermore, the vertical word on the right of Figure 2 is a $\mathfrak{j}$-word, as it is in the same connected component than the vertical $\mathfrak{j}$-word on the left of this same figure, and so it is a factor of S because, by hypothesis, its prefix of length n is a factor of S. Hence $\mathbf{u}$ has Property (**Ubiquity**). ◀

▶ **Remark 19.** *Note that, from Theorem 18, if* $\mathbf{u}$, $\mathbf{v}$ *are two stems such that* $\mathbf{v}$ *is a factor of some word in* $\mathbf{u}L_{\mathbf{u}}$, *then* $\mathbf{u}$ *is a factor of some word in* $\mathbf{v}L_{\mathbf{v}}$.

5.2 An equivalence on stems

Remember that $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ is a connected bireversible Mealy automaton such that $\mathfrak{t}(\mathcal{A})$ has no active self-liftable branch (and as a consequence all the elements of the semigroup $\langle \mathcal{A} \rangle_+$ have finite order). Let $\mathfrak{j}$ be a jungle tree of $\mathfrak{t}(\mathcal{A})$ with trunk of length n. All the stems considered from now on are stems of $\mathfrak{j}$.

In this subsection we prove several properties for the stems of the jungle tree $\mathfrak{j}$. Stems are used then in Section 6 to build a $\mathfrak{j}$-word inducing the same action than some given word.

Let us first introduce an equivalence relation on the set of stems.

▶ **Definition 20.** Let $\mathbf{u}$, $\mathbf{v}$ be two stems. We say that $\mathbf{u}$ is *equivalent* to $\mathbf{v}$, denoted by $\mathbf{u} \sim \mathbf{v}$, whenever there exists $\mathbf{s} \in Q^*$ such that $\mathbf{usv}$ is a j-word and $\rho_{\mathbf{us}}$ acts like the identity on Σ^*.

▶ **Lemma 21.** *The relation $\sim$ is an equivalence relation on stems.*

Proof. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three stems.

transitivity Suppose that $\mathbf{u} \sim \mathbf{v}$ and $\mathbf{v} \sim \mathbf{w}$: there exists $\mathbf{s}, \mathbf{t} \in Q^*$ such that $\mathbf{usv}$ and $\mathbf{vtw}$ are j-words, and $\rho_{\mathbf{us}}$ and $\rho_{\mathbf{vt}}$ act like the identity. As $\mathbf{v}$ is a stem, we obtain by Remark 14 that $\mathbf{usvtw}$ is a j-word, and $\rho_{\mathbf{usvt}}$ acts like the identity, so $\mathbf{u} \sim \mathbf{w}$.

reflexivity From Theorem 18, there exists $\mathbf{s} \in Q^*$ such that $\mathbf{usu}$ is a j-word (in fact from Theorem 18 one can even chose the beginning of $\mathbf{s}$, as long as we keep a j-word). As $\mathbf{u}$ is a stem, $\mathbf{usus}$ is also a j-word, and so are all the powers of $\mathbf{us}$. Now, by hypothesis and Theorem 12, $\mathbf{us}$ is of finite order, say α: $\mathbf{u}(\mathbf{su})^{\alpha-1}\mathbf{su}$ is a j-word and $\rho_{\mathbf{u}(\mathbf{su})^{\alpha-1}\mathbf{s}} = \rho_{(\mathbf{us})^\alpha}$ acts like the identity.

symmetry Suppose that $\mathbf{u} \sim \mathbf{v}$: there exists $\mathbf{s} \in Q^*$ such that $\mathbf{usv}$ is a j-word and $\rho_{\mathbf{us}}$ acts like the identity. From the reflexivity proof, there exists $\mathbf{t} \in Q^*$ such that $\mathbf{usvtu}$ is a j-word and $\rho_{\mathbf{usvt}}$ acts like the identity. Hence $\mathbf{vtu}$ is a j-word and $\rho_{\mathbf{vt}}$ acts like the identity, which proves the symmetry.

◀

Note that from reflexivity of $\sim$ and Theorem 18, if $\mathbf{u}$ and $\mathbf{v}$ are equivalent stems and $\mathbf{uw}$ is a j-word for some $\mathbf{w} \in Q^*$, then there exists $\mathbf{s} \in Q^*$ such that $\mathbf{uwsv}$ is a j-word and $\rho_{\mathbf{uws}}$ acts like the identity. So not only $\mathbf{v}$ can be reached from $\mathbf{u}$ by producing first the identity, but even if you walk in j after reading $\mathbf{u}$, you can still reach $\mathbf{v}$ and produce first the identity.

We can now consider the equivalence classes induced by $\sim$. The aim of this subsection is to prove that if $\mathcal{A}$ has a prime size, then for a given state q there is in each $\sim$-class a stem with prefix q (Theorem 30).

▶ **Proposition 22.** *All the equivalence classes of $\sim$ have the same size.*

Proof. Let $\mathbf{u}_0$ and $\mathbf{v}_0$ be two stems of j: they are states of the same connected component and so there exists $\mathbf{a} \in \Sigma^*$ such that $\delta_{\mathbf{a}}(\mathbf{u}_0) = \mathbf{v}_0$. Denote by $\{\mathbf{u}_0, \ldots, \mathbf{u}_k\}$ the $\sim$-class of $\mathbf{u}_0$: for any i, $1 \leq i \leq k$, there exists $\mathbf{s}_i \in Q^*$ such that $\mathbf{u}_0\mathbf{s}_i\mathbf{u}_i$ is a j-word and $\rho_{\mathbf{u}_0\mathbf{s}_i}$ acts like the identity. Define the words $\mathbf{v}_i \in Q^{|\mathbf{u}_i|}$ and $\mathbf{t}_i \in Q^{|\mathbf{s}_i|}$ in the following way: $\delta_{\mathbf{a}}(\mathbf{u}_0\mathbf{s}_i\mathbf{u}_i) = \mathbf{v}_0\mathbf{t}_i\mathbf{v}_i$. Note that $\mathbf{v}_0\mathbf{t}_i\mathbf{v}_i$ is also a j-word: any factor of size n of $\mathbf{v}_0\mathbf{t}_i\mathbf{v}_i$ is the image of a stem (the corresponding factor in $\mathbf{u}_0\mathbf{s}_i\mathbf{u}_i$) and therefore belongs to the connected component of $\mathbf{u}_0$ and $\mathbf{v}_0$, hence every prefix of $\mathbf{v}_0\mathbf{t}_i\mathbf{v}_i$ is on a 1-self-liftable path. Now $\rho_{\mathbf{v}_0\mathbf{t}_i}$ acts like the identity by the reversibility of $\mathcal{A}$, so $\mathbf{v}_i$ is $\sim$-equivalent to $\mathbf{v}_0$. Furthermore, as $\rho_{\mathbf{u}_0\mathbf{s}_i}$ acts like the identity, we know that $\mathbf{v}_i = \delta_{\mathbf{a}}(\mathbf{u}_i)$, and all the $\mathbf{v}_i$ are different. ◀

5.3 Combinatorial properties of stems

We now state several combinatorial properties for stems. Let $k_1, k_2, \ldots, k_n$ be the labels, from root to $\perp(\mathbf{e})$, of the jungle tree $\mathrm{j} = \mathrm{j}(\mathbf{e})$. Recall that, since $\mathcal{A}$ is connected, $k_1 = p$ and by construction of the jungle tree $k_n \geq 2$. For instance in Figure 3, $n = 4$, $k_1 = k_2 = 3$, and $k_3 = k_4 = 2$.

First if we consider no restriction then we can directly count stems by looking to the labels of the trunk:

▶ **Lemma 23.** *The number of stem with a given prefix depends only on length i of the prefix and is $k_{i+1}k_{i+2}\ldots k_n$.*

We are now interested in two somehow dual questions. Fix a j-word $\mathbf{u}$ of length less than n: (i) if $\mathbf{u}$ is the prefix of a stem in some $\sim$-class γ, in how many way can $\mathbf{u}$ be completed in γ (Proposition 24)? (ii) in how many $\sim$-classes is $\mathbf{u}$ the prefix of a stem (Corollary 29)?

▶ **Proposition 24.** *Fix some* j*-word* $\mathbf{u}$ *of length less than n, a $\sim$-class γ of stems including an element with prefix $\mathbf{u}$, and some integer k such that $|\mathbf{u}|+k \leq n$. The number of $\mathbf{v} \in Q^k$ such that $\mathbf{uv}$ is a prefix of a stem of γ depends only on $|\mathbf{u}|$ and k.*

Proof. By the same argument than in the proof of Proposition 22. ◀

Let $\mathbf{u} \in Q^*$ be a prefix of a stem in some $\sim$-class γ. Denote by $\mathcal{S}_{\text{eq}}(|\mathbf{u}|+1)$ the cardinality of the set $\{q \in Q \mid \mathbf{u}q \text{ is a prefix of some stem in } \gamma\}$ (from Proposition 24 it depends only of $|\mathbf{u}|$ and so it is well-defined).

In order to obtain a minimal bound on the size of a $\sim$-class, we introduce another equivalence relation between stems which is finer than $\sim$, as proved in Lemma 26:

▶ **Definition 25.** Let $\mathbf{u}, \mathbf{v}$ be two stems. Define the relation $\mathbf{u} \wedge_0 \mathbf{v}$ whenever there exists a stem $\mathbf{s}$ such that both $\mathbf{su}$ and $\mathbf{sv}$ are j-words. The equivalence relation $\wedge$ is defined as the transitive closure of $\wedge_0$.

Note that by the construction of the jungle tree, a $\wedge_0$-class contains k^n elements, where k stands for the arity of this jungle tree.

▶ **Lemma 26.** *The relation $\wedge$ is finer than the relation $\sim$: $\mathbf{u} \wedge \mathbf{v} \Rightarrow \mathbf{u} \sim \mathbf{v}$.*

Proof. By transitivity it is enough to prove that: $\mathbf{u} \wedge_0 \mathbf{v} \Rightarrow \mathbf{u} \sim \mathbf{v}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two stems such that $\mathbf{u} \wedge_0 \mathbf{v}$: there exists a stem $\mathbf{s}$ such that $\mathbf{su}$ and $\mathbf{sv}$ are j-words. From Theorem 18, there exists a word $\mathbf{w} \in Q^*$ such that $\mathbf{uws}$ is a j-word. As $\mathbf{u}$ and $\mathbf{s}$ are stems, and $\mathbf{su}$ is a j-word, $(\mathbf{uws})^2$ is a j-word by Remark 14, and so are all the powers of $\mathbf{uws}$. Now, by hypothesis and Theorem 12, the word $\mathbf{uws}$ has finite order, say α: $(\mathbf{uws})^\alpha\mathbf{v}$ is a j-word and $\rho_{(\mathbf{uws})^\alpha}$ acts like the identity. ◀

▶ **Corollary 27.** *For any i, $\mathcal{S}_{eq}(i) \geq 2$.*

Proof. For a stem $\mathbf{u}$, the set of words in $\wedge_0$-relation with $\mathbf{u}$, seen as a tree, has the same arity than j; so, by Lemma 26, for any i, $\mathcal{S}_{\text{eq}}(i)$ is greater than or equal to the arity of j. ◀

▶ **Proposition 28.** *Fix a* j*-word* $\mathbf{u}$ *of length less than n. The number of stems prefixed by $\mathbf{u}$ in a $\sim$-class is either 0 or depends only on $|\mathbf{u}|$.*

Proof. By the same argument than in the proof of Proposition 22. ◀

From Propositions 24 and 28 we obtain:

▶ **Corollary 29.** *Fix a* j*-word* $\mathbf{u}$ *of length less than n. The number of $\sim$-classes where $\mathbf{u}$ is the prefix of some stem depends only on $|\mathbf{u}|$.*

Denote by $\mathcal{P}_{\text{eq}}(|\mathbf{u}|+1)$ the number of $\sim$-classes containing a stem prefixed by $\mathbf{u}$ (it is correctly define by Corollary 29).

We can now prove the main result of this section:

▶ **Theorem 30.** *Let $\mathcal{A}$ be a connected bireversible Mealy automaton of prime size and without any active self-liftable branch. The set of states which appear as first letter of a stem in a fixed $\sim$-class is the whole stateset.*

Proof. Suppose $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ has prime size p, and let $\mathfrak{j}$ be a jungle tree of $\mathfrak{t}(\mathcal{A})$ whose trunk $\mathbf{e}$ has length n. We denote by $k_1,\dots,k_n$ the label of the edges of $\mathbf{e}$ (from top to bottom). By the connectivity of $\mathcal{A}$, $k_1 = p$.

Let γ be a $\sim$-class of stems for $\mathfrak{j}$ and $\mathbf{u} \in Q^*$ of length $i \leq n$ be the prefix of some stem in γ. Consider all the stems in γ with prefix $\mathbf{u}$.

From Lemma 23, the number of stems of $\mathfrak{j}$ prefixed by $\mathbf{u}$ is $k_{i+1} \times k_{i+2} \times \dots \times k_n$. On the other hand, it is also the number of stems with prefix $\mathbf{u}$ in γ, *i.e.* $\mathcal{S}_{\text{eq}}(i+1) \times \dots \times \mathcal{S}_{\text{eq}}(n)$, times the number of $\sim$-classes which has a stem prefixed by $\mathbf{u}$, *i.e.* $\mathcal{P}_{\text{eq}}(i+1) \times \dots \times \mathcal{P}_{\text{eq}}(n)$. Hence

$$k_{i+1} \times k_{i+2} \times \dots \times k_n = \mathcal{S}_{\text{eq}}(i+1) \times \mathcal{P}_{\text{eq}}(i+1) \times \mathcal{S}_{\text{eq}}(i+2) \times \mathcal{P}_{\text{eq}}(i+2) \times \dots \times \mathcal{S}_{\text{eq}}(n) \times \mathcal{P}_{\text{eq}}(n).$$

It is straightforward that $k_n = \mathcal{P}_{\text{eq}}(n) \times \mathcal{S}_{\text{eq}}(n)$ and by induction $\mathcal{P}_{\text{eq}}(\ell) \times \mathcal{S}_{\text{eq}}(\ell) = k_\ell$ for all ℓ. In particular for $\ell = 1$, we get that $\mathcal{S}_{\text{eq}}(1)$ devides k_1. Since $k_1 = p$ and, from Corollary 27, $\mathcal{S}_{\text{eq}}(1) \geq 2$, we obtain then $\mathcal{S}_{\text{eq}}(1) = p$. ◀

▶ **Corollary 31.** *Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a connected bireversible Mealy automaton of prime size, with no active self-liftable branch. Let $\mathfrak{j}$ be a jungle tree of $\mathfrak{t}(\mathcal{A})$ and $\mathbf{u}$ some $\mathfrak{j}$-word. Then for any state $x \in Q$, there exists $\mathbf{w} \in Q^*$ such that $\mathbf{uw}x$ is a $\mathfrak{j}$-word and $\rho_{\mathbf{w}}$ acts like the identity of Σ^*.*

Proof. Let $\mathbf{s}$ be a stem such that $\mathbf{us}$ is a $\mathfrak{j}$-word: there exists a stem $\mathbf{x}$ with first letter x in the $\sim$-class of $\mathbf{s}$, from Theorem 30, *i.e.* there exists $\mathbf{v} \in Q^*$ such that $\mathbf{svx}$ is a $\mathfrak{j}$-word and $\rho_{\mathbf{sv}}$ acts like the identity of Σ^*. Conclusion comes from Remark 14. ◀

Note that in the previous corollary, the word $\mathbf{u}$ can be empty.

6 Proof of the main theorem

We now have all elements to prove our main result.

▶ **Theorem 32.** *A connected invertible-reversible Mealy automaton of prime size cannot generate an infinite Burnside group.*

Proof. Let $\mathcal{A}$ be a connected invertible-reversible Mealy automaton of prime size. If $\mathcal{A}$ is not bireversible we can apply [1, 7] and we get that, on one hand, $\langle \mathcal{A} \rangle$ is necessarily infinite, but on the other hand, it cannot be Burnside. If $\mathcal{A}$ is bireversible and $\mathfrak{t}(\mathcal{A})$ has an active self-liftable branch, then $\langle \mathcal{A} \rangle$ has an element of infinite order by Theorem 12.

Therefore we can assume that $\mathcal{A}$ is bireversible and $\mathfrak{t}(\mathcal{A})$ has no active self-liftable branch. Let us show that $\langle \mathcal{A} \rangle$ is finite. Let $\mathfrak{j}$ be some jungle tree of $\mathfrak{t}(\mathcal{A})$. As in [13] we prove that for any word $\mathbf{u} \in Q^*$, $\rho_{\mathbf{u}}$ has some uniformly bounded power which acts like some cyclic $\mathfrak{j}$-word.

Let $\mathbf{u} \in Q^*$. We prove by induction that any prefix $\mathbf{u}$ induces the same action than some $\mathfrak{j}$-word. It is obviously true for the empty prefix. Fix some $k < |\mathbf{u}|$ and suppose that the prefix $\mathbf{v}$ of length k of $\mathbf{u}$ induces the same action than some $\mathfrak{j}$-word $\mathbf{s}$. Let $x \in Q$ be the $(k+1)$-th letter of $\mathbf{u}$. By Corollary 31, there exists a $\mathfrak{j}$-word $\mathbf{w}$ inducing the identity, such that $\mathbf{sw}x$ is a $\mathfrak{j}$-word. But $\mathbf{v}x$ and $\mathbf{sw}x$ induce the same action ; the result follows. Hence we obtain a $\mathfrak{j}$-word $\mathbf{u}^{(1)}$ inducing the same action than $\mathbf{u}$.

By the very same process, we can construct, for any $i \in \mathbb{N}$, a j-word $\mathbf{u}^{(i)}$ inducing the same action than $\mathbf{u}$, such that $\mathbf{u}^{(1)}\mathbf{u}^{(2)}\ldots\mathbf{u}^{(i)}$ is a j-word. Since the set Q^n is finite there exist $i < j$, $j - i \leq |Q|^n$, such that $\mathbf{u}^{(i)}$ and $\mathbf{u}^{(j)}$ have the same prefix of length n. Take $\mathbf{v} = \mathbf{u}^{(i)}\mathbf{u}^{(i+1)}\ldots\mathbf{u}^{(j-1)}$: $\mathbf{v}$ is a cyclic j-word and induces the same action than $\mathbf{u}^{j-i}$. By Proposition 15, the order of $\rho_{\mathbf{v}}$ is bounded by a constant depending only on j, hence so is the order of $\rho_{\mathbf{u}}$ (with a different constant, but still depending only on j). Consequently, every element of $\langle\mathcal{A}\rangle$ has a finite order, uniformly bounded by a constant, whence, as $\langle\mathcal{A}\rangle$ is residually finite, by Zelmanov's theorem [14, 15], $\langle\mathcal{A}\rangle$ is finite, which concludes the proof. ◀

The tools and techniques we have developed here enabled to bridge the gap between 3 and the set of all prime numbers. The next step is the extension of our result to any connected automaton. However, experiments suggest that there are strong similarities between the non prime case and the non connected case, bringing the hope to solve entirely the question of the (im)possible generation of an infinite Burnside group by a reversible Mealy automaton. Note that the primality of the stateset is not used here before Theorem 30. It is likely that the extension of Theorem 32 to more general statesets will require to choose carefully some k-self-liftable branches, with $k > 1$. In fact, there exist examples of automata for which the set of first letters in a $\sim$-class is not the whole stateset. However the $\sim$-classes seem to still play a crucial role in these examples. So our construction will certainly be a key element for a more general result.

References

1 A. Akhavi, I. Klimann, S. Lombardy, J. Mairesse, and M. Picantin. On the finiteness problem for automaton (semi)groups. *Int. J. Algebr. Comput.*, 22(6):26p., 2012.

2 S.V. Alešin. Finite automata and the Burnside problem for periodic groups. *Mat. Zametki*, 11:319–328, 1972.

3 W. Burnside. On an unsettled question in the theory of discontinuous groups. *Quart. J. Math.*, 33:230–238, 1902.

4 D. D'Angeli and E. Rodaro. A geometric approach to (semi)-groups defined by automata via dual transducers. *Geometriae Dedicata*, 174-1:375–400, 2015.

5 P.W. Gawron, V.V. Nekrashevych, and V.I. Sushchansky. Conjugation in tree automorphism groups. *Internat. J. Algebr. Comput.*, 11-5:529–547, 2001.

6 V.M. Gluškov. Abstract theory of automata. *Uspehi Mat. Nauk*, 16-5:3–62, 1961.

7 Th. Godin, I. Klimann, and M. Picantin. On torsion-free semigroups generated by invertible reversible Mealy automata. In *LATA'15*, volume 8977 of *LNCS*, pages 328–339, 2015.

8 E.S. Golod. On nil-algebras and finitely residual groups. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 28:273–276, 1964.

9 E.S. Golod and I. Shafarevich. On the class field tower. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:261–272, 1964.

10 R. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14-1:53–54, 1980.

11 R. Grigorchuk and D. Savchuk. Ergodic decomposition of group actions on rooted trees. *to appear in Proc. of Steklov Inst. of Math.*, 2015.

12 I. Klimann. Automaton semigroups: The two-state case. *Theor. Comput. Syst. (special issue STACS'13)*, pages 1–17, 2014. doi:10.1007/s00224-014-9594-0.

13 I. Klimann, M. Picantin, and D. Savchuk. A connected 3-state reversible Mealy automaton cannot generate an infinite Burnside group. In *DLT' 15*, volume 9168 of *LNCS*, pages 313–325, 2015.

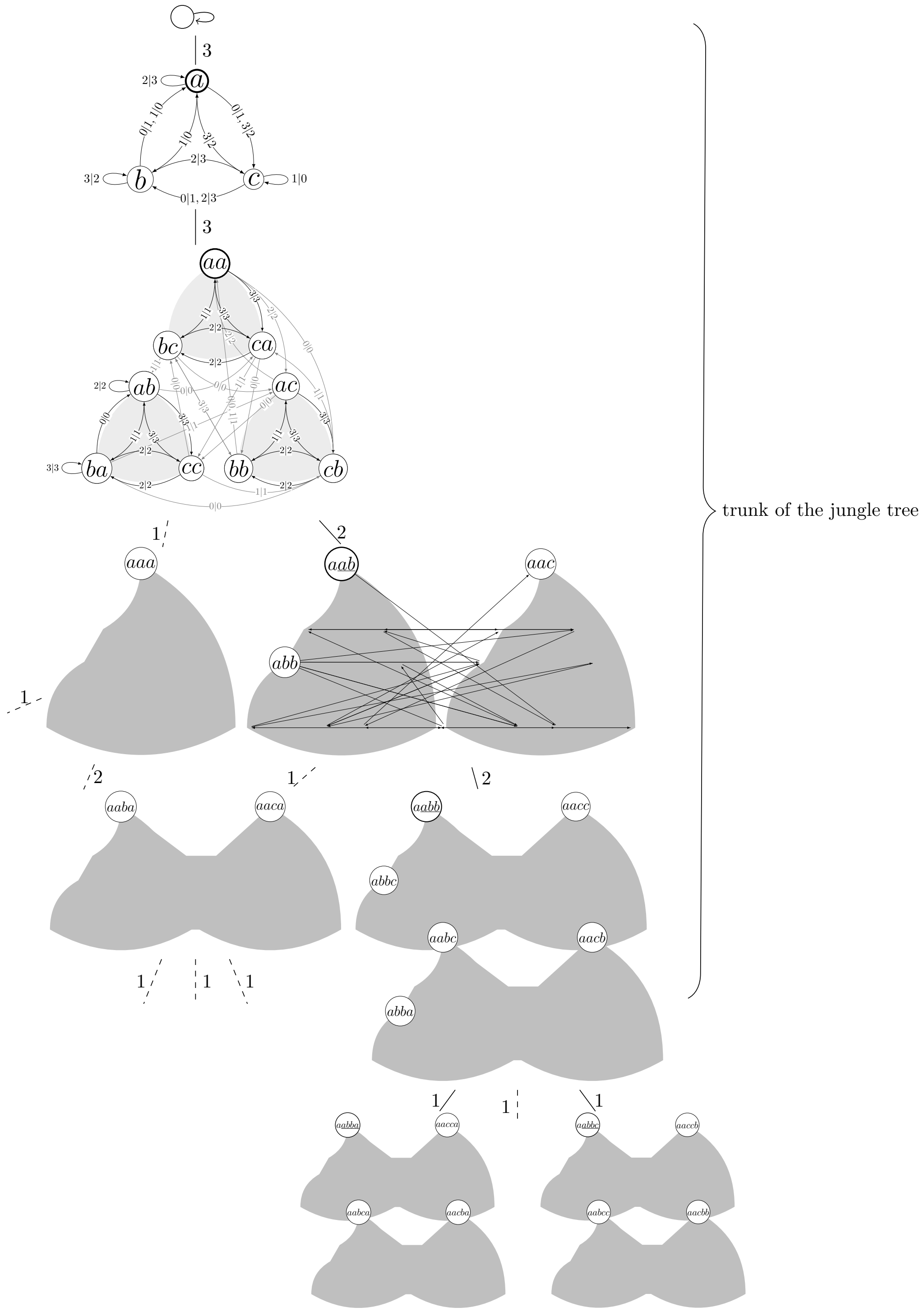

Figure 3 An exemple of the first levels of an orbit tree (all edges) and a jungle tree (plain edges). After the trunk the jungle tree consists in a regular binary tree (plain edges).

14 E.I. Zel′manov. Solution of the restricted Burnside problem for groups of odd exponent. *Izv. AN SSSR Math+*, 54-1:42–59, 221, 1990.
15 E.I. Zel′manov. Solution of the restricted Burnside problem for 2-groups. *Mat. Sb.*, 182-4:568–592, 1991.