Polynomial Space Randomness in Analysis*

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Abstract

We study the interaction between polynomial space randomness and a fundamental result of analysis, the Lebesgue differentiation theorem. We generalize Ko's framework for polynomial space computability in $\mathbb{R}^n$ to define *weakly pspace-random* points, a new variant of polynomial space randomness. We show that the Lebesgue differentiation theorem characterizes weakly pspace random points. That is, a point x is weakly pspace random if and only if the Lebesgue differentiation theorem holds for a point x for every pspace L_1-computable function.

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1 Introduction

The theory of computing allows for a meaningful definition of an individual point of Euclidean space being "random". Classically, such a notion would seem paradoxical, as any singleton set (indeed, any countably infinite set) has measure zero. Martin-Löf used computability to give the first mathematically robust definition of a point being random [10]. Since Martin-Löf's original definition, many notions of randomness have been introduced. In addition to Martin-Löf randomness, two of the most prominent variants are Schnorr randomness and computable randomness [4]. By developing a theory of resource-bounded measure, Lutz initiated the study of resource-bounded randomness [12, 13]. This allowed for research in algorithmic randomness to extend to resource-bounded computation [21].

Recently, research in algorithmic randomness has used computable analysis to study the connection between randomness and classical analysis [1, 5, 6, 7, 14, 15, 20]. With the rise of measure theory, many fundamental theorems of analysis have been "almost everywhere" results. Theorems of this type state that a certain property holds for almost every point; i.e., the set of points that does not satisfy the property is of measure zero. However, almost everywhere theorems typically give no information about which points satisfy the stated property. By adding computability restrictions, tools from algorithmic randomness are able to strengthen a theorem from a property simply holding almost everywhere, to one that holds for all random points. For example, an important classical result of analysis is Lebesgue's theorem on nondecreasing functions. Lebesgue showed that every nondecreasing continuous function $f : [0, 1] \to \mathbb{R}$ is differentiable almost everywhere. Brattka, Miller and

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Nies characterized computable randomness using Lebesgue's theorem by proving the following result [2].

▶ **Theorem.** *Let $z \in [0,1]$. Then z is computably random if and only if $f'(z)$ exists for every nondecreasing computable function $f : [0,1] \to \mathbb{R}$.*

This paper concerns a related theorem, also due to Lebesgue [9].

▶ **Lebesgue Differentiation Theorem.** *For each $f \in L_1([0,1]^n)$,*

$$f(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

for almost every $x \in [0,1]^n$. The limit is taken over all open cubes Q containing x as the diameter of Q tends to 0.

Pathak first studied the Lebesgue differentiation theorem in the context of Martin-Löf randomness [18]. Under the assumption that the function is L_1-computable, Pathak showed that the Lebesgue differentiation theorem holds for every Martin-Löf random point. Subsequently, Pathak, Rojas and Simpson improved this theorem [19]. They showed that the Lebesgue differentiation theorem holds at a point z for every L_1 computable function if and only if z is Schnorr random [19]. Independently, and using very different techniques, Rute also showed that the Lebesgue differentiation theorem holds for Schnorr random points [20].

This paper concerns the connection between *resource-bounded* randomness and analysis. While there has been work on this interaction [3, 11, 17], resource-bounded randomness in analysis is still poorly understood. Recently, Nies extended the result of Brattka, Miller and Nies to the polynomial time domain [17]. Specifically, Nies characterized polynomial time randomness using the differentiability of nondecreasing polynomial time computable functions. In this paper, we extend this research of the Lebesgue differentiation theorem to the context of resource-bounded randomness. We show that the Lebesgue differentiation theorem characterizes *weakly polynomial space randomness*. We note that the polynomial space variant of Nies' result implies our result in one dimension. However, as in classical analysis, the proof for arbitrary dimension requires significantly different tools.

In order to work with resource bounded computability over continuous domains, we use the framework for polynomial space computability in $\mathbb{R}^n$ developed by Ko [8]. Using generalizations of Ko's *polynomial space approximable* sets, we define weakly polynomial space randomness, a new variant of polynomial space randomness. We prove that Lutz's notion of polynomial space randomness implies weakly polynomial randomness. Weakly polynomial space randomness uses open covers, similar to Martin-Löf's original definition, unlike the martingale definitions commonly used in resource-bounded randomness. The use of open covers lends itself better to adapting many theorems of classical analysis. We believe that the notion of weakly polynomial space randomness will be useful in further investigations of resource-bounded randomness in analysis.

Using this definition of randomness, we extend the result of Pathak, et al, and Rute to polynomial space randomness. Specifically, we prove that a point x is weakly polynomial space random if and only if the Lebesgue differentiation theorem holds at x for every polynomial space L_1-computable function. Structurally, the proof of this theorem largely follows that of Pathak, et al. However, the restriction to polynomial space forces significant changes to the internal methods. To prove the converse of our theorem, we introduce *dyadic tree decompositions*. Intuitively, a dyadic tree decomposition partitions an open cover randomness test into a tree structure. This allows for the construction of a polynomial space

L_1-computable function so that the Lebesgue differentiation theorem fails for any point covered by the test. We believe that dyadic tree decompositions will be useful in further research.

2 Preliminaries

Throughout the paper, μ will always denote the Lebesgue measure on $\mathbb{R}^n$. We denote the set of all Lebesgue integrable functions $f : [0,1]^n \to \mathbb{R}$ by $L_1([0,1]^n)$. A *dyadic rational number* d is a rational number that has a finite binary expansion; that is $d = \frac{m}{2^r}$ for some integers m, r with $r \geq 0$. We denote the set of all dyadic rational numbers by $\mathbf{D}$. We denote the set of all dyadic rationals d of precision r by $\mathbf{D}_r$. Formally,

$$\mathbf{D}_r = \{\frac{m}{2^r} \,|\, m \in \mathbb{Z}\}.$$

We denote the set of dyadic rationals in the interval $[0,1]$ by $\mathbf{D}[0,1]$. We denote the set of dyadic rationals of precision r in the interval $[0,1]$ by $\mathbf{D}_r[0,1]$. An *open dyadic cube of precision* r is a subset $Q \subseteq \mathbb{R}^n$ such that

$$Q = (\frac{a_1}{2^r}, \frac{a_1+1}{2^r}) \times \ldots \times (\frac{a_n}{2^r}, \frac{a_n+1}{2^r}),$$

where $a_i \in \mathbb{Z}$, and $r \in \mathbb{N}$. We say that the points $\{\frac{a_1}{2^r}, \frac{a_1+1}{2^r}, \ldots \frac{a_n}{2^r}, \frac{a_n+1}{2^r}\}$ are the *endpoints of* Q. In the same manner, we define closed dyadic cubes, and half-open dyadic cubes. We denote the set of all open dyadic cubes of precision r by

$$\mathbf{B}_r = \{Q \,|\, Q \text{ is an open dyadic cube of precision } r\}.$$

For an open set $Q \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$, define the translation of Q by t to be the set

$$t + Q = \{t + x \,|\, x \in Q\}.$$

2.1 Resource-Bounded Randomness in Euclidean Space

Lutz and Lutz recently adapted resource-bounded randomness to arbitrary dimension [11]. In this section, we review their definition of polynomial space randomness in $\mathbb{R}^n$.

Let $r \in \mathbb{N}$, $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$. Define the *$r$-dyadic cube at* $\mathbf{u}$ to be the half-open dyadic cube of precision r,

$$Q_r(\mathbf{u}) = [u_1 \cdot 2^{-r}, (u_1+1) \cdot 2^{-r}) \times \ldots \times [u_n \cdot 2^{-r}, (u_n+1) \cdot 2^{-r}).$$

Define the family

$$\mathcal{Q}_r = \{Q_r(\mathbf{u}) \mid \mathbf{u} \in \{0, \ldots, 2^r - 1\}^n\}.$$

So then $\mathcal{Q}_r$ is a partition of the unit cube $[0,1)^n$. The family

$$\mathcal{Q} = \bigcup_{r=0}^{\infty} \mathcal{Q}_r,$$

is the set of all half-open dyadic cubes in $[0,1)^n$.

A *martingale* on $[0,1)^n$ is a function $d : \mathcal{Q} \to [0,\infty)$ satisfying

$$d(Q_r(\mathbf{u})) = 2^{-n} \sum_{\mathbf{a} \in \{0,1\}^n} d(Q_{r+1}(2\mathbf{u} + \mathbf{a})), \tag{1}$$

for all $Q_r(\mathbf{u}) \in \mathcal{Q}$. We may think of a martingale d as a strategy for placing successive bets on which cube contains x. After r bets have been placed, the bettor's capital is

$$d^{(r)}(\mathbf{x}) = d(Q_r(\mathbf{u})),$$

where $\mathbf{u}$ is the unique element of $\{0,\ldots,2^r-1\}^n$ such that $\mathbf{x} \in Q_r(\mathbf{u})$. A martingale d *succeeds* at a point $\mathbf{x} \in [0,1)^n$ if

$$\limsup_{r\to\infty} d^{(r)}(\mathbf{x}) = \infty.$$

Let

$$J = \{(r,\mathbf{u}) \in \mathbb{N} \times \mathbf{Z}^n \mid \mathbf{u} \in \{0,\ldots,2^r-1\}^n\}.$$

We say that a martingale $d : \mathcal{Q} \to [0,\infty)$ is *computable* if there is a computable function $\hat{d} : \mathbb{N} \times J \to \mathbb{Q} \cap [0,\infty)$ such that for all $(s,r,\mathbf{u}) \in \mathbb{N} \times J$,

$$|\hat{d}(s,r,\mathbf{u}) - d(Q_r(\mathbf{u}))| \le 2^{-s}. \tag{2}$$

A martingale $d : \mathcal{Q} \to [0,\infty)$ is *p-computable* (resp. *pspace-computable*) if there is a function $\hat{d} : \mathbb{N} \times J \to \mathbb{Q} \cap [0,\infty)$ that satisfies (2) and is computable in $(s+r)^{O(1)}$ time (resp. space). A point $x \in \mathbb{R}^n$ is *p-random* (resp. *pspace-random*) if no p-computable (resp. pspace-computable) martingale succeeds at x.

2.2 Polynomial Space Computability in Euclidean Space

In this section, we review Ko's framework for complexity theory in $\mathbb{R}^n$ [8]. For the remainder of the paper, we include the write tape when considering polynomial space bounds of Turing machines.

We first introduce the polynomial space L_1-computable functions, the class of functions we will be using in the proof of the Lebesgue differentiation theorem. This definition is equivalent to Ko's notion of *pspace approximable functions*. It is a direct analog of the L_1-computable functions used in computable analysis.

A function $f : [0,1]^n \to \mathbb{R}$ is a *simple step function* if f is a step function such that

1. $f(x) \in \mathbf{D}$ for all $x \in [0,1]^n$ and
2. there exists a finite number of (disjoint) dyadic boxes $Q_1,\ldots,Q_k$ and dyadic rationals $d_1,\ldots,d_k$ such that $f(x) = \sum\limits_{i=1}^{k} d_i \chi_{Q_i}(x)$, where χ_Q is the characteristic function of a set Q.

A function $f \in L_1([0,1]^n)$ is *polynomial space L_1-computable* if there exists a sequence of simple step functions, $\{f_m\}_{m\in\mathbb{N}}$, and a polynomial p such that for all $d \in \mathbf{D}^n$,

1. $f_m(x) = \sum\limits_{i=1}^{k} d_i \chi_{Q_i}(x)$, such that the endpoints of each Q_i are in $\mathbf{D}^n_{p(m)}$,
2. there is a polynomial space TM M computing f_m in the sense that
$M(0^m, d) = \begin{cases} f_m(d) & \text{if } d \text{ is not a breakpoint of } f_m \\ \# & \text{otherwise} \end{cases}$
3. $\|f - f_m\|_1 \le 2^{-n}$.

Note that we may assume that the polynomial p is increasing. We will frequently use the following nice property of polynomial space L_1-computable functions. If $f \in L_1([0,1]^n)$ is approximated by sequence of simple step function $\{f_m\}$ at precision p, then for every $i > 0$, f_i is a constant function on every $Q \in \mathbf{B}_{p(i)}$.

An infinite sequence $\{S_m\}_{m\in\mathbb{N}}$ of finite unions of open boxes is *polynomial space computable* if there exists a polynomial space TM M such that for all $m > 0$, and all $d \in \mathbf{D}^n$,

$$M(0^m, d) = \begin{cases} 1 & \text{if } d \in S_m \\ -1 & \text{if } d \text{ is a boundary point of } S_m \\ 0 & \text{otherwise} \end{cases}$$

A set $S \subseteq [0,1]^n$ is *polynomial space approximable* if S is measurable and there exists a polynomial space computable sequence of sets $\{S_m\}_{m\in\mathbb{N}}$ such that, for every $m > 0$,

1. there is a polynomial p such that all endpoints of S_m are in $\mathbf{D}^n_{p(m)}$ and
2. $\mu(S \Delta S_m) \leq 2^{-m}$.

Note that we may assume that the polynomial p is increasing; that is $p(i) \leq p(i+1)$, for all $i \in \mathbb{N}$.

3 Uniformly Approximable Sequences

We now generalize Ko's definition of approximable sets to approximable *arrays* of sets. We follow Ko in first defining computability, then leveraging this to define approximability.

▶ **Definition 1.** An infinite array $\{S^k_m\}_{k,m\in\mathbb{N}}$ of finite unions of open boxes is *uniformly polynomial space computable* if there exists a polynomial space TM M such that for all $k, m > 0$, and all $d \in \mathbf{D}^n$,

$$M(0^m, 0^k, d) = \begin{cases} 1 & \text{if } d \in S^k_m \\ -1 & \text{if } d \text{ is an boundary point of } S^k_m \\ 0 & \text{otherwise} \end{cases}$$

If $\{S^k_m\}$ is uniformly polynomial space computable and M is a TM satisfying the definition, we say M *computes* $\{S^k_m\}$.

▶ **Definition 2.** A sequence of sets $\{U_m\}_{m\in\mathbb{N}}$ is *uniformly polynomial space approximable* if there exists a uniformly polynomial space computable array of sets $\{S^k_m\}$ and a polynomial p such that

1. all endpoints of S^k_m are in $\mathbf{D}^n_{p(m+k)}$ and
2. $\mu(U_m \Delta S^k_m) \leq 2^{-k}$.

If a polynomial p and a uniformly polynomial space computable sequence $\{S^k_m\}$ satisfies (1) and (2), we say that $\{S^k_m\}_{k,m\in\mathbb{N}}$ *approximates* $\{U_m\}$ *at precision* p. Note that we may assume that the polynomial p is increasing.

We now show that we can construct *uniformly* pspace computable sequences from pspace computable sequences. This lemma will be useful, as polynomial space computability is an easier property to verify than its uniform counterpart.

▶ **Lemma 3.** *Let $\{T_i\}_{i\in\mathbb{N}}$ be a pspace computable sequence, and q_1, q_2 be polynomials. For every k, $m > 0$, define the set S^k_m by*

$$S^k_m = \bigcup_{i=q_1(m)}^{q_2(k)} T_i.$$

Then the array $\{S^k_m\}$ is uniformly polynomial space computable.

Similarly, we are able to construct uniformly pspace approximable sequences from other uniformly approximable sequences.

▶ **Lemma 4.** *Let q be a polynomial, $j \in \mathbb{N}$, and $\{V_i\}_{i\in\mathbb{N}}$ be a uniformly pspace approximable sequence, such that $\mu(V_i) \leq 2^{-i+j}$. Define the sequence $\{U_m\}_{m\in\mathbb{N}}$ by*

$$U_m = \bigcup_{i=q(m)}^{\infty} V_i.$$

Then $\{U_m\}_{m\in\mathbb{N}}$ is a uniformly pspace approximable sequence.

4 Weakly Polynomial Space Randomness

Using uniformly polynomial space approximable sequences, we give an open-cover definition of polynomial space randomness. This variant is intended to be similar to the open-cover definitions of the various computable randomness notions. However, the resource bounds force us to replace the typical enumerability requirements with approximability.

▶ **Definition 5.** Let $a, b \in \mathbb{Z}$. An infinite sequence of open sets $\{U_m\}_{m\in\mathbb{N}} \subseteq [a,b]^n$ is a *polynomial space $\mathcal{W}$-test (pspace $\mathcal{W}$-test)* if the following hold.

1. For every m, $\mu(U_m) \leq 2^{-m}$.
2. There is a uniformly pspace computable array $\{S_m^k\}$ approximating $\{U_m\}$ such that, for all m,

$$U_m \subseteq \liminf_{k\to\infty} S_m^k,$$

A point x *passes* a polynomial space $\mathcal{W}$-test $\{U_m\}_{m\in\mathbb{N}}$ if $x \notin \bigcap_{m=1}^{\infty} U_m$. We say that x is *weakly pspace random* if x passes every polynomial space $\mathcal{W}$-test.

The approximability of pspace $\mathcal{W}$-tests allows us to estimate the measure of the open covers in polynomial space.

▶ **Lemma 6.** *If $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test, then there exists a polynomial space TM M such that for every s, r, $m \in \mathbb{N}$ and $\mathbf{u} \in \{0,\ldots,2^r-1\}^n$*

$$|M(0^s, 0^r, \mathbf{u}, 0^m) - \mu(U_m \cap Q_r(\mathbf{u}))| \leq 2^{-s}.$$

We are now able to relate weakly polynomial space randomness with Lutz's pspace randomness. The following lemma shows that pspace randomness implies weakly pspace randomness.

▶ **Theorem 7.** *Let $\{U_m\}_{m\in\mathbb{N}}$ be a polynomial space $\mathcal{W}$-test. Then there exists a pspace martingale d succeeding on all points $x \in \bigcap_{m=1}^{\infty} U_m \bigcap [0,1]^n$.*

5 Randomness and the Lebesgue Differentiation Theorem

In this section we prove our main theorem, that the Lebesgue differentiation theorem characterizes weakly pspace-randomness. Recall the statement of Lebesgue's theorem.

▶ **Lebesgue Differentiation Theorem.** *For each $f \in L_1([0,1]^n)$,*

$$f(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

for almost every $x \in [0,1]^n$. The limit is taken over all open cubes Q containing x as the diameter of Q tends to 0.

A point x that satisfies the Lebesgue differentiation theorem is called a *Lebesgue point*. We will prove the following theorem,

▶ **Main Theorem.** *A point x is weakly pspace-random if and only if for every polynomial space L_1-computable $f \in L_1([0,1]^n)$, and every polynomial space computable sequence of simple functions $\{f_m\}_{m\in\mathbb{N}}$ approximating f,*

$$\lim_{m\to\infty} f_m(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)} \tag{3}$$

where the limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

We first make several remarks regarding the form of our main theorem. The use of polynomial space L_1-computability is not simply for the sake of generality. It is well-known that if a function is continuous, the Lebesgue differentiation theorem holds for *every* point. Thus, to get a non-trivial randomness result, we must allow the function to be discontinuous. Our second remark concerns the limit of the approximating functions. In the statement of the classical theorem, the integral limit is equal to $f(x)$; whereas in our main theorem, it is equal to $\lim_{m\to\infty} f_m(x)$. This concession is necessary. For any point x, it is trivial to construct a polynomial space L_1-computable function f such that

$$f(x) \neq \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}.$$

Consider the function f which is 0 for all points, except at the given point x, $f(x) = 1$. Clearly, f is polynomial space L_1-computable, but x does not satisfy the Lebesgue differentiation theorem.

5.1 Random points satisfy the Lebesgue differentiation theorem

The outline of our proof roughly follows that of the classical proof of the Lebesgue differentiation theorem [19, 22]. However, the restriction to polynomial space computation significantly changes the internal methods. We first show that if a point $x \in [0,1]^n$ is weakly pspace-random, then it must be contained in an open dyadic cube. This is a useful property of weakly pspace-random points that we take advantage of in later theorems.

▶ **Lemma 8.** *Let $x = (x_1, \ldots, x_n) \in [0,1]^n$ be weakly pspace-random. Then, for every i, x_i is not a dyadic rational.*

Let f be a polynomial space L_1-computable function, approximated by the pspace computable sequence of simple step functions $\{f_m\}_{m\in\mathbb{N}}$. We now show that for every weakly pspace-random point x, the limit $\lim\limits_{m\to\infty} f_m(x)$ exists. We will need the following inequality due to Chebyshev. For every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, define the set

$$S(f, \epsilon) = \{x \mid |f(x)| > \epsilon\}.$$

▶ **Chebyshev's Inequality.** *Let $f \in L_1([0,1]^n)$ and $\epsilon > 0$. Then $\mu(S(f,\epsilon)) \leq \frac{\|f\|_1}{\epsilon}$.*

▶ **Lemma 9.** *Let $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of simple step functions $\{f_m\}_{m \in \mathbb{N}}$. If x is weakly pspace-random, the limit $\lim\limits_{m\to\infty} f_m(x)$ exists.*

We now focus on the limit

$$\lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

on the right hand side of our main theorem (equation (3)). The restriction to polynomial space computation creates difficulties in considering arbitrary open cubes. Intuitively, we overcome this obstacle through the use of translations of dyadic cubes, which are more amenable to polynomial space computation. Formally, for $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$, define the set

$$\mathbf{B}_r^t = \{I_r^t \mid I_r^t = t + Q, \text{ where } Q \in \mathbf{B}_r\}.$$

That is, $\mathbf{B}_r^t$ is the set of all translations of dyadic cubes of precision r by points $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$. For every $x \in [0,1]^n$, let $I_r^t(x)$ denote the (unique) element of $\mathbf{B}_r^t$ containing x. The following theorem of Rute [20], using results due to Morayne and Solecki [16], shows that it suffices to prove that the right hand limit of equation (3) exists for these translations.

▶ **Theorem 10** ([20])**.** *Let $f \in L_1([0,1]^n)$, and $x \in [0,1]^n$. Then the following are equivalent,*

1. *the limit $\lim\limits_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}$ exists, where the limit is taken over all cubes containing x, as the diameter goes to* 0
2. *the limit $\lim\limits_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}$ exists, for all $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.*

We now show that the limit

$$\lim_{m\to\infty} \frac{\int_{I_r^t(x)} |f - f_m| d\mu}{\mu(I_r^t(x))}$$

exists, for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$. We will need the following inequality due to Hardy and Littlewood. For every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, define the set

$$T(f,\epsilon) = \{x \mid \sup_{r,t} \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t)} > \epsilon\},$$

where the supremum is taken over all $r > 0$ and $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

▶ **Hardy/Littlewood Inequality.** *There is a constant c such that, for every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, $\mu(T(f,\epsilon)) \leq \frac{c\|f\|_1}{\epsilon}$.*

▶ **Lemma 11.** *Let $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of step functions $\{f_m\}_{m\in\mathbb{N}}$. If x is weakly pspace-random, then*

$$\lim_{m\to\infty} \frac{\int_{I_r^t(x)} |f - f_m| d\mu}{\mu(I_r^t(x))} = 0,$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$.

We are now able to prove that weakly pspace random points satisfy the Lebesgue differentiation theorem.

▶ **Theorem 12.** *If x is weakly pspace-random, then for every polynomial space L_1-computable $f \in L_1([0,1]^n)$, and every polynomial space computable sequence of simple functions $\{f_m\}_{m\in\mathbb{N}}$ approximating f,*

$$\lim_{m\to\infty} f_m(x) = \lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

where the limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

Proof. Let x be weakly pspace-random. By Theorem 10, it suffices to show that

$$\lim_{m\to\infty} f_m(x) = \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}$$

for all $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

Let $\epsilon > 0$. By Lemmas 9 and 11, there exists an N such that for all $i > N$,

$$|f_i(x) - \lim_{m\to\infty} f_m(x)| < \frac{\epsilon}{2}, \tag{4}$$

and

$$\frac{\int_{I_k^t(x)} |f - f_i| d\mu}{\mu(I_k^t(x))} < \frac{\epsilon}{2}, \tag{5}$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $k > 0$. Let $i > N$. Then, using (4) we obtain

$$|\lim_{m\to\infty} f_m(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}| < \frac{\epsilon}{2} + |f_i(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}|. \tag{6}$$

By Lemma 8, for every $r > 0$, $x \in Q$ for some $Q \in \mathbf{B}_r$. Since f_i is a simple step function, f_i is constant on every $Q \in \mathbf{B}_{p(i)}$. So there exists an N' so that for all $r > N'$,

$$f_i(x) = \frac{\int_{I_r^t(x)} f_i d\mu}{\mu(I_r^t(x))},$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$. Therefore, by inequality (5), for every $r > N'$,

$$\begin{aligned}
|f_i(x) - \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t(x))}| &= |\frac{\int_{I_r^t(x)} f_i d\mu}{\mu(I_r^t(x))} - \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t(x))}| && (7)\\
&\leq \frac{\int_{I_r^t(x)} |f - f_i| d\mu}{\mu(I_r^t(x))} && (8)\\
&< \frac{\epsilon}{2}. && (9)
\end{aligned}$$

Combining inequalities (6) and (9) we have

$$|\lim_{m\to\infty} f_m(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}| < \epsilon.$$

Since ϵ was arbitrary, the proof is complete. ◀

5.2 Non-random points are not Lebesgue points

We now show that converse of our main theorem holds. That is, we show that if a point x is not weakly pspace random, the limit $\lim\limits_{Q\to x} \frac{1}{\mu(Q)}\int_Q f d\mu$ does not exist. Our approach is largely similar from the construction of Pathak, et al [19]. However, due to the restriction of polynomial space computation, the implementation is significantly different. To adapt the construction of Pathak et al, we first introduce a notion that will partition a pspace $\mathcal{W}$-test $\{U_m\}$ into a tree of dyadic cubes.

Recall that the *level* of a node in a rooted tree is the length of the (unique) path from the root to the node. We denote the set of all nodes of a tree $\mathbf{T}$ at level i by $Level_i(\mathbf{T})$.

▶ **Definition 13.** A *dyadic tree decomposition of* $[0,1]^n$ is a tree $\mathbf{T}$ of dyadic cubes rooted at $[0,1]^n$ such that the following hold:

1. For every cube $Q \in \mathbf{T}$, the children of Q, are subsets of Q.
2. For any two cubes $Q_1, Q_2 \in \mathbf{T}$, either Q_1 and Q_2 are disjoint, or one contains the other.
3. For any cube $Q \in \mathbf{T}$,

$$\mu(\bigcup_{B\in Child(Q)} B) \leq \frac{\mu(Q)}{4}.$$

A dyadic tree decomposition $\mathbf{T}$ is *polynomial space approximable* if there exists a polynomial p and uniformly pspace computable array $\{T_m^k\}_{k,m\in\mathbb{N}}$ such that the following hold.

1. For every $k, m \in \mathbb{N}$, T_m^k is a finite union of disjoint dyadic cubes.
2. For every $\mu(Level_m(\mathbf{T})\Delta T_m^k) \leq 2^{-(k+m)}$.

Intuitively, for every k and m, T_m^k is a good approximation of the mth level of the tree $\mathbf{T}$.

We now show that every pspace $\mathcal{W}$-test admits a pspace approximable dyadic tree decomposition. We build the tree inductively, using the uniformly pspace computable sequence of the previous lemma.

▶ **Lemma 14.** *Let $\{U_m\}_{m\in\mathbb{N}}$ be a pspace $\mathcal{W}$-test. Then there exists a pspace approximable dyadic tree decomposition $\mathbf{T}$ such that, for every non-dyadic $x \in \bigcap U_m$, x is contained in an infinite path in $\mathbf{T}$.*

We are now able to prove the converse of Theorem 12, thereby completing the proof of our main theorem. The proof of this theorem involves constructing a function that takes advantage of the dyadic tree decomposition of a pspace W-test succeeding on x. We construct the function so that it assigns different values to alternating levels of the tree. As we are guaranteed that x is in an infinite path of the tree, the function oscillates around x.

▶ **Theorem 15.** *If $x \in [0,1]^n$ is not weakly pspace random, then there exists a pspace L_1 computable function f such that the limit $\lim\limits_{Q\to x} \frac{1}{\mu(Q)}\int_Q f d\mu$ does not exist.*

Proof. We first assume that $x = (x_1, \ldots, x_n)$ so that some component x_i of x is a dyadic rational. Without loss of generality assume that $x_1 = d \in \mathbf{D}$. Define the function $f : [0,1]^n \to \mathbb{R}$ to be

$$f(y) = \begin{cases} 1 & \text{if } y \in [0,d] \times [0,1] \times \ldots \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that f is pspace L_1-computable, and that the limit $\lim\limits_{Q\to x} \frac{1}{\mu(Q)}\int_Q f d\mu$ does not exist.

Assume that $x = (x_1, \ldots, x_n)$ so that x_i is not a dyadic rational for all $i \leq n$. Let $\{U_m\}_{m\in\mathbb{N}}$ be a pspace $\mathcal{W}$-test succeeding on x. Let $\mathbf{T}$ be a pspace computable dyadic tree partition of $\{U_m\}_{m\in\mathbb{N}}$ given by Lemma 14. Define $f : [0,1]^n \to \mathbb{R}$ as follows. For every $Q \in \mathbf{T}$,

$$f(Q - \bigcup_{B\in Child(Q)} B) = \begin{cases} 1 & \text{if the level of } Q \text{ in } \mathbf{T} \text{ is even} \\ 0 & \text{if the level of } Q \text{ in } \mathbf{T} \text{ is odd} \end{cases}$$

It is clear that f is integrable and well defined for all points that are not in the intersection $\bigcap U_m$. We now show that f is pspace L_1-computable. Let $\{T_m^k\}_{k,m\in\mathbb{N}}$ be the uniformly pspace computable array approximating $\mathbf{T}$. For every $m \in \mathbb{N}$, define

$$\mathbf{T}_m = \bigcup_{i=1}^{m} T_i^{m+2}.$$

We can consider $\mathbf{T}_m$ as a finite subtree of $\mathbf{T}$ which well approximates $\mathbf{T}$. For every $m \in \mathbf{N}$ and every $Q \in \mathbf{T}_m$, define the set of children of Q in the approximation $\mathbf{T}_m$ by

$$C_m(Q) = Child(Q) \cap \mathbf{T}_m.$$

For every $m \in \mathbb{N}$, define $f_m : [0,1]^n \to \mathbb{R}$ as follows.

$$f_m(Q - \bigcup_{B\in C_m(Q)} B) = \begin{cases} 1 & \text{if the level of } Q \text{ in } \mathbf{T}_m \text{ is even} \\ 0 & \text{if the level of } Q \text{ in } \mathbf{T}_m \text{ is odd} \end{cases}$$

It is clear that f_m is a simple step function. Since the array $\{T_m^k\}$ approximating $\mathbf{T}$ is uniformly pspace computable, on input $(0^m, d)$ we are able to compute the level of the largest dyadic cube in $\mathbf{T}$ containing d in polynomial space. Therefore the sequence of functions $\{f_m\}$ is pspace computable.

We now prove that $\{f_m\}_{m\in\mathbb{N}}$ approximates f. For $m \in \mathbb{N}$, define the set $A = \mathbf{T} - \mathbf{T}_m$, the set of all cubes in $\mathbf{T}$ that are not in the approximation $\mathbf{T}_m$. We now bound the error of our approximation $\mathbf{T}_m$. From the definition of tree decompositions, we have

$$\begin{aligned} \mu(A) &= \mu(\mathbf{T} - \mathbf{T}_m) \\ &= \mu(\bigcup_{i=1}^{m} Level_i(\mathbf{T}) - T_i^{m+2}) + \mu(\bigcup_{i=m+1}^{\infty} Level_i(\mathbf{T})) \\ &\leq \sum_{i=1}^{m} \mu(Level_i(\mathbf{T}) - T_i^{m+2}) + \sum_{i=m+1}^{\infty} \mu(Level_i(\mathbf{T})) \\ &\leq \sum_{i=1}^{m} 2^{-(i+m+2)} + \sum_{i=m+1}^{\infty} 2^{-2i} \\ &\leq 2^{-m}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|f - f_m\|_1 &= \int_0^1 |f - f_m| \\ &= \int_A |f - f_m| \\ &\leq \mu(A) \\ &\leq 2^{-m}. \end{aligned}$$

Hence, f is a pspace L_1 computable function.

Finally, we show that the limit $\lim_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist. We first show that $\limsup_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu \geq \frac{3}{4}$. Let $N \in \mathbb{N}$. By Lemma 14, x is contained in an infinite path of $\mathbf{T}$. Choose a dyadic cube $Q \in \mathbf{T}$ containing x so that $\mu(Q) < 2^{-N}$ and the level of Q in $\mathbf{T}$ is even. Then, by our construction of f,

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q f d\mu &\geq \frac{1}{\mu(Q)} \int_{Q-Child(Q)} 1 d\mu \\ &= \frac{1}{\mu(Q)} \mu(Q - Child(Q)) \\ &\geq \frac{3}{4}. \end{aligned} \tag{10}$$

Similarly, we show that $\liminf_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu \leq \frac{1}{4}$. Let $N \in \mathbb{N}$. Choose a dyadic cube $Q \in \mathbf{T}$ containing x so that $\mu(Q) < 2^{-N}$ and the level of Q in $\mathbf{T}$ is odd. Then, by our construction of f,

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q f d\mu &\leq \frac{1}{\mu(Q)} \int_{Child(Q)} 1 d\mu \\ &= \frac{1}{\mu(Q)} \mu(Child(Q)) \\ &\leq \frac{1}{4}. \end{aligned} \tag{11}$$

Combining the equalities (10) and (11), we see that the limit $\lim_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist. ◀

Finally, by Theorems 12 and 15, the Lebesgue differentiation theorem characterizes weakly pspace randomness.

6 Conclusion and Open Problems

In the computable setting, there is a strong connection between randomness and classical theorems of analysis. However, this interaction is not as well understood in the context of resource-bounded randomness. An interesting direction is to characterize randomness for different computational resource bounds using the Lebesgue differentiation theorem. For example, what notion of polynomial time randomness is characterized by the Lebesgue differentiation theorem?

We believe the notion of weakly polynomial space randomness will be useful in further investigations into resource-bounded randomness in analysis. An interesting avenue of future research is to relate weakly pspace-randomness with other notions of polynomial space randomness. We showed that Lutz's definition of pspace-randomness implies weakly pspace randomness, but the converse is not known. We conjecture that weakly pspace randomness is strictly weaker than Lutz's notion of pspace-randomness.

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